Lesson 17

Infinite Decimal Expansions

Goals

- Compare and contrast (orally) decimal expansions for rational and irrational numbers.
- Coordinate (orally and in writing) repeating decimal expansions and rational numbers that represent the same number.

Learning Targets

- I can write a repeating decimal as a fraction.
- I understand that every number has a decimal expansion.

Lesson Narrative

In this lesson, students explore finding decimal expansions of rational numbers as well as irrational numbers. Students begin by using long division to find the decimal expansion of $\frac{3}{7}$, which starts to repeat after the seventh decimal place. Repeated reasoning allows students to stop the long division process and express the decimal expansion following the same steps.

Next, students learn how to take a repeating decimal expansion and rewrite it in fraction form by ordering a set of cards that show the steps and an explanation of the process. They then practice with different decimal expansions following the same steps.

In the last activity of this lesson, students investigate how to approximate decimal expansions of irrational numbers. Since values like $\sqrt{2}$ and π cannot be written as a fraction, students use successive approximations to find more and more digits of their decimal expansions. Students see that there is no easy way to keep zooming in on these irrational numbers since they are not predictable like repeating decimals, so we use symbols to name them. However, in practice, using approximations is usually good enough for a given purpose.

Student Learning Goal

Let's think about infinite decimals.

Access for Students with Diverse Abilities

- Engagement (Activity 1, Activity 2)

Access for Multilingual Learners

- MLR8: Discussion Supports (Activity 2)

Required Materials

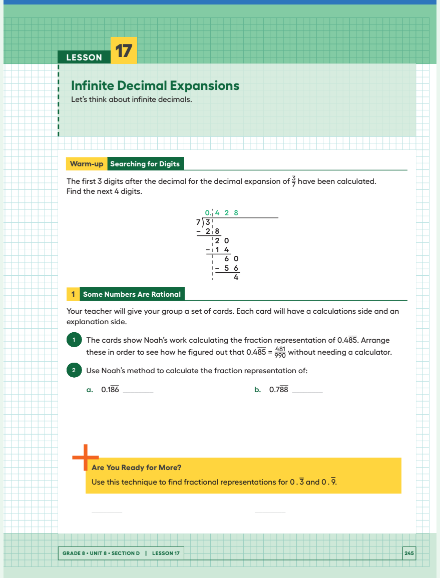
Materials to Copy

- Some Numbers Are Rational Cards (1 copy for every 2 students): Activity 2

Lesson Timeline

Warm-up	Activity 1	Activity 2	Lesson Synthesis	Cool-down (Assessment)
5 min	15 min	15 min	10 min	5 min

Student Workbook



Warm-up
Searching for Digits

5 min

Activity Narrative

In this activity, students practice using long division to calculate the decimal form of a rational number. Students are expected to notice a repeating pattern.

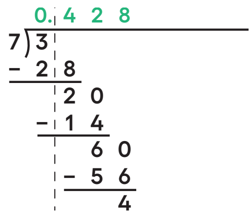
Launch

Arrange students in groups of 2. Since the goal is for students to notice a repeating pattern, do not provide access to calculators. If needed, remind them of the previous activity where the decimal expansion of $\frac{2}{11}$ was shown to be $0.\overline{18}$ using long division and repeated reasoning.

Give students 1–2 minutes of quiet work time, and follow with a whole-class discussion.

Student Task Statement

The first 3 digits after the decimal for the decimal expansion of $\frac{3}{7}$ have been calculated. Find the next 4 digits.



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Activity Synthesis

The goal of this discussion is to make sure students understand that *all* rational numbers have a decimal expansion that eventually repeats. Ask students to share the next 4 digits and record them on the long division calculation for all to see. Discuss:

- *"Without calculating, what number do you think will be next? Why?"*

  I think 2 will be the next digit because I can see the starting pattern has begun again.

Continue the calculation and verify that 2 comes next and continue until reaching 4 again. Point out that this cycle will continue indefinitely—we can predict what will happen at each step because it is exactly like what happened 6 steps ago.

