Lesson 8

A Proof of the Pythagorean Theorem

Goals

- Calculate an unknown side length of a right triangle using the Pythagorean Theorem, and explain (orally) the reasoning.
- Explain (orally) an area-based algebraic proof of the Pythagorean Theorem.

Learning Target

I can explain why the Pythagorean Theorem is true.

Lesson Narrative

In this lesson, students work through an area-based algebraic proof of the Pythagorean Theorem. This proof is based on two squares with the same area that are subdivided in different ways. By reasoning about the areas of both squares, students are able to see that $a^2 + b^2 = c^2$.

This lesson is also the first time students practice applying the Pythagorean Theorem to triangles with 1 unknown side length and not on a grid. Students will have more opportunities in later lessons to further practice and apply this understanding in different situations, so fluency applying the Pythagorean Theorem is not expected at this time.

An optional activity in this lesson works best when each student has access to a device that can run the applet, which allows students to experience a transformations-based proof without the additional preparation of paper manipulatives.

Student Learning Goal

Let's prove the Pythagorean Theorem.

Access for Students with Diverse Abilities

- Engagement (Activity 1)

Access for Multilingual Learners

- MLR5: Co-Craft Questions (Activity 1)

Instructional Routines

- MLR5: Co-Craft Questions
- Notice and Wonder

Required Materials

Materials to Gather

- Blank paper: Lesson
- Scissors: Activity 4

Materials to Copy

- A Transformational Proof Cutouts (1 copy for every 2 students): Activity 4

Activity 4:
For the digital version of the activity, acquire devices that can run the applet.

Lesson Timeline

5 min	20 min	10 min	15 min	10 min	5 min
Warm-up	Activity 1	Activity 2	Activity 3	Lesson Synthesis	Cool-down (Assessment)

Instructional Routines

Notice and Wonder

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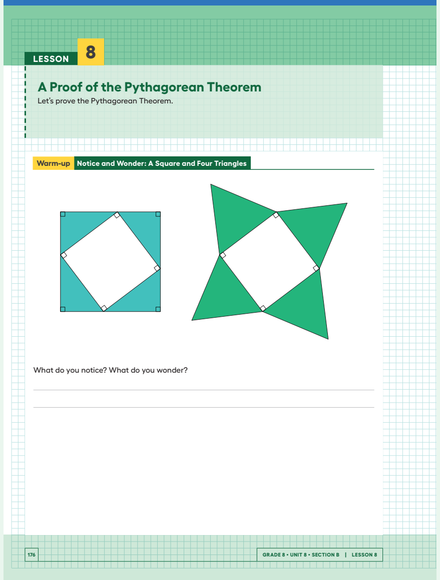
Please log in to the site before using the QR code or URL.

Student Workbook



Warm-up

Notice and Wonder: A Square and Four Triangles

5 min

Activity Narrative

The purpose of this *Warm-up* is for students to study a diagram in preparation for understanding a proof of the Pythagorean Theorem. While students may notice and wonder many things about these diagrams, the fact that the construction on the left has right triangles, while the construction on the right does not is the important discussion point.

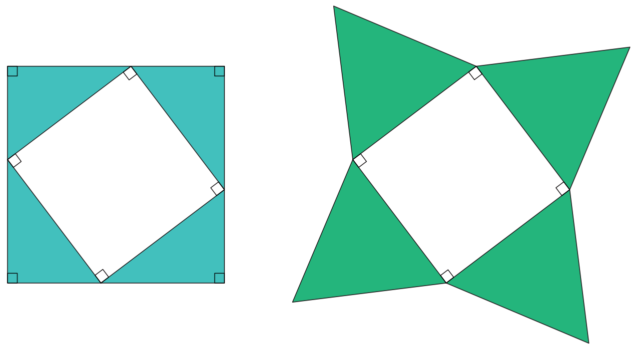
This prompt gives students opportunities to see and make use of structure. The specific structure they might notice is that the construction on the left leads to a composite figure of a square, while the composite figure on the right is not. This is due to whether or not the construction has right triangles.

Launch

Arrange students in groups of 2. Display the image for all to see. Ask students to think of at least one thing they notice and at least one thing they wonder.

Give students 1 minute of quiet think time and then 1 minute to discuss the things they notice and wonder with their partner.

Student Task Statement



What do you notice? What do you wonder?

