Lesson 15

The Volume of a Cone

Goals

- Calculate the volume of a cone and cylinder given the height and radius, and explain (orally) the solution method.
- Compare the volumes of a cone and a cylinder with the same base and height, and explain (orally and in writing) the relationship between the volumes.

Learning Targets

- I can find the volume of a cone in mathematical and real-world situations.
- I know the formula for the volume of a cone.

Lesson Narrative

In this lesson students start working with cones and learn that the volume of a cone is $\frac{1}{3}$ the volume of a cylinder with a congruent base and the same height. First, students learn a method for quickly sketching a cone and the meaning of the *radius and height* of a cone. Then they watch a video (or if possible, a live demonstration) showing that it takes three cones of water to fill a cylinder with the same radius and height. At this point, it is taken as a mysterious and beautiful fact that the volume of a cone is one third the volume of the associated cylinder. A proof of this fact requires mathematics beyond grade level.

Next, students write the volume of a cone given a specific volume of a cylinder with the same base and height, and vice versa. Then they use the formula for the volume of a cylinder learned in previous lessons to write the general formula $V = \frac{1}{3}\pi r^2 h$ for the volume, V, of a cone in terms of its height, h, and radius, r.

In the last activity, students practice computing the volumes of some cones. They also practice critiquing a response about calculating volume to clarify its meaning, correct an error, and adding more detail.

Student Learning Goal

Let's explore cones and their volumes.

Access for Students with Diverse Abilities

- Engagement (Activity 2)

Access for Multilingual Learners

- MLR3: Critique, Correct, Clarify (Activity 2)

Instructional Routines

- MLR3: Critique, Correct, Clarify
- Notice and Wonder

Required Preparation

Warm-up:
If possible, provide students access to geometric solids with the same base and height measurements.

Activity 1:
During the *Launch*, prepare to show a video. Alternatively, do a demonstration with a cone that could be filled with water and poured into a cylinder with matching base and height dimensions.

Lesson Timeline

Warm-up	Activity 1	Activity 2	Lesson Synthesis	Assessment: Cool-down
5 min	15 min	15 min	10 min	5 min

Warm-up

Which Has a Larger Volume?

5 min

Activity Narrative

The purpose of this activity is to get students thinking about how the volume of a cone might relate to the volume of a cylinder with the same base and height. Additionally, students learn one method for sketching a cone.

Launch

If you have access to appropriate geometric solids that include a cylinder and a cone with congruent bases and equal heights, consider showing these to students or passing them around for students to hold if time permits.

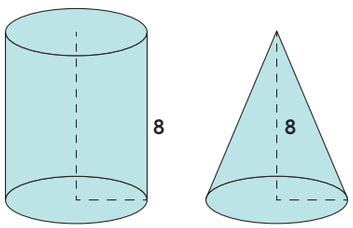
Arrange students in groups of 2.

Give students 2–3 minutes of quiet work time followed by time to discuss fractional amount with partner.

Follow with a whole-class discussion.

Student Task Statement

The cone and cylinder have the same height, and the radii of their bases are equal.



1. Which figure has a larger volume?

   The cylinder has a larger volume.

2. Do you think the volume of the cone is more or less than $\frac{1}{2}$ the volume of the cylinder? Explain your reasoning.

   Sample response:

   I don't think the cone is more than half the volume because I don't think I would fit two cones inside the cylinder.

3. Sketch two different sized cones. The oval doesn't have to be on the bottom! For each drawing, label the cone's radius with r and height with h.

   Answers vary.

Inspire Math

Drums video

Go Online

Before the lesson, show this video to reinforce the real-world connection.

