Lesson 2

Regular Tessellations

Goal

Justify (orally and in writing) that regular triangles, squares, and hexagons are the only regular polygons that can be used to create a regular tessellation.

Learning Target

I can create tessellations with regular polygons.

Lesson Narrative

In this second lesson in the sequence of three optional lessons, students look at regular tessellations, which are made of regular polygons. Students construct arguments about which regular polygons can create regular tessellations and reason about why others cannot. This reasoning requires an understanding of the structure of interior angles of regular polygons.

Through the lesson, students show in detail that triangles, squares, and hexagons give the only possible regular tessellations. Students begin by exploring triangles, squares, pentagons, hexagons, and octagons to determine which can create a regular tessellation. Then students focus on equilateral triangles and the angle measurements that guarantee equilateral triangles will tessellate. Finally, students experiment with shapes that have more sides to determine if any of those can create regular tessellations, bringing in their knowledge of angle measures to support their claims.

Student Learning Goal

Let's make some regular tessellations.

Lesson Timeline

15 min	15 min	15 min
Activity 1	Activity 2	Activity 3

Access for Students with Diverse Abilities

- Action and Expression (Activity 3)

Access for Multilingual Learners

- MLR8: Discussion Supports (Activity 2)

Required Materials

Materials to Gather

- Tracing paper: Activity 1, Activity 2, Activity 3
- Protractors: Activity 3

Activity 1:
For the digital version of the activity, acquire devices that can run the applet.

Activity 3:
For the digital version of the activity, acquire devices that can run the applet.

Activity 1

Regular Tessellations

15 min

Activity Narrative

There is a digital version of this activity.

The goal of this activity is to introduce a *regular* tessellation of the plane and conjecture which shapes give regular tessellations. Students construct arguments for which shapes can and cannot be used to make a regular tessellation. The focus is on experimenting with shapes and noticing that in order for a shape to make a regular tessellation, we need to be able to put a whole number of those shapes together at a single vertex with no gaps and no overlaps. This greatly limits what angles the polygons can have and, as a result, there are only three regular tessellations of the plane. This conjecture will be demonstrated in the other two activities of this lesson.

In the digital version of the activity, students use an applet to decide if regular polygons create tessellations. The applet allows students to work with many copies of each polygon without tracing. The digital version may be preferable if time is limited.

Launch

Display a table for all to see with at least two columns keeping track of which regular polygons make a tessellation and which do not. Students may need a reminder that regular polygons are polygons with all congruent sides and angles. A third column could be used for extra comments (for example, about angle size of the polygon or other remarks). Here is an example of a table that could be used:

regular polygon	tessellate?	notes
octagon		
hexagon		
pentagon		
square		
triangle		

Introduce the idea of a regular tessellation:

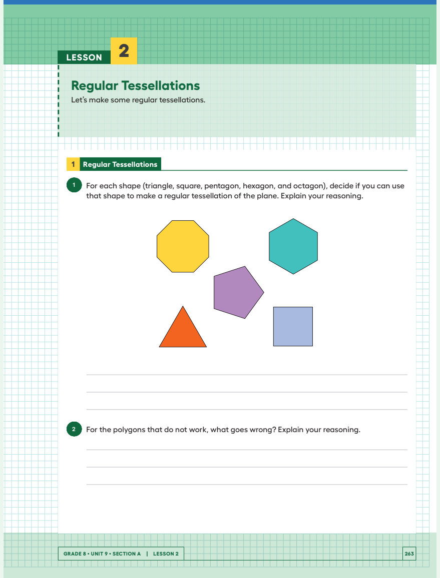
- Only one type and size of regular polygon is used.
- If polygons meet, they either share a single vertex or a single side.

Building on Student Thinking

If students are not sure how to explain their reasoning for why some of the shapes do not tessellate, consider asking:

> *"Tell me more about how you tried to tessellate the shape."*
> *"What is the same and what is different about this shape and one that does tessellate?"*

Student Workbook


LESSON 2

Regular Tessellations

Let's make some regular tessellations.

1 Regular Tessellations

1 For each shape (triangle, square, pentagon, hexagon, and octagon), decide if you can use that shape to make a regular tessellation of the plane. Explain your reasoning.

