Lesson 13

Congruence

Goals

- Determine whether shapes are congruent by measuring corresponding points.
- Draw and label corresponding points on congruent figures.
- Justify (orally and in writing) that congruent figures have equal corresponding distances between pairs of points.

Learning Target

I can use distances between points to decide if two figures are congruent.

Student Learning Goal

Let's find ways to test congruence of interesting figures.

Lesson Narrative

So far, we have mainly looked at congruence for polygons. Polygons are special because they are determined by line segments. These line segments give polygons easily defined distances and angles to measure and compare. For a more complex shape with curved sides, the situation is a little different (unless the shape has special properties, such as being a circle). The focus here is on the fact that the distance between *any* pair of corresponding points of congruent figures must be the same. Because there are too many pairs of points to consider, this is mainly a criterion for showing that two figures are *not* congruent: that is, if there is a pair of points on one figure that are a different distance apart than the corresponding points on another figure, then those figures are *not* congruent.

One of the mathematical practices that takes center stage in this lesson is attending to precision. For congruent figures built out of several different parts (for example, a collection of circles), the distances between *all* pairs of points must be the same. It is not enough that the constituent parts (circles for example) be congruent: they must also be in the same configuration, the same distance apart. This follows from the definition of congruence: Rigid motions do not change distances between points, so if figure 1 is congruent to figure 2 then the distance between *any* pair of points in figure 1 is equal to the distance between the corresponding pair of points in figure 2.

Access for Students with Diverse Abilities

- Engagement (Activity 1)
- Action and Expression (Activity 3)

Access for Multilingual Learners

- MLR7: Compare and Connect (Activity 2)
- MLR8: Discussion Supports (Activity 3)

Instructional Routines

- MLR5: Co-Craft Questions

Required Materials

Materials to Gather

- Chart paper: Warm-up
- Math Community Chart: Warm-up
- Geometry toolkits: Activity 1, Activity 2, Activity 3
- Rulers: Activity 2

Lesson Timeline

Warm-up	Activity 1	Activity 2	Activity 3	Lesson Synthesis	Cool-down (Assessment)
5 min	10 min	15 min	10 min	10 min	5 min

Congruence

Lesson Narrative (continued)

Math Community
In today's activities, students are introduced to the idea of math norms as expectations that help everyone in the room feel safe, comfortable, and productive doing math together. Students then consider what norms would connect and support the math actions the class recorded so far in the Math Community Chart.

Instructional Routines

MLR5: Co-Craft Questions

ilclass.com/r/10695544

Please log in to the site before using the QR code or URL.

Warm-up

Oval Questions

5 min

Activity Narrative

The purpose of this activity is for students to orient themselves to an image of ovals on a coordinate plane. This image will be used in an upcoming activity as students study congruence of curved shapes. Creating mathematical questions about an image gives students an opportunity to make sense of problems.

This is the first time Math Language Routine 5: *Co-Craft Questions* is suggested in this course. In this routine, students are given a context or situation, often in the form of a problem stem (for example, a story, image, video, or graph) with or without numerical values. Students develop mathematical questions that can be asked about the situation. A typical prompt is: “What mathematical questions could you ask about this situation?” The purpose of this routine is to allow students to make sense of a context before feeling pressure to produce answers, and to develop students’ awareness of the language used in mathematics problems.

This activity uses the *Co-Craft Questions* math language routine to advance reading and writing as students make sense of a context and practice generating mathematical questions.

Launch

Tell students to close their books or devices (or to keep them closed). Arrange students in groups of 2. Use *Co-Craft Questions* to give students an opportunity to familiarize themselves with the image, and to practice producing the language of mathematical questions.

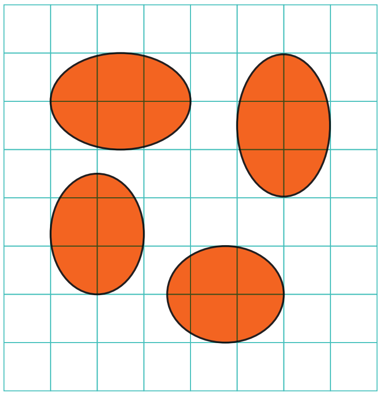
Display the image for all to see.
Ask students,

“What mathematical questions could you ask about this image?”

Give students 1–2 minutes to write a list of mathematical questions that could be asked about the image before comparing questions with a partner.

As partners discuss, support students in using conversation and collaboration skills to generate and refine their questions, for instance, by revoicing a question, seeking clarity, or referring to their written notes. Listen for how students use language about congruence and corresponding parts.

Student Task Statement



Sample responses:

- Which ovals are congruent to each other?
- What rigid transformation will take one congruent oval to another?
- Which ovals have the same area?
- What are the dimensions of each oval?

Activity Synthesis

Invite several partners to share one question with the class and record responses. Ask the class to make comparisons among the shared questions and their own. Ask,

"What do these questions have in common? How are they different?"

