Lesson 10

The Converse

Goals

- Determine whether a triangle with given side lengths is a right triangle using the converse of the Pythagorean Theorem.
- Generalize (orally) that if the side lengths of a triangle satisfy the equation $a^2 + b^2 = c^2$, then the triangle must be a right triangle.
- Justify (orally) that a triangle with side lengths 3, 4, and 5 must be a right triangle.

Learning Targets

- I can explain why it is true that if the side lengths of a triangle satisfy the equation $a^2 + b^2 = c^2$ then it must be a right triangle.
- If I know the side lengths of a triangle, I can determine if it is a right triangle or not.

Access for Students with Diverse Abilities

- Representation (Activity 1)

Access for Multilingual Learners

- MLR7: Compare and Connect (Activity 2)

Instructional Routines

- MLR7: Compare and Connect

Lesson Narrative

This lesson guides students through a proof of the converse of the Pythagorean Theorem. Students then use the converse to decide whether a triangle with three given side lengths is a right triangle or not.

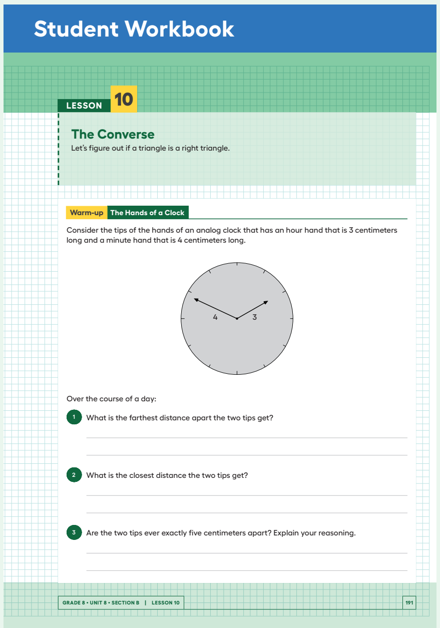
To understand the proof of the converse of the Pythagorean Theorem, students first consider the hands of a clock. As one hand rotates away from the other, the angle created between the two hands increases, as does the distance between the tips of each hand. This idea is carried over into the next activity where students are guided through a proof of the converse, which says that if $a^2 + b^2 = c^2$, then the side lengths a, b, and c must form a right triangle. In the last activity, students use this thinking around the structure of a right triangle to take a non-right triangle and determine how one of the sides can be changed to make it a right triangle.

Student Learning Goal

Let's figure out if a triangle is a right triangle.

Lesson Timeline

Warm-up	Activity 1	Activity 2	Lesson Synthesis	Cool-down (Assessment)
5 min	15 min	15 min	10 min	5 min


Student Workbook

LESSON 10

The Converse

Let's figure out if a triangle is a right triangle.

Warm-up The Hands of a Clock

Consider the tips of the hands of an analog clock that has an hour hand that is 3 centimeters long and a minute hand that is 4 centimeters long.

Over the course of a day:

1. What is the farthest distance apart the two tips get?
2. What is the closest distance the two tips get?
3. Are the two tips ever exactly five centimeters apart? Explain your reasoning.

Warm-up
The Hands of a Clock

5 min

Activity Narrative

This *Warm-up* prepares students for the argument of the converse of the Pythagorean Theorem that will be constructed in the next activity. The *Warm-up* relies on the Pythagorean Theorem and geometrically intuitive facts about how close or far apart the two hands of a clock can get from one another.

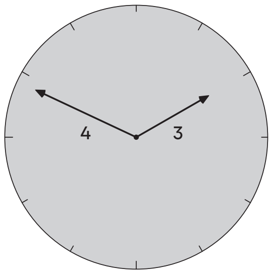
Launch

Arrange students in groups of 2.

Give students 1 minute of quiet think time, and follow with a whole-class discussion.

Student Task Statement

Consider the tips of the hands of an analog clock that has an hour hand that is 3 centimeters long and a minute hand that is 4 centimeters long.



Over the course of a day:

1. What is the farthest distance apart the two tips get?

   If the two hands are pointing in opposite directions, the tips will be 7 centimeters apart.

