Lesson 21

Cylinders, Cones, and Spheres

Goals

- Calculate the value of the radius of a sphere with a given volume using the structure of the equation, and explain (orally) the solution method.
- Determine what information is needed to solve a problem involving volumes of cones, cylinders, and spheres. Ask questions to elicit that information.

Learning Targets

- I can find the radius of a sphere if I know its volume.
- I can solve mathematical and real-world problems about the volume of cylinders, cones, and spheres.

Lesson Narrative

In this lesson, students use the formula for the volume of a sphere to solve various problems. The *Warm-up* begins with an opportunity to analyze common errors that people make when using this formula.

The main activity in this lesson is an *Information Gap*. In it, students must reason about a cone and sphere with matching dimensions and ask precise questions to get the information they need to solve the problem.

The second and last activities in the lesson are optional. In the second activity, students use the structure of an equation to find the radius of a sphere when they know its volume. Consider using this activity if students would benefit from additional practice working with the volume of a sphere before the *Information Gap*. In the last activity, they have opportunities to practice using all of the new volume formulas they have learned in this unit. Use this activity if time allows and students would benefit from extra practice working with volume formulas and equations involving π.

Display the poster with the volume formulas from this unit throughout the lesson for students to refer to.

Student Learning Goal

Let's find the volume of shapes.

Access for Students with Diverse Abilities

- Action and Expression (Activity 2)

Access for Multilingual Learners

- MLR4: Information Gap (Activity 2)
- MLR8: Discussion Supports (Activity 3)

Instructional Routines

- MLR4: Information Gap

Required Materials

Materials to Gather

- Tools for creating a visual display: Lesson
- Math Community Chart: Activity 2

Materials to Copy

- Unknown Dimensions Cards (1 copy for every 4 students): Activity 2

Required Preparation

Lesson:
Provide students with tools to make a visual display during the *Lesson Synthesis*.

Lesson Timeline

5 min	5 min	20 min	10 min	10 min	5 min
Warm-up	Activity 1	Activity 2	Activity 3	Lesson Synthesis	Cool-down

Assessment: Cool-down (5 min)

Warm-up
Sphere Arguments

5 min

Activity Narrative

The purpose of this *Warm-up* is for students to consider some common errors that happen when calculating the volume of a sphere. This work will help prepare students for the *Information Gap* in this lesson.

Monitor for students who use the formula for a cylinder or cone, who use r^2 instead of r^3, or who forget to include π as a factor in the computation.

Launch

Arrange students in groups of 2.

Give students 1–2 minutes of quiet work time followed by time to discuss their responses with their partner.

Student Task Statement

Four students each calculated the volume of a sphere with a radius of 9 centimeters, and they got four different answers.

- Han thinks it is 108 cubic centimeters.
- Jada gets 108π cubic centimeters.
- Tyler calculates 972 cubic centimeters.
- Mai says it is 972π cubic centimeters.

Do you agree with any of them? Explain your reasoning.

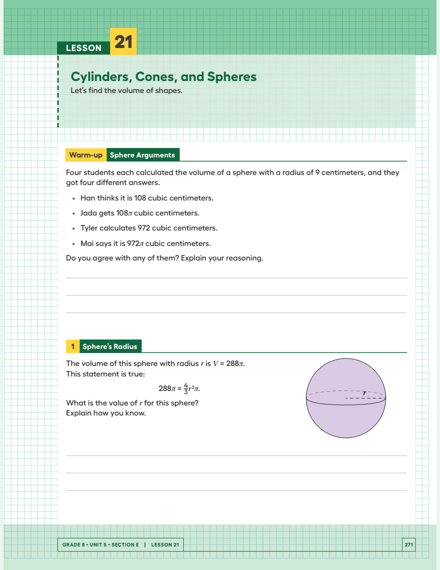
Mai's calculation is correct.

Sample explanation: The volume of a sphere is found with the formula $V = \frac{4}{3}\pi r^3$. Using 9 for the radius, the volume is $\frac{4}{3}\pi(9^3) = \frac{4}{3}\pi(729) = 972\pi$.

