Lesson 7

Measurement Error

Goals

- Comprehend that percent error is greater when the actual values are smaller than the measured values for an object.
- Determine the largest possible percent error of an estimated measurement.
- Generalize (orally) a process for calculating the maximum percent error of measurements that are added or multiplied together.

Lesson Narrative

The activities in this optional lesson all address the concept of measurement error. The activities build on each other and increase in difficulty, but each activity can stand on its own and not all activities must be completed for students to benefit from the work. As students move through the activities, they build from error in a single measurement to adding measurements to two- and three-dimensional measurements. Students look for patterns and use repeated reasoning as they complete similar mathematical work in increasingly challenging problems. As they make decisions about percent error, students attend to precision by evaluating the degree of certainty to use when reporting their answer. This lesson relies on skills developed in Unit 4 and Unit 6.

Student Learning Goal

Let's check how accurate our calculations are.

Lesson Timeline

Activity 1	Activity 2	Activity 3	Activity 4
15 min	15 min	15 min	25 min

Access for Students with Diverse Abilities

- Representation (Activity 2)

Access for Multilingual Learners

- MLR3: Critique, Correct, Clarify (Activity 3)
- MLR8: Discussion Supports (Activity 1)

Instructional Routines

- MLR3: Critique, Correct, Clarify

Required Materials

Materials to Gather

- Four-function calculators: Activity 1, Activity 2, Activity 3, Activity 4

Activity 1

How Long Are These Pencils?

15 min

Activity Narrative

In this activity, students measure approximate lengths and decide possibilities for actual lengths. There are two layers of attending to precision involved in this task:

- Deciding how accurately the pencils can be measured, probably to the nearest mm or to the nearest 2 mm, but this depends on the eyesight and confidence of the student
- Finding the possible percent error in the measurement chosen

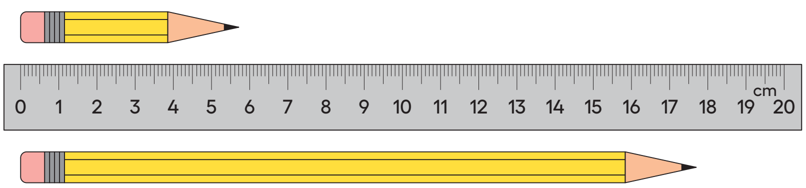
Launch

Arrange students in groups of 2.

Provide access to calculators.

Give students 4–5 minutes of quiet work time, followed by partner and whole-class discussion.

Student Task Statement



1. Estimate the length of each pencil.

   Sample response: The shorter pencil appears to be between 5.3 and 5.5 cm, perhaps 5.4 cm to the nearest mm, while the longer pencil appears to be 17.7 cm to the nearest mm (it is between 17.6 and 17.8 cm).

2. How accurate are your estimates?

   Sample response: The estimate is accurate to within 1 mm. For the short pencil, it is more than 5.3 cm and less than 5.5 cm, but it is not possible to tell which is closer. Similarly, the longer pencil is more than 17.6 cm and less than 17.8 cm.

3. For each estimate, what is the largest possible percent error?

   For the shorter pencil, taking 5.4 cm as the measured length, the actual length x is at least 5.3 cm and at most 5.5 cm. The percent error if x is as small as possible is $\frac{0.1}{5.3} \approx 2\%$, and if x is as big as possible, then the error is $\frac{0.1}{5.5} \approx 2\%$. The first of these gives the greatest percent error, although they are close. For the longer pencil, the percent error is smaller. The biggest it can be is $\frac{0.1}{17.7} \approx 0.6\%$. This makes sense because 0.1 cm is a bigger percentage of the length of the small pencil.