Tell students that all rational numbers have a decimal expansion that eventually repeats. Sometimes they eventually repeat 0s, like in $\frac{3}{8} = 0.3750000 \ldots$. Sometimes they repeat several digits like in $\frac{3}{7} = 0.\overline{428571}$. If necessary, remind students that in overline notation, the line goes over the digits that repeat.

Be careful in the use of the word "pattern," as it can be ambiguous. For example, there is a pattern to the digits of the number 0.12112111211112 … , but the number is not rational.

Activity 1

Some Numbers Are Rational

15 min

Activity Narrative

In this activity, students learn and practice a strategy for rewriting repeating decimals as a fraction. Students begin by arranging cards in an order that shows how a strategy is used to show $0.4\overline{85} = \frac{481}{990}$. Next, students use the strategy to calculate the fractional representations of two other values.

The extension problem asks students to repeat the task for decimal expansions of $0.\overline{3}$ and $0.\overline{9}$. It may be counter-intuitive for students to conclude that $0.\overline{9} = 1$. Students might insist that $0.\overline{9}$ must be strictly less than 1, which can prompt an interesting discussion. Invite students to make their argument more precise.

Launch

Arrange students in groups of 2. Do not provide access to calculators since the purpose of this activity is to rewrite a decimal representation of a number as a fraction.

Demonstrate the algorithm with an example such as converting $0.\overline{12} = \frac{12}{99}$.

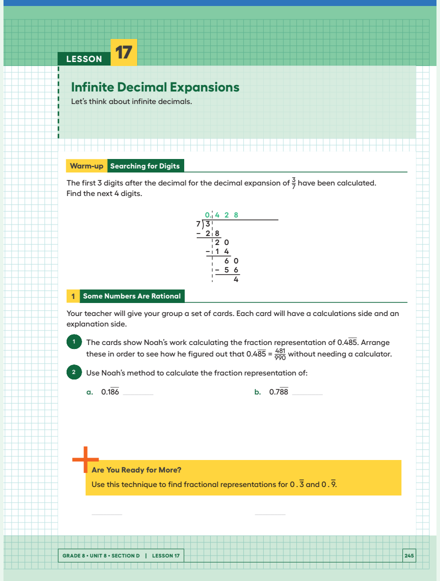
Distribute a set of the slips cut from the blackline master to each group. Tell students to order the cards with their partner, then work on the second question individually.

Give students 5 minutes of quiet work time and follow with a whole-class discussion.

Access for Students with Diverse Abilities (Activity 1, Launch)

Engagement: Develop Effort and Persistence.
Chunk this task into more manageable parts. Give students half of the cards to start with and introduce the remaining cards once students have completed their initial set of matches.
Supports accessibility for: Conceptual Processing, Organization, Memory

Student Workbook



Student Task Statement

Your teacher will give your group a set of cards. Each card will have a calculations side and an explanation side.

1. The cards show Noah's work calculating the fraction representation of $0.4\overline{85}$. Arrange these in order to see how he figured out that $0.4\overline{85} = \frac{481}{990}$ without needing a calculator.

   See blackline master for the correct order.

2. Use Noah's method to calculate the fraction representation of:

   a. $0.1\overline{86}$

   $\frac{185}{990}$ (or equivalent)

   Sample response:

   $x = 0.1\overline{86}$

   $100x = 18.6\overline{86}$

   $100x - x = 18.6\overline{86} - x$

   $99x = 18.5$

   $990x = 185$

   $x = \frac{185}{990}$

   b. $0.7\overline{88}$

   $\frac{71}{90}$ (or equivalent)

   Sample response:

   $x = 0.7\overline{88}$

   $10x = 7.8\overline{8}$

   $10x - x = 7.8\overline{8} - x$

   $9x = 7.1$

   $90x = 71$

   $x = \frac{71}{90}$

Are You Ready for More?

Use this technique to find fractional representations for $0.\overline{3}$ and $0.\overline{9}$.

