Lesson 20

The Volume of a Sphere

Goals

- Calculate the volume of a sphere, cylinder, and cone which all have a radius of r and height of $2r$, and explain (orally) the relationship between their volumes.
- Create an equation to represent the volume of a sphere as a function of its radius, and explain (orally and in writing) the reasoning.

Learning Target

I can find the volume of a sphere when I know the radius.

Student Learning Goal

Let's explore spheres and their volumes.

Lesson Narrative

The purpose of this lesson is for students to recognize that the volume of a sphere with radius r is $\frac{4}{3}\pi r^3$ and begin to use the formula. Students inspect an image of a sphere that snugly fits inside a cylinder (they each have the same radius, and the height of the cylinder is equal to the diameter of the sphere), and use their intuition to guess how the volume of the sphere relates to the volume of the cylinder, building on the work in the previous lesson.

Then, they watch a video that shows a sphere inside a cylinder, and the contents of a cone (with the same base and height as the cylinder) are poured into the remaining space, helping students make sense of the relationships between the volumes. This demonstration shows that for these figures, the cylinder contains the volumes of the sphere and cone together. From this observation, the volume of a specific sphere is computed.

Then, the formula $\frac{4}{3}\pi r^3$ for the volume of a sphere is derived. (At this point, this is taken to be true for any sphere even though we only saw a demonstration involving a particular sphere, cone, and cylinder. A general proof of the formula for the volume of a sphere would require mathematics beyond grade level.)

In the last activity, students reason about the relationship between the volumes for any cone, sphere, or cylinder group with matching dimensions by rewriting and using the structure of the volume formulas for a cone and cylinder to determine the formula for a sphere.

Access for Students with Diverse Abilities

- Engagement (Activity 2)

Access for Multilingual Learners

- MLR7: Compare and Connect (Activity 1)

Instructional Routines

- MLR5: Co-Craft Questions
- MLR7: Compare and Connect
- Notice and Wonder

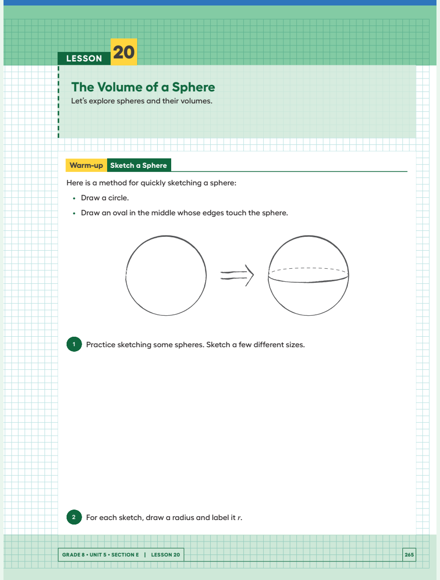
Required Preparation

Activity 1:
During the *Launch*, students will need to view a video.

Lesson Timeline

Warm-up	Activity 1	Activity 2	Lesson Synthesis	Cool-down (Assessment)
5 min	20 min	10 min	10 min	5 min

Student Workbook


LESSON 20

The Volume of a Sphere

Let's explore spheres and their volumes.

Warm-up Sketch a Sphere

Here is a method for quickly sketching a sphere:

- Draw a circle.
- Draw an oval in the middle whose edges touch the sphere.

1 Practice sketching some spheres. Sketch a few different sizes.

2 For each sketch, draw a radius and label it r.

GRADE 8 • UNIT 5 • SECTION E | LESSON 20

Warm-up

Sketch a Sphere

5 min

Activity Narrative

The purpose of this activity is for students to practice sketching spheres and labeling the radius and diameter of the sphere.

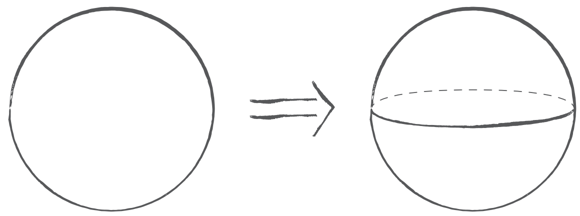
Launch

Give students 1–2 minutes of quiet work time, and follow with a whole-class discussion.