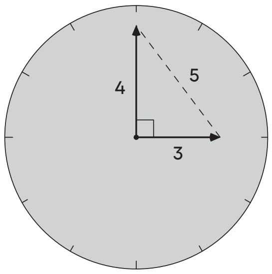
2. What is the closest distance the two tips get?

   If the two hands are pointing in the same direction (for example, at noon), the tips will be 1 centimeter apart.

3. Are the two tips ever exactly five centimeters apart? Explain your reasoning.

   Yes

   Sample reasoning: Whenever the two hands make a right angle (for example, at 3:00), then by the Pythagorean Theorem, the two tips will be 5 centimeters apart, since $3^2 + 4^2 = 5^2$.



Activity Synthesis

The focus of this discussion is to prepare students to follow a specific line of reasoning that is needed to make sense of the converse of the Pythagorean Theorem. First, invite 1–2 students to briefly share their answers to the three questions. Then ask students to consider the following line of reasoning:

Imagine two hands starting together, where one hand stays put and the other hand rotates around the face of the clock. As one hand rotates, the distance between its tip and the tip of the stationary hand continually increases until they are pointing in opposite directions. So from one moment to the next, the tips get farther apart.

(Proving this requires mathematics beyond grade 8, so for now we will just accept the results of the thought experiment as true.)

Activity 1

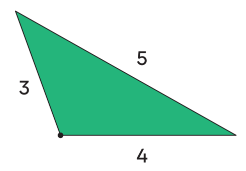
Proving the Converse

15 min

Activity Narrative

This activity introduces students to the converse of the Pythagorean Theorem: In a triangle with side lengths a, b, and c, if we have $a^2 + b^2 = c^2$, then the triangle must be a right triangle, and c must be its hypotenuse.

This may be the first time students have considered the idea that a theorem might work one way but not the other. For example, it is not clear at first glance that there is no such thing as an obtuse triangle with side lengths 3, 4, and 5, as in the image. But since $3^2 + 4^2 = 5^2$, the converse of the Pythagorean Theorem will say that the triangle must be a right triangle.

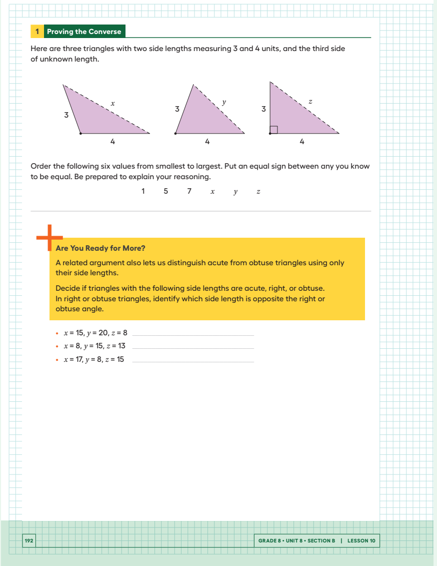


The argument this activity presents is based on the thought experiment introduced in the *Warm-up*. As the angle created by two sides of length 3 and 4 increases, the distance between their endpoints also increases, from a distance of 1 when they are pointing in the same direction to a distance of 7 when pointing in opposite directions. There is then only one angle along the way where the distance between them is 5, and by the Pythagorean Theorem, this happens when the angle between them is a right angle. This argument generalizes to any arbitrary right triangle, proving that the only angle that gives $a^2 + b^2 = c^2$ is precisely the right angle.

Access for Students with Diverse Abilities (Activity 1, Launch)

Representation: Develop Language and Symbols.
Use virtual or concrete manipulatives to connect symbols to concrete objects or values. For example, use GeoGebra or hands-on manipulatives to demonstrate how increasing or decreasing the angle between 2 sides affects the opposite side length.
Supports accessibility for: Visual-Spatial Processing, Conceptual Processing

Student Workbook



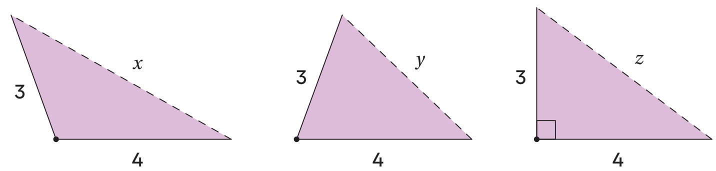
Launch

Arrange students in groups of 2.

