Lesson 16

Finding Cone Dimensions

Goals

- Calculate the value of one dimension of a cone, and explain (orally and in writing) the reasoning.
- Compare volumes of a cone and cylinder in context, and justify (orally) which volume is a better value for a given price.
- Create a table of dimensions of cones, and describe (orally) patterns that arise.

Learning Target

I can find missing information about a cone if I know its volume and some other information.

Access for Students with Diverse Abilities

- Action and Expression (Warm-up, Activity 2)

Access for Multilingual Learners

- MLR5: Co-Craft Questions (Activity 3)
- MLR8: Discussion Supports (Warm-up, Activity 1)

Instructional Routines

- Math Talk

Lesson Narrative

As they did with cylinders in a previous lesson, in this lesson students compute volumes given radii and heights and find the radius or height given a cylinder's volume and the other dimension by reasoning about the structure of the volume formula.

Next, they apply their understanding about the volumes of cylinders and cones to decide which popcorn container and price offers the best deal. This activity offers students the chance to reason about volume, out-of-scale images, and price per unit calculations. Students may use some or all of these ideas as they explain their reasoning and critique the reasoning of others for which container is the best deal.

Student Learning Goal

Let's figure out the dimensions of cones.

Lesson Timeline

Warm-up	Activity 1	Activity 2	Activity 3	Lesson Synthesis	Cool-down (Assessment)
5 min	5 min	15 min	10 min	10 min	5 min

Warm-up
Math Talk: Thirds

5 min

Activity Narrative

This *Math Talk* focuses on finding the unknown value in an equation. It encourages students to think about the fraction $\frac{1}{3}$ and to rely on the structure of the equations to mentally solve problems. The strategies elicited here will be helpful later in the lesson when students are solving for the unknown length of the radius or height of a cone given its volume.

To solve each equation mentally, students need to look for and make use of structure.

Launch

Tell students to close their student workbooks or devices (or to keep them closed). Reveal one problem at a time. For each problem:

Give students quiet think time, and ask them to give a signal when they have an answer and a strategy.

Invite students to share their strategies, and record and display their responses for all to see.

Use the questions in the *Activity Synthesis* to involve more students in the conversation before moving to the next problem.

Keep all previous problems and work displayed throughout the talk.

Student Task Statement

Solve each equation mentally.

A. $27 = \frac{1}{3}h$

$h = 81$. Sample reasoning: I know $\frac{1}{3} \cdot 3 = 1$, so I multiplied each side by 3.

B. $27 = \frac{1}{3}r^2$

$r = 9$. Sample reasoning: I used the previous answer and that $81 = 9^2$.

C. $12\pi = \frac{1}{3}\pi a$

$a = 36$. Sample reasoning: Since each side has π, I know $12 = \frac{1}{3}a$, so a must be 36.

D. $12\pi = \frac{1}{3}\pi b^2$

$b = 6$. Sample reasoning: I used the previous answer and that $36 = 6^2$.

Activity Synthesis

To involve more students in the conversation, consider asking:

"Who can restate ____'s reasoning in a different way?"

"Did anyone use the same strategy but would explain it differently?"

"Did anyone solve the problem in a different way?"

"Does anyone want to add on to ____'s strategy?"

"Do you agree or disagree? Why?"

"What connections to previous problems do you see?"

Instructional Routines

Math Talk

ilclass.com/r/10694967

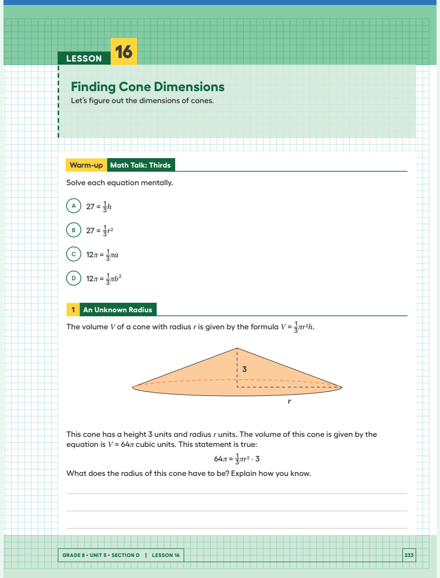
Please log in to the site before using the QR code or URL.

