Lesson 18

Scaling Two Dimensions (Optional)

Goals

- Compare and contrast (orally) graphs of linear and nonlinear functions.
- Create an equation and a graph representing the volume of a cone as a function of its radius, and describe (orally and in writing) how a change in radius affects the volume.
- Describe (orally and in writing) how changing the input of a certain nonlinear function affects the output.

Learning Targets

- I can create a graph representing the relationship between volume and radius for all cones (or cylinders) with a fixed height.
- I can explain in my own words why changing the radius by a scale factor changes the volume by the scale factor squared.

Access for Students with Diverse Abilities

- Action and Expression (Activity 1)

Access for Multilingual Learners

- MLR8: Discussion Supports (Activity 1)

Required Preparation

Activity 2:
For the digital version of the activity, acquire devices that can run the applet.

Lesson Narrative

The main purpose of this lesson is to understand that if two of the dimensions of a three-dimensional figure are scaled by the same factor, the volume scales by the square of that factor. A secondary purpose is to see some interesting examples of nonlinear functions arising from geometry.

Students begin the lesson by considering rectangular prisms with a square base and a fixed height. They see that if the side length of the square base of the prism is tripled, the volume is multiplied by 9, which is 3^2. The following activity invites students to explore how the volume changes as a function of the radius for cones with a fixed height. Using the structure of the formula for the volume of a cone, they see that tripling the radius leads to a similar result as with the prism—the volume scales by a factor of 3^2.

Student Learning Goal

Let's change more dimensions of shapes.

Lesson Timeline

Warm-up	Activity 1	Activity 2	Lesson Synthesis	Cool-down (Assessment)
5 min	15 min	15 min	10 min	5 min

Warm-up
Tripling Statements

5 min

Activity Narrative

The purpose of this *Warm-up* is for students to explore how scaling the addends or factors in an expression affects their sum or product. Students determine which statements are true and then create one statement of their own that is true. This will prepare students to see structure in the equations they will encounter in the lesson. Identify students who:

- Choose the correct statements (b, c).
- Pick numbers to test the validity of statements.
- Use algebraic structure to show that the statements are true.

Ask these students to share during the discussion.

Launch

Arrange students in groups of 2.

Give students 1–2 minutes of quiet work time followed by time to discuss their chosen statements with their partner.

Follow with a whole-class discussion.

Student Task Statement

m, n, a, b, and c all represent positive integers. Consider these two equations:

$$m = a + b + c \qquad\qquad n = abc$$

1. Which of these statements are true? Select **all** that apply.

   A. If a is tripled, m is tripled.

   B. If a, b, and c are all tripled, then m is tripled.

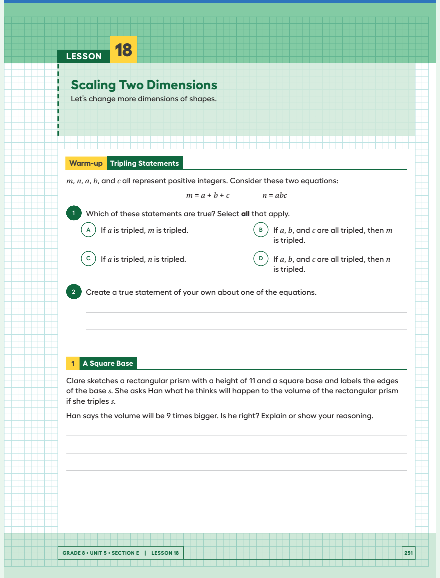
   C. If a is tripled, n is tripled.

   D. If a, b, and c are all tripled, then n is tripled.

2. Create a true statement of your own about one of the equations.

   Sample response: If a, b, and c are all tripled, then n is 27 times as large. When a, b, and c are tripled, the result is $3a \cdot 3b \cdot 3c$, which can be written as $(3 \cdot 3 \cdot 3)abc$ or $27abc$.

Student Workbook


LESSON 18

Scaling Two Dimensions

Let's change more dimensions of shapes.

Warm-up Tripling Statements

m, n, a, b, and c all represent positive integers. Consider these two equations:

$m = a + b + c$ $\quad$ $n = abc$

1 Which of these statements are true? Select **all** that apply.

A If a is tripled, m is tripled.

B If a, b, and c are all tripled, then m is tripled.