Access for Multilingual Learners (Activity 1, Student Task)

MLR8: Discussion Supports
Display sentence frames to support students as they respond to "How close to the actual lengths are your estimates?": "My estimate is within ____ mm of the actual length because …" "I agree because… " and "I disagree because …"
Advances: Speaking, Conversing

Building on Student Thinking

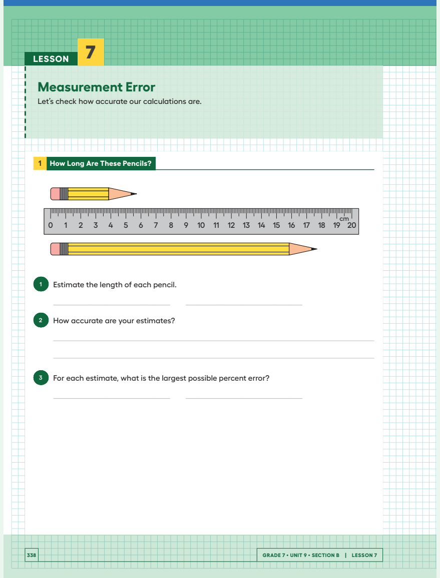
If students try to find an exact value for the length of each pencil, consider asking:

"How did you decide the length of the pencil?"
"Assuming you measured the pencil accurately to the nearest millimeter, what is the longest the actual length of the pencil could be? What is the shortest it could be?"

If students do not remember how to calculate percent error, consider asking:

"How did you estimate the length of the pencil?"
"What is the biggest difference possible between the estimated and actual lengths? What percentage of the actual length would that be?"

Student Workbook



Activity Synthesis

The goal of this discussion is for students to practice how they talk about precision.

Discussion questions include:

- *"How did you decide how accurately you can measure the pencils?"*

  I looked for a value that I was certain was less than the length of the pencil and a value that I was certain was bigger. My estimate was halfway in between.

- *"Were you sure which mm measurement the length is closest to?"*

Sample responses:

  Yes, I could tell that the short pencil is closest to 5.4 cm. No, the long pencil looks to be closest to 17.7 mm, but I'm not sure. I am sure it is between 17.6 cm and 17.8 cm.

- *"Were the percent errors the same for the small pencil and for the long pencil? Why or why not?"*

  No. I was able to measure each pencil to within 1 mm. This is a smaller percentage of the longer pencil length than it is of the smaller pencil length.

Other possible topics of conversation include noting that the level of accuracy of a measurement depends on the measuring device. If the ruler were marked in sixteenths of an inch, one would only be able to measure to the nearest sixteenth of an inch. If it were only marked in centimeters, one would only be able to measure to the nearest centimeter.

Activity 2
How Long Are These Floor Boards?

15 min

Activity Narrative

This activity examines how measurement errors behave when they are added together. In other words, if there is a measurement m with a maximum error of 1% and a measurement n with a maximum error of 1%, what percent error can $m + n$ have? In addition to examining accuracy of measurements carefully, students work through examples and look for patterns in order to hypothesize, and eventually show, how percent error behaves when measurements with error are added to one another.

Monitor for students who look for patterns, recognize the usefulness of the distributive property, or formulate the problem abstractly with variables.

Launch

Read the problem out loud and ask students what information they would need to know to be able to solve the problem. Students may say that they need to know what length the boards are supposed to be, because it is likely that they haven't realized that they can solve the problem without this information. Explain that floor boards come in many possible lengths: 18-inch and 36-inch lengths are both common, but the boards can be anywhere between 12 and 84 inches. Ask students to pick values for two actual lengths and figure out the error in that case. Then they can pick two different examples, make the calculations again, and look for patterns.
Provide access to calculators.