Give students 2 minutes of quiet work time, and follow with a whole-class discussion.

Student Task Statement

Here are three triangles with two side lengths measuring 3 and 4 units, and the third side of unknown length.



Order the following six values from smallest to largest. Put an equal sign between any you know to be equal. Be prepared to explain your reasoning.

1   5   7 x y z

1   y 5 z x 7

Are You Ready for More?

A related argument also lets us distinguish acute from obtuse triangles using only their side lengths.

Decide if triangles with the following side lengths are acute, right, or obtuse. In right or obtuse triangles, identify which side length is opposite the right or obtuse angle.

- $x = 15$, $y = 20$, $z = 8$
- $x = 8$, $y = 15$, $z = 13$
- $x = 17$, $y = 8$, $z = 15$

Take the two smaller sides and call them a and b. Call the longest side c. Compute $a^2 + b^2$. If this equals c^2, it's a right triangle. If c^2 is bigger than $a^2 + b^2$, the triangle is obtuse. If c^2 is smaller than $a^2 + b^2$, the triangle is acute.

- Obtuse, since $8^2 + 15^2$ is 17^2. The square of the length of the third side, 20^2, is too big, so the triangle is obtuse, and the side of length 20 is opposite the obtuse angle.
- Acute, since $8^2 + 13^2$ is 233. The square of the length of the third side, 15^2, or 225, is too small, so the triangle is acute.
- Right, since $8^2 + 15^2 = 17^2$. The side of length 17 is opposite the right angle.

Activity Synthesis

The goal of this discussion is for students to understand that given a triangle with two side lengths of 3 and 4, there is only one angle between 0 and 180 degrees where the third side has a length of 5. Begin by inviting students to share their order of the 6 values. Consider discussing the following questions:

"Why is 1 the smallest value?"

The distance between the two vertices of a triangle with legs of length 3 and 4 could only be 1 if the two legs were on top of each other.

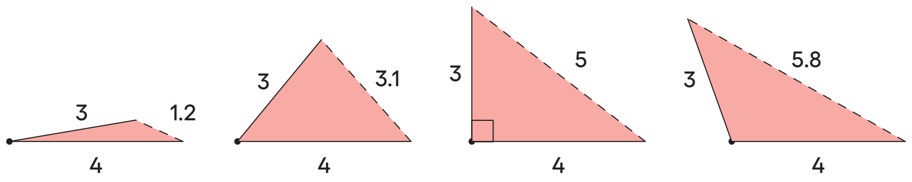
"Why is 7 the largest value?"

The distance between the two vertices of a triangle with legs of length 3 and 4 could only be 7 if the two legs were facing opposite directions.

"Why is $z = 5$?"

Since z is the hypotenuse of a right triangle, by the Pythagorean Theorem its length must be 5.

Then display this image for all to see.



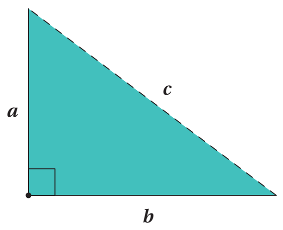
Make sure students understand the following sequence of ideas:

As we saw with the clock problem, the length of the third side continually increases from 1 to 7 as the angle increases between 0° and 180°.

Because of this, there is only one angle along the way that gives a third side length of 5.

By the Pythagorean Theorem, if the angle is a right angle, the third side length is 5.

Then display this image for all to see.



Discuss how the argument could be made with any two starting side lengths a and b by asking the following questions:

"What is the smallest possible distance between the tips of a and b?"

When sides a and b are on top of each other, or when the angle between the two sides is 0°, the distance c would be $a - b$ or $b - a$.

"What is the greatest possible distance between the tips of a and b?"

When the angle between a and b is 180°, the distance between the tips is $a + b$ or $b + a$.

Instructional Routines

MLR7: Compare and Connect

ilclass.com/r/10695592

Please log in to the site before using the QR code or URL.

Access for Multilingual Learners (Activity 2)

MLR7: Compare and Connect
This activity uses the *Compare and Connect* math language routine to advance representing and conversing as students use mathematically precise language in discussion.