Student Task Statement

Here is a method for quickly sketching a sphere:

- Draw a circle.
- Draw an oval in the middle whose edges touch the sphere.



1. Practice sketching some spheres. Sketch a few different sizes.

   Answers vary.

2. For each sketch, draw a radius and label it r.

   Answers vary.

Activity Synthesis

Invite students to share their sketches. Ask students to share what the diameter would look like if they did not already draw one in. Remind students that sketches can be used to help visualize a problem where an image might not be provided. In today's lesson, they will be working with activities that may or may not have images provided, and they should sketch images or label any images provided to use as a tool to help understand the problem thoroughly.

Activity 1
A Sphere in a Cylinder

20 min

Activity Narrative

The purpose of this activity is for students to reason about the volume of a sphere with a specific radius using a cone and cylinder with matching dimensions.

In the *Launch*, students begin with an image of a sphere in a cylinder. The sphere and cylinder have the same radius, and the height of the cylinder is equal to the diameter of the sphere. Students reason about how the volumes of the two figures compare in order to get a closer estimate of the volume of the sphere.

Then students watch a video that shows a sphere inside a cylinder set up like the image. A cone with the same base and height as the cylinder is introduced, and its contents are poured into the cylinder, completely filling the empty space inside the cylinder not taken up by the sphere. Students are asked to record anything they notice and wonder as they watch the video, and a list is created as a class. They then consider what must be true about the relationship between the volumes of a cone, cylinder, and sphere without doing any calculations.

In the *Task Statement*, students consider a cone, sphere, and cylinder with matching dimensions. With a partner, they reason about the volume of the sphere using the volumes they calculate for the cone and cylinder along with what they learned from the video.

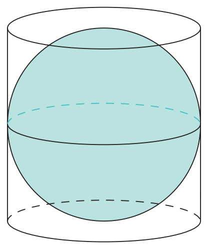
Monitor for students who discuss either method for calculating the volume of the sphere:

- Subtracting the volume of the cone from the volume of the cylinder
- Making the connection that a cone is $\frac{1}{3}$ of the cylinder, so the sphere must be the $\frac{2}{3}$ that fills up the rest of the cylinder

Launch

Arrange students in groups of 2. Display for all to see:

A sphere fits snugly into a cylinder so that its circumference touches the curved surface of the cylinder and the top and bottom touch the bases of the cylinder.



Instructional Routines

MLR7: Compare and Connect

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Please log in to the site before using the QR code or URL.

Access for Multilingual Learners (Activity 1)

MLR7: Compare and Connect
This activity uses the *Compare and Connect* math language routine to advance representing and conversing as students use mathematically precise language in discussion.

Instructional Routines

Notice and Wonder

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Ask,

"In the previous lesson we thought about hemispheres in cylinders. Here is a sphere in a cylinder. Which is bigger, the volume of the cylinder or the volume of the sphere? Do you think the bigger one is twice as big, more than twice as big, or less than twice as big?"

Then give students 1 minute of quiet think time. Invite students to share their responses, and keep their answers displayed for all to see throughout the lesson so that they can be referred to during the *Lesson Synthesis*.

Show the video (found at the end of this *Launch*), and tell students to write down anything they notice or wonder while watching. Ask students to share the things they noticed and wondered. Record and display their responses for all to see. If not mentioned by students, bring up the following points:

Notice:

- The sphere fits inside the cylinder.
- The sphere is filled up.
- There is space around the sphere inside the cylinder.
- It takes the volume of one cone to fill up the remaining spaces in the cylinder.

Wonder:

- Do the sphere and the cylinder have the same radius?
- Do the cone and cylinder have the same radius?
- Do the cone and cylinder have the same height?

Tell students that the sphere inside the cylinder seen in the video is the same as the one in the picture shown previously. Ask students:

"Does this give us any answers to the list of wonders?"

Yes, this tells us that the sphere and cylinder have the same radius.