$0.\overline{3} = \frac{1}{3}$ and $0.\overline{9} = \frac{9}{9} = 1$

Activity Synthesis

The goal of this discussion is to reinforce student understanding of the strategy introduced to rewrite a repeating decimal as a fraction. Begin the discussion by selecting 2–3 students to share their work for the second problem, displaying each step for all to see. Ask if anyone completed the problem in a different way and, if so, have those students also share.

If no students notice it, point out that when rewriting $0.7\overline{88}$, we can multiply by 100, but multiplying by 10 also works since the part that repeats is only 1 digit long.

Conclude the discussion by asking students to rewrite $0.3\overline{0}$ using this strategy.

Activity 2
Some Numbers Are Not Rational

15 min

Activity Narrative

In this activity, students revisit irrational numbers to emphasize the point that no fractional representation exists for these numbers. For all irrational numbers, long division doesn't work because there are no two integers to divide, and different methods to approximate the value need to be discussed.

Students will approximate the value of $\sqrt{2}$ using successive approximations and discuss how they might figure out a value for π using measurements of circles. They will also plot these values on number lines to reinforce the idea that irrational numbers are still numbers—they have a place on the number line even if they cannot be written as a positive or negative fraction.

Launch

Arrange students in groups of 2. Provide access to a calculator without a square root button. If that calculator does not have a button π, tell students that a calculator will give something similar to 3.141592654 as a value for π.

Discuss with students:

"In a previous activity, we used long division to find decimal representations of numbers. Why can't we do that with irrational numbers?"

There are no 2 numbers to divide since an irrational number cannot be written as a fraction.

Then give students 3–4 minutes of quiet work time.

Pause students and tell them to check in with their partner after the first set of questions. Continue work on the remaining questions, and follow with a whole-class discussion.

Student Task Statement

1. **a.** Why is $\sqrt{2}$ between 1 and 2 on the number line?

   Sample response: $1^2 = 1$ and $2^2 = 4$, so $\sqrt{2}$ must be between 1 and 2.

   b. Why is $\sqrt{2}$ between 1.4 and 1.5 on the number line?

   Sample response: $1.4^2 = 1.96$ and $1.5^2 = 2.25$, so $\sqrt{2}$ must be between 1.4 and 1.5.

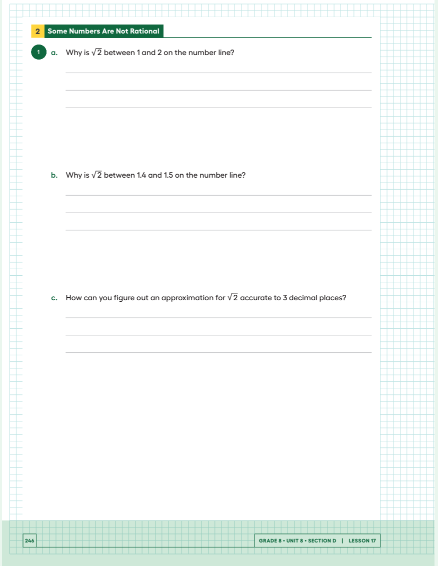
   c. How can you figure out an approximation for $\sqrt{2}$ accurate to 3 decimal places?

   Sample response: Checking the squares of values from 1.40 to 1.50, we find that 1.41^2 is the closest to $(\sqrt{2})^2$. Next, checking the squares of values from 1.410 to 1.420, we find that 1.414^2 is the closest to $(\sqrt{2})^2$. So, 1.414 is approximately equal to $\sqrt{2}$.

Access for Students with Diverse Abilities (Activity 2, Launch)

Engagement: Internalize Self Regulation.
Chunk this task into more manageable parts to differentiate the degree of difficulty or complexity. Invite students to respond to all parts of the first question and as much of the second question as they have time for.
Supports accessibility for: Organization, Attention

Student Workbook


2 Some Numbers Are Not Rational

1. a. Why is $\sqrt{2}$ between 1 and 2 on the number line?

   b. Why is $\sqrt{2}$ between 1.4 and 1.5 on the number line?

   c. How can you figure out an approximation for $\sqrt{2}$ accurate to 3 decimal places?