"Between 0° and 180°, how many times will the distance between the tips of a and b be the same?"

The distance between the tips will never be the same.

"Between 0° and 180°, how many times will $a^2 + b^2 = c^2$?"

Only once, when the angle between a and b is 90°.

Conclude that the only for $a^2 + b^2 = c^2$ is if the triangle is a right triangle, with hypotenuse c. This result is called "the converse of the Pythagorean Theorem."

Activity 2

Calculating Legs of Right Triangles

15 min

Activity Narrative

The purpose of this activity is for students to apply the Pythagorean Theorem and its converse in mathematical contexts. In the first problem, students apply the Pythagorean Theorem to find unknown side lengths. In the second problem, students use the fact that by changing side lengths so that they satisfy $a^2 + b^2 = c^2$, the resulting triangle is guaranteed to be a right triangle.

- Monitor for students who solve the second problem by:
- Using 4 and 6 as legs and solving for the hypotenuse.
- Using 4 and 7 as legs and solving for the hypotenuse.
- Using 6 and 7 as legs and solving for the hypotenuse.
- Using 7 as the hypotenuse and either 4 or 6 as one of the legs and solving for the other leg.
- Using 6 as the hypotenuse and 4 as the leg and solving for the other leg.

While it is not necessary to discuss all of these different approaches, monitor for at least one where the value changed is a leg and one where it is the hypotenuse.

Launch

Provide students with access to calculators.

Display the image from the first question. Tell students that the right angle is not labeled, but the prompt tells us these are both right triangles. Ask,

"How do we know which side of a right triangle is the hypotenuse?"

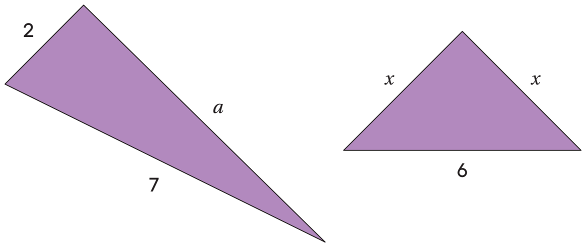
It is the longest side of the right triangle.

Give students 4–5 minutes of quiet work time, and follow with a whole-class discussion.

Select work from students with different strategies, such as those described in the *Activity Narrative*, to share later.

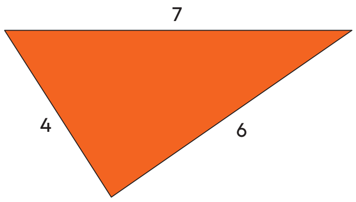
Student Task Statement

1. Given the information provided for the right triangles shown here, find the unknown leg lengths to the nearest tenth.



$a = 6.7$ $x = 4.2$

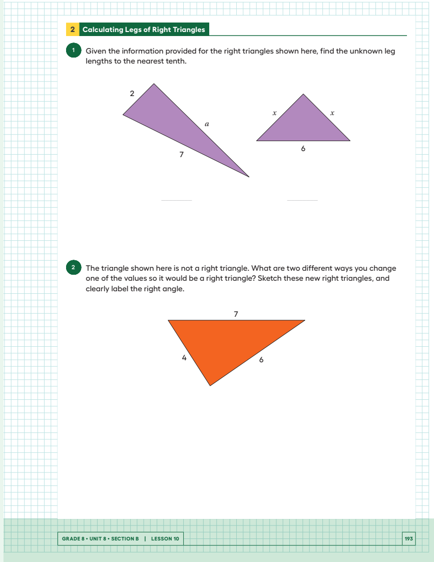
2. The triangle shown here is not a right triangle. What are two different ways you change *one* of the values so it would be a right triangle? Sketch these new right triangles, and clearly label the right angle.



Triangles sketched with the right angle labeled opposite the hypotenuse.

Sample response: If 4 and 6 were legs of a right triangle, then the hypotenuse would be the value of c in the equation $4^2 + 6^2 = c^2$. This means $c^2 = 52$ and $c = \sqrt{52} \approx 7.2$.