Listen for and amplify language related to the learning goal, such as testing the congruence of the shapes.

Math Community

At the end of the Warm-up, display the Math Community Chart. Tell students that norms are expectations that help everyone in the room feel safe, comfortable, and productive doing math together. Using the Math Community Chart, offer an example of how the "Doing Math" actions can be used to create norms. For example, "In the last exercise, many of you said that our math community sounds like 'sharing ideas.' A norm that supports that is 'We listen as others share their ideas.' For a teacher norm, 'questioning vs telling' is very important to me, so a norm to support that is 'Ask questions first to make sure I understand how someone is thinking.'"

Invite students to reflect on both individual and group actions. Ask,

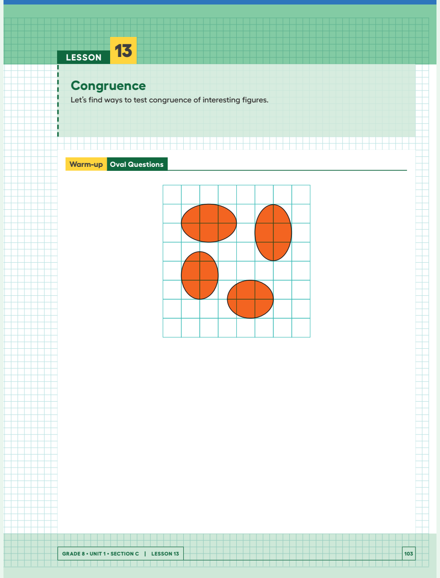
"As we work together in our mathematical community, what norms, or expectations, should we keep in mind?"

Give 1–2 minutes of quiet think time and then invite as many students as time allows to share either their own norm suggestion or to "+1" another student's suggestion.

Record student thinking in the student and teacher "Norms" sections on the Math Community Chart.

Conclude the discussion by telling students that what they made today is only a first draft of math community norms and that they can suggest other additions during the *Cool-down*. Throughout the year, students will revise, add, or remove norms based on those that are and are not supporting the community.

Student Workbook

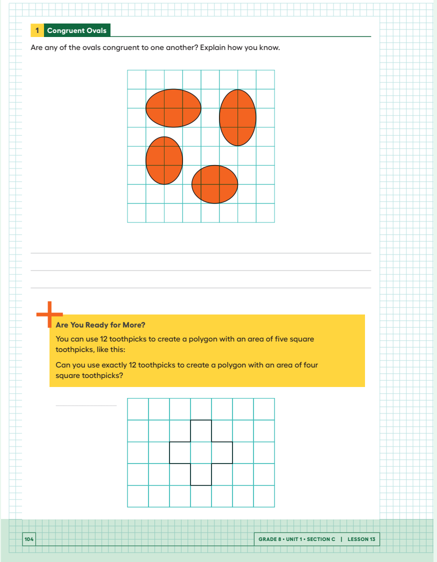


Access for Students with Diverse Abilities (Activity 1, Launch)

Engagement: Develop Effort and Persistence.
Connect a new concept to one with which students have experienced success. For example, review the criteria used to determine congruence for polygons so that students can transfer these strategies in determining congruence for curved shapes.

Supports accessibility for: Social-Emotional Functioning, Conceptual Processing

Student Workbook


1 Congruent Ovals

Are any of the ovals congruent to one another? Explain how you know.

Are You Ready for More?

You can use 12 toothpicks to create a polygon with an area of five square toothpicks, like this:

Can you use exactly 12 toothpicks to create a polygon with an area of four square toothpicks?

Activity 1

Congruent Ovals

10 min

Activity Narrative

In this activity, students consider how to determine whether curved shapes are congruent. Unlike a polygon, a curved shape is not completely determined by a set of vertices. Students consider the features of ovals as a way to identify that two shapes are not congruent, such as the length of the widest part of each oval. In order to provide a viable argument that two curved shapes are congruent, students must apply the definition of congruence by identifying a rigid transformation so that one shape matches up exactly with the other.

Monitor for students who use precise language of transformations as they attempt to move one traced oval to match up perfectly with another. Select them to share their reasoning during the discussion. Also monitor for arguments based on specific features for why neither of the upper ovals can be congruent to either of the lower ones and select a few students to share during the whole-class discussion.

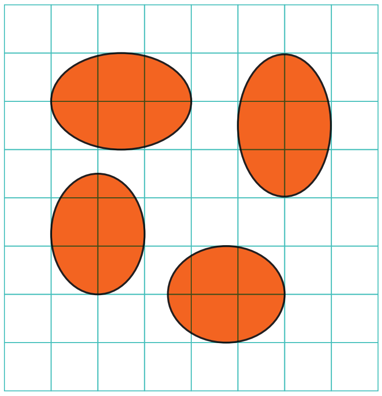
Launch

Arrange students in groups of 2. Provide access to geometry toolkits.

