Lesson 8

How Many Solutions?

Goals

- Describe (orally) a linear equation as having “one solution,” “no solutions,” or “an infinite number of solutions,” and solve equations in one variable with one solution.
- Describe (orally) features of linear equations with one solution, no solution, or an infinite number of solutions.

Learning Target

I can solve equations with different numbers of solutions.

Lesson Narrative

The purpose of this lesson is to help students identify structural features of an equation that indicate the number of solutions it has. For example, when they reach an equation like $6x + 2 = 6x + 5$ (no solution) or $6x + 2 = 6x + 2$ (infinitely many solutions), they notice that they can stop solving the equation. To articulate the conditions that indicate the number of solutions an equation has, students must be precise in their language and use the structure of the equations.

Student Learning Goal

Let’s solve equations with different numbers of solutions.

Access for Students with Diverse Abilities

- Engagement (Activity 1)

Access for Multilingual Learners

- MLR1: Stronger and Clearer Each Time (Activity 2)
- MLR8: Discussion Supports (Activity 1)

Instructional Routines

- Card Sort
- MLR1: Stronger and Clearer Each Time
- MLR8: Discussion Supports

Required Materials

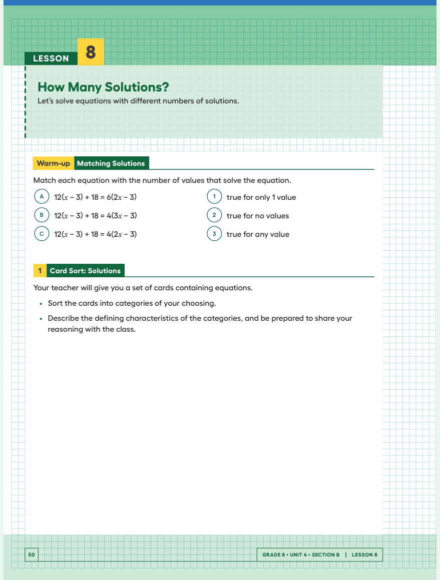
Materials to Copy

- Thinking About Solutions Some More Cards (1 copy for every 3 students): Activity 1

Lesson Timeline

Warm-up	Activity 1	Activity 2	Lesson Synthesis	Assessment: Cool-down
5 min	15 min	20 min	10 min	5 min

Student Workbook



Warm-up

Matching Solutions

5 min

Activity Narrative

Students extend their understanding from the previous lessons to recognize the structure of a linear equation for all possible types of solutions: one solution, no solution, or infinitely many solutions. Students are still using language such as "true for one value of x," "always true" or "true for any value of x," and "never true." Students should be able to articulate that this depends both on the coefficient of the variable and on the **constant term** on each side of the equation.

Launch

Give students 2–3 minutes of quiet think time followed by a whole-class discussion.

Student Task Statement

Match each equation with the number of values that solve the equation.

A. $12(x - 3) + 18 = 6(2x - 3)$

B. $12(x - 3) + 18 = 4(3x - 3)$

C. $12(x - 3) + 18 = 4(2x - 3)$

1. true for only 1 value

2. true for no values

3. true for any value

Activity Synthesis

In order to highlight the structure of these equations, ask students:

"What do you notice about equations that are true for no values?"

These equations have equal or equivalent coefficients for the variable, but unequal values for the constants on each side of the equation.

"What do you notice about equations that are true for all values?"

These equations have equivalent expressions on each side of the equation, so the coefficients are equal and the constants are equal or equivalent on each side.

"What do you notice about equations that have exactly one value that makes them true?"

These equations have different values for the coefficients on each side of the equation and it doesn't matter what the constant term says.

Display the equation $x = 12$ for all to see. Ask students how this fits with their explanations.

We can see that there is one solution. Another way to think of this is that the coefficient of x is 1 on the left side of the equation, and the coefficient of x is 0 on the right side of the equation. So the coefficients of x are different, just as the explanation states.

Activity 1
Card Sort: Solutions

15 min

Activity Narrative

Students sort different equations during this activity. A sorting task gives students opportunities to analyze representations, statements, and structures closely and make connections.