Tell students that the cone and cylinder have the same height and base area. Ask students:

"Does this give us any more answers to the list of wonders?"

Yes, the cone and cylinder have the same height and radius.

"What does that mean about the volume of the cone and the volume of the cylinder in the video?"

The volume of the cone is $\frac{1}{3}$ the volume of the cylinder.

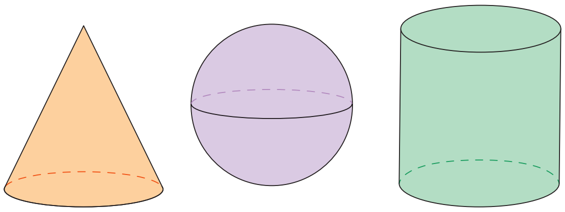
Show the video one more time, and ask students to think about how we might calculate the volume of the sphere if we know the radius of the cone or cylinder. Give students 30 seconds of quiet think time followed by time for a partner discussion to share their ideas before beginning work on the questions.

Select work from students with different strategies, such as those described in the *Activity Narrative*, to share later.

Video 'Volume of a Cylinder, Sphere, and Cone' available here: **player.vimeo.com/video/304138133**.

Student Task Statement

Here are a cone, a sphere, and a cylinder that all have the same radii and heights. The radius of the cylinder is 5 units. When necessary, express all answers in terms of π.



1. What is the height of the cylinder?

   10 units

   The top of the sphere touches the top of the cylinder, so the diameter of the sphere is the height of the cylinder.

2. What is the volume of the cylinder?

   250π cubic units, because $V = \pi 5^2 \cdot 10$

3. What is the volume of the cone?

   $\frac{250}{3}\pi$ cubic units, because $V = \frac{1}{3}\pi 5^2 \cdot 10$

4. What is the volume of the sphere? Explain your reasoning.

   $\frac{500}{3}\pi$ cubic units

   Sample response: Subtracting the volume of the cone from the volume of the cylinder gives the volume of the sphere. So the volume of the sphere is $\frac{500}{3}\pi$ cubic units because $250\pi - \frac{250}{3}\pi = \frac{500}{3}\pi$.

Activity Synthesis

The goal of this discussion is to compare and contrast the different ways students calculated the volume of the sphere.

Display 2–3 approaches or representations from previously selected students for all to see. If time allows, invite students to briefly describe their approach, then use *Compare and Connect* to help students compare, contrast, and connect the different approaches. Here are some questions for discussion:

- *"What do the approaches have in common? How are they different?"*

  They both use the volume of the formulas for a cylinder. One is using the fact that the volume of the cone is $\frac{1}{3}$ of the volume of the cylinder, so the sphere's volume must make up the other $\frac{2}{3}$. The other subtracts the volumes we know in order to get the unknown volume of the sphere.

- *"Why do the different approaches lead to the same outcome?"*

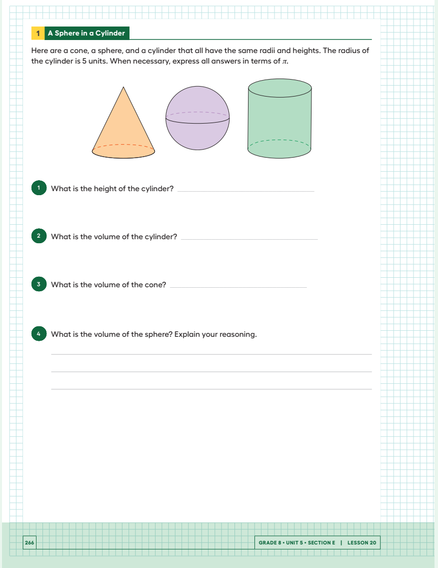
  They are both ways of calculating the volume of the same shape using the fact that the volume of a cone is $\frac{1}{3}$ the volume of a cylinder with matching dimensions.

Building on Student Thinking

If students struggle to keep track of all the dimensions of the different figures, consider asking:

"Can you tell me some things you know are true about the three shapes?"
"How could adding dimensions and≈labels to the figures help you reason about their areas?"