C If a is tripled, n is tripled.

D If a, b, and c are all tripled, then n is tripled.

2 Create a true statement of your own about one of the equations.

1 A Square Base

Clare sketches a rectangular prism with a height of 11 and a square base and labels the edges of the base s. She asks Han what he thinks will happen to the volume of the rectangular prism if she triples s.

Han says the volume will be 9 times bigger. Is he right? Explain or show your reasoning.

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Access for Students with Diverse Abilities (Activity 1, Student Task)

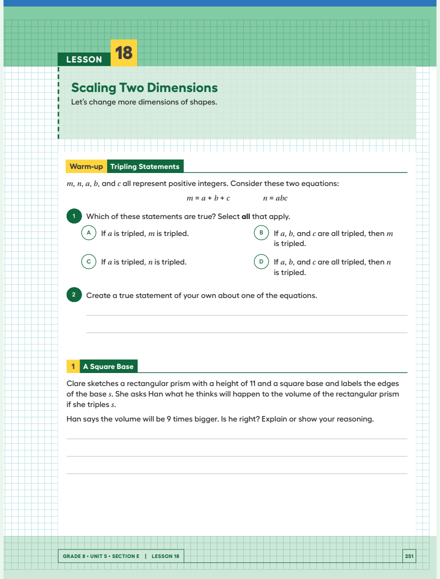
Action and Expression: Develop Expression and Communication.
To help get students started, display sentence frames such as "Han is ____ because____" or "I agree with ____ because____."
Supports accessibility for: Language, Organization

Access for Multilingual Learners (Activity 1, Student Task)

MLR8: Discussion Supports.
Prior to solving the problem, invite students to make sense of the situation and take turns sharing their understanding with their partner about how the shape of the rectangular prism changes when the value s is tripled. Listen for and clarify any questions about the context.

Advances: Reading, Representing

Student Workbook


LESSON 18

Scaling Two Dimensions

Let's change more dimensions of shapes.

Warm-up Tripling Statements

m, n, a, b, and c all represent positive integers. Consider these two equations:

$m = a + b + c$ $\quad$ $n = abc$

1. Which of these statements are true? Select **all** that apply.
   - A. If a is tripled, m is tripled.
   - B. If a, b, and c are all tripled, then m is tripled.
   - C. If a is tripled, n is tripled.
   - D. If a, b, and c are all tripled, then n is tripled.
2. Create a true statement of your own about one of the equations.

1 A Square Base

Clare sketches a rectangular prism with a height of 11 and a square base and labels the edges of the base s. She asks Han what he thinks will happen to the volume of the rectangular prism if she triples s.

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GRADE 8 • UNIT 5 • SECTION E | LESSON 18 251

Activity Synthesis

Ask previously identified students to share their reasoning about which statements are true (or not true). Display any examples (or counterexamples) for all to see, and ask students to refer to them while sharing. If using the algebraic structure is not brought up in students' explanations, display for all to see:

- If a, b, and c are all tripled, the expression becomes $3a + 3b + 3c$, which can be written as $3(a + b + c)$ by using the distributive property to factor out the 3. So if all the addends are tripled, their sum, m, is also tripled.
- Looking at the third statement, if a is tripled, the expression becomes $(3a)bc$, which, by using the associative property, can be written as $3(abc)$. So if just a is tripled, then n, the product of a, b, and c is also tripled.

Activity 1
A Square Base

15 min

Activity Narrative

The purpose of this activity is for students to examine how changing the input of a nonlinear function changes the output. In this activity, students consider how the volume of a rectangular prism with a square base and a known height of 11 units changes if the edge lengths of the base triple. By studying the structure of the equation representing the volume function, students see that tripling the input leads to an output that is 9 times greater. Students conclude the activity by creating a graph of the situation and notice that it, unlike previous graphs, is nonlinear.

Identify students who make sketches of the two rectangular prisms or write expressions of the form $99\,s^2$ or $11\,(3s)^2$ to describe the volume of the tripled rectangular prism.

Launch

Give students quiet work time. Leave 5–10 minutes for a whole-class discussion and follow-up questions.

Student Task Statement

Clare sketches a rectangular prism with a height of 11 and a square base and labels the edges of the base s. She asks Han what he thinks will happen to the volume of the rectangular prism if she triples s.