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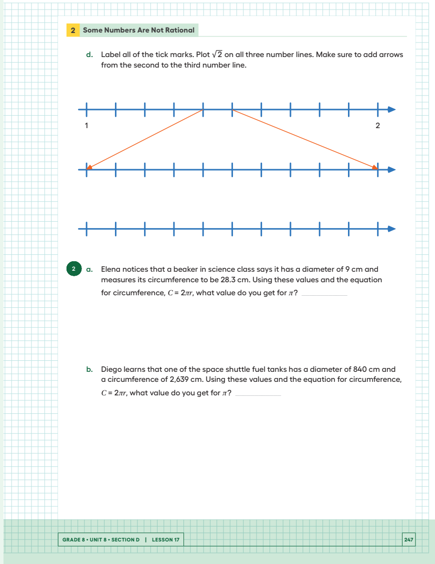
Student Workbook



2 Some Numbers Are Not Rational

d. Label all of the tick marks. Plot $\sqrt{2}$ on all three number lines. Make sure to add arrows from the second to the third number line.

1 2

2 a. Elena notices that a beaker in science class says it has a diameter of 9 cm and measures its circumference to be 28.3 cm. Using these values and the equation for circumference, $C = 2\pi r$, what value do you get for π? ____

b. Diego learns that one of the space shuttle fuel tanks has a diameter of 840 cm and a circumference of 2,639 cm. Using these values and the equation for circumference, $C = 2\pi r$, what value do you get for π? ____

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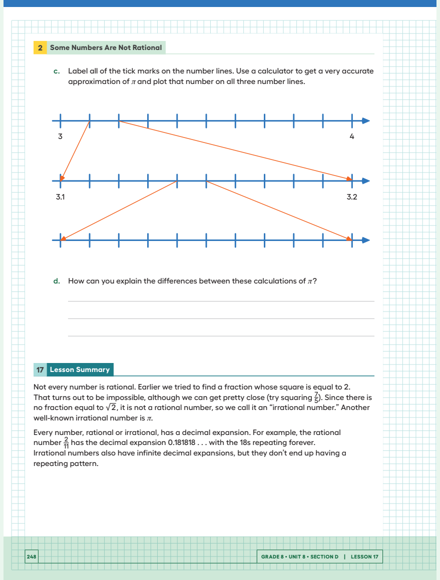
Student Workbook



2 Some Numbers Are Not Rational

c. Label all of the tick marks on the number lines. Use a calculator to get a very accurate approximation of π and plot that number on all three number lines.

3 4

3.1 3.2

d. How can you explain the differences between these calculations of π?

17 Lesson Summary

Not every number is rational. Earlier we tried to find a fraction whose square is equal to 2. That turns out to be impossible, although we can get pretty close (try squaring $\frac{7}{5}$). Since there is no fraction equal to $\sqrt{2}$, it is not a rational number, so we call it an "irrational number." Another well-known irrational number is π.

Every number, rational or irrational, has a decimal expansion. For example, the rational number $\frac{2}{11}$ has the decimal expansion 0.181818 . . . with the 18s repeating forever. Irrational numbers also have infinite decimal expansions, but they don't end up having a repeating pattern.

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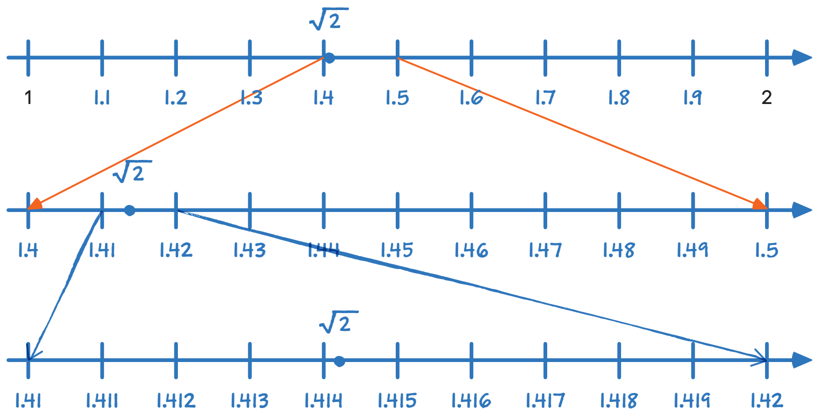
d. Label all of the tick marks. Plot $\sqrt{2}$ on all three number lines. Make sure to add arrows from the second to the third number line.