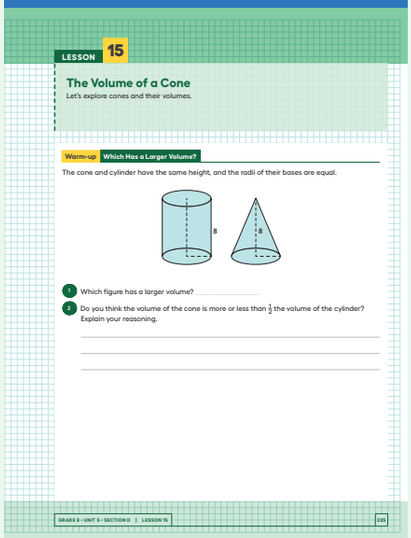
ilclass.com/l/614212

Please log in to the site before using the QR code or URL.

Building on Student Thinking

If students think the two shapes will have the same volume, ask them to imagine dropping the cone into the cylinder and having extra space around the cone and still inside the cylinder.

Student Workbook


LESSON 15

The Volume of a Cone

Let's explore cones and their volumes.

Warm-up Which Has a Larger Volume?

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8 8

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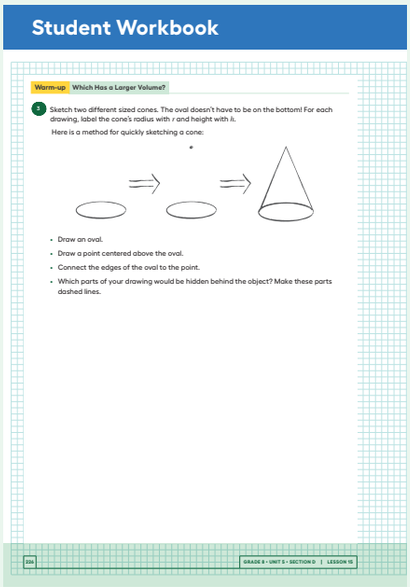


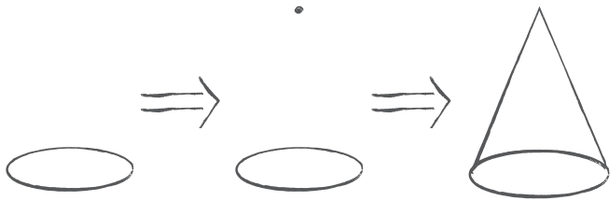
Warm-up Which Has a Larger Volume?

1. Sketch two different sized cones. The oval doesn't have to be on the bottom! For each drawing, label the cone's radius with r and height with h.

Here is a method for quickly sketching a cone:

- Draw an oval.
- Draw a point centered above the oval.
- Connect the edges of the oval to the point.
- Which parts of your drawing would be hidden behind the object? Make these parts dashed lines.

Here is a method for quickly sketching a cone:

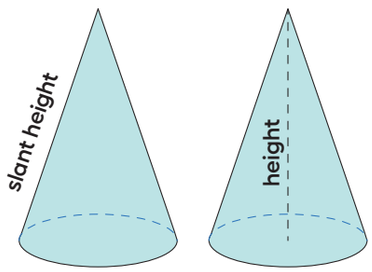


- Draw an oval.
- Draw a point centered above the oval.
- Connect the edges of the oval to the point.
- Which parts of your drawing would be hidden behind the object? Make these parts dashed lines.

Activity Synthesis

Invite students to share their answers to the first two questions. The next activity includes a video that shows that it takes 3 cones to fill a cylinder that has the same base and height as the cone, so it is not necessary that students come to an agreement about the second question, just solicit students' best guesses, and tell them that we will find out the actual fractional amount in the next activity.

If time allows, end the discussion by selecting 2–3 students to share their sketches. Otherwise, display these for all to see and compare the different heights and radii. If no student draws a perpendicular height or slant height, use these images to remind students that height creates a right angle with something in the figure. In the case of the cones, the height is perpendicular to the circular base.