Building on Student Thinking

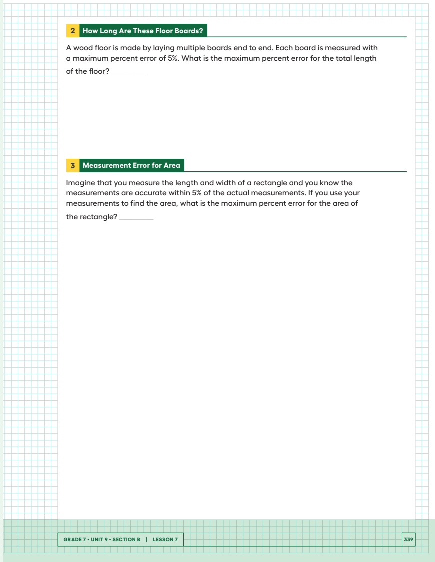
If students pick example lengths but struggle with what to do with them, consider asking:

> *"What would be the maximum and minimum measured lengths?"*
> *"What would the percent error be if both measurements were maximum? What if they were both minimum?"*

Access for Students with Diverse Abilities (Activity 2, Synthesis)

Representation: Develop Language and Symbols.
Invite students to explain their thinking orally, using a diagram to illustrate the situation.
Supports accessibility for: Language, Fine Motor Skills

Student Workbook


2 How Long Are These Floor Boards?

A wood floor is made by laying multiple boards end to end. Each board is measured with a maximum percent error of 5%. What is the maximum percent error for the total length of the floor? ________

3 Measurement Error for Area

Imagine that you measure the length and width of a rectangle and you know the measurements are accurate within 5% of the actual measurements. If you use your measurements to find the area, what is the maximum percent error for the area of the rectangle? ________

Student Task Statement

A wood floor is made by laying multiple boards end to end. Each board is measured with a maximum percent error of 5%. What is the maximum percent error for the total length of the floor?

The maximum percent error is 5%. Sample explanation: If x is the actual length and m is the measured length of one board, then $0.95x < m$ and $m < 1.05x$: I know this because the measurement m has a maximum error of 5%. If is the actual length and n is the measured length of a second board, then $0.95y < n$ and $n < 1.05y$. If both boards have maximum length, the total length would be $1.05x + 1.05y = 1.05(x + y)$. If they are both minimum, the total length would be $0.95x + 0.95y = 0.95(x + y)$. So the maximum percent error would be 5%. This same argument works for any number of boards because the distributive property works for any number of addends.

Activity Synthesis

The goal of this discussion is for students to generalize from their specific examples of measurements to understand the general pattern and express it algebraically.

Poll the class on the measurements they tried and the maximum percent error they calculated. Invite students to share any patterns they noticed, especially students who recognized the usefulness of the distributive property for making sense of the general pattern.

Guide students to use variables to talk about the patterns more generally.

- If a board is supposed to have length x with a maximum percent error of 5%, then the shortest it could be is $0.95x$ and the longest it could be is $1.05x$.
- If another board is supposed to have length y, it could be between $0.95y$ and $1.05y$.
- When the boards are laid end-to-end, the shortest the total length could be is $0.95x + 0.95y$, which is equivalent to $0.95(x + y)$.
- The longest the total length could be is $1.05x + 1.05y$, or $1.05(x + y)$.
- Because of the distributive property, it is seen that the maximum percent error is still 5% after the board lengths are added together.

One interesting point to make, if students have also done the previous activity about measuring pencils, is that the sum of the board lengths could be measured with a lower percent error than each individual board (assuming the tape measure is long enough), just like an error of 1 mm was a smaller percentage of the length of the longer pencil.

Activity 3

Measurement Error for Area

15 min

Activity Narrative

This activity examines how measurement errors behave when quantities are multiplied. In other words, if there is a measurement m with a maximum error of 5% and a measurement n with a maximum error of 5%, what percent error can $m \cdot n$ have? Students reason abstractly and quantitatively when they decide how to represent this situation and interpret the meaning of their solution.

Monitor for students who use different methods to solve the problem, such as trying out sample numbers or using expressions with variables.

In this activity, students critique a statement or response that is intentionally unclear, incorrect, or incomplete and improve it by clarifying meaning, correcting errors, and adding details.

Launch

Arrange students in groups of 2.

Provide access to calculators.

If desired, suggest that students try out several different sample numbers for the length and width of the rectangle, calculate the maximum percent error, and look for a pattern.

Give students 4–5 minutes of quiet work time, followed by time to discuss their work with their partner, followed by whole-class discussion.

