Lesson 19

Estimating a Hemisphere

Goals

- Calculate the volumes of a cylinder and cone with the same radius and height, and justify (orally and in writing) that the volumes are an upper and lower bound for the volume of a hemisphere of the same radius.
- Estimate the volume of a hemisphere by using the formulas for volume of a cone and volume of a cylinder, and explain (orally) the estima tion strategy.

Learning Target

I can estimate the volume of a hemisphere by calculating the volume of a shape I know is larger and the volume of a shape I know is smaller.

Student Learning Goal

Let's estimate volumes of hemispheres with figures we know.

Lesson Narrative

The purpose of this lesson is to introduce students to working with spheres by using shapes they are now familiar with—prisms, cones, and cylinders—to estimate the volume of a hemisphere. The work done in this lesson prepares students to reason about the formula for the volume of a sphere in a future lesson.

The *Warm-up* primes students to think about the structure of the volume formulas for cones and cylinders when the height and radius have the same value.

Next, they consider the relationship between the volume of a hemisphere and a prism that have matching dimensions. By comparing two different situations where the dimensions of one are double that of the other, this activity sets the expectation that when the radius of a sphere doubles, the volume increases by a factor of 8 since the length, width, and height of the sphere also double.

In the second activity, students fit a hemisphere inside a cylinder and use the volume of the cylinder to make an estimate of the volume of the hemisphere. Then they do the same thing with a cone that fits inside the hemisphere. The volume of the hemisphere has to be between the volume of the cone and the volume of the cylinder, and students consider what this means by studying the volume equations for cylinders and cones whose radius and height are the same value.

Access for Students with Diverse Abilities

- Representation (Activity 2)

Access for Multilingual Learners

- MLR6: Three Reads (Activity 1)

Instructional Routines

- Notice and Wonder

Required Materials

Materials to Gather

- Spherical objects: Activity 2

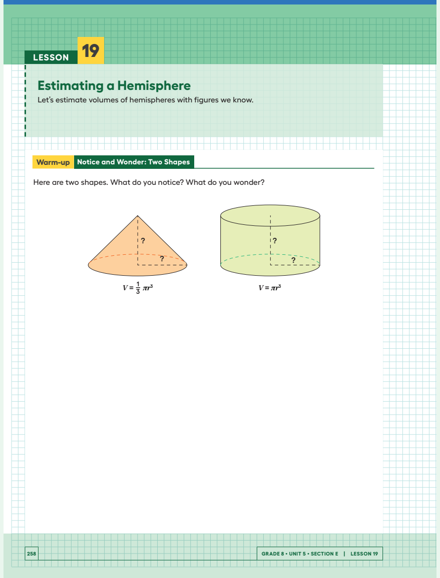
Required Preparation

Activity 1:
If possible, have some physical examples of hemispheres on hand for students to see. Examples could be glass paperweights or dome lids. Alternatively, have a sphere, such as a globe or basketball, with a marked equator to clearly divide it into two hemispheres.

Lesson Timeline

Warm-up	Activity 1	Activity 2	Lesson Synthesis	Cool-down (Assessment)
5 min	15 min	15 min	10 min	5 min

Student Workbook



Warm-up

Notice and Wonder: Two Shapes

5 min

Activity Narrative

The purpose of this *Warm-up* is for students to review how to manipulate the formulas for volume of a cylinder and cone and consider what they look like when the height and radius are the same, which will be useful when students encounter these shapes throughout the lesson.

This prompt gives students opportunities to see and make use of structure. The specific structure they might notice is the fact that $h = r$.

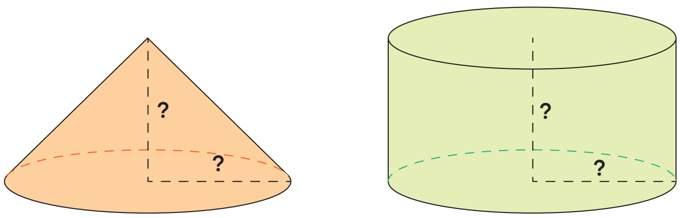
Launch

Arrange students in groups of 2. Display the shapes for all to see. Ask students to think of at least one thing they notice and at least one thing they wonder.

Give students 1 minute of quiet think time and then 1 minute to discuss with their partner the things they notice and wonder.

Student Task Statement

Here are two shapes. What do you notice? What do you wonder?