Activity 1
From Cylinders to Cones

15 min

Activity Narrative

The purpose of this activity is for students to learn the relationship that the volume of a cone is $\frac{1}{3}$ of the volume of a cylinder and use it to calculate the volume of various cones. Students start by watching a video (or demonstration) that shows that it takes the contents of 3 cones to fill the cylinder when they have congruent bases and equal heights. Next, they practice using this information starting from either a known cylinder volume or a known cone volume before generalizing the relationship between any cone and cylinder pair with matching dimensions.

Launch

Video 'How Many Cones Does it Take to Fill a Cylinder with the Same Base and Height?' available here: **player.vimeo.com/video/309581286**.

Tell students to close their student workbooks or devices (or to keep them closed). Display the video, or conduct a demonstration for all to see. Give students 1 minute of quiet think time, and ask them to be prepared to share at least one thing they notice and one thing they wonder. Record and display responses without editing or commentary for all to see.

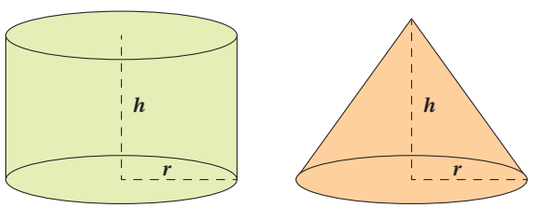
If the fact that it takes the contents of 3 cones to fill the cylinder or that the volume of the cone is $\frac{1}{3}$ the volume of the cylinder does not come up during the conversation, ask students to discuss this idea.

Tell students to open their student workbooks or devices and begin work on the questions. As they work, select students who do either of the following for the last question to share later:

- Write the equation as $\frac{1}{3}V$ (or $V \div 3$), where V represents the volume of a cylinder with the same base and height as the cone.
- Write the equation in terms of r and h ($V = \frac{1}{3}\pi r^2 h$).

Student Task Statement

A cone and cylinder have the same height, and their bases are congruent circles.



1. If the volume of the cylinder is 90 cm^3, what is the volume of the cone?

30 cm^3

The volume of a cone is $\frac{1}{3}$ the volume of a cylinder, and $\frac{1}{3} \cdot 90$.

Instructional Routines

Notice and Wonder

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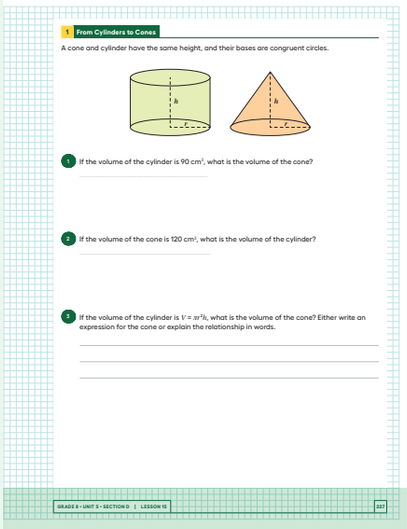
Student Workbook


1 From Cylinders to Cones

A cone and cylinder have the same height, and their bases are congruent circles.

1. If the volume of the cylinder is 90 cm^3, what is the volume of the cone?
2. If the volume of the cone is 120 cm^3, what is the volume of the cylinder?
3. If the volume of the cylinder is $V = \pi r^2 h$, what is the volume of the cone? Either write an expression for the cone or explain the relationship in words.

Instructional Routines

MLR3: Critique, Correct, Clarify

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Access for Multilingual Learners (Activity 2)

MLR3: Critique, Correct, Clarify
This activity uses the *Critique, Correct, Clarify* math language routine to advance representing and conversing as students critique and revise mathematical arguments.

2. If the volume of the cone is 120 cm^3, what is the volume of the cylinder?

360 cm^3

The volume of the cylinder is 3 times larger than the volume of a cone, and $3 \cdot 120$.

3. If the volume of the cylinder is $V = \pi r^2 h$, what is the volume of the cone? Either write an expression for the cone or explain the relationship in words.

Sample responses: $V = \frac{1}{3}\pi r^2 h$ (or equivalent) or "the volume of the cone is $\frac{1}{3}$ that of the volume of the cylinder."