Give students 3 minutes of quiet work time, then 3–5 minutes to share their reasoning with a partner, followed by a whole-class discussion.

Student Task Statement

Are any of the ovals congruent to one another? Explain how you know.



Sample response: All four shapes are ovals. For the top two shapes, they can be surrounded by rectangles that measure 3 units by 2 units. For the bottom two shapes, they measure about $2\frac{1}{2}$ (possibly a little less) units by 2 units. This means that the top shapes are not congruent to the bottom shapes.

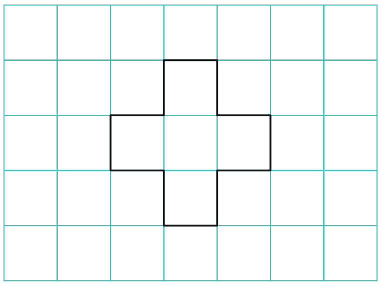
The top left oval can be taken to the top right oval with a 90-degree clockwise rotation around its center point followed by a translation 3.5 units right and .5 unit down. This means that the top shapes are congruent to each other.

The bottom left oval can be taken to the bottom right oval by a 90-degree counterclockwise rotation about the point in the center of the coordinate grid. This means that the bottom shapes are congruent to each other.

Are You Ready for More?

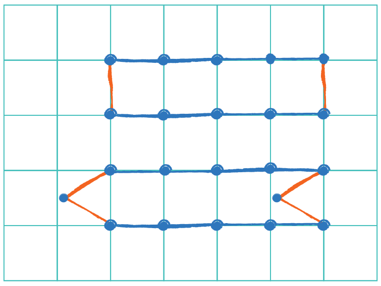
You can use 12 toothpicks to create a polygon with an area of five square toothpicks, like this:

Can you use exactly 12 toothpicks to create a polygon with an area of four square toothpicks?



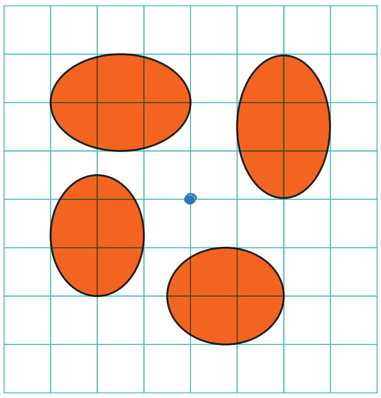
There is more than one solution, but here is one approach:

The first shape has a perimeter of 10 units and an area of 4 square units. To get two more toothpicks in without changing the area, an indentation can be put on one side which is balanced by the part that "sticks out" on the other side:



Activity Synthesis

Invite selected students to share the rigid transformations they used to show which pairs of ovals are congruent. If no students suggest it, show this image for all to see:



Ask students to describe the rigid transformation from the top left oval to the top right oval using the marked point as the center of rotation, and demonstrate if needed.

Invite previously selected students to share the features they noticed that showed the top ovals could not be congruent to the bottom ovals.

Ask students,

"How is showing that two ovals are congruent different than showing two polygons are congruent?"

There aren't angle measures and side lengths to measure that can show the shapes are congruent. The only way is to use a transformation to take one shape to the other.

Activity 2

Corresponding Points in Congruent Figures

15 min

Activity Narrative

In this activity, students compare corresponding parts of congruent figures with curved parts. Students have used the property that corresponding sides of congruent polygons have the same length in previous activities, but curved shapes like ovals do not have sides. Students connect corresponding side lengths on a polygon with corresponding points on a curved shape as they draw corresponding line segments on congruent figures. They use the property that translations, rotations, and reflections do not change distances between points to conclude that corresponding parts of the curved figures must stay the same distance apart.

Monitor for students who use these strategies to identify corresponding points on the two corresponding figures:

- Looking for corresponding parts of the figures, including line segments and points
- Performing rigid transformations with tracing paper to match the figures to each other

Select students using these strategies and select them to share during the discussion.

Launch

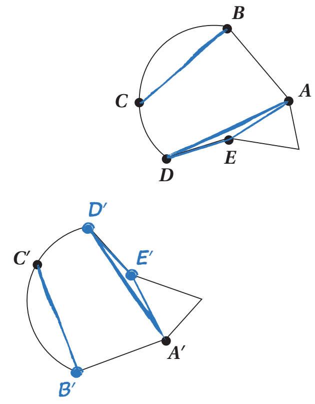
Arrange students in groups of 2.

Give 5 minutes of quiet work time followed by sharing with a partner and a whole-class discussion.

Provide access to geometry toolkits. Each student will need a ruler.

Student Task Statement

Here are two congruent shapes with some corresponding points labeled:



1. On the bottom figure, draw the points corresponding to B, D, and E, and label them B', D', and E'.
2. Draw line segments AD and $A'D'$ and measure them. Do the same for segments BC and $B'C'$ and for segments AE and $A'E'$. What do you notice?