Things students may notice:

- There are two figures both made up of a square and four triangles.
- The triangles in the figure on the left are right triangles and the triangles in the figure on the right are not.
- The figure on the left is a square.
- The figure on the right is not a square.
- The smaller squares in the middle look the same size.

Things students may wonder:

- Are the squares in the middle the same size?
- Are the triangles in the left figure all the same size?
- Are the triangles in the right figure all the same size?
- Does it matter if the triangles are right triangles?
- Does it matter if they are equilateral triangles?
- What are these figures going to be used for?

Activity Synthesis

Ask students to share the things they noticed and wondered. Record and display their responses without editing or commentary for all to see. If possible, record the relevant reasoning on or near the image. Next, ask students,

"Is there anything on this list that you are wondering about now?"

Encourage students to observe what is on display and respectfully ask for clarification, point out contradicting information, or voice any disagreement.

Tell students that when there is a square with congruent right triangles on each side, as shown on the left, they form a larger square (they will be able to prove this in high school). But it doesn't work if the triangles are not right triangles. This construction will be used in the next activity.

Activity 1

Adding Up Areas

20 min

Activity Narrative

The purpose of this activity is for students to work through an area-based algebraic proof of the Pythagorean Theorem. Figure G was first encountered by students in an earlier unit on transformations.

While there are many proofs of the Pythagorean Theorem similar to the one in this activity, they often rely on $(a + b)^2 = a^2 + 2ab + b^2$, which is beyond the scope of grade 8. For this proof, students reason about the areas of the two squares with the same dimensions. Each square is divided into smaller regions in different ways. Using the equality of the total area of each square, they uncover the Pythagorean Theorem. The extension uses this same division to solve a challenging area composition and decomposition problem.

Launch

Arrange students in groups of 2. Tell students to close their books or devices (or to keep them closed). Display the images from the *Task Statement* for all to see and begin by explaining how the two figures are constructed.

Each figure starts with a square with side length $a + b$.

Figure F partitions the square into two squares and two rectangles.

Figure G takes a right triangle with legs a and b and puts one identical copy of it in each corner of the square. The copies touch each other because the short leg of one and the long leg of the one next to it add up to $a + b$, so they fit exactly into a side. The four triangles form a quadrilateral in the middle. We know the quadrilateral is a square because

- The corners must be 90 degree angles:
  - The two acute angles in each triangle must sum to 90 degrees because the sum of the angles in a triangle is 180 degrees, and the third angle is 90 degrees.

Instructional Routines

MLR5: Co-Craft Questions

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Please log in to the site before using the QR code or URL.

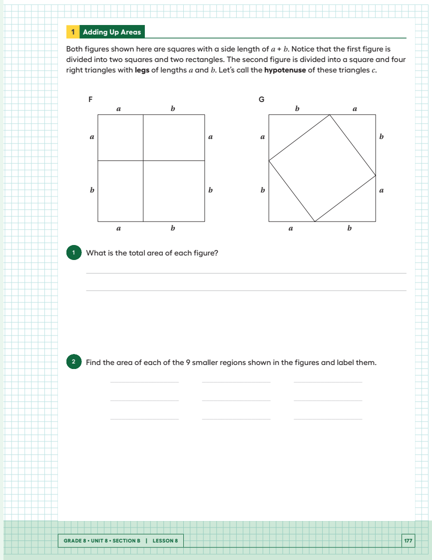
Access for Multilingual Learners (Activity 1)

MLR5: Co-Craft Questions
This activity uses the *Co-Craft Questions* math language routine to advance reading and writing as students make sense of a context and practice generating mathematical questions.

Access for Students with Diverse Abilities (Activity 1, Student Task)

Engagement: Develop Effort and Persistence.
Chunk this task into more manageable parts. Have students find the area of Figure F before working on Figure G. Check in with students to provide feedback and encouragement after each chunk.
Supports accessibility for: Attention, Social-Emotional Functioning

Student Workbook



- ◦ The two smaller angles along with one of the corners of the quadrilateral form a straight angle with a measure of 180 degrees, that means that the angle at the corner must also be 90 degrees.
- All four sides are the same length: They all correspond to a hypotenuse of one of the congruent right triangles.

Use *Co-Craft Questions* to orient students to the context and elicit possible mathematical questions.

Without revealing the questions, give students 1–2 minutes to write a list of mathematical questions that could be asked about the images before comparing questions with a partner.

Invite several partners to share one question with the class and record responses. Ask the class to make comparisons among the shared questions and their own. Ask,

"What do these questions have in common? How are they different?"