Activity Synthesis

The purpose of this discussion is to make sure that students understand the relationship between the volume of a cone and the volume of a cylinder when their base area and heights are the same.

Invite previously selected students to share the volume equation they wrote for the last question. Display examples for all to see, and ask,

"Are these equations the same? How can you know for sure?"

The calculated volume is the same when both equations are used.

If no student suggests it, connect $\frac{1}{3}V$, where V represents the volume of a cylinder with the same base and height as the cone, to the volume of the cone, $\frac{1}{3}\pi r^2 h$. Reinforce that these are equivalent expressions.

Add the formula $V = \frac{1}{3}\pi r^2 h$ and a diagram of a cone to your classroom displays of the formulas being developed in this unit.

Activity 2

Calculate That Cone

15 min

Activity Narrative

The purpose of this activity is for students to calculate the volume of cones given their height and radius. Students are given a cylinder with the same height and radius and use the volume relationship they learned earlier to calculate the volume of the cone. They then use the newly learned formula to calculate the volume of a cone given a height and radius. For the last problem, an image is not provided so that students have the opportunity to sketch one if they need it.

In this activity, students critique a statement or response that is intentionally unclear, incorrect, or incomplete and improve it by clarifying meaning, correcting errors, and adding details.

Launch

Give students 2–3 minutes to complete the first question.

Use *Critique, Correct, Clarify* to give students an opportunity to improve a sample written response to the first question by correcting errors, clarifying meaning, and adding details.

Display this first draft:

💬 *"The volume is 4000π because $\pi \cdot 4 \cdot 10^2$."*

Ask,

💬 *"What parts of this response are unclear, incorrect, or incomplete?"*

As students respond, annotate the display with 2–3 ideas to indicate the parts of the writing that could use improvement.

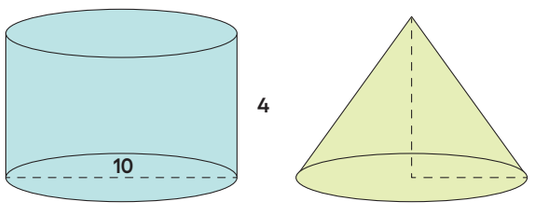
Give students 2–4 minutes to work with a partner to revise the first draft.

Select 1–2 students or groups to slowly read aloud their draft. Record for all to see as each draft is shared. Then invite the whole class to contribute additional language and edits to make the final draft even more clear and more convincing.

Student Task Statement

1. Here is a cylinder and cone that have the same height and the same base area.

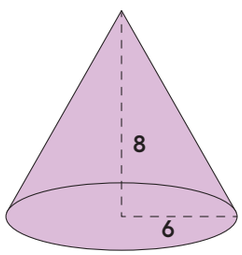
What is the volume of each figure? Express your answers in terms of π.



Cylinder: 100π; cone: $\frac{100}{3}\pi$

To calculate the volume of the cylinder, find the area of the base and multiply it by the height of the cylinder. The area of the base is πr^2, and the height is 4, giving $\pi 5^2 4 = 100\pi$. The volume of the cone is $\frac{1}{3}$ of the cylinder's volume, which is $\frac{100}{3}\pi$.

2. Here is a cone.



a. What is the area of the base? Express your answer in terms of π.

36π square units because $A = \pi 6^2$

b. What is the volume of the cone? Express your answer in terms of π.