2 For the polygons that do not work, what goes wrong? Explain your reasoning.

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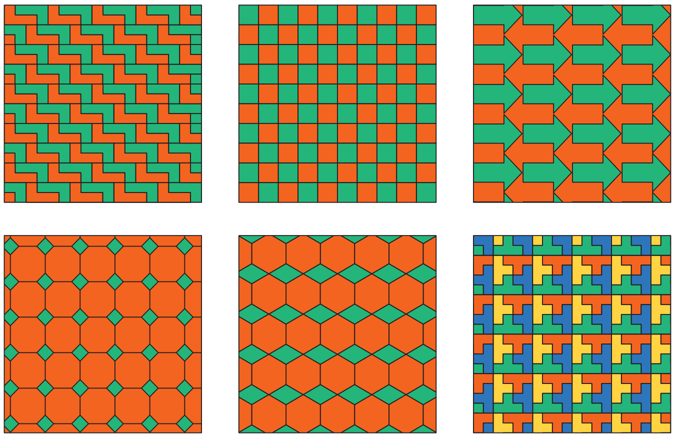
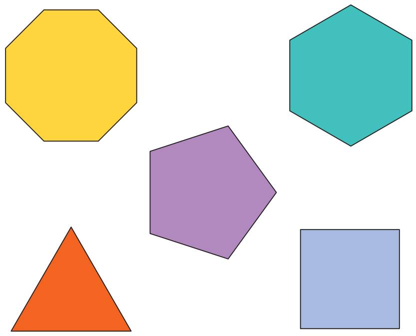
Show some pictures of tessellations that are not regular, and ask students to identify why they are not (for example, several different polygons are used, edges of polygons do not match up completely). Ask students which of the tessellations pictured here are regular tessellations (only the one with squares):



Make tracing paper available to all students. Tell students that they can use the tracing paper to put together several copies of the polygons.

Student Task Statement

1. For each shape (triangle, square, pentagon, hexagon, and octagon), decide if you can use that shape to make a regular tessellation of the plane. Explain your reasoning.



The triangles, squares, and hexagons all tessellate the plane. Three hexagons meet at each vertex, four squares meet at each vertex, and six triangles meet at each vertex.

2. For the polygons that do not work, what goes wrong? Explain your reasoning.

For the pentagons, three can fit together at a vertex. There is a little extra space left, but it's not enough to fit another pentagon. Two octagons can fit together at a vertex with space left over, but cannot tessellate the plane.

Activity Synthesis

To help students think more about what shapes do and do not tessellate and why, ask:

"Which polygons appear to tessellate the plane?"

square, equilateral triangle, hexagon

"How did you decide?"

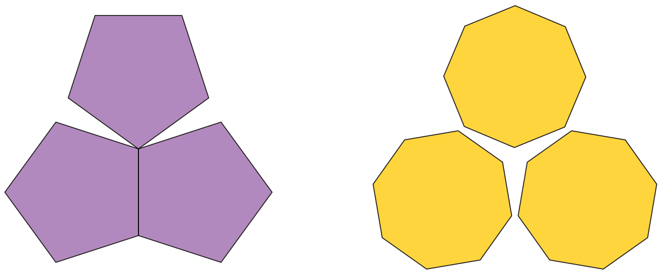
I put as many together as I could at 1 vertex and checked to see if there was extra space left over.

"Why does the pentagon not work to tessellate the plane?"

Three fit together at one vertex, but there is extra space, not enough for a fourth.

"Why does the octagon not work?"

Two fit together, but there is not enough space for a third.



During the discussion, fill out the table, indicating that it is possible to make a tessellation with equilateral triangles, squares, and hexagons, but not with pentagons or octagons.

Activity 2

Equilateral Triangle Tessellation

15 min

Activity Narrative

The goal of this activity is to verify, via angle calculations, that equilateral triangles and regular hexagons can be used to make regular tessellations of the plane. Students have encountered the equilateral triangle plane tessellations earlier in grade 8 when working on an isometric grid. In order to complete their investigation of regular tessellations of the plane, it remains to be shown that no other polygons work. This will be done in the next activity.

Students are required to reason abstractly and quantitatively in this activity. Tracing paper indicates that six equilateral triangles can be put together sharing a single vertex. Showing that this is true for abstract equilateral triangles requires careful reasoning about angle measures.