Activity Synthesis

For each answer, ask students to indicate whether or not they agree. Display the number of students who agree with each answer all to see. Invite someone who agreed with 972π to explain their reasoning. Ask students if they think they know what the other students did incorrectly to get their answers. (To get 108, Han and Jada likely used r^2 instead of r^3, and Tyler may have forgotten to write π as part of his answer.)

Student Workbook


LESSON 21

Cylinders, Cones, and Spheres

Let's find the volume of shapes.

Warm-up Sphere Arguments

Four students each calculated the volume of a sphere with a radius of 9 centimeters, and they got four different answers.

- Han thinks it is 108 cubic centimeters.
- Jada gets 108π cubic centimeters.
- Tyler calculates 972 cubic centimeters.
- Mai says it is 972π cubic centimeters.

Do you agree with any of them? Explain your reasoning.

1 Sphere's Radius

The volume of this sphere with radius r is $V = 288\pi$.
This statement is true:

$288\pi = \frac{4}{3}r^3\pi$.

What is the value of r for this sphere?
Explain how you know.

Building on Student Thinking

Students who substituted a value for π and solved the resulting equation may have rounded along the way, making the value for the radius slightly less than 6, while the actual value is exactly 6. This is a good opportunity to talk about the effects of rounding and how to minimize the error that rounding introduces.

Activity 1 : Optional

Sphere's Radius

5 min

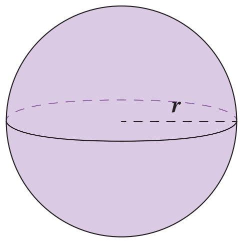
Activity Narrative

The purpose of this activity is for students to think about how to find the radius of a sphere when its volume is known. Students can examine the structure of the equation for volume and reason about a number that makes the equation true. They can also notice that π is a factor on each side of the equation, so the equation is still true if rewritten without π on each side. Both strategies simplify the solution process and minimize the need for rounding. Watch for students who substitute a value for π to each side of the equation, who use the structure of the equation to reason about the solution, or who solve another way so that these strategies can be shared and compared in the whole-class discussion.

Launch

Allow students 3–4 minutes work time, and follow with a whole-class discussion.

Student Task Statement



The volume of this sphere with radius r is $V = 288\pi$.

This statement is true:

$$288\pi = \frac{4}{3}r^3\pi.$$

What is the value of r for this sphere? Explain how you know.

6 units

Sample responses: Examine the equation $288\pi = \frac{4}{3}r^3\pi$. The value π appears on both sides of the equation. This means that the remaining factors on each side must be equal, so $288 = \frac{4}{3}r^3$. Multiplying each side by $\frac{3}{4}$ gives $216 = r^3$. The number 6 yields 216 when cubed.

Activity Synthesis

The purpose of the discussion is to examine how students reasoned through each step in solving for the unknown radius. Ask previously identified students to share their responses.

Consider asking students the following questions to help clarify the different approaches students took:

"π appears on both sides of the volume equation. Did you deal with this as a first step or later in the solution process?"

"How did you deal with the fraction in the equation?"

"If the final step in your solution was solving for r when $r^3 = 216$, how did you solve? If you found that r^3 was a different number, how did you solve?"

Activity 2

Info Gap: Unknown Dimensions

20 min

Activity Narrative

In this activity, students calculate the volumes of spheres but do not initially have enough information to do so. To bridge the gap, they need to exchange questions and ideas.

The *Information Gap* structure requires students to make sense of problems by determining what information is necessary and then to ask for information they need to solve it. This may take several rounds of discussion if their first requests do not yield the information they need. It also allows them to refine the language they use and ask increasingly more precise questions until they get the information they need.

If any partners finish quickly, perhaps because they recall that the volume of a cone is half the volume of a sphere with matching dimensions, encourage them to collaborate to find a different way to solve the problem.

Launch

Math Community

Display the Math Community Chart for all to see. Give students a brief quiet think time to read the norms, or invite a student to read them out loud. Tell students that during this activity they are going to practice looking for their classmates putting the norms into action. At the end of the activity, students can share what norms they saw and how the norm supported the mathematical community during the activity.