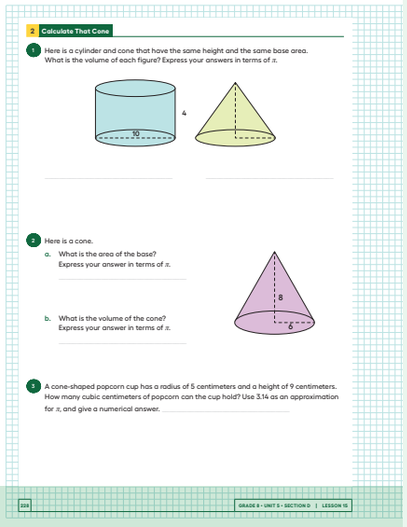
96π cubic units because the volume is $\frac{1}{3}$ of the area of the base multiplied by the height of the cone ($\frac{1}{3} \cdot 36\pi \cdot 8 = 96\pi$)

Building on Student Thinking

If students use 10 as the radius in the first problem, consider asking:

"Can you explain how you figured out the volume for the cylinder and cone?"
"Where do you see the height and radius in the picture of the cone?"

Student Workbook


2 Calculate That Cone

1. Here is a cylinder and cone that have the same height and the same base area. What is the volume of each figure? Express your answers in terms of π.

10 4

2. Here is a cone.
   a. What is the area of the base? Express your answer in terms of π.
   b. What is the volume of the cone? Express your answer in terms of π.

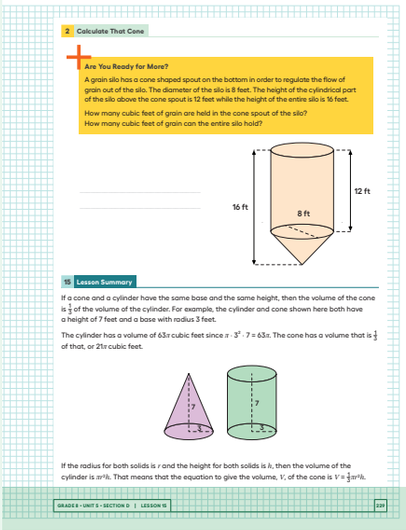
8 6

3. A cone-shaped popcorn cup has a radius of 5 centimeters and a height of 9 centimeters. How many cubic centimeters of popcorn can the cup hold? Use 3.14 as an approximation for π, and give a numerical answer.

Access for Students with Diverse Abilities

Engagement: Develop Effort and Persistence.
Encourage and support opportunities for peer interactions. Prior to the whole-class discussion, invite students to share their work with a partner. Display sentence frames to support student conversation, such as "First I ____ because ..." "How did you know ...?" and "How did you get ...?"
Supports accessibility for: Language, Social-Emotional Functioning

Student Workbook


2 Calculate That Cone

Are You Ready for More?

A grain silo has a cone shaped spout on the bottom in order to regulate the flow of grain out of the silo. The diameter of the silo is 8 feet. The height of the cylindrical part of the silo above the cone spout is 12 feet while the height of the entire silo is 16 feet.

How many cubic feet of grain are held in the cone spout of the silo?
How many cubic feet of grain can the entire silo hold?

16 ft
8 ft
12 ft

15 Lesson Summary

If a cone and a cylinder have the same base and the same height, then the volume of the cone is $\frac{1}{3}$ of the volume of the cylinder. For example, the cylinder and cone shown here both have a height of 7 feet and a base with radius 3 feet.

The cylinder has a volume of 63π cubic feet since $\pi \cdot 3^2 \cdot 7 = 63\pi$. The cone has a volume that is $\frac{1}{3}$ of that, or 21π cubic feet.

7
3
7
3

If the radius for both solids is r and the height for both solids is h, then the volume of the cylinder is $\pi r^2 h$. That means that the equation to give the volume, V, of the cone is $V = \frac{1}{3}\pi r^2 h$.

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3. A cone-shaped popcorn cup has a radius of 5 centimeters and a height of 9 centimeters. How many cubic centimeters of popcorn can the cup hold? Use 3.14 as an approximation for π, and give a numerical answer.

235.5 cm^3 because with a radius of 5 cm and a height of 9 cm, the volume is calculated with the equation $V = \frac{1}{3} \cdot 3.14 \cdot 5^2 \cdot 9$

Are You Ready for More?