2. a. Elena notices that a beaker in science class says it has a diameter of 9 cm and measures its circumference to be 28.3 cm. Using these values and the equation for circumference, $C = 2\pi r$, what value do you get for π?

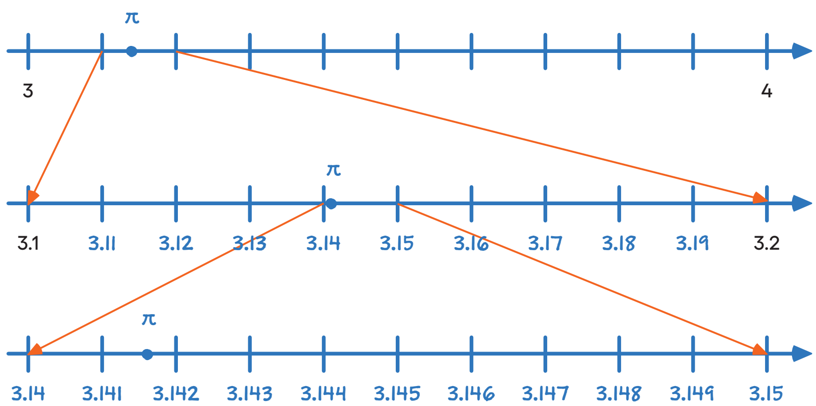
$28.3 \div 9 = 3.1\overline{4}$

b. Diego learns that one of the space shuttle fuel tanks has a diameter of 840 cm and a circumference of 2,639 cm. Using these values and the equation for circumference, $C = 2\pi r$, what value do you get for π?

$2639 \div 840 = 3.14\overline{16}$

c. Label all of the tick marks on the number lines. Use a calculator to get a very accurate approximation of π and plot that number on all three number lines.

A calculator will give something similar to 3.141592654 as a value for π.



d. How can you explain the differences between these calculations of π?

Sample response: Diego and Elena's calculations use measurements made from actual objects, which means they are approximated with things like tape measures. The objects are probably also not perfectly round. This makes their calculations less accurate than my calculator, which has many digits of π programmed into it.

Activity Synthesis

The purpose of this discussion is to deepen students' understanding that irrational numbers cannot be written as fractions, but they are still numbers with a location on the number line. As such, their values are approximated using different methods than for rational numbers. Discuss with students:

- ***"How long do you think you could keep using the method in the first problem to find more digits of the decimal representation of $\sqrt{2}$?"***

  My calculator only shows 9 digits to the right of the decimal, so that's as far as I could go.

- ***"Would this method work for any root?"***

  Yes, I could find the two perfect squares that the square of the number we are taking the square root of is between and then keep approximating the value of the root from there.

Tell students that the strategy they used in the first problem is called "successive approximation." It takes time, but successive approximation works for finding more and more precise approximations of irrational numbers, as long as there is a clear value to them check against. In the case of $\sqrt{2}$, since $\sqrt{2}^2 = 2$, there is a clear value to test whether approximations are too high or too low.

Lastly, select 2–3 students to share their reasoning to the last part of the second problem. Make sure students understand that since measuring has limitations of accuracy, any calculation of π using measurement, such as with the beaker and the fuel tank, will have limited accuracy.