Tell students they will practice calculating the volume of spheres. Display the *Information Gap* graphic that illustrates a framework for the routine for all to see.

Remind students of the structure of the *Information Gap* routine, and consider demonstrating the protocol if students are unfamiliar with it.

Arrange students in groups of 2. In each group, give a problem card to one student and a data card to the other student. After reviewing their work on the first problem, give students the cards for a second problem and instruct them to switch roles.

Instructional Routines

MLR4: Information Gap

ilclass.com/r/10695522
Please log in to the site before using the QR code or URL.

Access for Multilingual Learners (Activity 2)

MLR4: Information Gap
This activity uses the *Information Gap* math language routine, which facilitates meaningful interactions by positioning some students as holders of information that is needed by other students, creating a need to communicate.

Access for Students with Diverse Abilities (Activity 2, Launch)

Action and Expression: Internalize Executive Functions.
Check for understanding by inviting students to rephrase directions in their own words. Keep a display of the *Information Gap* graphic visible throughout the activity, or provide students with a physical copy.

Supports accessibility for: Memory, Organization

Student Workbook


2 Info Gap: Unknown Dimensions
Your teacher will give you either a problem card or a data card. Do not show or read your card to your partner.
If your teacher gives you the problem card:
1 Silently read your card and think about what information you need to answer the question.
2 Ask your partner for the specific information that you need. "Can you tell me ____?"
3 Explain to your partner how you are using the information to solve the problem. "I need to know ____ because . . ." Continue to ask questions until you have enough information to solve the problem.
4 Once you have enough information, share the problem card with your partner, and solve the problem independently.
5 Read the data card, and discuss your reasoning.
If your teacher gives you the data card:
1 Silently read your card. Wait for your partner to ask for information.
2 Before telling your partner any information, ask, "Why do you need to know ____?"
3 Listen to your partner's reasoning and ask clarifying questions. Only give information that is on your card. Do not figure out anything for your partner! These steps may be repeated.
4 Once your partner says they have enough information to solve the problem, read the problem card, and solve the problem independently.
5 Share the data card, and discuss your reasoning.
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Student Task Statement

Your teacher will give you either a problem card or a data card. Do not show or read your card to your partner.

If your teacher gives you the problem card:

1. Silently read your card and think about what information you need to answer the question.
2. Ask your partner for the specific information that you need. "Can you tell me _____ ?"
3. Explain to your partner how you are using the information to solve the problem. "I need to know _____ because …"
   Continue to ask questions until you have enough information to solve the problem.
4. Once you have enough information, share the problem card with your partner, and solve the problem independently.
5. Read the data card, and discuss your reasoning.

If your teacher gives you the data card:

1. Silently read your card. Wait for your partner to ask for information.
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3. Listen to your partner's reasoning and ask clarifying questions.
   Only give information that is on your card. Do not figure out anything for your partner!
   These steps may be repeated.
4. Once your partner says they have enough information to solve the problem, read the problem card, and solve the problem independently.
5. Share the data card, and discuss your reasoning.

Problem card 1: The volume of the sphere is 288π cm^3. Students can calculate this value by finding the radius of the cone and then using the volume formula for a sphere, or they can use the fact that the volume of a sphere is twice that of the volume of a cone with the same dimensions.

Problem Card 2: Possible solution paths:

- Use the volume of the cone and $h = 2r$ to find $r = 3$, so the volume of the sphere is 36π cm^3.
- Since the radius and height of the cone and sphere are equal, the volume of the sphere must be twice the volume of the cone, or 36π cm^3.

Activity Synthesis

After students have completed their work, share the correct answers, and ask students to discuss the process of solving the problems. Here are some questions for discussion:

- *"How did you decide what information to ask for first? How did the information on your card help?"*

  My card said the cone and the sphere have the same dimensions, so I started by asking for those dimensions so I could use the volume formula.

- *"Give an example of a question that you asked, the clue you received, and how you made use of it."*

  I asked for the volume of the cone and was told it was 144π cm³. I used that value, the fact that the height of the cone is double the radius, and the volume formula for a cone to figure out the radius of the cone. Since the radius of the sphere is the same, I could then calculate the volume of the sphere.