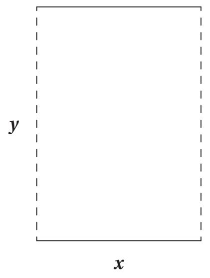
Han says the volume will be 9 times bigger. Is he right? Explain or show your reasoning.

Yes, Han is right.

Sample reasoning: When the edge length of the square base is tripled, the area of the base is multiplied by 9. This makes the volume equation, $V = 99s^2$, nine times as large as the original: $V = 11(3s)^2 = 11 \cdot 3s \cdot 3s = 11 \cdot 9s^2 = 99s^2$.

Are You Ready for More?

A cylinder can be constructed from a piece of paper by curling it so that two opposite edges (the dashed edges in the figure) can be glued together.



1. If we wanted to increase the volume inside the resulting cylinder, would it make more sense to double x or double y, or does it not matter? Explain your reasoning.

   Double length x. Since x represents the circumference of the circular base, this would result in doubling the radius of the cylinder as well. Doubling the radius will result in 4 times the volume. Doubling y, the height of the cylinder, results in doubling the volume.

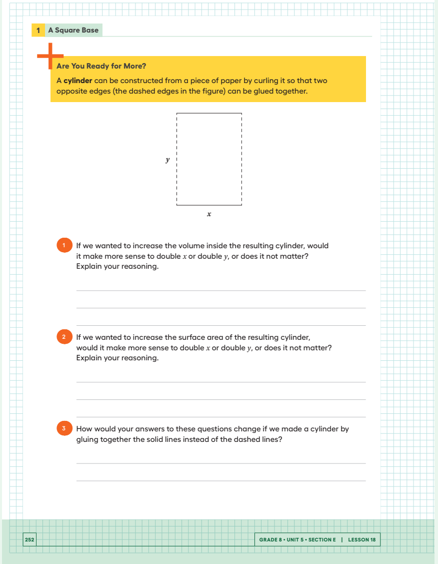
2. If we wanted to increase the surface area of the resulting cylinder, would it make more sense to double x or double y, or does it not matter? Explain your reasoning.

   Double length x. Doubling x will quadruple the area of the circular bases as well as double the area of the curved surface. Doubling y will only double the curved surface.

3. How would your answers to these questions change if we made a cylinder by gluing together the solid lines instead of the dashed lines?

   If the cylinder was created by connecting the solid edges, doubling length y would result in both a larger volume and surface area because y would be the length related to the radius of the cylinder.

Student Workbook



Activity Synthesis

The purpose of this discussion is for students to create a graph representing the relationship between the volume V and side s, noticing that unlike previous graphs, this one is nonlinear.

Select previously identified students to share whether they think Han is correct. If possible, begin with students who made sketches of the two rectangular prisms to make sense of the problem. If not brought up by students, connect Han's reasoning to the equation for the volume of the prism, $V = 99\,s^2$.

Tell students that they are now going to think about what the graph of this volume equation looks like. Ask students:

"If this equation were graphed with edge length s on the horizontal axis and the volume of the prism on the vertical axis, what would the graph look like?"

Give 3–5 minutes for students to make a graph.

If needed, suggest students first make a table showing the volume of the rectangular prism when s equals 1, 2, 3, 4, and 5 units (the corresponding values of volume are 11, 44, 99, 176, and 275 cubic units) and then sketch a graph using these points.

Select 1–2 students to display their table and graph for all to see. Ask students what they notice about the graph when compared to the graphs from previous lessons (the graph is nonlinear—the volume increases by the square of whatever the base edge length increases by).

Activity 2
Playing with Cones

15 min

Activity Narrative

There is a digital version of this activity.

In this activity, students continue working with function representations to investigate how changing dimensions affects the volume of a shape. Students start by representing the relationship between volume and radius for cones with a fixed height with an equation and graph. They use these representations to justify what they think will happen when the radius of the cylinder is tripled.

Identify students who use the equation to answer the last question and students who use the graph to answer the last question.

In the digital version of the activity, students use an applet to make the graph. The applet allows students to graph accurately and quickly. Use the digital version if time is limited and students can access the applet readily on a device.

Launch

Arrange students in groups of 2.

Give students 4–7 minutes to work with their partner. Follow with a whole-class discussion.