Listen for and amplify questions about the total area for each square or the area of each of the nine smaller regions of the squares.

Reveal the question,

"What is the total area of each figure?"

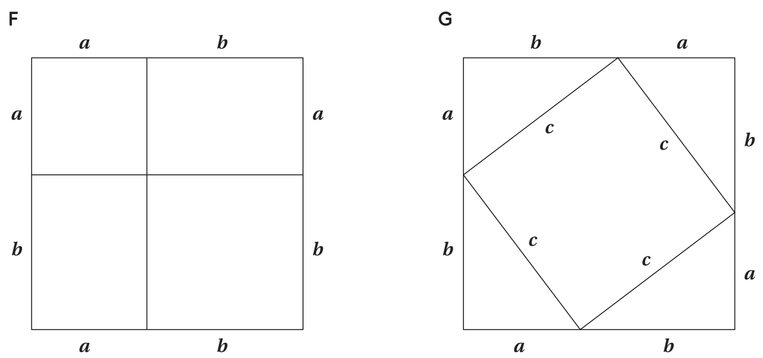
and give students 1–2 minutes to compare it to their own question and those of their classmates. Invite students to identify similarities and differences by asking:

"Which of your questions is most similar to or different from the ones provided? Why?"

Give students 3 minutes of quiet work time for the first two problems. Ask partners to share their work and come to an agreement on the area of each figure and region before moving on to the last problem. Follow with a whole-class discussion.

Student Task Statement

Both figures shown here are squares with a side length of $a + b$. Notice that the first figure is divided into two squares and two rectangles. The second figure is divided into a square and four right triangles with **legs** of lengths a and b. Let's call the **hypotenuse** of these triangles c.



1. What is the total area of each figure?

$(a + b)^2$

2. Find the area of each of the 9 smaller regions shown in the figures and label them.

- Figure F: a^2, ab, ab, b^2
- Figure G: $\frac{1}{2}ab$, $\frac{1}{2}ab$, $\frac{1}{2}ab$, $\frac{1}{2}ab$, c^2.

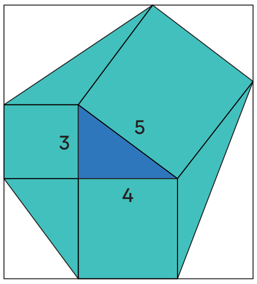
3. Add up the area of the 4 regions in Figure F and set this expression equal to the sum of the areas of the 5 regions in Figure G. If you rewrite this equation using as few terms as possible, what do you have?

$a^2 + b^2 = c^2$

The sum of the area of regions in F is $a^2 + 2ab + b^2$, and the sum of the area of regions in G is $4\left(\frac{1}{2}ab\right) + c^2 = 2ab + c^2$. Then $a^2 + 2ab + b^2 = 2ab + c^2$ implies $a^2 + b^2 = c^2$.

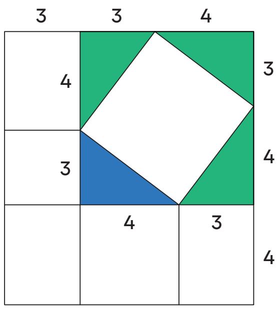
Are You Ready for More?

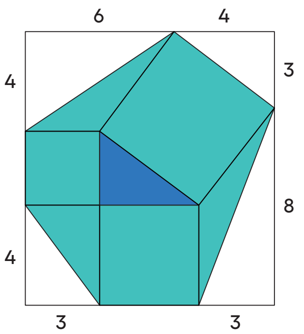
Take a 3-4-5 right triangle, add on the squares of the side lengths, and form a hexagon by connecting vertices of the squares as in the image. What is the area of this hexagon?



74

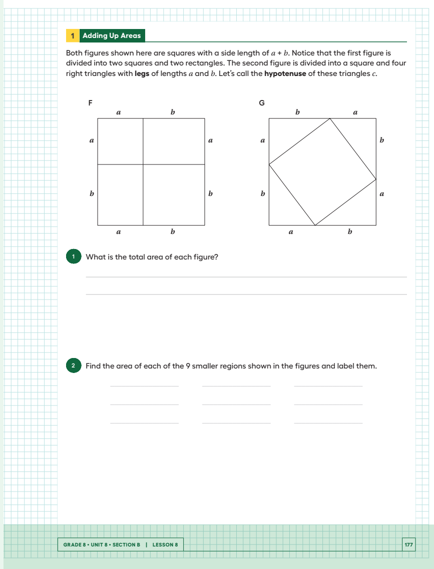
We can find the area of the shaded region if we can find the area of the surrounding rectangle and subtract the area of the four unshaded triangles. To find the dimensions of the rectangle, add in three more copies of the right triangle as in our proof of the Pythagorean Theorem (see image on the left). By writing down all the side lengths, we can see that the width of the rectangle is 10 and its height is 11, and so has an area of 110.