Student Workbook


1 A Sphere in a Cylinder

Here are a cone, a sphere, and a cylinder that all have the same radii and heights. The radius of the cylinder is 5 units. When necessary, express all answers in terms of π.

1. What is the height of the cylinder?
2. What is the volume of the cylinder?
3. What is the volume of the cone?
4. What is the volume of the sphere? Explain your reasoning.

Instructional Routines

MLR5: Co-Craft Questions

ilclass.com/r/10695544
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Access for Students with Diverse Abilities (Activity 2, Student Task)

Engagement: Develop Effort and Persistence.
Chunk this task into more manageable parts. For example, present one question at a time. Check in with students to provide feedback and encouragement after each chunk.
Supports accessibility for: Attention, Social-Emotional Functioning

"Did anyone solve the problem the same way but would explain it differently?"

Display this expression for all to see: $\pi 5^2 \cdot 10 - \frac{1}{3}\pi 5^2 \cdot 10$

Ask students:

"What does this expression represent?"

the volume of the cylinder minus the volume of the cone

Draw students' attention back to the guesses they made at the start of the activity about how much bigger the cylinder's volume is than the sphere. Ask students if we can answer that question now. (Note: If students do not explicitly make the connection that the sphere's volume is $\frac{2}{3}$ the volume of the cylinder, they will have another chance to look at the relationship in the next activity.)

Activity 2

Spheres in Cylinders

10 min

Activity Narrative

The purpose of this activity is to build from the concrete version in the previous activity to a generalized formula of a sphere with an unknown radius. The previous activity prepared students with strategies to work through this task in which they must manipulate the variables in the volume equations. Students first calculate the volumes of the cylinder and cone in the activity and use what they learned in the previous activity to calculate the volume of the sphere. Finally, they are asked about the relationship between the volumes of the cylinder and sphere, which connects back to the discussion of the previous activity.

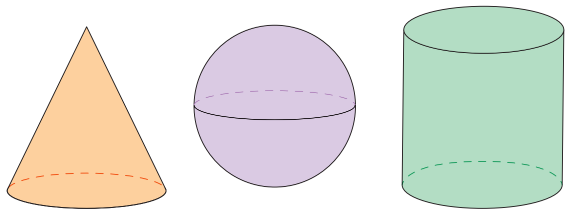
As students rearrange the volume equations, they are making use of algebraic structure to reason about the formula for the volume of a sphere.

Launch

Select students who recognize that the volume of the sphere is $\frac{2}{3}$ the volume of the cylinder and use that to come up with the general formula for volume of a sphere, $\frac{4}{3}\pi r^3$, or who use the subtraction method discussed in the previous activity to share later.

Student Task Statement

A cone, a sphere, and a cylinder that all have the same radius and height are shown here. Let the radius of the cylinder be r units. When necessary, express answers in terms of π.



1. What is the height of the cylinder in terms of r?

   $2r$, because the diameter of the sphere is the height of the cylinder

2. What is the volume of the cylinder in terms of r?

   $2\pi r^3$, because $V = \pi r^2 2r$

3. What is the volume of the cone in terms of r?

   $\frac{2}{3}\pi r^3$, because $V = \frac{1}{3}\pi r^2 2r$

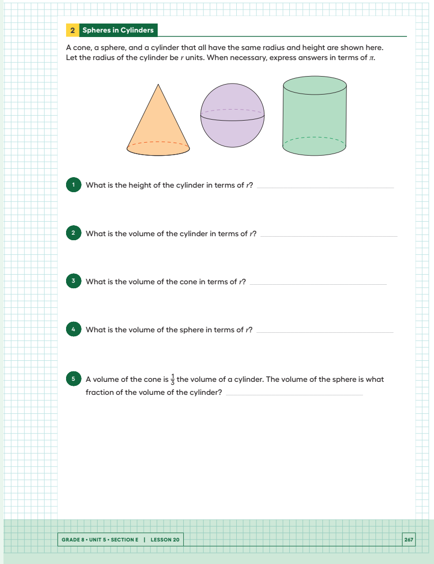
4. What is the volume of the sphere in terms of r?