Student Workbook


2 Calculating Legs of Right Triangles

1 Given the information provided for the right triangles shown here, find the unknown leg lengths to the nearest tenth.

2 The triangle shown here is not a right triangle. What are two different ways you change one of the values so it would be a right triangle? Sketch these new right triangles, and clearly label the right angle.

Activity Synthesis

The goal of this discussion is for students to share how they used the Pythagorean Theorem to reason about the side lengths of right triangles. Select 1–2 students to share their work for the first problem.

For the second problem, display 2–3 approaches from previously selected students for all to see. Use *Compare and Connect* to help students compare, contrast, and connect the different approaches. It is not necessary to go through all of the possibilities—the focus here is to see how the various equations of the form $a^2 + b^2 = c^2$ informed how they sketched their right triangles. Here are some questions for discussion:

- *"What do the approaches have in common? How are they different?"*

  "What kinds of additional details or language helped you understand the displays? Are there any additional details or language that you have questions about?"

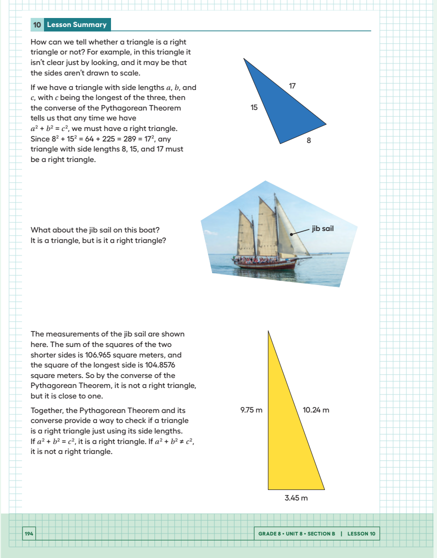
  "Did anyone solve the problem the same way, but would explain it differently?"

If time allows:

- *"How many different ways are there to solve this problem?"*

  6 ways

Student Workbook


10 Lesson Summary

How can we tell whether a triangle is a right triangle or not? For example, in this triangle it isn't clear just by looking, and it may be that the sides aren't drawn to scale.

If we have a triangle with side lengths a, b, and c, with c being the longest of the three, then the converse of the Pythagorean Theorem tells us that any time we have $a^2 + b^2 = c^2$, we must have a right triangle. Since $8^2 + 15^2 = 64 + 225 = 289 = 17^2$, any triangle with side lengths 8, 15, and 17 must be a right triangle.

What about the jib sail on this boat? It is a triangle, but is it a right triangle?

The measurements of the jib sail are shown here. The sum of the squares of the two shorter sides is 106.965 square meters, and the square of the longest side is 104.8576 square meters. So by the converse of the Pythagorean Theorem, it is not a right triangle, but it is close to one.

Together, the Pythagorean Theorem and its converse provide a way to check if a triangle is a right triangle just using its side lengths. If $a^2 + b^2 = c^2$, it is a right triangle. If $a^2 + b^2 \neq c^2$, it is not a right triangle.

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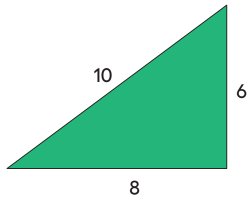
Lesson Synthesis

The purpose of this discussion is to summarize the Pythagorean Theorem and its converse. Begin by displaying this image:



Discuss:

- *"Do you think this triangle is a right triangle or not? Explain your reasoning."*

  Answers vary.

- *"How can we tell for sure if this is a right triangle, even though no right angle is indicated?"*

  We can use the Pythagorean Theorem and see if the sum of the squares of the two shorter sides equals the square of the longest side.

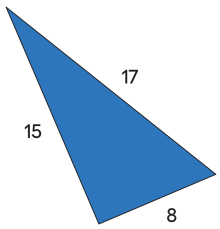
Give students 1–2 minutes of quiet work time to see if the Pythagorean Theorem holds for this triangle. Emphasize that while the angle across from the side of length 10 looks like a right angle, we can't be sure it is just from the image.

Invite 1–2 students to share their work showing that the triangle is a right triangle, or display the following for all to see:

Since $6^2 + 8^2 = 10^2$ is true, we know, by the converse of the Pythagorean Theorem, that the triangle is a right triangle and that the angle across from the side of length 10 is a right angle.