Access for Students with Diverse Abilities (Warm-up, Launch)

Action and Expression: Internalize Executive Functions.
To support working memory, provide students with sticky notes or mini whiteboards.

Supports accessibility for: Memory, Organization

Student Workbook


LESSON 16

Finding Cone Dimensions

Let's figure out the dimensions of cones.

Warm-up Math Talk: Thirds

Solve each equation mentally.

A $27 = \frac{1}{3}h$

B $27 = \frac{1}{3}r^2$

C $12\pi = \frac{1}{3}\pi a$

D $12\pi = \frac{1}{3}\pi b^2$

1 An Unknown Radius

The volume V of a cone with radius r is given by the formula $V = \frac{1}{3}\pi r^2 h$.

This cone has a height 3 units and radius r units. The volume of this cone is given by the equation is $V = 64\pi$ cubic units. This statement is true:

$$64\pi = \frac{1}{3}\pi r^2 \cdot 3$$

What does the radius of this cone have to be? Explain how you know.

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Access for Multilingual Learners (Warm-up, Synthesis)

MLR8: Discussion Supports.
Display sentence frames to support students when they explain their strategy. For example, "First, I ____ because ..." or "I noticed ____, so I ...." Some students may benefit from the opportunity to rehearse what they will say with a partner before they share with the whole class.

Advances: Speaking, Representing

Access for Multilingual Learners (Activity 1, Launch)

MLR8: Discussion Supports. Display sentence frames for students to use when they share their strategies with their partners. Examples: "The radius is ___ because ___" or "To find the radius, first I ___. Then, I ___." or "I used the $\frac{1}{3}$ by___."

Advances: Speaking, Conversing

Building on Student Thinking

Students might struggle with the $\frac{1}{3}$ while solving for the unknown radius length. Encourage students to think about the strategies they used in the *Warm-up* to work with the $\frac{1}{3}$.

Activity 1: Optional

An Unknown Radius

5 min

Activity Narrative

The purpose of this activity is for students to calculate the radius of the cone given the volume and height. This activity is similar to an activity in a previous lesson in which students calculated the radius of a cylinder given the volume and height. The difference here is that solving for the unknown takes an additional step since there is also a $\frac{1}{3}$ in the equation.

Launch

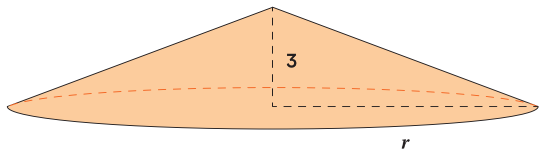
Arrange students in groups of 2.

Give students 2–3 minutes of quiet work time followed by time to share their strategies with their partner.

Follow with a whole-class discussion.

Student Task Statement

The volume V of a cone with radius r is given by the formula $V = \frac{1}{3}\pi r^2 h$.



This cone has a height 3 units and radius r units. The volume of this cone is given by the equation is $V = 64\pi$ cubic units. This statement is true:

$$64\pi = \frac{1}{3}\pi r^2 \cdot 3$$

What does the radius of this cone have to be? Explain how you know.

8 units

Sample reasoning: The equation simplifies to $64 = r^2$, so r must be 8.

Activity Synthesis

The purpose of this discussion is for students to share strategies they used to calculate the radius. Invite 1–3 students to share their thinking. Ask students,

"Which do you think is more challenging: calculating the radius of a cone or a cylinder when the heights and volumes of the shapes are known? Why?"

I think they are the same: I just have to work with $\frac{1}{3}$ when calculating the radius of the cone, which adds an extra step.

Activity 2

Cones with Unknown Dimensions

15 min

Activity Narrative

The purpose of this activity is for students to use the structure of the volume formula for cones to calculate missing dimensions of a cone given other dimensions. Students are given the image of a generic cone with marked dimensions for the radius, diameter, and height to help their reasoning about the different rows in the table.