Student Task Statement

Imagine that you measure the length and width of a rectangle and you know the measurements are accurate within 5% of the actual measurements. If you use your measurements to find the area, what is the maximum percent error for the area of the rectangle?

The maximum percent error would be 10.25%. If x is the actual length and m is the measured length, then $0.95x < m$ and $m < 1.05x$ since $0.05x$ is 5% of x. If y is the actual width and n is the measured width, then the biggest possible error is $0.05y$, so $0.95y < n$ and $n < 1.05y$. If they are both maximum, the area would be $1.05^2 xy = 1.1025xy$. If they are both minimum, the area would be $0.95^2 xy = 0.9025xy$. So the maximum percent error would be when they are both at the maximum possible error, and the percent error would be 10.25%.

Instructional Routines

MLR3: Critique, Correct, Clarify

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Access for Multilingual Learners (Activity 3)

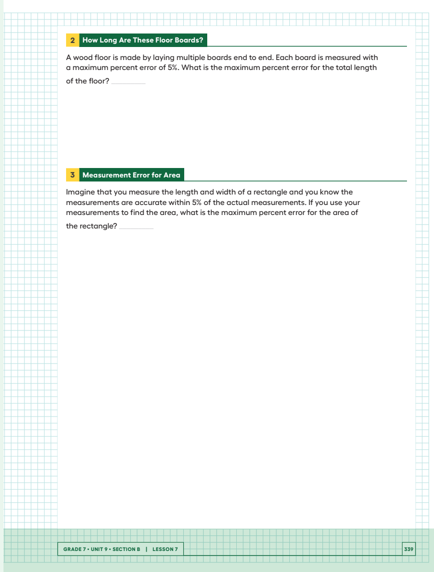
MLR3: Critique, Correct, Clarify
This activity uses the *Critique, Correct, Clarify* math language routine to advance representing and conversing as students critique and revise mathematical arguments.

Building on Student Thinking

If students think the maximum error for the area is 5% because both the length and width are within 5% of the actual values, consider asking:

"Can you explain how you calculated your answer?"
"What is different about length and area?"

Student Workbook


2 How Long Are These Floor Boards?

A wood floor is made by laying multiple boards end to end. Each board is measured with a maximum percent error of 5%. What is the maximum percent error for the total length of the floor? ________

3 Measurement Error for Area

Imagine that you measure the length and width of a rectangle and you know the measurements are accurate within 5% of the actual measurements. If you use your measurements to find the area, what is the maximum percent error for the area of the rectangle? ________

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Activity Synthesis

Use Critique, Correct, Clarify to give students an opportunity to improve a sample written response by correcting errors, clarifying meaning, and adding details.

- Display this first draft:

"Both the length and width have maximum errors of 5%. I know that area is length times width, so I multiplied 5% by 5% to get 25%."

Ask,

"What parts of this response are unclear, incorrect, or incomplete?"

As students respond, annotate the display with 2–3 ideas to indicate the parts of the writing that could use improvement.

Give students 2–4 minutes to work with a partner to revise the first draft.

- Select 1–2 individuals to read their revised draft aloud slowly enough to record for all to see. Scribe as each student shares, then invite the whole class to contribute additional language and edits to make the final draft even more clear and more convincing.

Activity 4: Optional

Measurement Error for Volume

25 min

Activity Narrative

This challenging activity examines how measurement errors behave when 3 quantities are multiplied (versus 2 quantities in the previous activity). In other words, if I have measurements a, b, and c each with a maximum error of 5%, what percent error can $a \cdot b \cdot c$ have? The arithmetic and algebraic demands of this task are high because students take a product of three quantities that each have a maximum percent error of 5%. Students repeat their process for each measurement and use this repetition to build their understanding of measurement errors.

Launch

Arrange students in groups of 2.

Provide access to calculators. Make sure students realize that the first question gives the measured values, not the actual values, for each dimension.

Give students 10 minutes to discuss with their partners, followed by whole-class discussion.