Lesson Summary

How can we tell whether a triangle is a right triangle or not? For example, in this triangle it isn't clear just by looking, and it may be that the sides aren't drawn to scale.



If we have a triangle with side lengths a, b, and c, with c being the longest of the three, then the converse of the Pythagorean Theorem tells us that any time we have $a^2 + b^2 = c^2$, we must have a right triangle. Since $8^2 + 15^2 = 64 + 225 = 289 = 17^2$, any triangle with side lengths 8, 15, and 17 must be a right triangle.

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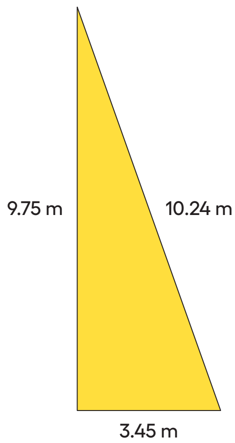


Together, the Pythagorean Theorem and its converse provide a way to check if a triangle is a right triangle just using its side lengths. If $a^2 + b^2 = c^2$, it is a right triangle. If $a^2 + b^2 \neq c^2$, it is not a right triangle.

Cool-down

Is It a Right Triangle?

5 min

Student Task Statement

The triangle has side lengths 7, 10, and 12. Is it a right triangle? Explain your reasoning.

No

If this were a right triangle, then $7^2 + 10^2$ would equal 12^2. However, this is not the case.

Responding To Student Thinking

Points to Emphasize

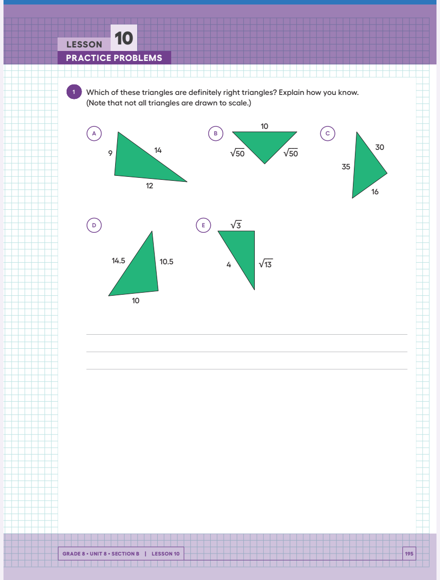
If most students struggle with determining whether or not a triangle is a right triangle, focus on identifying situations when the Pythagorean Theorem can be used. For example, in the activity referred to here, ask students if the Pythagorean Theorem could be used if the field Mai and Tyler are at were not a rectangle.