Monitor for the different ways in which groups choose to categorize the equations, but especially for categories that distinguish between the number of solutions.

Students should look for ways to sort the equations without solving them. In particular, they should pay attention to the coefficients of the variable term as well as any constant terms.

Launch

Tell students to close their books or devices (or to keep them closed). Arrange students in groups of 2 and distribute pre-cut cards. Allow students to familiarize themselves with the representations on the cards:

Give students 1 minute to place all the cards face up and to start thinking about possible ways to sort the cards into categories.

Pause the class, and select 1–3 students to share the categories that they identified.

Discuss as many different categories as time allows.

Attend to the language that students use to describe their categories and equations, giving them opportunities to describe their equations more precisely. Highlight the use of terms like “solution,” “coefficient,” and “term.” After a brief discussion, invite students to continue with the activity.

Student Task Statement

Your teacher will give you a set of cards containing equations.

- Sort the cards into categories of your choosing.
- Describe the defining characteristics of the categories, and be prepared to share your reasoning with the class.

Sample response:

- One solution: A, C, G, H, I
- No solution: B, D, F
- Infinite number of solutions: E, J

Instructional Routines

Card Sort

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Instructional Routines

MLR8: Discussion Supports

ilclass.com/r/10695617

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Access for Students with Diverse Abilities (Activity 1, Launch)

Engagement: Develop Effort and Persistence.
Chunk this task into more manageable parts. Give students a subset of the cards to start with, and introduce the remaining cards after students have completed their initial set of matches.

Supports accessibility for: Conceptual Processing, Organization, Memory

Access for Multilingual Learners (Activity 1, Activity Synthesis)

MLR8: Discussion Supports. Display sentence frames to support whole-class discussion: "We grouped these equations into the category ____ because_____." and "Some characteristics of this category are ____." Invite students to press for details by asking clarifying questions such as, "What other features do those equations have in common?" and "Can you explain how to create a different equation that would fall into that category?"

Advances: Conversing, Representing

Activity Synthesis

Select 2–3 groups to share one of their categories' defining characteristics and which equations they sorted into that category. Given the *Warm-up*, categories based on the number of values that make the equations true labeled "one value," "no values," or "any values" are likely. Introduce students to the language "infinite number of solutions" if it has not already come up in discussion.

Ask students if anyone noticed a way to categorize the cards without completely solving them. If not, ask students to look for what the equations have in common when the equations got to the point where they looked like $ax + b = cx + d$.

During the discussion, it is likely that students will want to refer to specific parts of an expression. Encourage students to use the words "coefficient" and "variable." Define the word **constant term** as the term in an expression that doesn't change, or the term that does not have a variable part. For example, in the expression $2x - 3$, the 2 is the coefficient of x, the 3 is a constant, and x is the variable.

Activity 2

Make Use of Structure

20 min

Activity Narrative

The purpose of this activity is to give students an opportunity to compare the structure of equations that have no solution, one solution, and infinitely many solutions. This may be particularly useful for students who need more practice identifying which equations will have each of these types of solutions before they attempt to solve the equations.

Launch

Arrange students in groups of 2.

Give students 3–5 minutes of quiet think time to work on the problems, followed by 2–3 minutes of partner discussion to work together. Follow with a whole-class discussion.

Tell partners to divide the sets of questions so that they each solve 2 sets on their own and 1 set together, and then to discuss the last question together.

Student Task Statement

For each equation, determine whether it has no solutions, exactly one solution, or is true for all values of x (and has infinitely many solutions). If an equation has one solution, solve the equation to find the value of x that makes the statement true.