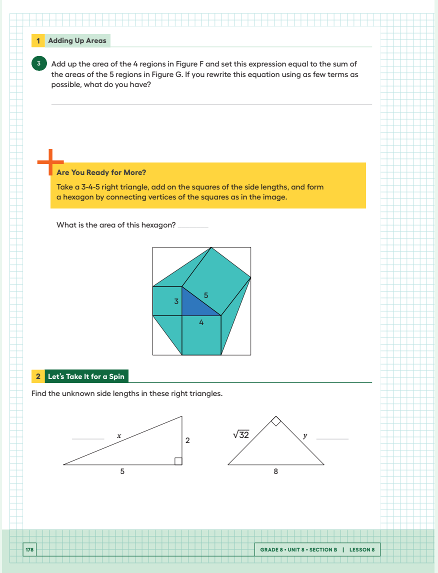


Returning to the original image (the image on the right), we see that the four triangles we have to subtract have areas of 12, 6, 6, and 12. The area of the shaded region is $110 - 12 - 6 - 6 - 12 = 74$.

Student Workbook

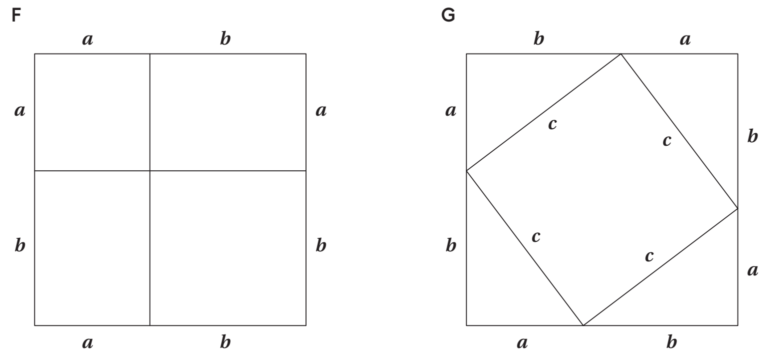


Student Workbook



Activity Synthesis

The goal of this discussion is to make sure students understand this area-based algebraic proof of the Pythagorean Theorem. Display the image from the *Task Statement* for all to see.



Select 2–3 groups to share their work for the third question. Make sure the last group presenting concludes with $a^2 + b^2 = c^2$ or something close enough that the class can get there with a little prompting. For example, if groups are stuck with the equation looking something like $a^2 + ab + b^2 + ab = \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}ab + c^2$, encourage them to try and combine like terms and remove any quantities both sides have in common.

Here are some questions for discussion:

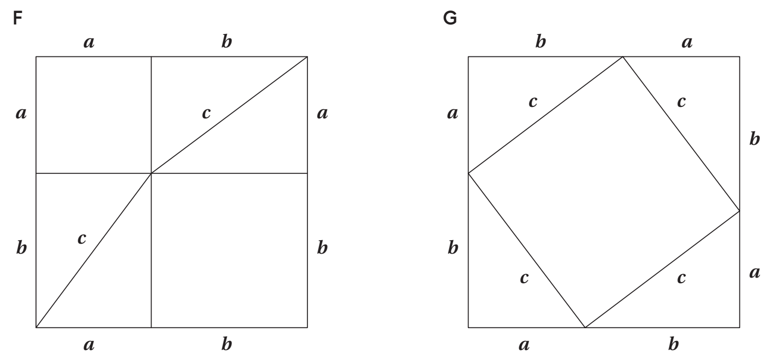
- *"How do you see regions from one figure matching regions in the other figure?"*

  The two small squares in Figure F match the one large square in Figure G.

- *"How do the rectangles and triangles match?"*

  The area of the two rectangles is the same as the area of four of the triangles, since two of the triangles together make a rectangle that is a wide and b long.

If needed, show students an image with the diagonals added in, such as the one shown here, to help make the connection between the two figures clearer.