A grain silo has a cone shaped spout on the bottom in order to regulate the flow of grain out of the silo. The diameter of the silo is 8 feet. The height of the cylindrical part of the silo above the cone spout is 12 feet while the height of the entire silo is 16 feet.

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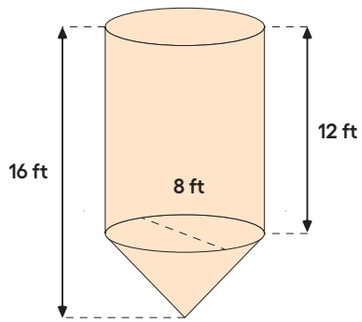


The entire grain silo holds $\frac{640}{3}\pi$ cubic feet of grain.

The cone holds $\frac{64}{3}\pi$ cubic feet of grain. Since the radius is 4 feet and the height of the cone is also 4 feet (16 – 12), the volume is $\frac{1}{3}\pi 4^2 \cdot 4$. Calculate the volume of the cylinder ($\pi 4^2 \cdot 12$), and add it to the volume of the cone to get the volume of the entire silo.

Activity Synthesis

Focus this discussion on the final problem, which is the first time students are asked to calculate the volume of a cone in a context.

Begin by asking students to share any sketches they came up with to help them calculate the answer. Explain to students that sometimes we encounter problems that don't have a visual example and only a written description. By using sketches and labeling relevant parts to help to visualize what is described in a problem, we can better understand what is being asked.

Lesson Synthesis

Prompt students to summarize the highlights of the lesson by asking:

- *"What is the relationship between the volume of a cylinder and the volume of a cone?"*

  The volume of a cone is $\frac{1}{3}$ of the volume of a cylinder, or the volume of the cylinder is 3 times the volume of the cone.

- *"If we know the volume of a cone, how do we calculate the volume of a cylinder that has the same height and base area?"*

  We can multiply the volume of the cone by 3.

- *"If we know the volume of a cylinder, how do we calculate the volume of a cone that has the same height and base area?"*

  We can multiply the volume of the cylinder by $\frac{1}{3}$.

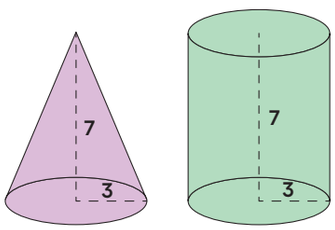
- *"If a cylinder and a cone have the same base, how tall does the cone have to be relative to the cylinder so that they both have the same volume?"*

  The cone needs to have a height 3 times the height of the cylinder for the two shapes to have the same volume.

Lesson Summary

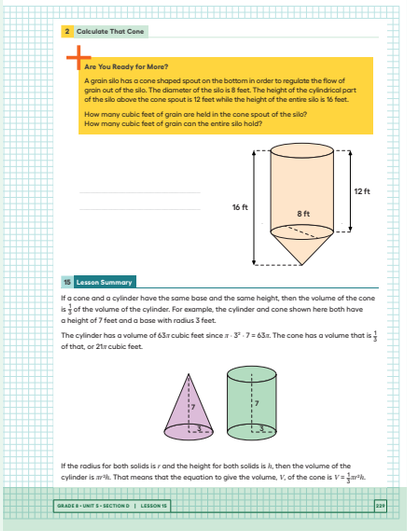
If a cone and a cylinder have the same base and the same height, then the volume of the cone is $\frac{1}{3}$ of the volume of the cylinder. For example, the cylinder and cone shown here both have a height of 7 feet and a base with radius 3 feet.

The cylinder has a volume of 63π cubic feet since $\pi \cdot 3^2 \cdot 7 = 63\pi$. The cone has a volume that is $\frac{1}{3}$ of that, or 21π cubic feet.



If the radius for both solids is r and the height for both solids is h, then the volume of the cylinder is $\pi r^2 h$. That means that the equation to give the volume, V, of the cone is $V = \frac{1}{3}\pi r^2 h$.