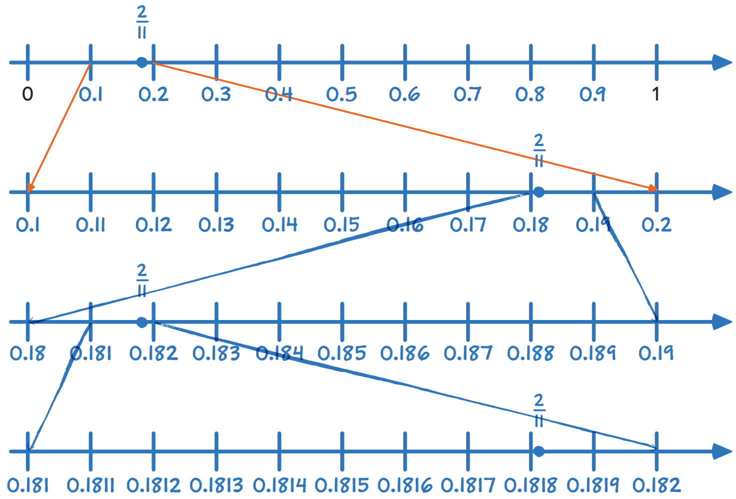
Access for Multilingual Learners (Activity 2, Synthesis)

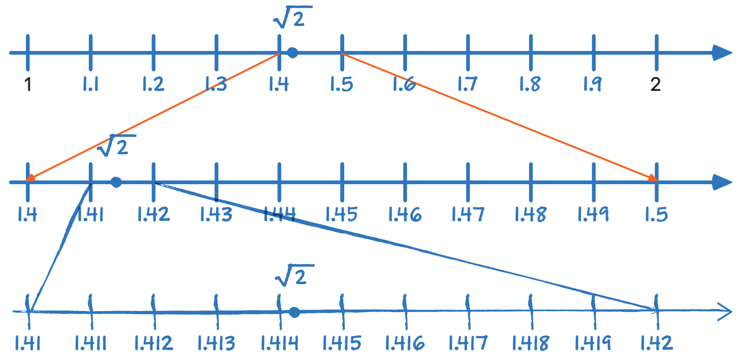
MLR8: Discussion Supports. For each response to the last part of the second problem that is shared, invite students to turn to a partner and restate what they heard using precise mathematical language.
Advances: Listening, Speaking

Lesson Synthesis

The main goal of this discussion is to emphasize that $\sqrt{2}$ and π are irrational numbers. Their decimal expansions never end and never repeat like rational numbers do. But they are still numbers!

Display the number lines for $\frac{2}{11}$ and $\sqrt{2}$ for all to see.





Ask students what is the same and what is different about the two images (Both images show number lines that are each partitioned into 10 parts. Each successive number line is labeled with one additional digit of accuracy. One image has 4 number lines while the other only has 3. The image for $\frac{2}{11}$ has a repeating pattern and would go on forever. The image for $\sqrt{2}$ would also go on forever but without a predictable pattern.)

Explain that the number $\sqrt{2}$ has to be a number because a square with an area of 2 square units has $\sqrt{2}$ as a side length. Similarly, π has to be a number because it relates the radius of a circle to its area. The main takeaway is that there exist numbers that are real, but cannot be written as positive or negative fractions. Here are some questions for discussion:

- *"What are some decimals for which our method of rewriting decimals as fractions will work?"*

  any repeating decimal

- *"We've seen that the decimal expansion of $\sqrt{2}$ does not repeat. What would happen if we tried to use Noah's method on $\sqrt{2}$? On π?"*

  It would not work because those numbers are irrational.

Lesson Summary

Not every number is rational. Earlier we tried to find a fraction whose square is equal to 2. That turns out to be impossible, although we can get pretty close (try squaring $\frac{7}{5}$). Since there is no fraction equal to $\sqrt{2}$, it is not a rational number, so we call it an "irrational number." Another well-known irrational number is π.

Every number, rational or irrational, has a decimal expansion. For example, the rational number $\frac{2}{11}$ has the decimal expansion 0.181818 ... with the 18s repeating forever. Irrational numbers also have infinite decimal expansions, but they don't end up having a repeating pattern.

Cool-down

Repeating in Different Ways

5 min

Student Task Statement

Let $x = 0.147$ and let $y = 0.\overline{147}$.

1. Is x a rational number?

   Yes, $0.147 = \frac{147}{1{,}000}$ is rational.