   $\frac{4}{3}\pi r^3$, because $V = 2\pi r^3 - \frac{2}{3}\pi r^3$

5. A volume of the cone is $\frac{1}{3}$ the volume of a cylinder. The volume of the sphere is what fraction of the volume of the cylinder?

   $\frac{2}{3}$

Student Workbook



2 Spheres in Cylinders

A cone, a sphere, and a cylinder that all have the same radius and height are shown here. Let the radius of the cylinder be r units. When necessary, express answers in terms of π.

1. What is the height of the cylinder in terms of r?
2. What is the volume of the cylinder in terms of r?
3. What is the volume of the cone in terms of r?
4. What is the volume of the sphere in terms of r?
5. A volume of the cone is $\frac{1}{3}$ the volume of a cylinder. The volume of the sphere is what fraction of the volume of the cylinder?

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Activity Synthesis

Use this discussion to help students see the different ways to reason about calculating the volume of a sphere.

Here is a summary of the two approaches.

The $\frac{2}{3}$ method:

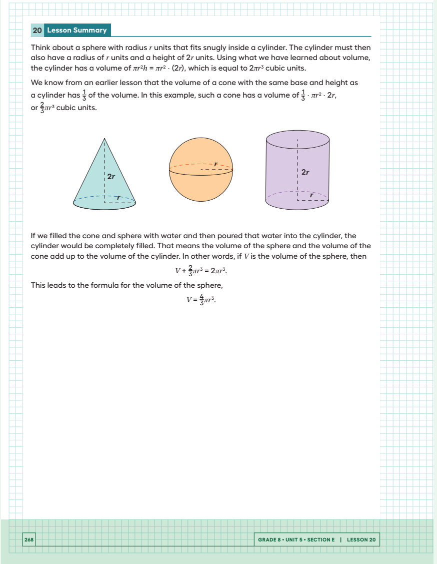
$$\begin{aligned}\text{volume of the sphere} &= \tfrac{2}{3}(\text{volume of the cylinder})\\ &= \tfrac{2}{3}(2\pi r^3)\\ &= \tfrac{4}{3}\pi r^3\end{aligned}$$

Subtraction method:

$$\begin{aligned}\text{volume of the sphere} &= \text{volume of the cylinder} - \text{volume of the cone}\\ &= 2\pi r^3 - \tfrac{2}{3}\pi r^3\\ &= \left(2 - \tfrac{2}{3}\right)\pi r^3\\ &= \tfrac{4}{3}\pi r^3\end{aligned}$$

Although either method works, there are reasons students may choose one over the other. The subtraction method is a bit more involved, as it requires the distributive property to combine like terms and the subtraction of $\frac{2}{3}$ from 2. It may make more sense to students, however, since it describes the video demonstrating that the volume of the sphere is the difference between the volumes of the cylinder and cone. The $\frac{2}{3}$ method is a bit simpler in terms of manipulating expressions, but students might not fully understand why the volume of the sphere is $\frac{2}{3}$ the volume of the cylinder.

Student Workbook


20 Lesson Summary

Think about a sphere with radius r units that fits snugly inside a cylinder. The cylinder must then also have a radius of r units and a height of $2r$ units. Using what we have learned about volume, the cylinder has a volume of $\pi r^2 h = \pi r^2 \cdot (2r)$, which is equal to $2\pi r^3$ cubic units.

We know from an earlier lesson that the volume of a cone with the same base and height as a cylinder has $\frac{1}{3}$ of the volume. In this example, such a cone has a volume of $\frac{1}{3} \cdot \pi r^2 \cdot 2r$, or $\frac{2}{3}\pi r^3$ cubic units.

If we filled the cone and sphere with water and then poured that water into the cylinder, the cylinder would be completely filled. That means the volume of the sphere and the volume of the cone add up to the volume of the cylinder. In other words, if V is the volume of the sphere, then

$V + \frac{2}{3}\pi r^3 = 2\pi r^3$.

This leads to the formula for the volume of the sphere,

$V = \frac{4}{3}\pi r^3$.