While completing the table, students work with exact values of π as well as statements that require reasoning about squared values. The final row of the table asks students to find missing dimensions given an expression representing volume that uses letters to represent the height and the radius. This requires students to manipulate expressions consisting only of variables representing dimensions.

Encourage students to make use of work done in some rows to help find missing information in other rows. By paying attention to what rows have values in common, students can use the structure of the table and their knowledge of the volume formula to calculate related values more efficiently.

Launch

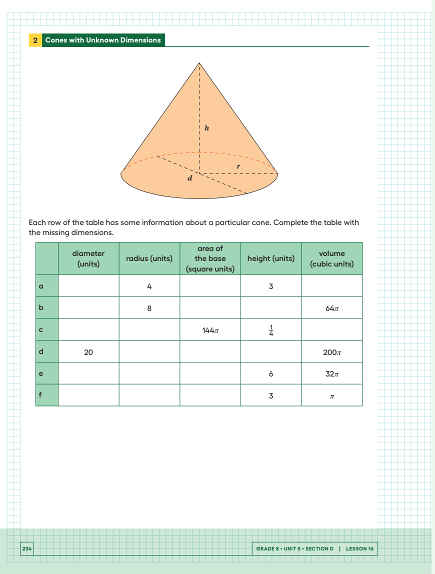
Give students 6–8 minutes of quiet work time, followed by a whole-class discussion.

As they work, select students who are using the information given in some rows to find missing information in other rows to share during the *Activity Synthesis*.

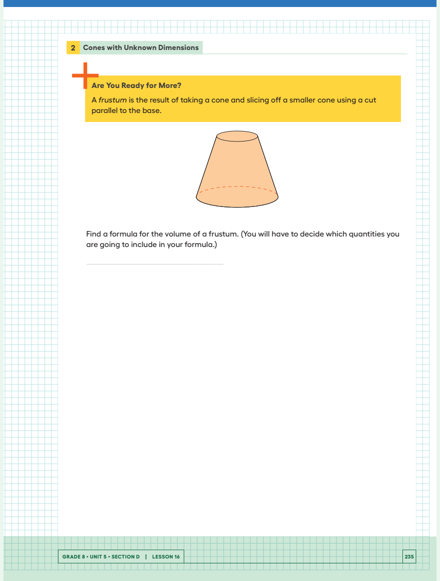
Access for Students with Diverse Abilities (Activity 2, Student Task)

Action and Expression: Internalize Executive Functions.
To support development of organizational skills in problem-solving, chunk this task into more manageable parts. For example, ask students to complete two lines of the table at a time and then assess for accuracy and comprehension.
Supports accessibility for: Organization, Attention

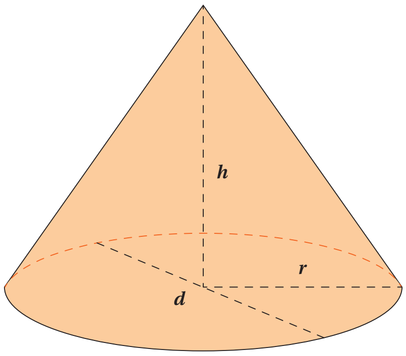
Student Workbook



Student Workbook



Student Task Statement

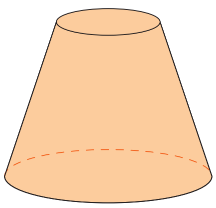


Each row of the table has some information about a particular cone. Complete the table with the missing dimensions.

	diameter (units)	radius (units)	area of the base (square units)	height (units)	volume of cone (cubic units)
a	8	4	16π	3	16π
b	16	8	64π	3	64π
c	24	12	144π	$\frac{1}{4}$	12π
d	20	10	100π	6	200π
e	8	4	16π	6	32π
f	2	1	π	3	π

Are You Ready for More?

A *frustum* is the result of taking a cone and slicing off a smaller cone using a cut parallel to the base.