   Sample response: The lengths are the same between segment AD and segment $A'D'$. They are also the same between BC and $B'C$ as well as between AE and $A'E'$.
3. Do you think there could be a pair of corresponding segments with different lengths? Explain.

   no

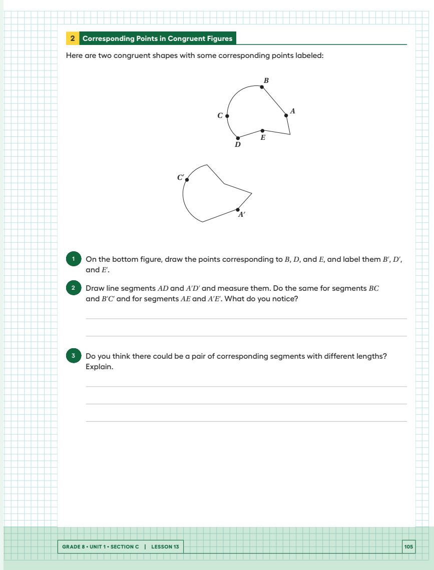
   Sample reasoning: Rigid transformations do not change distances between points. Corresponding segments in the two congruent figures must have the same length.

Access for Students with Diverse Abilities (Activity 2, Student Task)

Action and Expression: Internalize Executive Functions.
Invite students to plan a strategy, including the tools they will use, for drawing the corresponding parts and measuring line segments. If time allows, invite students to share their plan with a partner before they begin.

Supports accessibility for: Attention, Social-Emotional Functioning

Student Workbook


2 Corresponding Points in Congruent Figures

Here are two congruent shapes with some corresponding points labeled:

1 On the bottom figure, draw the points corresponding to B, D, and E, and label them B', D', and E'.

2 Draw line segments AD and $A'D'$ and measure them. Do the same for segments BC and $B'C'$ and for segments AE and $A'E'$. What do you notice?

3 Do you think there could be a pair of corresponding segments with different lengths? Explain.

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Access for Multilingual Learners (Activity 2, Synthesis)

MLR7: Compare and Connect. Lead a discussion comparing, contrasting, and connecting the different strategies. Ask, "How are the different approaches to matching parts of the figures the same?" "How are they different?" "Are there any benefits or drawbacks to one approach compared to another?"

Advances: Representing, Conversing

Building on Student Thinking

Students may think the two faces are congruent if all of the pieces of the faces match up, however the translations for each to match up are different. This may happen when students use tracing paper to test each individual piece. Ask students to find the distance between a pair of corresponding points that is not the same in the two faces. For example, ask them to measure the distance between a point on the left eye and point on the right eye and the corresponding distance in the second face.

Activity Synthesis

Ask selected students to show how they determined the points corresponding to B, D, and E, highlighting different strategies, such as identifying key features of the shapes and performing rigid motions. Ask students if these strategies would work for finding C' if it had not been marked. Performing rigid motions matches the shapes up perfectly, and so this method allows us to find the corresponding point for any point on the figure.

While it is challenging to test "by eye" whether or not complex shapes like these are congruent, the mathematical meaning of the word "congruent" is the same as with polygons: Two shapes are congruent when there is a sequence of translations, reflections, and rotations that match up one shape exactly with the other. Any pair of corresponding segments in congruent figures will have the same length because translations, reflections, and rotations do not change distances between points.

If time allows, have students use tracing paper to make a new figure that is either congruent to the shape in the activity or slightly different. Display several for all to see and poll the class to see if students think the figure is congruent or not. Check to see how the class did by lining up the new figure with one of the originals. Work with these complex shapes is important because we tend to rely heavily on visual intuition to check whether or not two polygons are congruent. This intuition may not be reliable if the polygons are complex or have very subtle differences that cannot be easily seen. The meaning of congruence in terms of rigid transformations and our visual intuition of congruence can effectively reinforce one another:

If shapes look congruent, then we can try to find the rigid transformation to demonstrate that they are congruent.

Activity 3: Optional
Astonished Faces

10 min

Activity Narrative

This activity is optional because it provides an opportunity for additional practice applying the property that all corresponding parts of congruent figures are congruent. In this activity, students examine two figures made out of disjoint pieces: the outline of a face, two eyes, and a mouth. The disjoint pieces are congruent to one another taken in isolation. But they are not congruent taken as a collective whole because the relative positions of the eyes, mouth, and face outline are different in the two images.

In order for two shapes to be congruent, all pairs of corresponding points in the two images must be the same distance apart. Strategically selecting corresponding points whose distance is not the same in the two figures requires a solid understanding of the meaning of congruence and corresponding parts.

Monitor for students who:

- Identify that the parts of the two faces are congruent, such as the face outline, eyes, and mouth.
- Identify distances between parts of each face that are not the same.