Note how these figures can be made for any right triangle with legs a and b and hypotenuse c.

Activity 2
Let's Take It for a Spin

10 min

Activity Narrative

The purpose of this activity is to practice using the Pythagorean Theorem. Prior to this lesson, students could only find the length of a segment by using a grid. The Pythagorean Theorem now makes it possible to find the length of any segment that is a side of a right triangle.

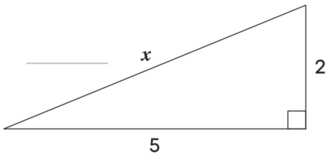
Launch

Arrange students in groups of 2.

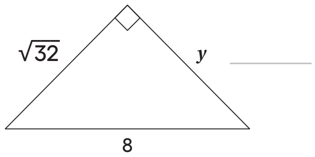
Give students 3 minutes of quiet work time, and follow with a whole-class discussion.

Student Task Statement

Find the unknown side lengths in these right triangles.



$x = \sqrt{29}$



$y = \sqrt{32}$

Activity Synthesis

The goal of this discussion is to make sure students understand how to use the Pythagorean Theorem when finding a leg versus when finding the hypotenuse. Students will have more practice using the theorem in future lessons.

Before students share their methods for finding the side length of a right triangle, present an incorrect solution based on a common error you observe in the class. For example, "I know that $a = \sqrt{32}$, $b = 8$, and $c = y$, so when I use the Pythagorean Theorem, I get the equation

$(\sqrt{32})^2 + 8^2 = y^2$.

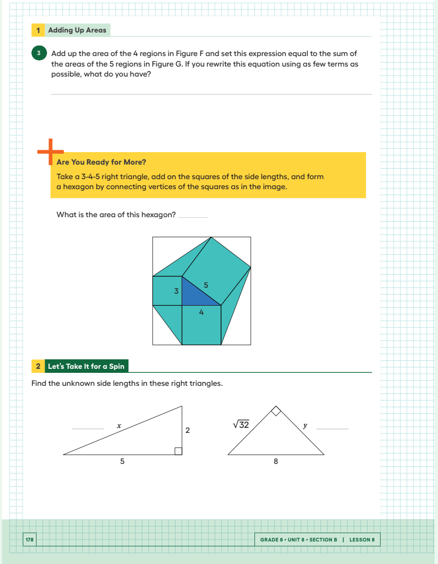
This equation simplifies to $32 + 64 = y^2$.

When I solve for y, I get $y = \sqrt{96}$."

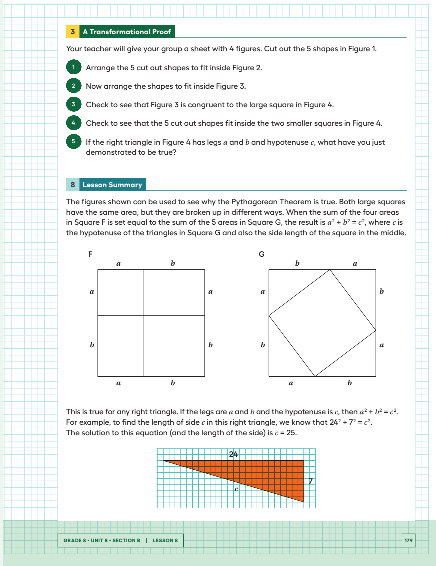
Ask students to identify the error and revise the original statement. Remind students that in the Pythagorean Theorem, a^2 and b^2 represent the squares of the legs of the right triangle, and c^2 represents the square of the hypotenuse.

Invite a few students to share their reasoning with the class for each unknown side length. As students share, record their steps for all to see, clearly showing the initial setup with $a^2 + b^2 = c^2$.

Student Workbook



Student Workbook



3 A Transformational Proof

Your teacher will give your group a sheet with 4 figures. Cut out the 5 shapes in Figure 1.

1. Arrange the 5 cut out shapes to fit inside Figure 2.
2. Now arrange the shapes to fit inside Figure 3.
3. Check to see that Figure 3 is congruent to the large square in Figure 4.
4. Check to see that the 5 cut out shapes fit inside the two smaller squares in Figure 4.
5. If the right triangle in Figure 4 has legs a and b and hypotenuse c, what have you just demonstrated to be true?