1. a. $6x + 8 = 7x + 13$ one solution: $x = -5$
   b. $6x + 8 = 2(3x + 4)$ infinitely many solutions, or true for every x
   c. $6x + 8 = 6x + 13$ no solution
2. a. $\frac{1}{4}(12 - 4x) = 3 - x$ infinitely many solutions, or true for every x
   b. $x - 3 = 3 - x$ one solution: $x = 3$
   c. $x - 3 = 3 + x$ no solution
3. a. $-5x - 3x + 2 = -8x + 2$ infinitely many solutions, or true for every x
   b. $-5x - 3x - 4 = -8x + 2$ no solutions
   c. $-5x - 4x - 2 = -8x + 2$ one solution: $x = -4$
4. a. $4(2x - 2) + 2 = 4(x - 2)$ one solution: $x = -\frac{1}{2}$
   b. $4x + 2(2x - 3) = 8(x - 1)$ no solution
   c. $4x + 2(2x - 3) = 4(2x - 2) + 2$ infinitely many solutions, or true for every x
5. a. $x - 3(2 - 3x) = 2(5x + 3)$ no solution
   b. $x - 3(2 + 3x) = 2(5x - 3)$ one solution: $x = 0$
   c. $x - 3(2 - 3x) = 2(5x - 3)$ infinitely many solutions, or true for every x
6. What do you notice about equations with one solution? How is this different from equations with no solutions and equations that are true for every x?

   Sample response: Equations with only one solution have a different amount of xs on each side, or the coefficients of x are not equal. Equations with no solution have the same coefficients of x but a different constant on each side, and equations with infinitely many solutions have equivalent expressions on each side of the equation.

Are You Ready for More?

Consecutive numbers follow one right after the other. An example of three consecutive numbers is 17, 18, and 19. Another example is -100, -99, -98.

1. Choose any set of three consecutive numbers. Find their average. What do you notice?

   The average is the second number of the three.
2. Find the average of another set of three consecutive numbers. What do you notice?

   The average of another set of three consecutive numbers is again the second of the three consecutive numbers.
3. Explain why the thing you noticed must always work, or find a counterexample.

   Sample reasoning: Any three consecutive numbers can be represented as x, $x + 1$, and $x + 2$. The average of the three numbers is $\frac{x + (x + 1) + (x + 2)}{3} = x + 1$, or the second of the three consecutive numbers.

Instructional Routines

MLR1: Stronger and Clearer Each Time

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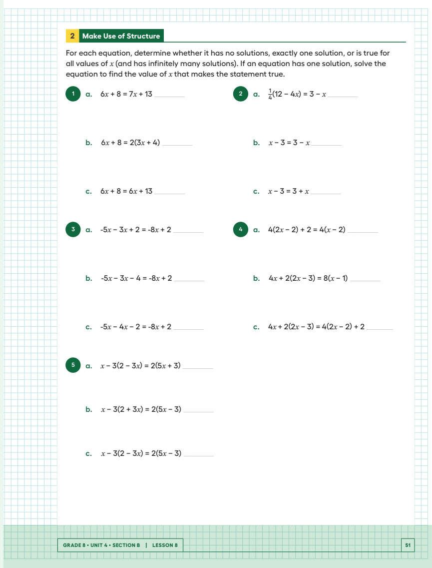
Access for Multilingual Learners (Activity 2)

MLR1: Stronger and Clearer Each Time
This activity uses the *Stronger and Clearer Each Time* math language routine to advance writing, speaking, and listening as students refine mathematical language and ideas.

Building on Student Thinking

On the second question, students may think that $x - 3 = 3 - x$ have the same coefficients of x. Ask students to rewrite each side in the form $ax + b = cx + d$ where the term with x is first and it is added to the constant term.

Student Workbook


2 Make Use of Structure

For each equation, determine whether it has no solutions, exactly one solution, or is true for all values of x (and has infinitely many solutions). If an equation has one solution, solve the equation to find the value of x that makes the statement true.

1. a. $6x + 8 = 7x + 13$ ____
   b. $6x + 8 = 2(3x + 4)$ ____
   c. $6x + 8 = 6x + 13$ ____
2. a. $\frac{1}{4}(12 - 4x) = 3 - x$ ____
   b. $x - 3 = 3 - x$ ____
   c. $x - 3 = 3 + x$ ____
3. a. $-5x - 3x + 2 = -8x + 2$ ____
   b. $-5x - 3x - 4 = -8x + 2$ ____
   c. $-5x - 4x - 2 = -8x + 2$ ____
4. a. $4(2x - 2) + 2 = 4(x - 2)$ ____
   b. $4x + 2(2x - 3) = 8(x - 1)$ ____
   c. $4x + 2(2x - 3) = 4(2x - 2) + 2$ ____
5. a. $x - 3(2 - 3x) = 2(5x + 3)$ ____
   b. $x - 3(2 + 3x) = 2(5x - 3)$ ____
   c. $x - 3(2 - 3x) = 2(5x - 3)$ ____

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Activity Synthesis

Briefly review each of the equations and how many solutions it has. If students disagree, ask them to explain their thinking about the equation and work to reach agreement.