Select students to share these observations during the whole-class discussion.

Launch

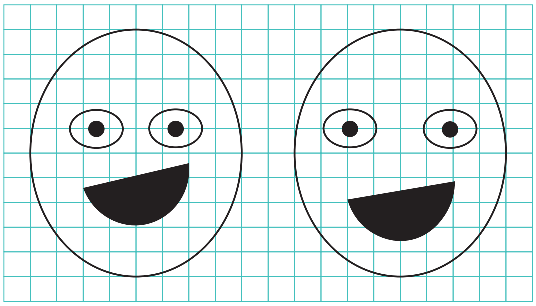
Arrange students in groups of 2. Provide access to geometry toolkits.

Give 2–3 minutes quiet work time followed by 2–3 minutes to share with their partner, followed by a whole-class discussion.

Provide access to geometry toolkits, including tracing paper. As students work, encourage them to look for other ways to know the shapes are not congruent beyond saying that they do not match up when students try to place one on top of the other.

Student Task Statement

Are these faces congruent? Explain your reasoning.



No

Sample reasoning: The faces are not congruent although the parts of the faces, like the outline, eyes, and mouth are congruent taken one by one. The two oval outlines of the faces sit in 8-unit-by-10-unit rectangles and they are congruent as can be seen by translating the left face outline 10 units to the right (or reflecting over the center vertical line on the grid). All 4 eyes in the two faces are also congruent. This can be shown using horizontal translations. The two mouths are congruent, also using a translation, but this time the translation has a horizontal component (10 units to the right) and a vertical component (about a half of a unit down).

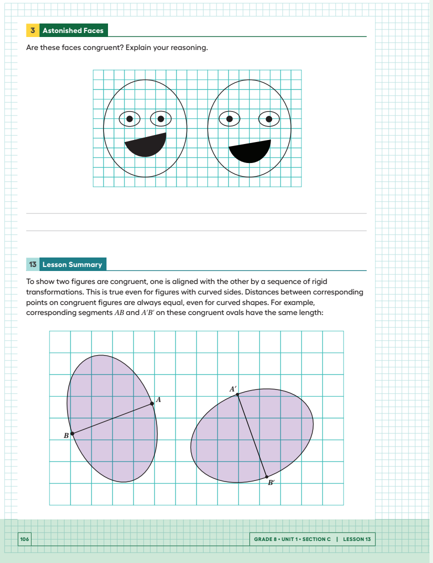
While the individual parts are congruent, the faces as wholes are not. In the figure on the left, the eyes are one unit apart. For the figure on the right they are almost 2 units apart. The mouth on the left is about one unit below each eye while the mouth on the right is more than one unit below each eye. The mouth on the right is also closer to the outline of the face than the mouth on the left. In each case, there is a pair of corresponding points in the two figures whose distances are different. This means that the figures are not congruent.

Access for Multilingual Learners (Activity 3, Student Task)

MLR8: Discussion Supports.
Revoice student ideas to demonstrate and amplify mathematical language use. For example, revoice the student statement "the eyes are the same" as "the eyes are congruent."

Advances: Speaking

Student Workbook



3 Astonished Faces

Are these faces congruent? Explain your reasoning.

13 Lesson Summary

To show two figures are congruent, one is aligned with the other by a sequence of rigid transformations. This is true even for figures with curved sides. Distances between corresponding points on congruent figures are always equal, even for curved shapes. For example, corresponding segments AB and $A'B'$ on these congruent ovals have the same length:

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Activity Synthesis

Ask selected students to share which parts of the faces are congruent and their reasoning. Emphasize precise language that uses rigid transformations, such as a translation.

Ask selected students to share how they determined whether the two face images are congruent. If no students suggest it, ask,

"What corresponding points were you able to identify that were not the same distance apart in the two faces?"

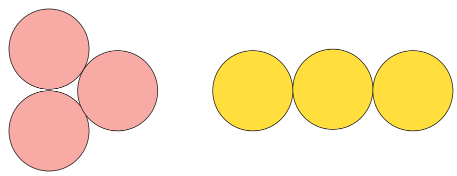
Distance between the eyes, from the eyes to the mouth, from the eyes to the face outline, or from the mouth to face outline.

Even though the individual parts of the two faces are congruent, the two faces are not congruent. We could find one translation that takes the outline of one face to the outline of the other and similarly we could find a translation taking the left eye, right eye, and mouth of one figure to the left eye, right eye, and mouth of the other. But these translations are different. In order for two figures to be congruent, there must be one sequence of transformations that match all parts of one figure perfectly with the other.

Lesson Synthesis

The purpose of this discussion is for students to articulate the conditions for congruence of figures.

Display this image for all to see:



Ask students how they could show whether the circles are congruent to each other.