Building on Student Thinking

Students may know that an equilateral triangle has 60-degree angles but may not be able to explain why or how this connects to regular hexagons. If students are having trouble explaining their thinking, consider asking:

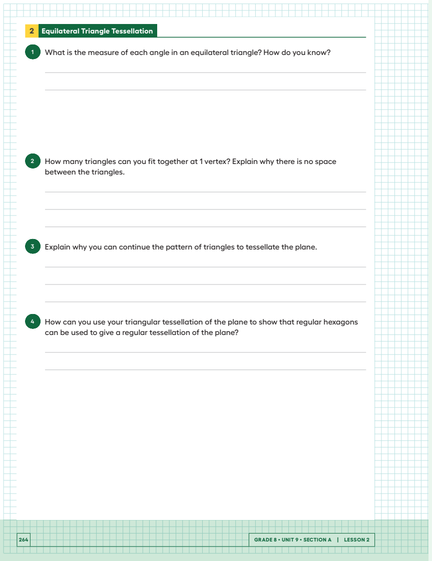
> *"What do you know about the angles in a triangle?"*
> *"How does the angle measure of an equilateral triangle connect to how many you can fit together at 1 vertex?"*

Access for Multilingual Learners (Activity 2, Synthesis)

MLR8: Discussion Supports
For each response to a discussion question that is shared, invite students to turn to a partner and restate what they heard using precise mathematical language.

Advances: Listening, Speaking

Student Workbook


2 Equilateral Triangle Tessellation

1 What is the measure of each angle in an equilateral triangle? How do you know?

2 How many triangles can you fit together at 1 vertex? Explain why there is no space between the triangles.

3 Explain why you can continue the pattern of triangles to tessellate the plane.

4 How can you use your triangular tessellation of the plane to show that regular hexagons can be used to give a regular tessellation of the plane?

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Launch

In the previous task, equilateral triangles, squares, and hexagons appeared to make regular tessellations of the plane. Tell students that the goal of this activity is to use geometry to verify that they do.

Refer students to regular polygons printed in the previous activity for a visual representation of an equilateral triangle.

Student Task Statement

1. What is the measure of each angle in an equilateral triangle? How do you know?

   60 degrees
   The sum of the angles in any triangle is 180 degrees, and they are all congruent, so each must be a 60-degree angle.

2. How many triangles can you fit together at 1 vertex? Explain why there is no space between the triangles.

   6, because $6 \cdot 60 = 360$

3. Explain why you can continue the pattern of triangles to tessellate the plane.

   Each place where 2 or more triangles meet in the pattern, the rest of the 6 triangles at that vertex can be filled out.

4. How can you use your triangular tessellation of the plane to show that regular hexagons can be used to give a regular tessellation of the plane?

   Each set of 6 triangles meeting in a single vertex makes a regular hexagon. These hexagons tile the plane.

Activity Synthesis

Consider asking the following questions to lead the discussion of this activity:

- *"How did you find the angle measures in an equilateral triangle?"*

  The sum of the angles is 180 degrees, and they are all congruent so each is 60 degrees.

- *"Why is there no space between 6 triangles meeting at a vertex?"*

  The angles total 360 degrees, which is a full circle.

- *"How does your tessellation with triangles relate to hexagons?"*

  The triangles meeting at certain vertices can be grouped into hexagons, which tessellate the plane.

- *"Are there other tessellations of the plane with triangles?"*

  Yes, infinite rows of triangles can be placed on top of one another—and displaced relative to one another.

Consider showing students an isometric grid, used earlier in grade 8 for experimenting with transformations, and ask them how this relates to tessellations.

It shows a tessellation with equilateral triangles.

Point out that this activity provides a mathematical justification for the "yes" in the table for triangles and hexagons.

Activity 3

Regular Tessellation for Other Polygons

15 min

Activity Narrative

There is a digital version of this activity.

The goal of this activity is to show that only triangles, squares, and hexagons give regular tessellations of the plane. The method used is experimentation with other regular polygons. The key observation is that the angles on regular polygons get larger as we add more sides, which is a good example of observing structure. Since three is the smallest number of polygons that can meet at a vertex in a regular tessellation, this means that once we pass six sides (hexagons), we will not find any further regular tessellations. The activities in this lesson now show that there are three and only three regular tessellations of the plane: triangles, squares, and hexagons.

In the digital version of the activity, students use an applet to decide if other regular polygons create tessellations. The applet allows students to work with many copies of each polygon without tracing. The digital version may be preferable if time is limited.

Launch

Ask students,

"Are there some other regular polygons, other than equilateral triangles, squares, and hexagons, that can be used to give regular tessellations of the plane?"

Some students may suggest regular polygons with more sides than the ones they have seen already, others may think that there are no other possibilities. Tell students that for this activity, they are going to investigate polygons with 7, 9, 10, 11, and 12 sides to see if they do or do not tessellate and why.