Use *Stronger and Clearer Each Time* to give students an opportunity to revise and refine their response to what they notice about equations with 1, 0, or infinitely many solutions. In this structured pairing strategy, students bring their first draft response into conversations with 2–3 different partners. They take turns being the speaker and the listener. As the speaker, students share their initial ideas and read their first draft. As the listener, students ask questions and give feedback that will help their partner clarify and strengthen their ideas and writing.

If time allows, display these prompts for feedback:

"____ makes sense, but what do you mean when you say …?"

"Can you describe that another way?"

"How do you know …? What else do you know is true?"

Close the partner conversations, and give students 3–5 minutes to revise their first draft. Encourage students to incorporate any good ideas and words that they got from their partners to make their next draft stronger and clearer.

If time allows, invite students to compare their first and final drafts. Select 2–3 students to share how their drafts changed and why they made the changes they did.

After *Stronger and Clearer Each Time*, make clear for students how the form of the expressions on each side of the equation can help make it clear to identify the number of solutions. In particular, it is helpful to rewrite the original equation into an equivalent equation of the form $ax + b = cx + d$. For example, it is not as easy to see how many solutions there are for the equation $2(x + 3) + 6 = 9x - (7x + 4)$. Rewriting the equation using the distributive property and combining like terms on each side, we can get $2x + 12 = 2x - 4$ and can more easily see that there are no solutions to this equation.

Students should notice that, when the original equation is rewritten into an equivalent equation of the form $ax + b = cx + d$,

- There is 1 solution if the coefficients a and c are not equal.
- There is no solution when the coefficients of x are equal ($a = c$) and the constant terms are not equal ($b \neq d$).
- There are infinitely many solutions when the coefficients of x are equal ($a = c$) and the constant terms are also equal ($b = d$).

Lesson Synthesis

Instruct students to write three equations with a variable term and a constant term on each side of the equation. Their equations should be one with no solution, one with infinitely many solutions, and one with exactly one solution. When they think they have three equations that meet these requirements, tell students to trade with a partner, and then identify which equation is each type.

Give partners 2–3 minutes to check their solutions and to discuss how they came up with their equations.

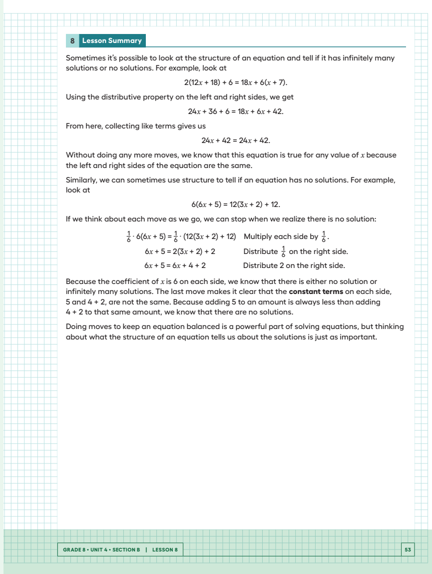
Ask students,

"How did you know how to make each type of equation?"

> I knew that the single-solution equation should have different coefficients for the variable terms, I knew that the many-solution equation should have equivalent expressions on each side, and I knew that the no-solution equation should differ only by a constant term on each side.

If time allows, consider making a poster for permanent display that shows an equation with coefficients, variables, and constant terms labeled.

Student Workbook



Lesson Summary

Sometimes it's possible to look at the structure of an equation and tell if it has infinitely many solutions or no solutions. For example, look at

$$2(12x + 18) + 6 = 18x + 6(x + 7).$$

Using the distributive property on the left and right sides, we get

$$24x + 36 + 6 = 18x + 6x + 42.$$

From here, collecting like terms gives us

$$24x + 42 = 24x + 42.$$

Without doing any more moves, we know that this equation is true for any value of x because the left and right sides of the equation are the same.