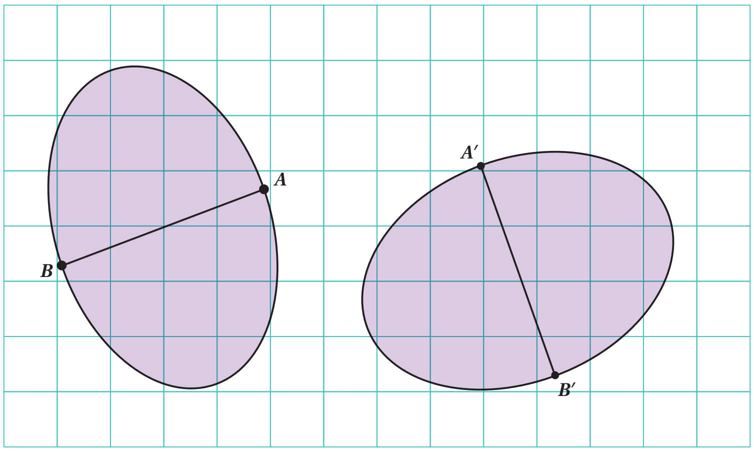
Find a translation that takes one circle to the others.

Ask students whether the image of circles on the left is congruent to the image of circles on the right, and how they know.

The circles on the left form a triangle, while the circles on the right form a line. This means that their corresponding parts are not all the same distance apart.

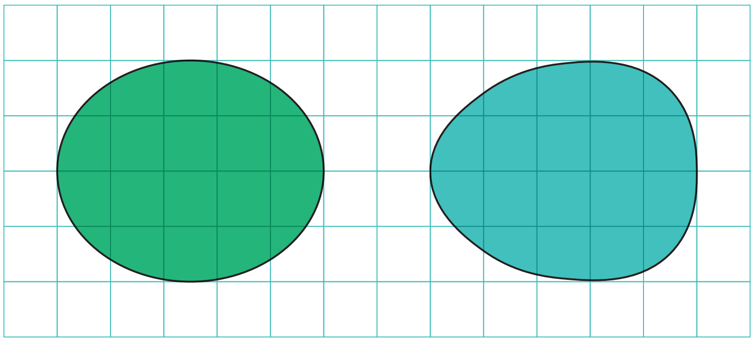
Lesson Summary

To show two figures are congruent, one is aligned with the other by a sequence of rigid transformations. This is true even for figures with curved sides. Distances between corresponding points on congruent figures are always equal, even for curved shapes. For example, corresponding segments AB and $A'B'$ on these congruent ovals have the same length:

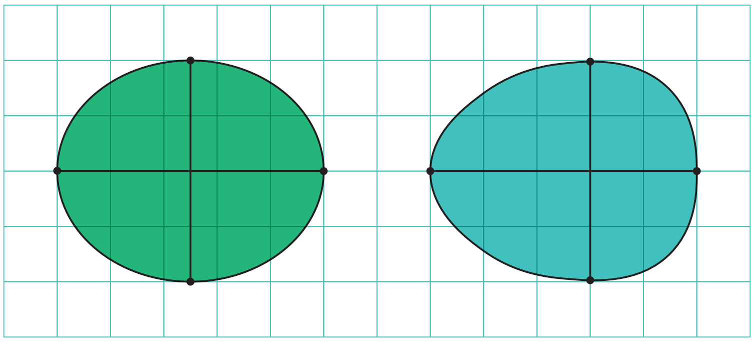


To show two figures are not congruent, you can find parts of the figures that should correspond but that have different measurements.

For example, these two ovals don't look congruent.



On both, the longest distance is 5 units across, and the longest distance from top to bottom is 4 units. The line segment from the highest to lowest point is in the middle of the left oval, but in the right oval, it's 2 units from the right end and 3 units from the left end. This shows they are not congruent.

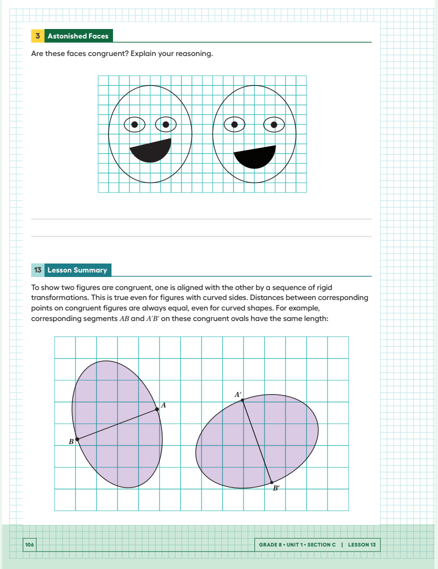


Responding To Student Thinking

Press Pause
By this point in the unit, there should be some student mastery of the concepts in this *Cool-down*. If most students struggle, make time to examine related work in Activities 2 and 3 of the lesson referred to here. The Course Guide provides additional ideas for revisiting earlier work.