Unit 8, Lesson 11, Activity 2 Cutting Corners

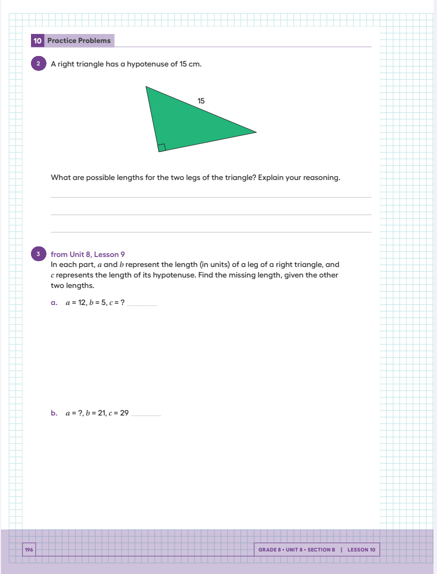
Practice Problems

5 Problems

Student Workbook

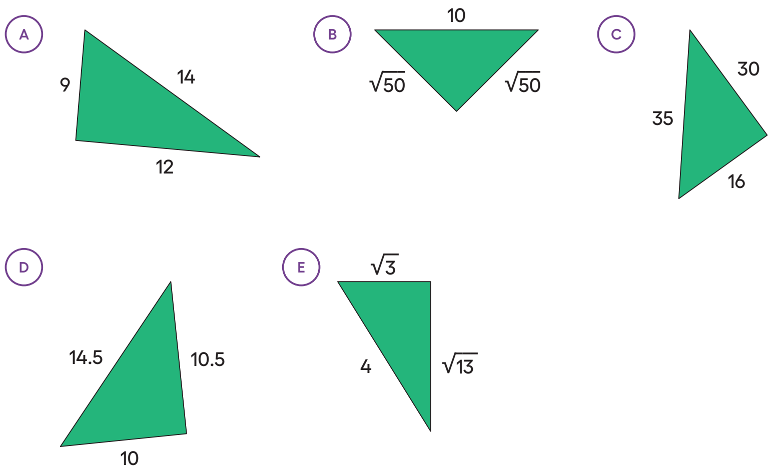


Student Workbook



Problem 1

Which of these triangles are definitely right triangles? Explain how you know. (Note that not all triangles are drawn to scale.)

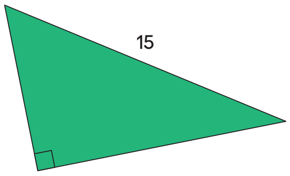


B, D, and E are right triangles. A and C are not.

- A: $9^2 + 12^2 = 14^2$ is false
- B: $\sqrt{50}^2 + \sqrt{50}^2 = 10^2$ is true
- C: $16^2 + 30^2 = 35^2$ is false
- D: $10^2 + 10.5^2 = 14.5^2$ is true
- E: $\sqrt{3}^2 + \sqrt{13}^2 = 4^2$ is true

Problem 2

A right triangle has a hypotenuse of 15 cm.



What are possible lengths for the two legs of the triangle? Explain your reasoning.

Sample responses: $\sqrt{200}$ cm and 5 cm, $\sqrt{125}$ cm and 10 cm. Sample reasoning: If the legs of the triangle are a and b, then we can set up the equation $a^2 + b^2 = 15^2$.
This means a^2 and b^2 must sum to 225 cm. If $a^2 = 25$ cm², then $b = 200$ cm. If $a^2 = 100$ cm², then $b^2 = 125$ cm².

Problem 3

from Unit 8, Lesson 9

In each part, a and b represent the length (in units) of a leg of a right triangle, and c represents the length of its hypotenuse. Find the missing length, given the other two lengths.

a. $a = 12$, $b = 5$, $c = ?$

$c = 13$ units

Sample reasoning: If $a = 12$ and $b = 5$, then $12^2 + 5^2 = c^2$.

b. $a = ?$, $b = 21$, $c = 29$

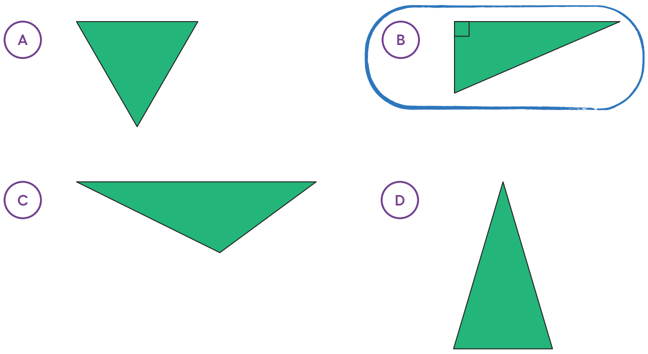
$a = 20$ units

Sample reasoning: If $b = 21$ and $c = 29$, then $a^2 + 21^2 = 29^2$.

Problem 4

from Unit 8, Lesson 7

For which triangle does the Pythagorean Theorem express the relationship between the lengths of its three sides?



Problem 5

from Unit 4, Lesson 5

Andre makes a trip to Mexico. He exchanges some dollars for pesos at a rate of 20 pesos per dollar. While in Mexico, he spends 9,000 pesos. When he returns, he exchanges his pesos for dollars (still at 20 pesos per dollar). He gets back $\frac{1}{10}$ the amount he started with. Find how many dollars Andre exchanges for pesos and explain your reasoning. If you get stuck, try writing an equation representing Andre's trip using a variable for the number of dollars he exchanges.

500 dollars

Sample reasoning: $\frac{20x - 9,000}{20} = \frac{x}{10}$, where x represents the number of dollars he exchanged. Rewrite the equation as $20x - 9,000 = 2x$, and then solve to find $x = 500$.

Student Workbook