Student Workbook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Responding To Student Thinking

More Chances
Students will have more opportunities to understand the mathematical ideas addressed here. There is no need to slow down or add additional work to the next lessons.

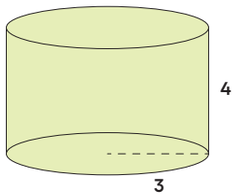
Cool-down

Calculate Volumes of Two Figures

5 min

Student Task Statement

There is a cone with the same base as the given cylinder but with a height that is 3 times taller. What is the volume of each figure? Express your answers in terms of π.



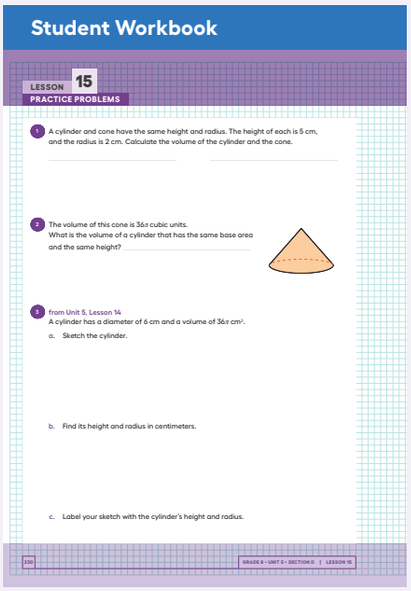
Cylinder: 36π cubic units, because $\pi \cdot 3^2 \cdot 4 = 36\pi$

Cone: 36π cubic units, because $\frac{1}{3}\pi \cdot 3^2 \cdot 12 = 36\pi$

Lesson 15

Practice Problems

5 Problems



Problem 1

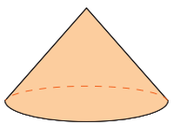
A cylinder and cone have the same height and radius. The height of each is 5 cm, and the radius is 2 cm. Calculate the volume of the cylinder and the cone.

Cylinder: 20π cm^3 Cone: $\frac{20}{3}\pi$ cm^3

Problem 2

The volume of this cone is 36π cubic units.

What is the volume of a cylinder that has the same base area and the same height?



108π cubic units, or about 339 cubic units

Sample reasoning: The volume of the cylinder is exactly 3 times the volume of the corresponding cone.

Problem 3

from Unit 5, Lesson 14

A cylinder has a diameter of 6 cm and a volume of 36π cm^3.

a. Sketch the cylinder.

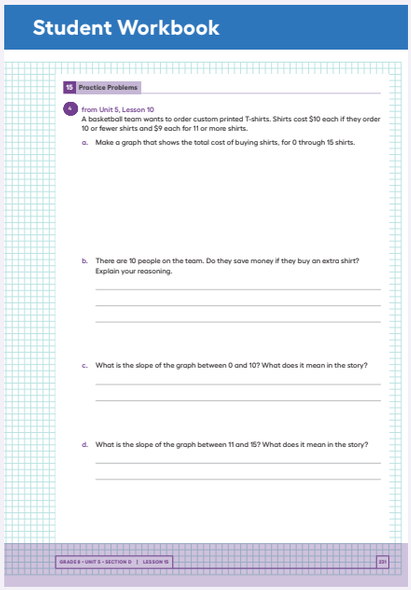
Cylinder is sketched.

b. Find its height and radius in centimeters.

height = 4 cm, radius = 3 cm

c. Label your sketch with the cylinder's height and radius.

Cylinder is sketched with height and radius labeled.


Student Workbook

15 Practice Problems

4 from Unit 5, Lesson 10

A basketball team wants to order custom printed T-shirts. Shirts cost $10 each if they order 10 or fewer shirts and $9 each for 11 or more shirts.

a. Make a graph that shows the total cost of buying shirts, for 0 through 15 shirts.

b. There are 10 people on the team. Do they save money if they buy an extra shirt? Explain your reasoning.

c. What is the slope of the graph between 0 and 10? What does it mean in the story?

d. What is the slope of the graph between 11 and 15? What does it mean in the story?