- *"When you asked your partner why they needed a specific piece of information, what kind of explanations did you consider acceptable?"*

As students respond, highlight any student sketches that include labeled dimensions, and display these for all to see. In particular, contrast students who used volume formulas to determine the value of the radius with those who remembered that the volume of a cone is half the volume of a sphere with the same dimensions (radius and height).

Math Community

Conclude the discussion by inviting 2–3 students to share a norm they identified in action. Provide this sentence frame to help students organize their thoughts in a clear, precise way:

- *"I noticed our norm '___' in action today, and it really helped me/my group because ___."*

Activity 3: Optional
The Right Fit

10 min

Activity Narrative

In this activity, students once again consider different figures with given dimensions, this time comparing their capacity to contain a certain amount of water. The goal is for students to not only apply the correct volume formulas, but to slow down and think about how the dimensions of the figures compare and how those measurements affect the volume of the figures.

Launch

Arrange students in groups of 2.

Give students 5 minutes of quiet work time, then time for a partner discussion.

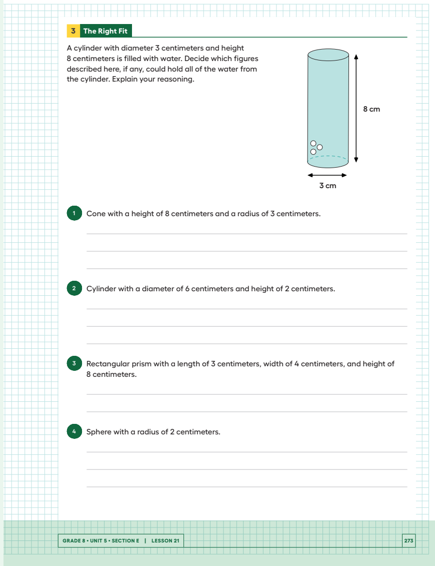
Follow with a whole-class discussion.

Access for Multilingual Learners (Activity 3, Launch)

MLR8: Discussion Supports. Display sentence frames to support students as they work with their partner and explain their reasoning. Examples: "I think ____ because _____." or "I (agree/disagree) because ______." or "The volume is (greater than/less than) _____, which means _____."

Advances: Speaking, Conversing

Student Workbook


3 The Right Fit

A cylinder with diameter 3 centimeters and height 8 centimeters is filled with water. Decide which figures described here, if any, could hold all of the water from the cylinder. Explain your reasoning.

1. Cone with a height of 8 centimeters and a radius of 3 centimeters.
2. Cylinder with a diameter of 6 centimeters and height of 2 centimeters.
3. Rectangular prism with a length of 3 centimeters, width of 4 centimeters, and height of 8 centimeters.
4. Sphere with a radius of 2 centimeters.

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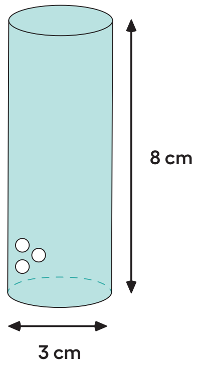
Student Task Statement



A cylinder with diameter 3 centimeters and height 8 centimeters is filled with water. Decide which figures described here, if any, could hold all of the water from the cylinder. Explain your reasoning.

The cone, cylinder, and rectangular prism can hold the water. The sphere cannot.

Sample response: The volume of the given cylinder is 18π, or around 56.5 cubic centimeters: $V = \pi\left(\frac{3}{2}\right)^2(8) = \pi\left(\frac{9}{4}\right)(8) = 18\pi$.

1. Cone with a height of 8 centimeters and a radius of 3 centimeters.

   Cone: $V = \frac{1}{3}\pi(3^2)(8) = \frac{1}{3}\pi(9)(8) = 24\pi$

   Since the volume is greater than 18π, the cone can hold the water.

2. Cylinder with a diameter of 6 centimeters and height of 2 centimeters.

   Cylinder: $V = \pi(3^2)(2) = \pi(9)(2) = 18\pi$

   Since the volume is equal to 18π, the cylinder can hold the water.