Find a formula for the volume of a frustum. (You will have to decide which quantities you are going to include in your formula.)

Sample response: Imagine the original cone before the top piece is cut off. Then we will let R be the larger radius of the frustum, r be the smaller radius of the frustum, H be the height of the original cone, and h be the height of the conical piece cut off. Then the formula for the frustum is the volume of a cone with radius R and height H minus the volume of the removed top, which has radius r and height h: $V=\frac{1}{3}(\pi R^2 H - \pi r^2 h)$

Alternatively, let x be the height of the piece cut off, and h be the height of the frustum. Then we can calculate volume using $V=\frac{1}{3}\pi(R^2(h+x) - r^2 x)$.

Activity Synthesis

The purpose of this discussion is to make visible the different strategies students used to calculate the values in the table and to highlight some key relationships between radius, height, and volume.

Display the table from the *Task Statement* for all to see. Invite 2–3 previously selected students to share the strategies they used to fill in the missing information. Ask students:

"Which information, in your opinion, was the hardest to calculate?"

"If you had to pick two pieces of information given in the table which would you want? Why?"

If not brought up by students, highlight the following rows:

- Rows a and e have the same radius, which means they have the same base area. The height in row e is double the height in row a, and the volume in row e is double the volume in row a. The height and volume increase proportionally.
- Rows a and b have the same height, but the radius for row b is double the radius for row a, and the volume for row b is quadruple the volume for row a. The radius and volume do not increase proportionally.

Give students 30 seconds of quiet think time, then invite 1–2 students to share why they think this is happening. (The volume formula for a cylinder is $V = \pi r^2 h$. Doubling the value of h doubles the volume , but doubling the value of r quadruples the volume because the radius is squared and $(2r)^2 = 4r$.

If time allows, select a few rows of the table, and ask students how they might find the volume of a cylinder with the same radius and height as the cone. (Multiply by 3.)

Conclude the discussion by making sure students understand that when working with the volume formula for either a cylinder or cone, if they know two out of three values for the radius, height, and volume, they can always calculate the third.

Access for Multilingual Learners (Activity 3, Launch)

MLR5: Co-Craft Questions.
Keep student workbooks or devices closed. Display only the problem stem and the image, without revealing the question. Give students 2–3 minutes to write a list of mathematical questions that could be asked about this situation, before comparing their questions with a partner. Invite each group to contribute one written question to a whole-class display. Ask the class to make comparisons among the shared questions and their own. Reveal the intended question for this task, and invite additional connections.
Advances: Reading, Writing

Activity 3

Popcorn Deals

10 min

Activity Narrative

The purpose of this activity is for students to reason about the volume of popcorn that cone- and cylinder-shaped popcorn cups hold and the price they pay for it. Students start by picking the popcorn container they would buy (without doing any calculations) and then work with a partner to answer the question of which container is a better value. The fact that one container (the cone) has a wider diameter and looks like it has more coming out of the top might sway students to think that the cone is a better deal. However, the difference in the volume amounts should support the fact that even with a taller height and a wider diameter, the cone still holds less volume than the cylinder.

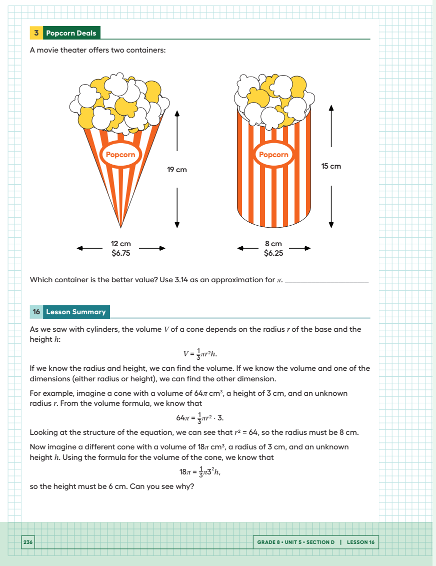
After having time to complete some calculations to support or reject their initial choice, groups have opportunities to explain their reasoning and critique the reasoning of others during the discussion.