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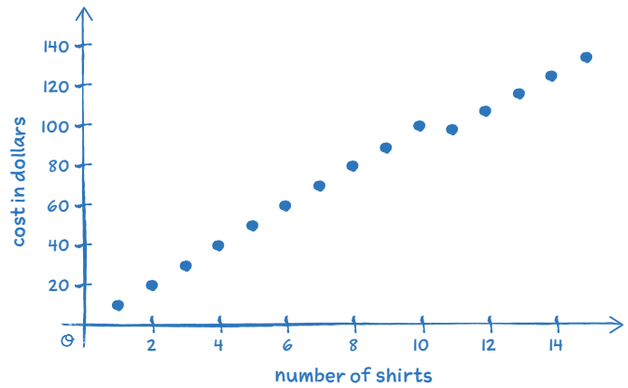
Problem 4

from Unit 5, Lesson 10

A basketball team wants to order custom printed T-shirts. Shirts cost $10 each if they order 10 or fewer shirts and $9 each for 11 or more shirts.

a. Make a graph that shows the total cost of buying shirts, for 0 through 15 shirts.



b. There are 10 people on the team. Do they save money if they buy an extra shirt? Explain your reasoning.

Yes

Sample reasoning: 11 shirts cost $99, and 10 shirts cost $100. Even if the team splits the cost of the extra shirt, they still save $1 altogether, or $0.10 apiece. If they can find someone to buy the extra shirt, they save $1 each.

c. What is the slope of the graph between 0 and 10? What does it mean in the story?

10 dollars per shirt: the price per shirt when they buy 10 or fewer

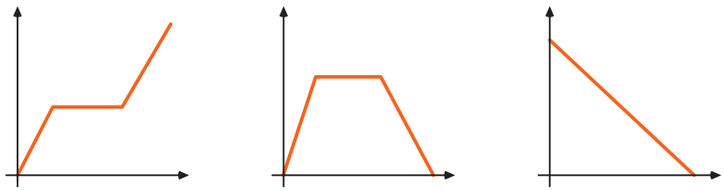
d. What is the slope of the graph between 11 and 15? What does it mean in the story?

9 dollars per shirt: the price per shirt when they buy 11 or more

Problem 5

from Unit 5, Lesson 6

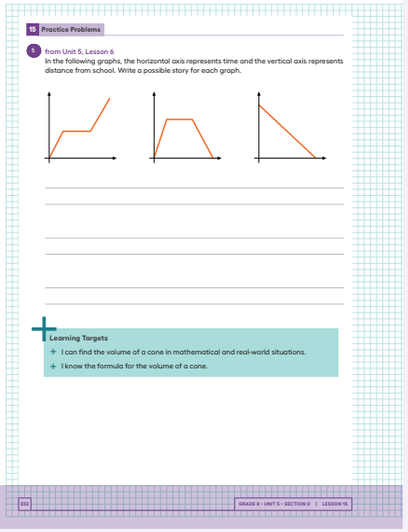
In the following graphs, the horizontal axis represents time and the vertical axis represents distance from school. Write a possible story for each graph.



Sample responses:

- Graph that is increasing, constant, then increasing again: A student leaves school and walks to a friend's house that is halfway between school and their house. After spending some time at a friend's house, the student continues walking home.
- Graph that is increasing, constant, then decreasing: An athlete leaves school to go home for a while, then returns to school for a game later in the evening.
- Graph that is decreasing: A parent starts from their workplace and drives directly to school to pick up their child.

Student Workbook


15 Practice Problems

5 from Unit 5, Lesson 6

In the following graphs, the horizontal axis represents time and the vertical axis represents distance from school. Write a possible story for each graph.

Learning Targets

+ I can find the volume of a cone in mathematical and real-world situations.
+ I know the formula for the volume of a cone.