Student Task Statement

There are many cones with a height of 7 units. Let r represent the radius and V represent the volume of these cones.

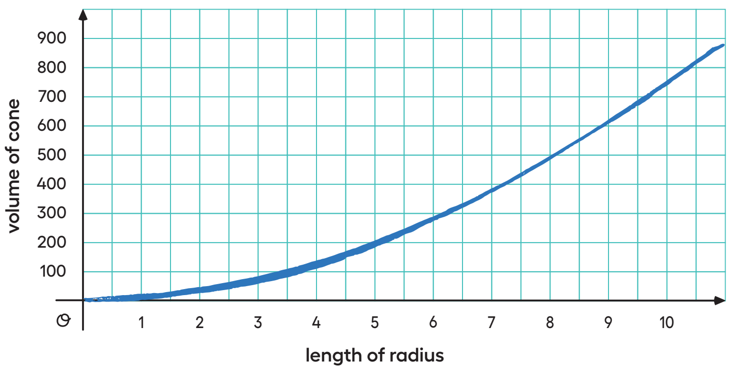
1. Write an equation that expresses the relationship between V and r. Use 3.14 as an approximation for π.

   $V = 7.33\, r^2$

2. Predict what happens to the volume if the value of r is tripled.

   Sample response: The volume is 9 times bigger if the value of r is tripled.

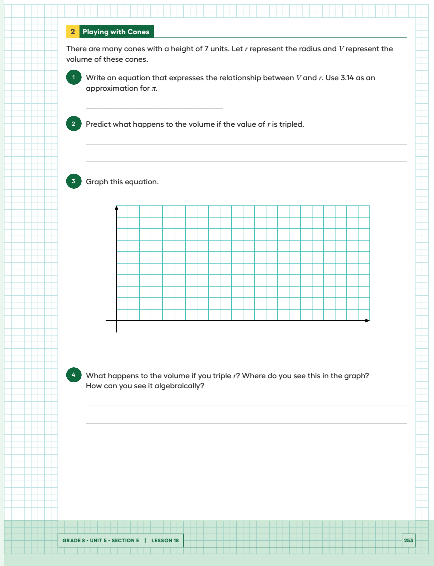
3. Graph this equation.



4. What happens to the volume if you triple r? Where do you see this in the graph? How can you see it algebraically?

   Sample response: When the radius, r, is tripled, the volume is 9 times as large. Since this is not a proportional relationship, in the graph it can be seen that when the radius triples from 1 cm to 3 cm, the volume changes from 7.33 cm^3 to 65.94 cm^3, a value 9 times as large. Algebraically, if the radius triples from r to $3r$, then the volume changes from $7.33\, r^2$ to $7.33\,(3r)^2 = 9 \cdot 7.33\, r^2 = 65.94\, r^2$.

Student Workbook



Activity Synthesis

The purpose of this discussion is for students to use the graph and equation to see that when the radius is tripled, the result is a volume that is 9 times as large.

Ask previously identified students to share their graphs and equations. Display both representations for all to see, and ask students to point out where in each representation it can be seen that the volume is 9 times as large. Ask students:

- *"If the radius was quadrupled (made 4 times as large), how many times as large would the volume be?"*

  The volume would be 16 times as large since $\frac{1}{3}\pi(4r)^2 = \frac{1}{3}\pi r^2 4^2 = 16 \cdot \frac{1}{3}\pi r^2$.

- *"If the radius was halved, how many times as large would the volume be?"*

  The volume would be $\frac{1}{4}$ times as large since $\frac{1}{3}\pi\left(\frac{1}{2}r\right)^2 = \frac{1}{3}\pi r^2\left(\frac{1}{2}\right)^2 = \frac{1}{4} \cdot \frac{1}{3}\pi r^2$.