Unit 1, Lesson 12 Congruent Polygons

Student Workbook



3 Astonished Faces

Are these faces congruent? Explain your reasoning.

13 Lesson Summary

To show two figures are congruent, one is aligned with the other by a sequence of rigid transformations. This is true even for figures with curved sides. Distances between corresponding points on congruent figures are always equal, even for curved shapes. For example, corresponding segments AB and $A'B'$ on these congruent ovals have the same length:

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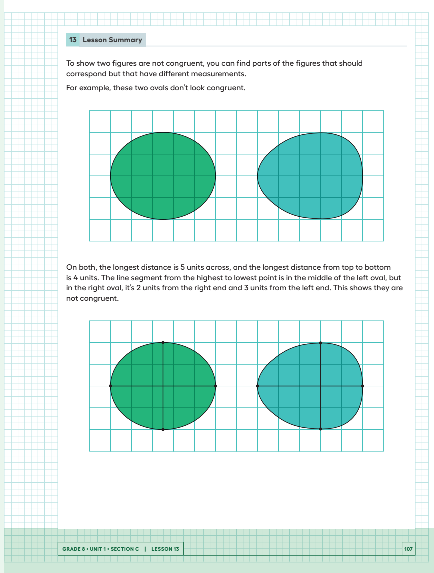
Student Workbook



13 Lesson Summary

To show two figures are not congruent, you can find parts of the figures that should correspond but that have different measurements.

For example, these two ovals don't look congruent.

On both, the longest distance is 5 units across, and the longest distance from top to bottom is 4 units. The line segment from the highest to lowest point is in the middle of the left oval, but in the right oval, it's 2 units from the right end and 3 units from the left end. This shows they are not congruent.

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Math Community

Before distributing the *Cool-downs*, display the Math Community Chart and the norms question "Which norm has not already been listed that you'd like to add to our chart?" Ask students to respond to the question after completing the *Cool-down* on the same sheet.

After collecting the *Cool-downs*, identify themes from the norms question. Use that information to add to the initial draft of the "Norms" sections of the Math Community Chart.

Cool-down

Explaining Congruence

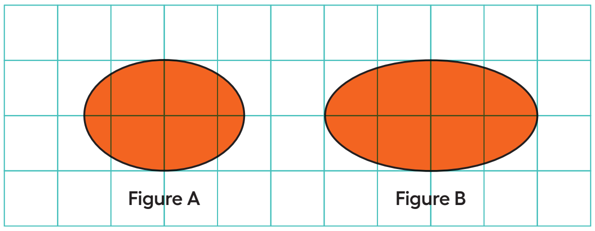
5 min

Launch

Provide access to tracing paper.

Student Task Statement

Are Figures A and B congruent? Explain your reasoning.



These figures are not congruent.

Sample reasoning: If they were congruent, the longest horizontal distances between two points would be the same. However, for A it is less than 4 units, and for B it is about 4 units.

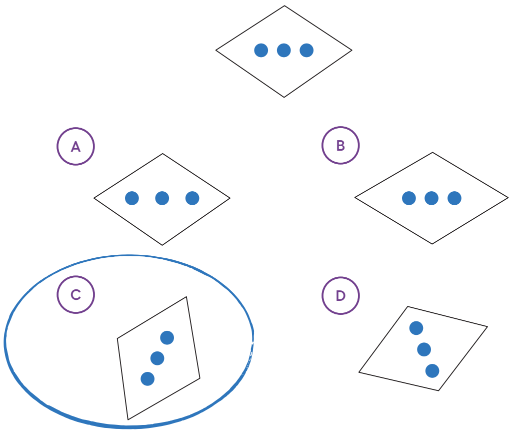
Lesson 13

Practice Problems

4 Problems

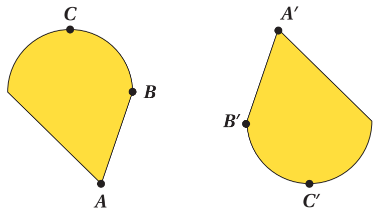
Problem 1

Which of these four figures are congruent to the top figure?



Problem 2

These two figures are congruent and have corresponding points marked.



a. Are angles ABC and $A'B'C'$ congruent? Explain your reasoning.

yes

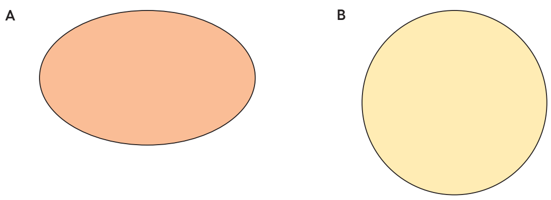
Sample reasoning: They are angles made by corresponding points on congruent figures so they are congruent.

b. Measure angles ABC and $A'B'C'$ to check your answer.

Both angles measure about 110 degrees.