8 Lesson Summary

The figures shown can be used to see why the Pythagorean Theorem is true. Both large squares have the same area, but they are broken up in different ways. When the sum of the four areas in Square F is set equal to the sum of the 5 areas in Square G, the result is $a^2 + b^2 = c^2$, where c is the hypotenuse of the triangles in Square G and also the side length of the square in the middle.

This is true for any right triangle. If the legs are a and b and the hypotenuse is c, then $a^2 + b^2 = c^2$. For example, to find the length of side c in this right triangle, we know that $24^2 + 7^2 = c^2$. The solution to this equation (and the length of the side) is $c = 25$.

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Activity 3: Optional

A Transformational Proof

15 min

Activity Narrative

There is a digital version of this activity.

In this optional activity, students explore a transformations-based proof of the Pythagorean Theorem. Since this proof is not one students are expected to derive on their own, the focus is on understanding why the steps are possible from a transformations perspective.

In the digital version of the activity, students use an applet to compare the areas of two smaller squares to the area of one larger square. The applet allows students to decompose the two smaller squares into smaller pieces and quickly rearrange them. This activity works best when each student has access to the applet and when having students cut shapes from the handout may take more time than is available. If students don't have individual access, displaying the applet for all to see would be helpful during the *Synthesis*.

Launch

Arrange students in groups of 2. Tell them that they are going to explore the areas of 3 different squares.

Instruct students to cut out Figure 1 into 5 shapes.

Student Task Statement

Your teacher will give your group a sheet with 4 figures. Cut out the 5 shapes in Figure 1.

1. Arrange the 5 cut out shapes to fit inside Figure 2.

   No response necessary.

2. Now arrange the shapes to fit inside Figure 3.

   No response necessary.

3. Check to see that Figure 3 is congruent to the large square in Figure 4.

   No response necessary.

4. Check to see that the 5 cut out shapes fit inside the two smaller squares in Figure 4.

   No response necessary.

5. If the right triangle in Figure 4 has legs a and b and hypotenuse c, what have you just demonstrated to be true?

   Sample response: Since the 5 pieces can be arranged to be either the area $a^2 + b^2$ of the two smaller squares or the area c^2 of the larger square, we have demonstrated that for a right triangle with legs a and b and hypotenuse c, $a^2 + b^2 = c^2$ is true.

Activity Synthesis

The goal of this discussion is for students to see that the sum of the areas of the 2 smaller squares is congruent to the area of the larger square. Invite several groups to share their answers to the last question.

Lesson Synthesis

The goal of this discussion is to review what students know about triangles where the Pythagorean Theorem does and does not work and to reinforce for students how to use it. Arrange students in groups of 2 and ask each student to draw 2 triangles on a blank sheet of paper: one where the Pythagorean Theorem works and one where it does not. Tell students to compare their triangles with their partner. Discuss:

- *"What is the same and different about the triangles you and your partner drew where the Pythagorean Theorem does work?"*

  They were both right triangles. Our right triangles were different sizes and were facing different directions.

- *"What is different between the triangles where the Pythagorean Theorem works and the triangles where it doesn't?"*

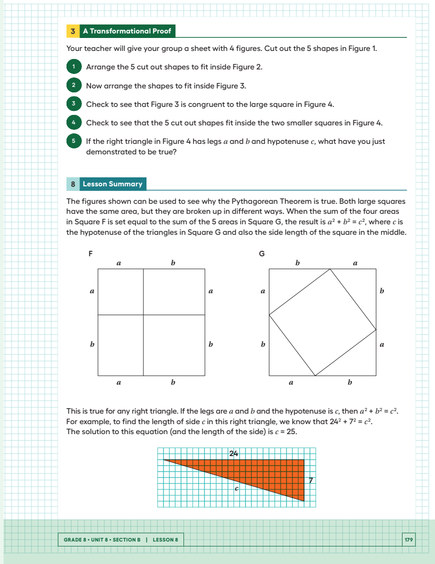
  The triangles where it works have to have a right angle, and the triangles where it doesn't work can be any other kind of triangle.

Next, ask students to label their triangle where the Pythagorean Theorem works with a and b for the legs and c for the hypotenuse and then share with their partner. Ask students if it matters which side is a, b, or c. (The hypotenuse, or side opposite the right angle, has to be c, but it does not matter which leg is a or b.)

Responding To Student Thinking

More Chances
Students will have more opportunities to understand the mathematical ideas addressed here. There is no need to slow down or add additional work to the next lessons.