- *"If the radius was scaled by an unknown factor a, how many times as large would the volume be?"*

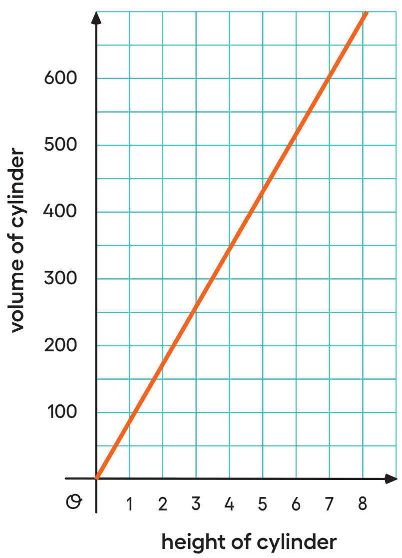
  The volume would be a^2 times as large since $\frac{1}{3}\pi(ar)^2 = \frac{1}{3}\pi r^2 a^2 = a^2\frac{1}{3}\pi r^2$.

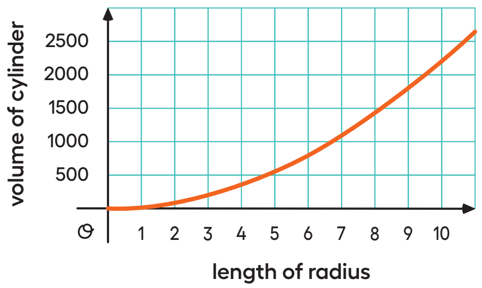
If students do not see the connection between scaling the radius length with a known value, like 4, and an unknown value a, use several known values to help students generalize that scaling the radius by a scales the volume by a^2.

If time allows, ask students to compare this activity to the previous. How do the equations compare? How do the graphs compare? (In the last activity, the graph was sketched during the discussion.)

Lesson Synthesis

Display these graphs for all to see, and give students 1 minute to consider what they represent.





Ask students:

- *"What do these graphs represent? How are these graphs similar? Different?"*

  The first graph shows the relationship between the height and volume of all the cylinders with a fixed radius. The second graph shows the relationship between the radius and volume of all the cylinders with a fixed height. The first is linear, and the second is nonlinear.

- *"Think about what happens when a cube's edge lengths are doubled or tripled. What happens to the volume?"*

  The volume is increased by $2^3 = 8$ or by $3^3 = 27$.

- *"Why do you think changing the radius of a cylinder results in a graph that is not proportional?"*

  Two dimensions change when the radius of a cylinder is changed.

Lesson Summary

There are many rectangular prisms that both have a length of 4 units and width of 5 units but have different heights. If h represents the height, then the volume V of such a prism is

$$V = 20h.$$

The equation shows us that the volume of a prism with a base area of 20 square units is a linear function of the height. Because this is a proportional relationship, if the height gets multiplied by a factor of a, then the volume is also multiplied by a factor of a:

$$V = 20(ah).$$

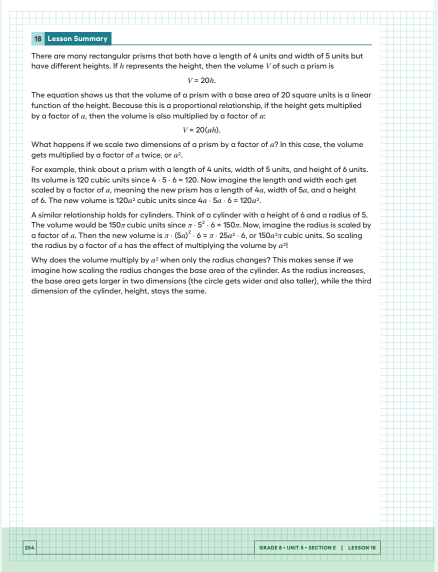
What happens if we scale *two* dimensions of a prism by a factor of a? In this case, the volume gets multiplied by a factor of a twice, or a^2.

For example, think about a prism with a length of 4 units, width of 5 units, and height of 6 units. Its volume is 120 cubic units since $4 \cdot 5 \cdot 6 = 120$. Now imagine the length and width each get scaled by a factor of a, meaning the new prism has a length of $4a$, width of $5a$, and a height of 6. The new volume is $120a^2$ cubic units since $4a \cdot 5a \cdot 6 = 120a^2$.

A similar relationship holds for cylinders. Think of a cylinder with a height of 6 and a radius of 5. The volume would be 150π cubic units since $\pi \cdot 5^2 \cdot 6 = 150\pi$. Now, imagine the radius is scaled by a factor of a. Then the new volume is $\pi \cdot (5a)^2 \cdot 6 = \pi \cdot 25\,a^2 \cdot 6$, or $150\,a^2\,\pi$ cubic units. So scaling the radius by a factor of a has the effect of multiplying the volume by a^2!