Building on Student Thinking

If students try to use just the image to reason about which is a better deal without noticing the images are not in the same scale, consider asking:

"How did you choose the cone or cylinder as the better deal?"
"How could the measurements given help you make your choice?"

Student Workbook



3 Popcorn Deals

A movie theater offers two containers:

Which container is the better value? Use 3.14 as an approximation for π.

16 Lesson Summary

As we saw with cylinders, the volume V of a cone depends on the radius r of the base and the height h:

$$V = \frac{1}{3}\pi r^2 h.$$

If we know the radius and height, we can find the volume. If we know the volume and one of the dimensions (either radius or height), we can find the other dimension.

For example, imagine a cone with a volume of 64π cm^3, a height of 3 cm, and an unknown radius r. From the volume formula, we know that

$$64\pi = \frac{1}{3}\pi r^2 \cdot 3.$$

Looking at the structure of the equation, we can see that $r^2 = 64$, so the radius must be 8 cm.

Now imagine a different cone with a volume of 18π cm^3, a radius of 3 cm, and an unknown height h. Using the formula for the volume of the cone, we know that

$$18\pi = \frac{1}{3}\pi 3^2 h,$$

so the height must be 6 cm. Can you see why?

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Launch

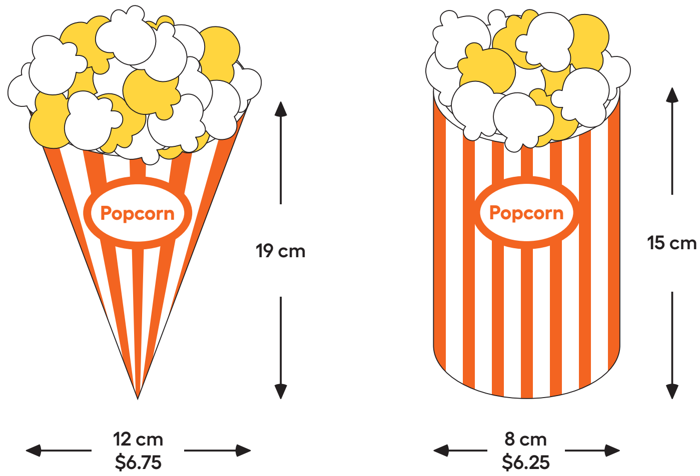
Arrange students in groups of 2. Give students 1 minute of quiet think time to decide which popcorn container they would purchase. Students are not asked to use any written calculations to determine this answer. Survey the class, and display the results for all to see. Keep this information displayed to refer back to during the *Activity Synthesis*.

Give students 3–4 minutes to work with their partner to determine which container is a better value.

As they work, select groups reasoning about the choice using both volume formulas and calculating the cost per unit to share during the whole-class discussion.

Student Task Statement

A movie theater offers two containers:



Which container is the better value? Use 3.14 as an approximation for π.

The cylinder cup is a better value. The cone's volume is about 715.92 cubic centimeters, and using the cost of the conical cup, it is about 106.07 cubic centimeters per dollar ($715.92 \div 6.75 \approx 106.07$). The cylinder's volume is about 753.6 cubic centimeters, and using the cost of the cylindrical cup, it is about 120.58 cubic centimeters per dollar ($753.6 \div 6.25 \approx 120.58$). Since the cylinder gives you more volume per dollar, it is a better value.

Activity Synthesis

Survey the class again for which container they would purchase. Display the new results next to the original results for all to see during the discussion.

Invite previously selected groups to share their arguments for which container has a better value. Encourage groups to share details of their calculations, and record these for all to see in order to mark any similarities and differences between the groups' arguments. After each group shares, ask the class if they agree or disagree with any of the statements made, and give students opportunities to justify their reasoning.
Here are some questions to choose from to help students think deeper about the situation:

- *"Do you think your volume calculations overestimate or underestimate the amount of popcorn each container can hold?"*

  I think the calculations underestimate the amount because the popcorn piles higher than the lip of the container.

- *"Why do you think movie theaters charged more for the cone?"*

  It may look like it has more volume to some people since it has a larger diameter and height.