3. Rectangular prism with a length of 3 centimeters, width of 4 centimeters, and height of 8 centimeters.

   Rectangular prism: $V = (3)(4)(8) = 96$

   Since the volume is greater than 56.5, the rectangular prism can hold the water.

4. Sphere with a radius of 2 centimeters.

   Sphere: $V = \frac{4}{3}\pi(2^3) = \frac{4}{3}\pi(8) = \frac{32}{3}\pi$

   Since the volume is less than 18π, the sphere cannot hold the water.

Are You Ready for More?

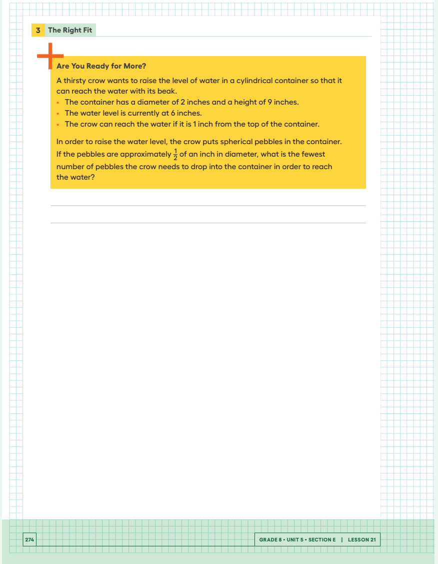
A thirsty crow wants to raise the level of water in a cylindrical container so that it can reach the water with its beak.

- The container has a diameter of 2 inches and a height of 9 inches.
- The water level is currently at 6 inches.
- The crow can reach the water if it is 1 inch from the top of the container.

In order to raise the water level, the crow puts spherical pebbles in the container. If the pebbles are approximately $\frac{1}{2}$ of an inch in diameter, what is the fewest number of pebbles the crow needs to drop into the container in order to reach the water?

The current volume of water is $V_1 = \pi \cdot 1^2 \cdot 6$ cubic inches, and the volume required is $V_2 = \pi \cdot 1^2 \cdot 8$ cubic inches. This means that the difference in volumes is $\pi \cdot 1^2 \cdot 2 \cong 6.28$ cubic inches. Each pebble has volume $V_p = \frac{4}{3} \cdot \pi \cdot \left(\frac{1}{4}\right)^3 \cong 0.065$ cubic inches. In order to raise the water by a volume of 6.28 cubic inches, 96.62 pebbles need to be added because $6.28/0.065 \cong 96.62$. Since fractional pebbles are not possible, the crow needs to add 97 pebbles.

Student Workbook


3 The Right Fit

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Activity Synthesis

The purpose of this discussion is to compare volumes of different figures by computation and also by considering the effect that different dimensions have on volume.

Ask students if they made any predictions about the volumes before directly computing them and, if yes, how they were able to predict. Students might have reasoned, for example, that the second cylinder had double the radius of the first, which would make the volume 4 times as great, but the height was only $\frac{1}{4}$ as great, so the volume would be the same. Or they might have seen that the cone would have a greater volume since the radius was double and the height was the same, making the volume (if it were another cylinder) 4 times as great, so the factor of $\frac{1}{3}$ for the cone didn't bring the volume down below the volume of the cylinder.

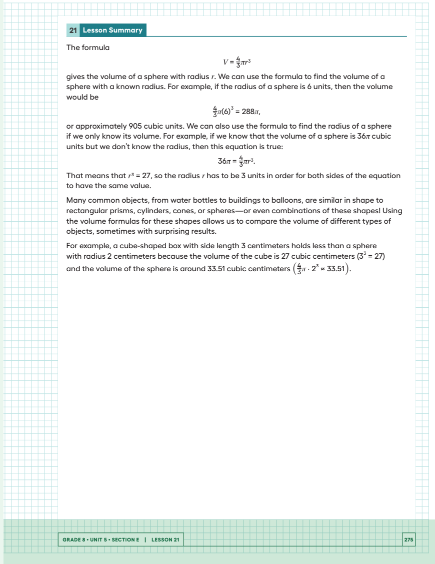
Lesson Synthesis

In this unit, students have learned how to find the volume of cylinders, cones, and spheres, how to find an unknown dimension when the volume and another dimension are known, and how to reason about the effects of different dimensions of volume. Assign groups of 2–3 students one of the questions shown here, and provide them with the tools to make a visual display explaining their response. Encourage students to make their displays as if they were explaining the answer to the question to someone who is not in the class, and encourage them to make up values for dimensions to illustrate their ideas. Suggest sketches of figures where appropriate.