Invite previously selected students to share their methods. Display for all to see the two different strategies side by side, and ask students:

"Which method did you use to calculate the volume of the sphere?"

"Which method do you prefer? Why?"

If time allows, conclude the discussion by inviting students to explain to their partner why the method they didn't use worked.

Add the formula $V = \frac{4}{3}\pi r^3$ and a diagram of a sphere to your classroom display of the formulas being developed in this unit.

Lesson Synthesis

Display the equation $V \approx 4r^3$ for all to see. Tell students,

"A quick estimate for the volume of a sphere of radius r that you can use if you don't have a calculator is $V \approx 4r^3$. (No fraction or π!) How good of an approximation do you think this is? Can you come up with a better one?"

Ask students to calculate the volume of a sphere with a radius of 10 inches using:

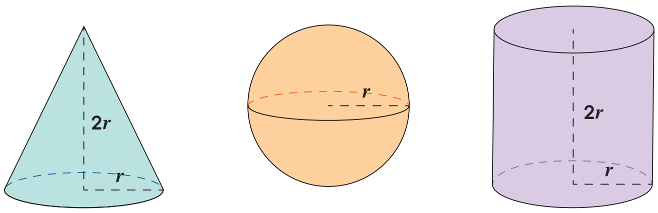
- The actual volume formula $V = \frac{4}{3}\pi r^3$. (4,188.79 cubic inches)
- The approximation formula $V \approx 4r^3$. (4,000 cubic inches)
- Their own approximation formula. (Possible formula: $V \approx 4 \cdot r^3 \cdot 1.05$, or 4,200 cubic inches)

Give students quiet think time, then time to compare their improved approximations with a partner and decide which of their formulas is the best approximation. Invite partners to share their choices with the class. Record and display student-created formulas for approximating the volume of a sphere for all to see.

Lesson Summary

Think about a sphere with radius r units that fits snugly inside a cylinder. The cylinder must then also have a radius of r units and a height of $2r$ units. Using what we have learned about volume, the cylinder has a volume of $\pi r^2 h = \pi r^2 \cdot (2r)$, which is equal to $2\pi r^3$ cubic units.

We know from an earlier lesson that the volume of a cone with the same base and height as a cylinder has $\frac{1}{3}$ of the volume. In this example, such a cone has a volume of $\frac{1}{3} \cdot \pi r^2 \cdot 2r$, or $\frac{2}{3}\pi r^3$ cubic units.



If we filled the cone and sphere with water and then poured that water into the cylinder, the cylinder would be completely filled. That means the volume of the sphere and the volume of the cone add up to the volume of the cylinder. In other words, if V is the volume of the sphere, then

$$V + \frac{2}{3}\pi r^3 = 2\pi r^3.$$

This leads to the formula for the volume of the sphere,

$$V = \frac{4}{3}\pi r^3.$$

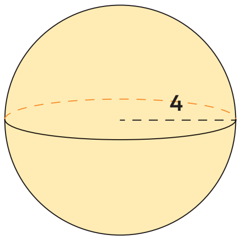
Cool-down

Volumes of Spheres

5 min

Student Task Statement

Recall that the volume of a sphere is given by the formula $V = \frac{4}{3}\pi r^3$.



1. Here is a sphere with radius 4 feet. What is the volume of the sphere? Express your answer in terms of π.

   $\frac{256}{3}\pi$ (or 85.33π) cubic feet, because $V = \frac{4}{3}\pi(4)^3$

2. A spherical balloon has a diameter of 4 feet. Approximate how many cubic feet of air this balloon holds. Use 3.14 as an approximation for π, and give a numerical answer.