- *"Do you think a lot of people would buy the cone over the cylinder?"*

  Yes. The 50-cent difference is not a lot, and since it is taller and wider, people might think the cone is bigger. Although the cylinder is a better value, there may be other considerations, like how easy the cup is to hold or put in the cup holder in the seat or whether you have time in line to calculate the value.

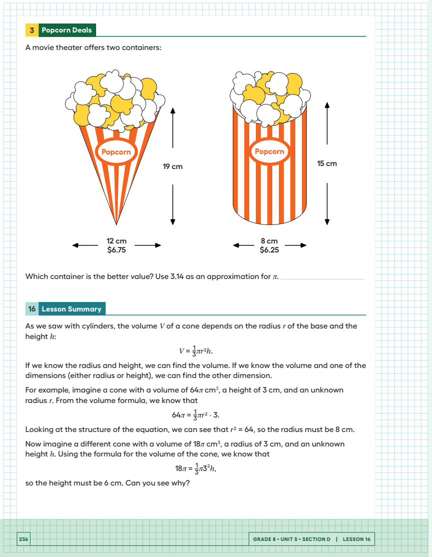
Lesson Synthesis

The purpose of this discussion is to for students to engage with the volume equation and notice that since there are two variables, r and h, that are unknown, there are many possible solutions.

Display the image of the popcorn cone from the activity "Popcorn Deals," including the dimensions of the cone. Ask students, "What size of cylinder cup would you need to have the same volume as the cone?"

Working in groups of 2, tell partners to determine the height and radius of a possible cylinder of equivalent volume and to make a sketch with labels on the dimensions. Display sketches, and invite students to share the strategies they used to find their cylinders. Highlight that there are many possible cylinders that would meet the requirements of having the same volume as the cone.

Student Workbook



3 Popcorn Deals

A movie theater offers two containers:

Which container is the better value? Use 3.14 as an approximation for π.

16 Lesson Summary

As we saw with cylinders, the volume V of a cone depends on the radius r of the base and the height h:

$$V = \frac{1}{3}\pi r^2 h.$$

If we know the radius and height, we can find the volume. If we know the volume and one of the dimensions (either radius or height), we can find the other dimension.

For example, imagine a cone with a volume of 64π cm^3, a height of 3 cm, and an unknown radius r. From the volume formula, we know that

$$64\pi = \frac{1}{3}\pi r^2 \cdot 3.$$

Looking at the structure of the equation, we can see that $r^2 = 64$, so the radius must be 8 cm.

Now imagine a different cone with a volume of 18π cm^3, a radius of 3 cm, and an unknown height h. Using the formula for the volume of the cone, we know that

$$18\pi = \frac{1}{3}\pi 3^2 h,$$

so the height must be 6 cm. Can you see why?

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Responding To Student Thinking

Points to Emphasize

If most students struggle using the formula for the volume, consider using the optional activity referred to here to give students more practice working with volume formulas.

Grade 8, Unit 5, Lesson 21, Activity 3
The Right Fit

Lesson Summary

As we saw with cylinders, the volume V of a cone depends on the radius r of the base and the height h:

$$V = \frac{1}{3}\pi r^2 h.$$

If we know the radius and height, we can find the volume. If we know the volume and one of the dimensions (either radius or height), we can find the other dimension.

For example, imagine a cone with a volume of 64π cm^3, a height of 3 cm, and an unknown radius r. From the volume formula, we know that

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Now imagine a different cone with a volume of 18π cm^3, a radius of 3 cm, and an unknown height h. Using the formula for the volume of the cone, we know that

$$18\pi = \frac{1}{3}\pi 3^2 h,$$

so the height must be 6 cm. Can you see why?

Cool-down

A Square Radius

5 min

Launch

Provide students with access to calculators.

Student Task Statement

Noah and Lin are making paper cones to hold popcorn to hand out at a family math night. They want the cones to hold 9π cubic inches of popcorn. What are two different possible values for height h and radius r for the cones?