"Describe some relationships between the volumes of cylinders, cones, and spheres."

"How do we find a missing dimension of a cylinder, cone, or sphere when we know the volume and another dimension (or just the volume in the case of the sphere)?"

Student Workbook



"What happens to the volume of a cylinder or cone when its height is doubled? Tripled?"

"What happens to the volume of a sphere when its height is doubled? Tripled?"

"What happens to the volume of a cylinder, cone, or sphere when its radius is doubled? Tripled?"

"What happens to the volume of a cylinder or cone when its height is doubled and its radius is halved?"

"What happens to the volume of a cylinder or cone when its radius is doubled and its height is halved?"

Lesson Summary

The formula

$$V = \frac{4}{3}\pi r^3$$

gives the volume of a sphere with radius r. We can use the formula to find the volume of a sphere with a known radius. For example, if the radius of a sphere is 6 units, then the volume would be

$$\frac{4}{3}\pi(6)^3 = 288\pi,$$

or approximately 905 cubic units. We can also use the formula to find the radius of a sphere if we only know its volume. For example, if we know that the volume of a sphere is 36π cubic units but we don't know the radius, then this equation is true:

$$36\pi = \frac{4}{3}\pi r^3.$$

That means that $r^3 = 27$, so the radius r has to be 3 units in order for both sides of the equation to have the same value.

Many common objects, from water bottles to buildings to balloons, are similar in shape to rectangular prisms, cylinders, cones, or spheres—or even combinations of these shapes! Using the volume formulas for these shapes allows us to compare the volume of different types of objects, sometimes with surprising results.

For example, a cube-shaped box with side length 3 centimeters holds less than a sphere with radius 2 centimeters because the volume of the cube is 27 cubic centimeters ($3^3 = 27$) and the volume of the sphere is around 33.51 cubic centimeters ($\frac{4}{3}\pi \cdot 2^3 \approx 33.51$).

Cool-down

New Four Spheres

5 min

Student Task Statement

Some information is given about each sphere. Order them from least volume to greatest volume. You may sketch a sphere to help you visualize if you prefer.

Sphere A has a radius of 4.

Sphere B has as a diameter of 6.

Sphere C has a volume of 64π.

Sphere D has a radius double that of sphere B.

B, C, A, D

Sphere A has a radius of 4, so its volume is $\frac{256}{3}\pi$.

Sphere B has a diameter of 6, so its radius is 3, and its volume is 36π.

Sphere C has a volume of 64π.

Sphere D has a radius twice as large as sphere B, so its radius is 6, and its volume is 288π.

Responding To Student Thinking

Press Pause

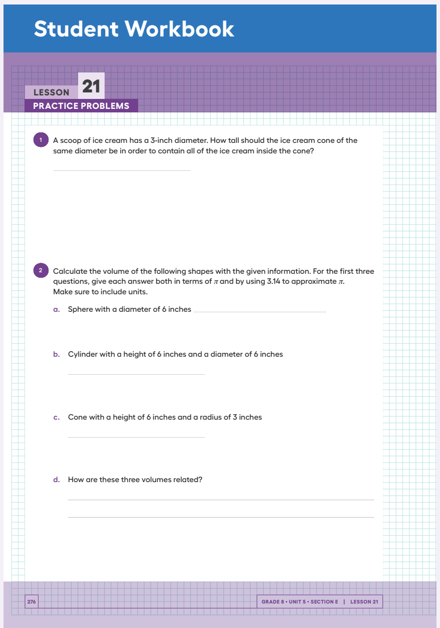
By this point in the unit, there should be some student mastery of working with volume formulas. If most students struggle, make time to complete some or all of the optional lesson referred to here:

Unit 5, Lesson 22 Volume as a Function of ... Grade 8, Unit 5, Lesson 22 Volume as a Function of ...