Why does the volume multiply by a^2 when only the radius changes? This makes sense if we imagine how scaling the radius changes the base area of the cylinder. As the radius increases, the base area gets larger in two dimensions (the circle gets wider and also taller), while the third dimension of the cylinder, height, stays the same.

Student Workbook



Responding To Student Thinking

More Chances
Students will have more opportunities to develop this understanding in later courses. There is no need to slow down or add additional work to review this concept at this time.

Cool-down

Halving Dimensions

5 min

Student Task Statement

There are many cylinders for which the height and radius are the same value. Let c represent the height and radius of a cylinder and V represent the volume of the cylinder.

1. Write an equation that expresses the relationship between the volume, height, and radius of this cylinder using c and V.

   $V = \pi c^3$

2. If the value of c is halved, what must happen to the value of the volume V?

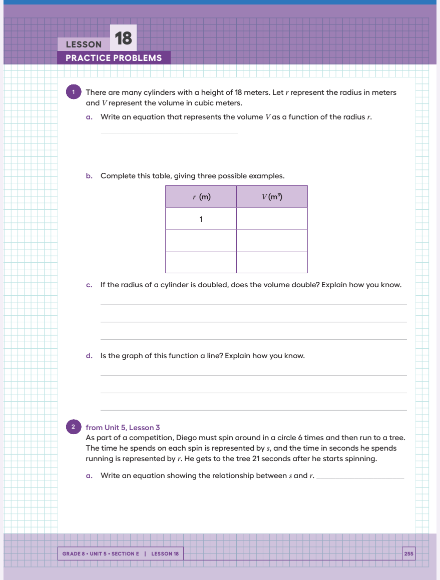
   If the value of c is halved, then the value of the volume would be $\frac{1}{8}$ of the original volume since $\pi\left(\frac{1}{2}c\right)^3 = \pi c^3\left(\frac{1}{2}\right)^3 = \frac{1}{8}\pi c^3$.

Lesson 18

Practice Problems

4 Problems

Student Workbook


LESSON 18
PRACTICE PROBLEMS

1 There are many cylinders with a height of 18 meters. Let r represent the radius in meters and V represent the volume in cubic meters.

a. Write an equation that represents the volume V as a function of the radius r.

b. Complete this table, giving three possible examples.

r (m)	V (m³)
1	

c. If the radius of a cylinder is doubled, does the volume double? Explain how you know.

d. Is the graph of this function a line? Explain how you know.

2 from Unit 5, Lesson 3
As part of a competition, Diego must spin around in a circle 6 times and then run to a tree. The time he spends on each spin is represented by s, and the time in seconds he spends running is represented by r. He gets to the tree 21 seconds after he starts spinning.

a. Write an equation showing the relationship between s and r.

Problem 1

There are many cylinders with a height of 18 meters. Let r represent the radius in meters and V represent the volume in cubic meters.

a. Write an equation that represents the volume V as a function of the radius r.

$V = 18\pi r^2$

b. Complete this table, giving three possible examples.

Sample response:

r (m)	V (m³)
1	18π
2	72π
4	288π

c. If the radius of a cylinder is doubled, does the volume double? Explain how you know.

No

Sample reasoning: The volume does not double, it is multiplied by 4.

d. Is the graph of this function a line? Explain how you know.

No

Sample reasoning: The three points in the table do not lie on a straight line.

Problem 2

from Unit 5, Lesson 3

As part of a competition, Diego must spin around in a circle 6 times and then run to a tree. The time he spends on each spin is represented by s, and the time in seconds he spends running is represented by r. He gets to the tree 21 seconds after he starts spinning.

a. Write an equation showing the relationship between s and r.

$6s + r = 21$ (or equivalent)

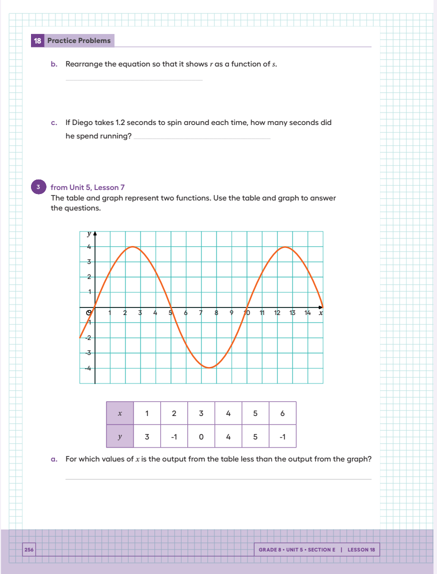
b. Rearrange the equation so that it shows r as a function of s.