Similarly, we can sometimes use structure to tell if an equation has no solutions. For example, look at

$$6(6x + 5) = 12(3x + 2) + 12.$$

If we think about each move as we go, we can stop when we realize there is no solution:

$\frac{1}{6} \cdot 6(6x + 5) = \frac{1}{6} \cdot (12(3x + 2) + 12)$	Multiply each side by $\frac{1}{6}$.
$6x + 5 = 2(3x + 2) + 2$	Distribute $\frac{1}{6}$ on the right side.
$6x + 5 = 6x + 4 + 2$	Distribute 2 on the right side.

Because the coefficient of x is 6 on each side, we know that there is either no solution or infinitely many solutions. The last move makes it clear that the **constant terms** on each side, 5 and 4 + 2, are not the same. Because adding 5 to an amount is always less than adding 4 + 2 to that same amount, we know that there are no solutions.

Doing moves to keep an equation balanced is a powerful part of solving equations, but thinking about what the structure of an equation tells us about the solutions is just as important.

Responding To Student Thinking

Points to Emphasize
If students struggle to recognize and define the structure of equations that makes them true for 1, 0, or all x, share examples of student reasoning that is incomplete (for example, it's true because it would be $36x$ on each side) and ask students to generate a counterexample (like $36x + 5 = 36x + 4$ is not true for any x). Allow students to revise their initial thinking on their *Cool-downs*.

Cool-down

How Does She Know?

5 min

Student Task Statement

Elena begins to solve this equation:

$$\frac{12x + 6(4x+3)}{3} = 2(6x + 4) - 2$$
$$12x + 6(4x + 3) = 3(2(6x + 4) - 2)$$
$$12x + 6(4x + 3) = 6(6x + 4) - 6$$
$$12x + 24x + 18 = 36x + 24 - 6$$

When she gets to the last line she stops and says the equation is true for all values of x. How can Elena tell?

Sample response: Elena can see that there are the same number of xs and the same constant terms on each side of the equation.

Lesson 8

Practice Problems

6 Problems

Problem 1

Lin is looking at the equation $2x - 32 + 4(3x - 2462) = 14x$. She said, “I can tell right away that there are no solutions, because on the left side, you will have $2x + 12x$ and a bunch of constants, but you have just $14x$ on the right side.” Do you agree with Lin? Explain your reasoning.

partially agree

Sample reasoning: She is right, but she does not mention checking the constant terms. Ignoring everything but the terms with x on the left side, we have $2x$ and $4(3x)$. In total, this will give $14x$. All of the constant terms on the left side are negative, so they won’t cancel to 0. Therefore, we have $14x$ + non-zero terms = $4x$, which will have no solutions.

Problem 2

Han is looking at the equation $6x - 4 + 2(5x + 2) = 16x$. He says, “I can tell right away there are no solutions, because on the left side, you will have $6x + 10x$ and a bunch of constants, but you have just $16x$ on the right side.” Do you agree with Han? Explain your reasoning.

disagree

Sample reasoning: Han checked the coefficients of x, but he did not check the constant terms. Ignoring everything but the terms with x on the left side, we have $6x$ and $2(5x)$. In total, this will give $16x$. Collecting all the constant terms on the left side will give $-4 + 2(2)$, which is 0. Therefore, we have $16x + 0 = 16x$, which is true for all values of x.

Problem 3

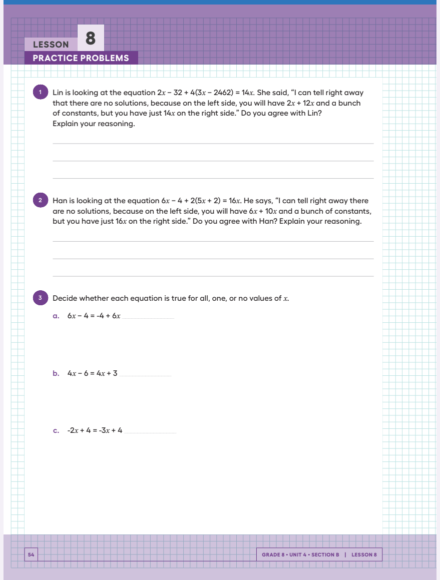
Decide whether each equation is true for **all**, one, or no values of x.