   33.49 cubic feet, because $V = \frac{4}{3}\pi(2)^3$

Responding To Student Thinking

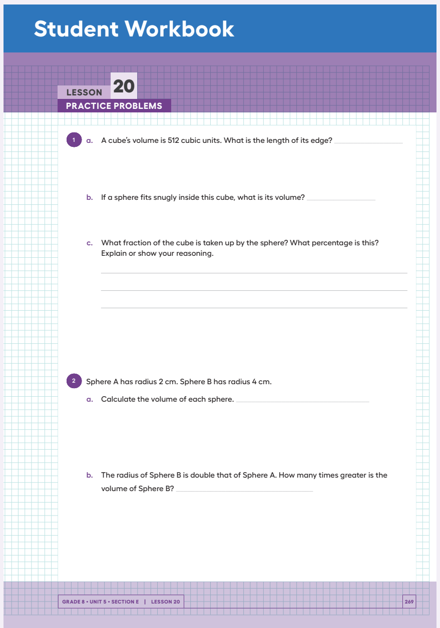
Points to Emphasize
If most students struggle with calculating the volume, focus on these calculations as opportunities arise. For example, use the *Activity Synthesis* of the activity referred to here, invite students to share the calculations they used to determine who had the correct value for a sphere of radius 9.

Grade 8, Unit 5, Lesson 21, Warm-up
Sphere Arguments

Lesson 20

Practice Problems

4 problems


Student Workbook

LESSON 20
PRACTICE PROBLEMS

1 a. A cube's volume is 512 cubic units. What is the length of its edge?

b. If a sphere fits snugly inside this cube, what is its volume?

c. What fraction of the cube is taken up by the sphere? What percentage is this? Explain or show your reasoning.

2 Sphere A has radius 2 cm. Sphere B has radius 4 cm.

a. Calculate the volume of each sphere.

b. The radius of Sphere B is double that of Sphere A. How many times greater is the volume of Sphere B?

Problem 1

a. A cube's volume is 512 cubic units. What is the length of its edge?

8 units

b. If a sphere fits snugly inside this cube, what is its volume?

$\frac{256}{3}\pi$ cubic units

c. What fraction of the cube is taken up by the sphere? What percentage is this? Explain or show your reasoning.

$\frac{\pi}{6}$, which is slightly more than 50%

Sample reasoning: The volume of the sphere as a fraction of the volume of the cube is $\frac{\frac{4}{3}\pi \cdot 4^3}{512}$. To make this fraction easier to work with, note that $\frac{\frac{4}{3}\pi \cdot 4^3}{512} = \frac{\frac{4}{3}\pi \cdot 4^3}{8^3} = \frac{4}{3}\pi \cdot \left(\frac{1}{2}\right)^3 = \frac{4\pi}{24} = \frac{\pi}{6}$. Since π is slightly more than 3, the fraction $\frac{\pi}{6}$ is slightly more than 50%.

Problem 2

Sphere A has radius 2 cm. Sphere B has radius 4 cm.

a. Calculate the volume of each sphere.

Sphere A: $\frac{32}{3}\pi$ cm^3, Sphere B: $\frac{256}{3}\pi$ cm^3

b. The radius of Sphere B is double that of Sphere A. How many times greater is the volume of Sphere B?

8 times greater

Problem 3

from Unit 5, Lesson 16

Three cones have a volume of 192π cm^3. Cone A has a radius of 2 cm. Cone B has a radius of 3 cm. Cone C has a radius of 4 cm. Find the height of each cone.

Cone A has a height of 144 cm. Cone B has a height of 64 cm. Cone C has a height of 36 cm.

Match the description of each sphere to its correct volume.

A. Sphere A: radius of 4 cm	**1.** 288π cm^3
B. Sphere B: diameter of 6 cm	**2.** $\frac{256}{3}\pi$ cm^3
C. Sphere C: radius of 8 cm	**3.** 36π cm^3
D. Sphere D: radius of 6 cm	**4.** $\frac{2,048}{3}\pi$ cm^3

Problem 4

from Unit 4, Lesson 15

While conducting an inventory in their bicycle shop, the owner noticed the number of bicycles is 2 fewer than 10 times the number of tricycles. They also know there are 410 wheels on all the bicycles and tricycles in the store. Write and solve a system of equations to find the number of bicycles in the store.

$b = 10t - 2$, $3t + 2b = 410$ (or equivalent)

178 bicycles

Student Workbook