Student Workbook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This is true for any right triangle. If the legs are a and b and the hypotenuse is c, then $a^2 + b^2 = c^2$. For example, to find the length of side c in this right triangle, we know that $24^2 + 7^2 = c^2$. The solution to this equation (and the length of the side) is $c = 25$.

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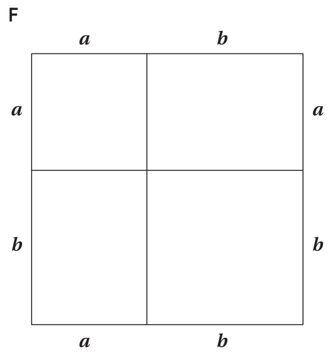

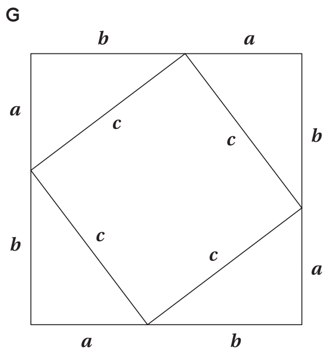

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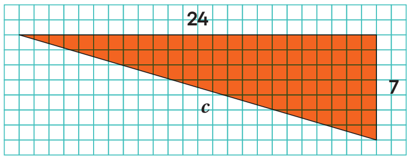


Cool-down

What Is the Hypotenuse?

5 min

Student Task Statement

Find the length of the hypotenuse in a right triangle if a is 5 cm and b is 8 cm.

$c = \sqrt{89}$ cm or $c \approx 9.4$ cm

Lesson **8**

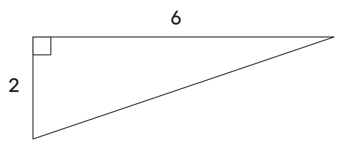
Practice Problems

6 Problems

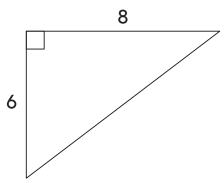
Problem 1

a. Find the lengths (in units) of the unlabeled sides.

i.



ii.

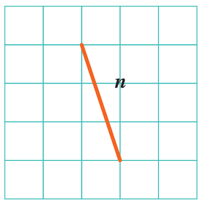


i. $\sqrt{40}$ units

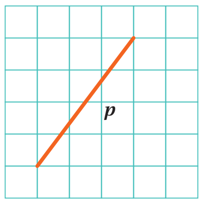
ii. $\sqrt{100}$ (or 10) units

b. One segment is n units long and the other is p units long. Find the value of n and p. (Each small grid square is 1 square unit.)

i.



ii.

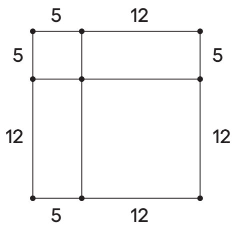


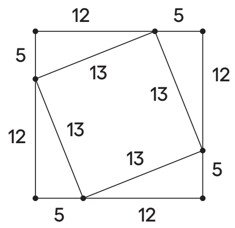
i. $\sqrt{10}$ units, because $1^2 + 3^2 = 10$

ii. 5 units, because $3^2 + 4^2 = 25$

Problem 2

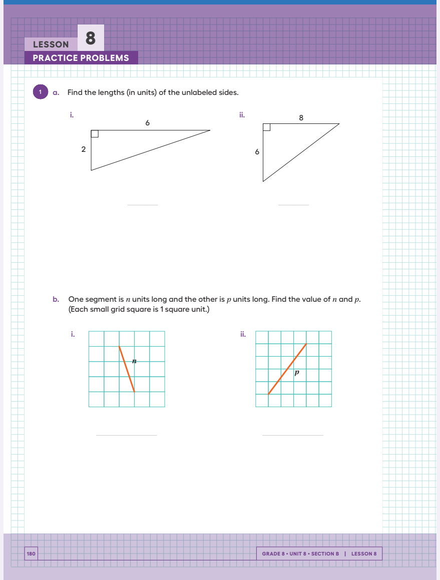
Use the areas of the two identical squares to explain why $5^2 + 12^2 = 13^2$ without doing any calculations.