a. $6x - 4 = -4 + 6x$ true for all values of x

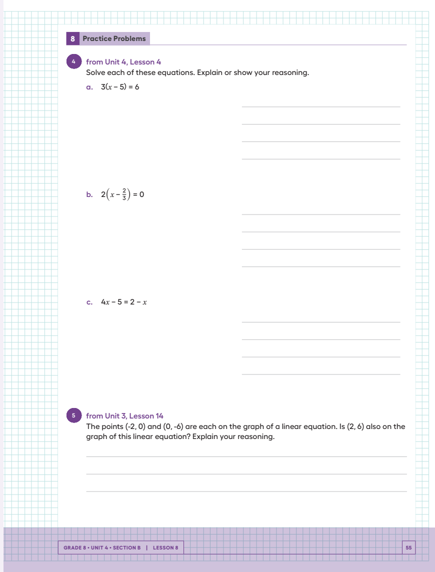
b. $4x - 6 = 4x + 3$ true for no values of x

c. $-2x + 4 = -3x + 4$ true for one value of x

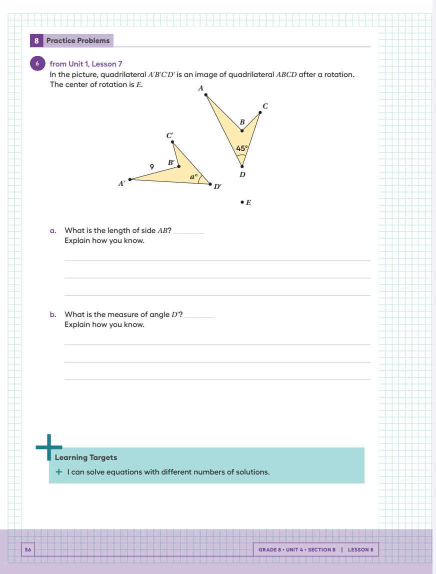
Student Workbook



Student Workbook



Student Workbook



Problem 4

from Unit 4, Lesson 4

Solve each of these equations. Explain or show your reasoning.

a. $3(x - 5) = 6$

$x = 7$

Sample reasoning: Multiply both sides by $\frac{1}{3}$, then add 5.

b. $2\left(x - \frac{2}{3}\right) = 0$

$x = \frac{2}{3}$

Sample reasoning: Multiply both sides by $\frac{1}{2}$, then add $\frac{2}{3}$.

c. $4x - 5 = 2 - x$

$x = \frac{7}{5}$

Sample reasoning: Add x and 5 to both sides, then multiply by $\frac{1}{5}$.

Problem 5

from Unit 3, Lesson 14

The points (-2, 0) and (0, -6) are each on the graph of a linear equation.
Is (2, 6) also on the graph of this linear equation? Explain your reasoning.

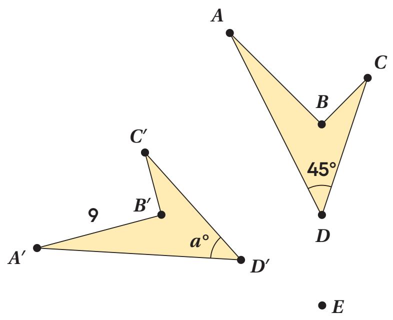
no

Sample reasoning: If the two points are graphed with the line that goes through both of them, the line does not pass through the first quadrant where (2, 6) is plotted.

Problem 6

from Unit 1, Lesson 7

In the picture, quadrilateral $A'B'C'D'$ is an image of quadrilateral $ABCD$ after a rotation. The center of rotation is E.



a. What is the length of side AB? Explain how you know.

9 units

Sample reasoning: Rotations preserve side lengths, and side $A'B'$ corresponds to side AB under this rotation.

b. What is the measure of angle D'? Explain how you know.

45 degrees

Sample reasoning: Rotations preserve angle measures, and angles D and D' correspond to each other under this rotation.