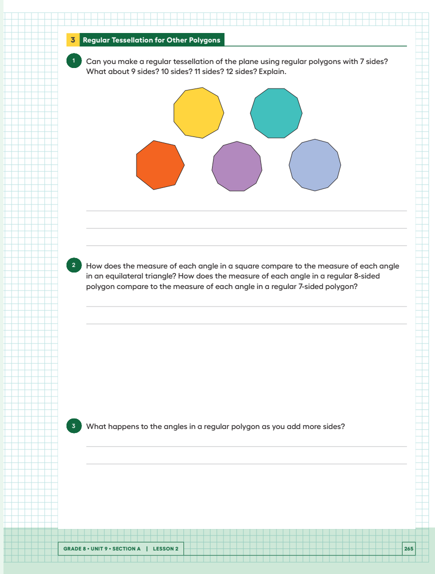
Provide access to tracing paper and protractors and tell students that they can use these to explore their conjectures.

Access for Students with Diverse Abilities (Activity 3, Student Task)

Action and Expression: Internalize Executive Functions.
To support development of organizational skills in problem-solving, chunk this task into more manageable parts. For example, present one question at a time and monitor students to ensure they are making progress throughout the activity.

Supports accessibility for: Organization, Attention

Student Workbook


3 Regular Tessellation for Other Polygons

1 Can you make a regular tessellation of the plane using regular polygons with 7 sides? What about 9 sides? 10 sides? 11 sides? 12 sides? Explain.

2 How does the measure of each angle in a square compare to the measure of each angle in an equilateral triangle? How does the measure of each angle in a regular 8-sided polygon compare to the measure of each angle in a regular 7-sided polygon?

3 What happens to the angles in a regular polygon as you add more sides?

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Student Workbook


4 Which polygons can be used to make regular tessellations of the plane?

Learning Targets

+ I can create tessellations with regular polygons.

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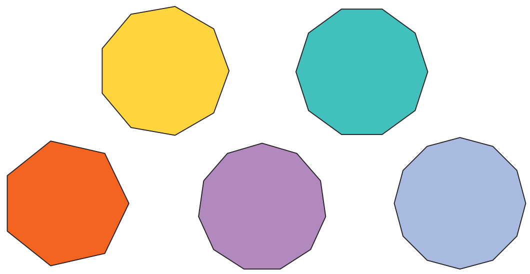
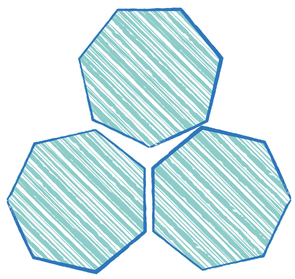
Student Task Statement

1. Can you make a regular tessellation of the plane using regular polygons with 7 sides? What about 9 sides? 10 sides? 11 sides? 12 sides? Explain.



None of the polygons will tessellate the plane. For each polygon, 3 cannot be brought together at 1 vertex. Here is the image of trying to do so with the regular polygon with 7 sides.



2. How does the measure of each angle in a square compare to the measure of each angle in an equilateral triangle? How does the measure of each angle in a regular 8-sided polygon compare to the measure of each angle in a regular 7-sided polygon?

   The angles in a square are 90 degrees, greater than the 60-degree angles in a triangle. The angles in a regular 8-sided polygon have greater measure than the angles in a regular 7-sided polygon.

3. What happens to the angles in a regular polygon as you add more sides?

   The angles increase in measure. Imagine opening up a polygon with 6 sides to add a 7th equal side. In order to fit the new side, all of the other sides must be spread out or opened up, increasing the measure of the angles.

4. Which polygons can be used to make regular tessellations of the plane?

   Only the triangle, square, and hexagon. As more sides are added, the angle measures get larger. 120 degrees is the biggest divisor of 360 that can be the measure of an interior angle of a regular polygon.

Activity Synthesis

Consider asking the following questions:

- *"How many triangles meet at each vertex in a regular tessellation with triangles?"*

  6

- *"What about squares?"*

  4

- *"Hexagons?"*

  3

- *"Why can't there be any regular tessellations with polygons of more than 6 sides?"*

  Only 2 could meet at a vertex, but this isn't possible since the angles have to add up to 360 degrees.

There are only 3 regular tessellations of the plane. Ask students if they have encountered these tessellations before and if so, where. For example:

- Triangles

  isometric grid

- Squares

  checkerboard, coordinate grid, floor and ceiling tiles

- Hexagons

  beehives, tiles