Student Task Statement

1. The length, width, and height of a rectangular prism were measured to be 10 cm, 12 cm, and 25 cm.

   The actual measurements could be as much as 0.5 cm over or under the measurements given.

   Assuming that these measurements are accurate to the nearest cm, what is the largest percent error possible for:

   a. each of the dimensions?

   The largest percent error occurs with the *smallest* measurement, so it will only be checked for the minimum possible actual lengths.

   - $0.5 \div 9.5 \approx 0.05$ (or about 5%)
   - $0.5 \div 11.5 \approx 0.04$ (or about 4%)
   - $0.5 \div 24.5 \approx 0.02$ (or about 2%)

   b. the volume of the prism?

   The measured volume is 3,000 cubic cm. The smallest the actual volume could be is $9.5 \cdot 11.5 \cdot 24.5 = 2{,}676.625$. The percent error in the case of the smallest one is $(3{,}000 - 2{,}676.625) \div 2{,}676.625$, or about 12%. The biggest the actual volume could be is $10.5 \cdot 12.5 \cdot 25.5 = 3{,}346.875$, which gives a percent error of about 10%. Again, it is found that the biggest possible error happens when the actual measurement is as small as possible.

2. For a different rectangular prism, the length, width, and height each have a maximum percent error of 1%. What is the largest percent error possible for the volume of the prism?

   If the actual dimensions are x, y, and z, then the minimum measured volume would be $(0.99)^3\, xyz \approx 0.97xyz$ and the maximum measured volume would be $(1.01)^3\, xyz \approx 1.03xyz$. So the largest percent error in the volume is about 3%.

Activity Synthesis

Some discussion points include:

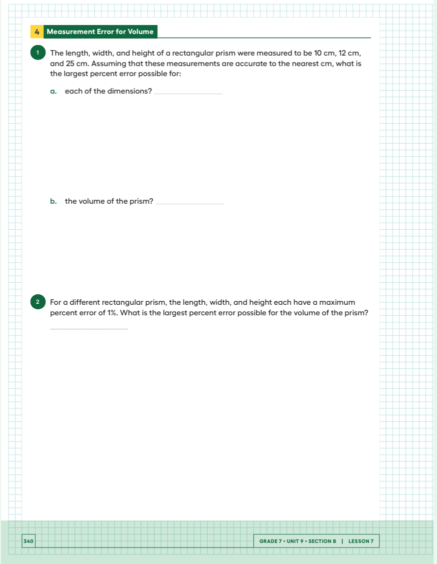
- The first problem gives measurements and errors (but no percent error), while the second problem gives no measurements but does give the percent error. This makes the calculations notably different for the two problems.
- In the first problem, measurements and the possible size of error are given. The task is to find the greatest percent error and, as seen in other cases, this happens for the smallest possible value of the measurement. If one were asked to find the percent error of each measurement, they would find that the error for the volume is a larger percent error than for any of the individual measurements.
- In the second problem, the maximum possible percent error is given, but no measurements, and the task is to find the largest possible error for the volume, that is, for the product of the three unknown measurements. The greatest percent error possible for the volume occurs when the measured value is as large as possible.
- A unifying feature in these two problems is that the largest percent errors occur when the actual measurements are smaller than the measured values.

Building on Student Thinking

If students think the maximum error possible for the volume is 1% because the measurements are all within 1% of the actual values, consider asking:

"Tell me more about your reasoning."
"What is the same and what is different about lengths and volume?"
"How could calculating the least and greatest possible values for length, width, and height help you figure out the least and greatest volumes?"

Student Workbook


4 Measurement Error for Volume

1 The length, width, and height of a rectangular prism were measured to be 10 cm, 12 cm, and 25 cm. Assuming that these measurements are accurate to the nearest cm, what is the largest percent error possible for:

a. each of the dimensions?

b. the volume of the prism?

2 For a different rectangular prism, the length, width, and height each have a maximum percent error of 1%. What is the largest percent error possible for the volume of the prism?