Lesson 21

Practice Problems

4 Problems


Student Workbook

LESSON 21
PRACTICE PROBLEMS

1 A scoop of ice cream has a 3-inch diameter. How tall should the ice cream cone of the same diameter be in order to contain all of the ice cream inside the cone?

2 Calculate the volume of the following shapes with the given information. For the first three questions, give each answer both in terms of π and by using 3.14 to approximate π. Make sure to include units.

a. Sphere with a diameter of 6 inches

b. Cylinder with a height of 6 inches and a diameter of 6 inches

c. Cone with a height of 6 inches and a radius of 3 inches

d. How are these three volumes related?

Problem 1

A scoop of ice cream has a 3-inch diameter. How tall should the ice cream cone of the same diameter be in order to contain all of the ice cream inside the cone?

6 inches, because the volume of the ice cream is $\frac{4}{3}\pi(1.5)^3$ must be the same as the volume of the cone given by $\frac{1}{3}\pi(1.5)^2 h$

Problem 2

Calculate the volume of the following shapes with the given information. For the first three questions, give each answer both in terms of π and by using 3.14 to approximate π. Make sure to include units.

a. Sphere with a diameter of 6 inches

36π, or about 113.04, cubic inches

b. Cylinder with a height of 6 inches and a diameter of 6 inches

54π, or about 169.56, cubic inches

c. Cone with a height of 6 inches and a radius of 3 inches

18π, or about 56.52, cubic inches

d. How are these three volumes related?

Sample response: The volume of the cone plus the volume of the sphere equals the volume of the cylinder. This is like when a sphere fits snugly inside of a cylinder, and after pouring in a cone full of water, the cylinder fills completely to the top. The cone takes up $\frac{1}{3}$ of the cylinder, and the sphere takes up the other $\frac{2}{3}$.

Problem 3

A coin-operated bouncy ball dispenser has a large glass sphere that holds many spherical balls. The large glass sphere has a radius of 9 inches. Each bouncy ball has radius of 1 inch and sits inside the dispenser.

If there are 243 bouncy balls in the large glass sphere, what proportion of the large glass sphere's volume is taken up by bouncy balls? Explain how you know.

About 33% (or $\frac{1}{3}$) of the sphere's volume is taken up by bouncy balls. Sample reasoning: The large glass sphere has radius 9 inches, so its volume in cubic inches is $\frac{4}{3}\pi(9)^3$. This is about 3,054 cubic inches. Each bouncy ball has radius 1 inch. The volume of 243 bouncy balls in cubic inches is $243 \cdot \frac{4}{3}\pi(1)^3$. This is about 1,018 cubic inches. To find the proportion taken up by bouncy balls, divide $\frac{1018}{3,054} \approx 0.33$. About 33% of the space is taken up by bouncy balls.

Problem 4

from Unit 5, Lesson 13

A farmer has a water tank for cows in the shape of a cylinder with radius of 7 feet and a height of 3 feet. The tank comes equipped with a sensor to alert the farmer to fill it up when the water falls to 20% capacity. What is the volume of the tank when the sensor turns on?

About 92 cubic feet, because the volume of the cylinder is approximately 462 cubic feet and the sensor turns on when 20% of this volume remains

Student Workbook


21 Practice Problems

3 A coin-operated bouncy ball dispenser has a large glass sphere that holds many spherical balls. The large glass sphere has a radius of 9 inches. Each bouncy ball has radius of 1 inch and sits inside the dispenser.

If there are 243 bouncy balls in the large glass sphere, what proportion of the large glass sphere's volume is taken up by bouncy balls? Explain how you know.

4 from Unit 5, Lesson 13

A farmer has a water tank for cows in the shape of a cylinder with radius of 7 feet and a height of 3 feet. The tank comes equipped with a sensor to alert the farmer to fill it up when the water falls to 20% capacity. What is the volume of the tank when the sensor turns on? ____________

Learning Targets

+ I can find the radius of a sphere if I know its volume.
+ I can solve mathematical and real-world problems about the volume of cylinders, cones, and spheres.

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