Problem 3

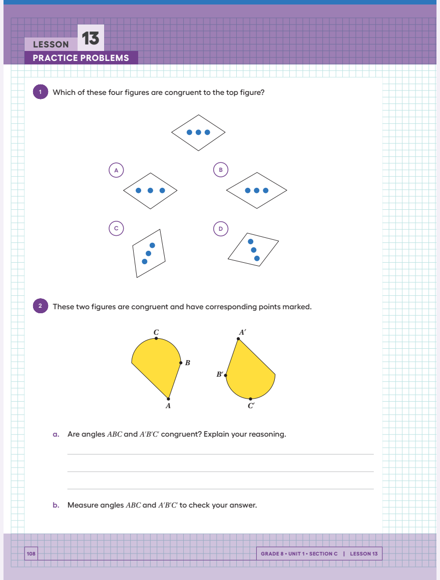
Here are two figures.



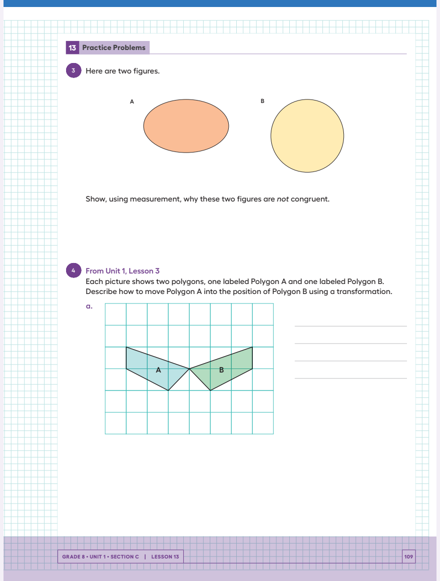
Explain why these two figures are *not* congruent.

Sample response: The rightmost and leftmost points on Figure A are further apart than any pair of points on Figure B. So these two points can not correspond to any pair of points on Figure B and the two figures are not congruent.

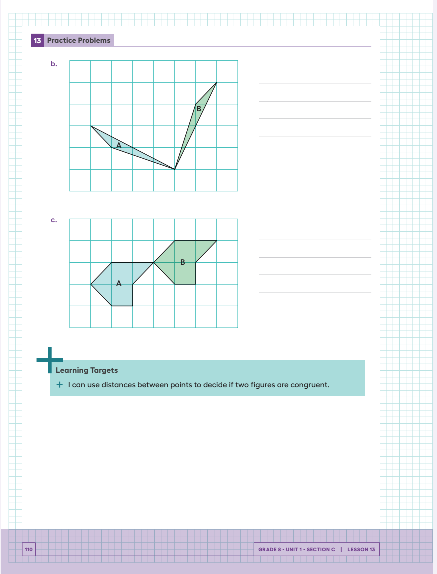
Student Workbook



Student Workbook



Student Workbook

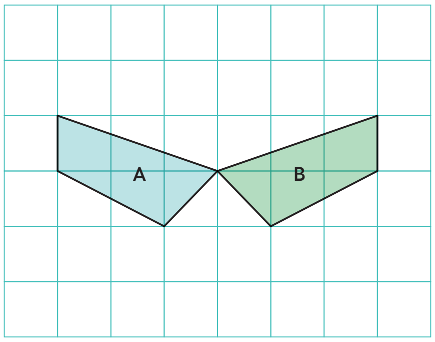


Problem 4

from Unit 1, Lesson 3

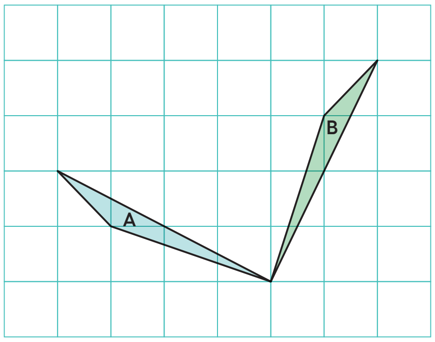
Each picture shows two polygons, one labeled Polygon A and one labeled Polygon B. Describe how to move Polygon A into the position of Polygon B using a transformation.

a.



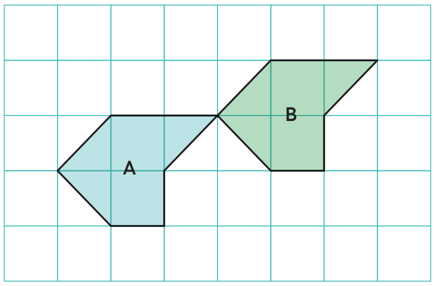
Sample response: Flip A over the vertical line through the vertex shared by A and B.

b.



Sample response: Rotate A in a clockwise direction around the vertex shared by the 2 polygons.

c.



Sample response: Translate A up and to the right. It needs to go up 1 unit and right 3 units.