Sample responses:

- Height and radius both 3 inches since $\frac{1}{3}\pi \cdot 3^2 \cdot 3 = 9\pi$.
- Radius 2 inches and height 6.75 inches since $\frac{1}{3}\pi \cdot 2^2 \cdot 6.75 = 9\pi$.
- Radius 1 inch and height 27 inches since $\frac{1}{3}\pi \cdot 1^2 \cdot 27 = 9\pi$.
- Radius 9 inches and height $\frac{1}{3}$ inches since $\frac{1}{3}\pi \cdot 9^2 \cdot \frac{1}{3} = 9\pi$. (This cone may look more like a plate, but it solves the problem.)

Lesson **16**

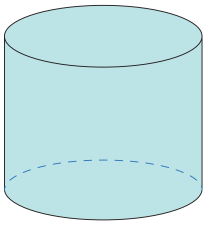
Practice Problems

5 Problems

Problem 1

from Unit 5, Lesson 15

The volume of this cylinder is 175π cubic units.



What is the volume of a cone that has the same base area and the same height? Explain how you know.

$\frac{175}{3}\pi$ cubic units, or about 183 cubic units

Sample reasoning: The volume of the cone is exactly $\frac{1}{3}$ the volume of the corresponding cylinder.

Problem 2

A cone has volume 12π cubic inches. Its height is 4 inches. What is its radius? Explain how you know.

3 inches

Sample reasoning: The volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$. With the information given, we know that $12\pi = \frac{1}{3}\pi r^2 \cdot 4$. So $12 = \frac{4}{3}r^2$, and $9 = r^2$. Then $r = 3$.

Problem 3

A cone has volume 3π cubic units.

a. If the cone's radius is 1 unit, what is its height?

9 units

b. If the cone's radius is 2 units, what is its height?

$\frac{9}{4}$ units

c. If the cone's radius is 5 units, what is its height?

$\frac{9}{25}$ unit

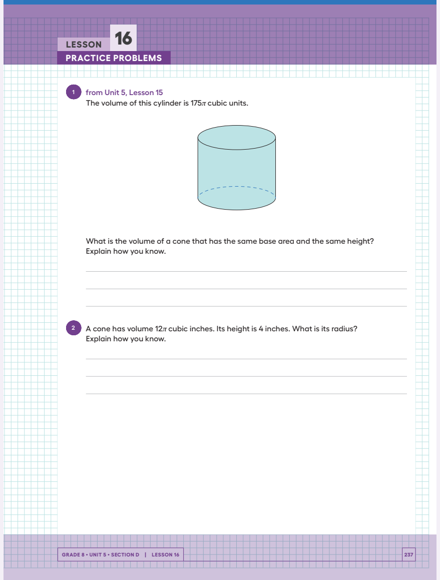
d. If the cone's radius is $\frac{1}{2}$ unit, what is its height?

36 units

e. If the cone's radius is r units, what is its height?

$\frac{9}{r^2}$ units

Student Workbook



LESSON 16 PRACTICE PROBLEMS

1 from Unit 5, Lesson 15

The volume of this cylinder is 175π cubic units.

What is the volume of a cone that has the same base area and the same height? Explain how you know.

2 A cone has volume 12π cubic inches. Its height is 4 inches. What is its radius? Explain how you know.

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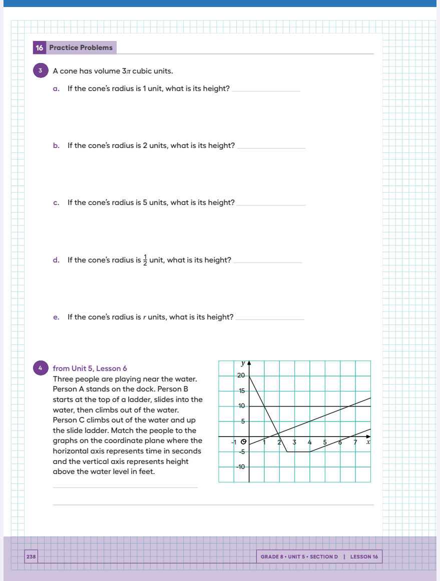
Student Workbook



16 Practice Problems

3 A cone has volume 3π cubic units.

a. If the cone's radius is 1 unit, what is its height?

b. If the cone's radius is 2 units, what is its height?

c. If the cone's radius is 5 units, what is its height?

d. If the cone's radius is $\frac{1}{2}$ unit, what is its height?

e. If the cone's radius is r units, what is its height?