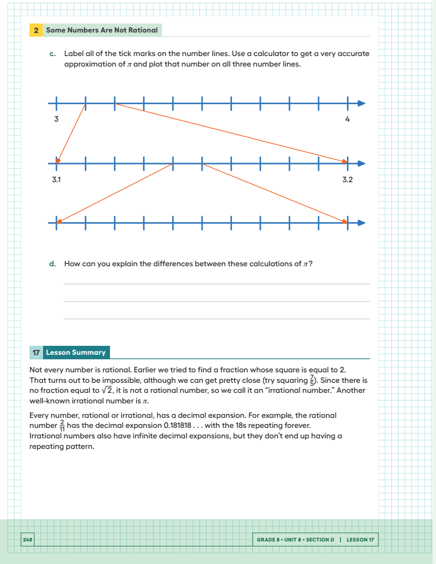
2. Is y a rational number?

   Yes, $0.\overline{147} = \frac{49}{333}$ is rational.

3. Which is larger, x or y?

   y is larger than x since $y = 0.147147147\ldots$ and $x = 0.147000000\ldots$

Student Workbook



2 Some Numbers Are Not Rational

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Responding To Student Thinking

Press Pause

If most students struggle with identifying rational and irrational numbers, make time to revisit this. For example, in the lesson referred to here, invite multiple students to share their thinking about how to determine whether or not a number is rational or irrational using long division.

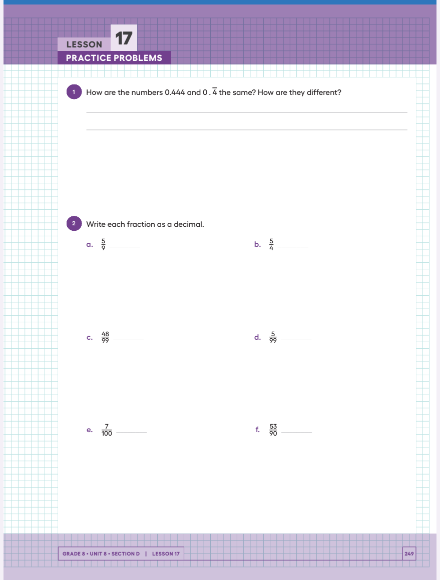
Unit 8, Lesson 17 Infinite Decimal Expansions

Lesson 17

Practice Problems

6 Problems

Student Workbook


LESSON 17
PRACTICE PROBLEMS

1 How are the numbers 0.444 and $0.\overline{4}$ the same? How are they different?

2 Write each fraction as a decimal.

Problem 1

How are the numbers 0.444 and $0.\overline{4}$ the same? How are they different?

Sample response: They are the same in that they are both rational numbers between 0.4 and 0.5, and the first three digits in their decimal expansions are the same. They are different in that $0.\overline{4}$ is greater than 0.444 because it has a greater digit in the ten-thousandths place. 0.444 is a terminating decimal, while $0.\overline{4}$ is an infinitely **repeating decimal.**

Problem 2

Write each fraction as a decimal.

a. $\frac{5}{9}$

$0.\overline{5}$

b. $\frac{5}{4}$

1.25

c. $\frac{48}{99}$

$0.\overline{48}$

d. $\frac{5}{99}$

$0.\overline{05}$

e. $\frac{7}{100}$

0.07

f. $\frac{53}{90}$

$0.5\overline{8}$

Problem 3

Write each decimal as a fraction.

a. $0.\overline{7}$

$\frac{7}{9}$ (or equivalent)

b. $0.\overline{2}$

$\frac{2}{9}$ (or equivalent)

c. $0.1\overline{3}$

$\frac{2}{15}$ (or equivalent)

d. $0.\overline{14}$

$\frac{14}{99}$ (or equivalent)

e. $0.\overline{03}$

$\frac{3}{99}$ (or equivalent)

f. $0.6\overline{38}$

$\frac{632}{990}$ (or equivalent)

g. $0.52\overline{4}$

$\frac{472}{900}$ (or equivalent)