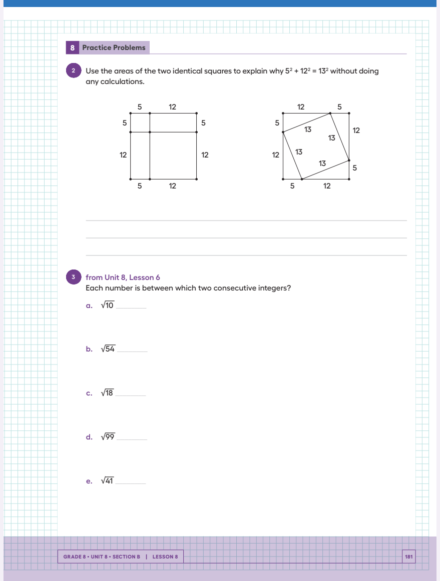


Sample response: The areas of the two large squares are the same since they are both 17 by 17 units. The area of the two rectangles on the left square are the same as the area of the 4 triangles in the right square (each pair of triangles makes a rectangle). So the area of the two smaller squares on the left must be the same as the area of the smaller square on the right. This means $5^2 + 12^2 = 13^2$.

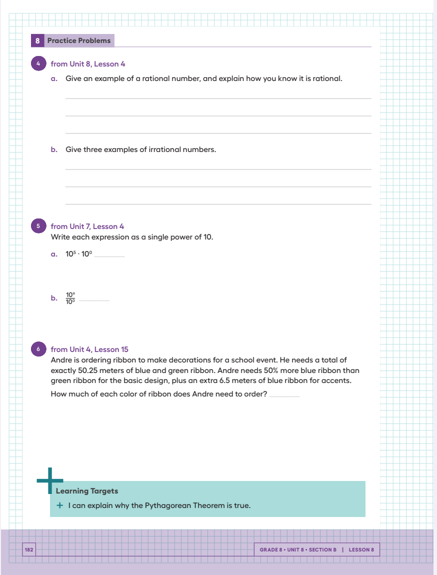
Student Workbook



Student Workbook



Student Workbook


8 Practice Problems

4 from Unit 8, Lesson 4

a. Give an example of a rational number, and explain how you know it is rational.

b. Give three examples of irrational numbers.

5 from Unit 7, Lesson 4

Write each expression as a single power of 10.

a. $10^5 \cdot 10^0$

b. $\frac{10^9}{10^0}$

6 from Unit 4, Lesson 15

Andre is ordering ribbon to make decorations for a school event. He needs a total of exactly 50.25 meters of blue and green ribbon. Andre needs 50% more blue ribbon than green ribbon for the basic design, plus an extra 6.5 meters of blue ribbon for accents.

How much of each color of ribbon does Andre need to order?

Learning Targets

+ I can explain why the Pythagorean Theorem is true.

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Problem 3

from Unit 8, Lesson 6

Each number is between which two consecutive integers?

a. $\sqrt{10}$

3 and 4

b. $\sqrt{54}$

7 and 8

c. $\sqrt{18}$

4 and 5

d. $\sqrt{99}$

9 and 10

e. $\sqrt{41}$

6 and 7

Problem 4

from Unit 8, Lesson 4

a. Give an example of a rational number, and explain how you know it is rational.

Sample response: $\frac{2}{3}$ is a rational number because rational numbers can be written as positive or negative fractions, and $\frac{2}{3}$ is a fraction.

b. Give three examples of irrational numbers.

Sample response: $\sqrt{2}$, $-\sqrt{12}$, $\sqrt{1.5}$

Problem 5

from Unit 7, Lesson 4

Write each expression as a single power of 10.

a. $10^5 \cdot 10^0$

10^5

b. $\frac{10^9}{10^0}$

10^9

Problem 6

from Unit 4, Lesson 15

Andre is ordering ribbon to make decorations for a school event. He needs a total of exactly 50.25 meters of blue and green ribbon. Andre needs 50% more blue ribbon than green ribbon for the basic design, plus an extra 6.5 meters of blue ribbon for accents.

How much of each color of ribbon does Andre need to order?

Andre needs to order 17.5 meters of green ribbon and 32.5 meters of blue ribbon.

Sample reasoning: Let b represent the length (in meters) of blue ribbon and g represent the length (in meters) of green ribbon. Then $b = 1.5g + 6.25$ and $b + g = 50.25$. Substituting $1.5g + 6.25$ in for b in the second equation gives $(1.5g + 6.25) + g = 50.25$. Solving for g gives $g = 17.5$. Since the two kinds of ribbon must combine to make 50.25 meters, the amount of blue ribbon can be found using $b = 50.25 - 17.5$. So $b = 32.5$ meters.