4 from Unit 5, Lesson 6

Three people are playing near the water. Person A stands on the dock. Person B starts at the top of a ladder, slides into the water, then climbs out of the water. Person C climbs out of the water and up the slide ladder. Match the people to the graphs on the coordinate plane where the horizontal axis represents time in seconds and the vertical axis represents height above the water level in feet.

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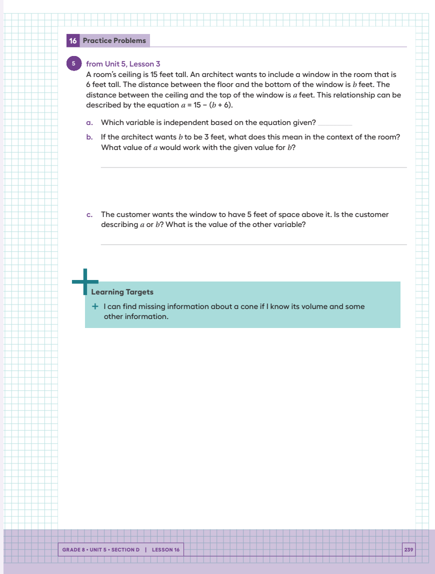
Student Workbook



16 Practice Problems

5 from Unit 5, Lesson 3

A room's ceiling is 15 feet tall. An architect wants to include a window in the room that is 6 feet tall. The distance between the floor and the bottom of the window is b feet. The distance between the ceiling and the top of the window is a feet. This relationship can be described by the equation $a = 15 - (b + 6)$.

a. Which variable is independent based on the equation given? ________

b. If the architect wants b to be 3 feet, what does this mean in the context of the room? What value of a would work with the given value for b?

c. The customer wants the window to have 5 feet of space above it. Is the customer describing a or b? What is the value of the other variable?

Learning Targets

+ I can find missing information about a cone if I know its volume and some other information.

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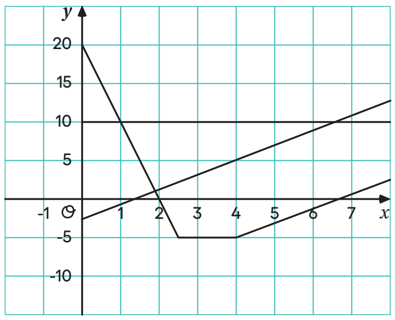
Problem 4

from Unit 5, Lesson 6

Three people are playing near the water. Person A stands on the dock. Person B starts at the top of a ladder, slides into the water, then climbs out of the water. Person C climbs out of the water and up the slide ladder. Match the people to the graphs on the coordinate plane where the horizontal axis represents time in seconds and the vertical axis represents height above the water level in feet.



Sample response: Person A is the constant graph of $y = 10$. Person B is the graph that includes the point $(0, 20)$. Person C is the graph that starts negative and increases.

Problem 5

from Unit 5, Lesson 3

A room's ceiling is 15 feet tall. An architect wants to include a window in the room that is 6 feet tall. The distance between the floor and the bottom of the window is b feet. The distance between the ceiling and the top of the window is a feet. This relationship can be described by the equation $a = 15 - (b + 6)$.

a. Which variable is independent based on the equation given?

b

b. If the architect wants b to be 3 feet, what does this mean in the context of the room? What value of a would work with the given value for b?

Sample response: The architect wants the bottom of the window to be 3 feet above the floor; a is 6 feet.

c. The customer wants the window to have 5 feet of space above it. Is the customer describing a or b? What is the value of the other variable?

a; b is 4 feet when $a = 5$