$r = 21 - 6s$ (or equivalent)

c. If Diego takes 1.2 seconds to spin around each time, how many seconds did he spend running?

13.8 seconds

Student Workbook


18 Practice Problems

b. Rearrange the equation so that it shows r as a function of s.

c. If Diego takes 1.2 seconds to spin around each time, how many seconds did he spend running?

3 from Unit 5, Lesson 7

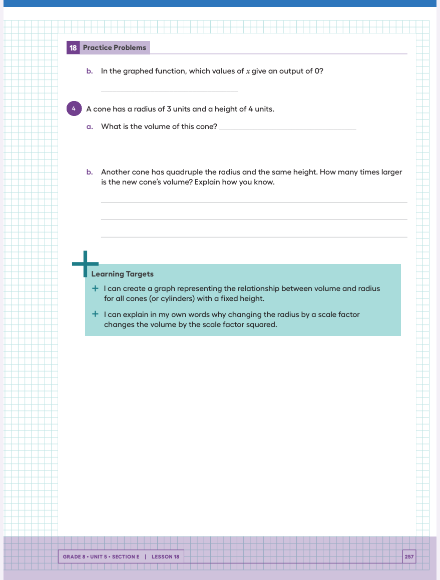
The table and graph represent two functions. Use the table and graph to answer the questions.

x	1	2	3	4	5	6
y	3	-1	0	4	5	-1

a. For which values of x is the output from the table less than the output from the graph?

256 GRADE 8 • UNIT 5 • SECTION E | LESSON 18

Student Workbook


18 Practice Problems

b. In the graphed function, which values of x give an output of 0?

4 A cone has a radius of 3 units and a height of 4 units.

a. What is the volume of this cone?

b. Another cone has quadruple the radius and the same height. How many times larger is the new cone's volume? Explain how you know.

Learning Targets

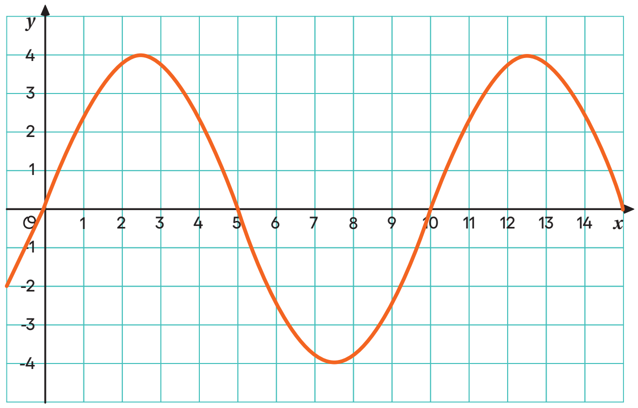
+ I can create a graph representing the relationship between volume and radius for all cones (or cylinders) with a fixed height.
+ I can explain in my own words why changing the radius by a scale factor changes the volume by the scale factor squared.

GRADE 8 • UNIT 5 • SECTION E | LESSON 18 257

Problem 3

from Unit 5, Lesson 7

The table and graph represent two functions. Use the table and graph to answer the questions.



x	1	2	3	4	5	6
y	3	-1	0	4	5	-1

a. For which values of x is the output from the table less than the output from the graph?

2 and 3

b. In the graphed function, which values of x give an output of 0?

0, 5, 10, and 15

Problem 4

A cone has a radius of 3 units and a height of 4 units.

a. What is the volume of this cone?

12π cubic units

b. Another cone has quadruple the radius and the same height. How many times larger is the new cone's volume? Explain how you know.

16 times larger

Sample reasoning: The original cone's volume is 12π. The new cone's volume is $V = \frac{1}{3} \cdot \pi \cdot (3 \cdot 4)^2 \cdot 4 = 192\pi = 16 \cdot 12\pi$. Quadrupling the radius makes the volume 4^2 times larger.