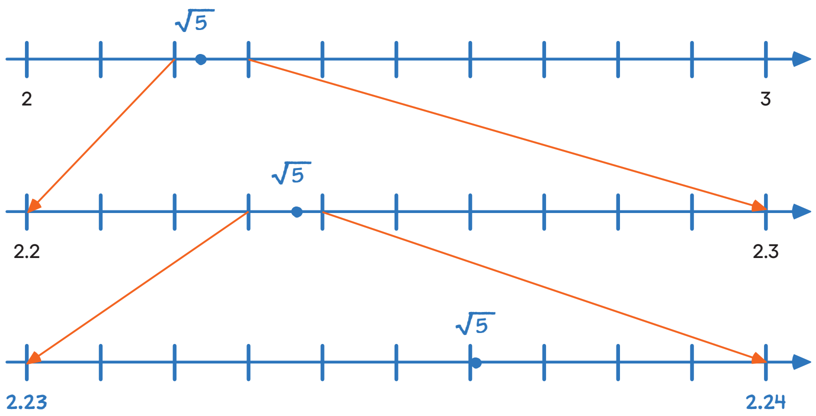
h. $0.1\overline{5}$

$\frac{14}{90}$ (or equivalent)

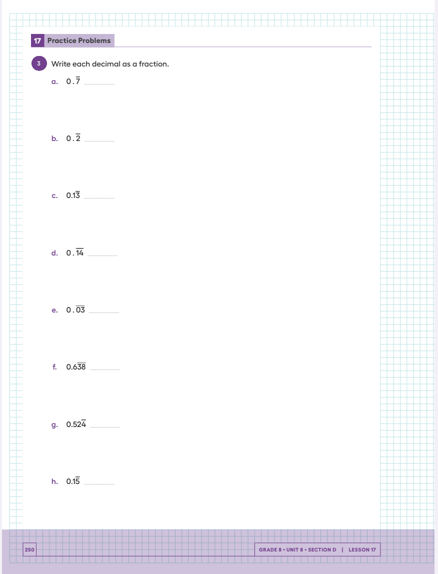
Problem 4

$2.2^2 = 4.84$ and $2.3^2 = 5.29$. This gives some information about $\sqrt{5}$.

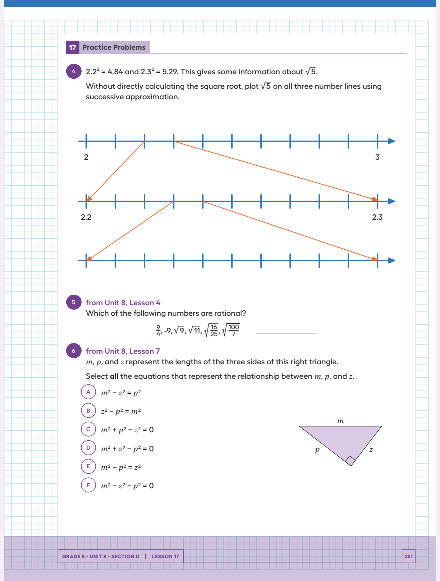
Without directly calculating the square root, plot $\sqrt{5}$ on all three number lines using successive approximation.



Student Workbook



Student Workbook



Problem 5 from Unit 8, Lesson 4

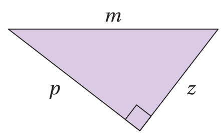
Which of the following numbers are rational?

$\frac{9}{4}$, -9, $\sqrt{9}$, $\sqrt{11}$, $\sqrt{\frac{16}{25}}$, $\sqrt{\frac{100}{7}}$

$\frac{9}{4}$ -9 $\sqrt{9}$ $\sqrt{\frac{16}{25}}$

Problem 6 from Unit 8, Lesson 7

m, p, and z represent the lengths of the three sides of this right triangle.



Select **all** the equations that represent the relationship between m, p, and z.

A. $m^2 - z^2 = p^2$

B. $z^2 - p^2 = m^2$

C. $m^2 + p^2 - z^2 = 0$

D. $m^2 + z^2 - p^2 = 0$

E. $m^2 - p^2 = z^2$

F. $m^2 - z^2 - p^2 = 0$