$V = \frac{1}{3}\pi r^3$ $\quad\quad$ $V = \pi r^3$

Students may notice:

- If the height and radius are the same for both the cylinder and cone, then the volume of the cone is one-third the volume of the cylinder.
- When the radius is the same as the height, the cone and cylinder seem much wider than tall.
- When the height and radius are the same, the volume acts as a function of one variable.

Students may wonder:

- Why would we want a cone or cylinder where the height and radius are the same?
- Does the volume of this type of cone or cylinder change in the same way a cube's volume does?
- Since their dimensions match, could we put the cone inside the cylinder to create a new shape?

Activity Synthesis

Ask students to share the things they noticed and wondered. Record and display their responses without editing or commentary. If possible, record the relevant reasoning on or near the shapes. Next, ask students, "Is there anything on this list that you are wondering about now?" Encourage students to observe what is on display and respectfully ask for clarification, point out contradicting information, or voice any disagreement.

If the "missing" variable for height, h, does not come up during the conversation, ask students to discuss this idea. Ensure students understand that the two equations have no variable h for height since the h was replaced by r due to the height and radius being the same for both shapes.

Activity 1
Hemispheres in Boxes

15 min

Activity Narrative

In this activity, students think about a hemisphere fitting inside the smallest possible box (or rectangular prism). Using the structure of volume formulas, students reason that the smallest box in which a hemisphere can fit has a square base with an edge length that is the same as the diameter of the hemisphere and that the height of the box will be the radius of the hemisphere.

A key takeaway from this activity is that the volume of the box gives an upper bound for the volume of the hemisphere. This activity prepares students for later work determining a lower bound and a new upper bound for the volume of a hemisphere using cylinders and cones.

Launch

Begin the discussion by inviting students to share examples of spheres they have seen. Some examples they might come up with are ping pong balls, soap bubbles, bath bombs, paper lanterns, baseballs, volleyballs, and a globe. If students don't come up with many examples, offer your own, or display results from an internet image search for "sphere." If possible, acquire spheres to display or pass around the class. Explain to students that the radius of a sphere is the distance from the center of the sphere (the point in the exact middle) to any point on the sphere.

Next, ask students if they are familiar with the word "hemisphere" and if they can think of examples of hemispheres. Some examples are hemispherical (or "half") balance boards, half of Earth, a dome or planetarium, and molds used in baking or to make bath bombs. Display an example, illustration, or diagram of a hemisphere for all to see, pointing out how the radius is the distance from the center of the flat side of the sphere to any point on the curved surface of the sphere.

Access for Multilingual Learners (Activity 1, Launch)

MLR6: Three Reads.
Keep student workbooks or devices closed. Display only the first problem stem, without revealing the questions. Say,

> *"We are going to read this statement 3 times."*

After the 1st read:

> *"Tell your partner what this situation is about."*

After the 2nd read:

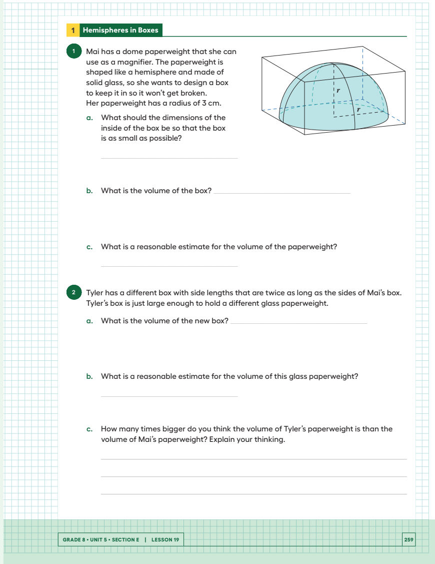
> *"List the quantities. What can be counted or measured?"*

For the 3rd read: Reveal and read the questions. Ask,

> *"What are some ways we might get started on this?"*

Advances: Reading, Representing

Student Workbook


1 Hemispheres in Boxes

1. Mai has a dome paperweight that she can use as a magnifier. The paperweight is shaped like a hemisphere and made of solid glass, so she wants to design a box to keep it in so it won't get broken. Her paperweight has a radius of 3 cm.
   a. What should the dimensions of the inside of the box be so that the box is as small as possible?
   b. What is the volume of the box?
   c. What is a reasonable estimate for the volume of the paperweight?
2. Tyler has a different box with side lengths that are twice as long as the sides of Mai's box. Tyler's box is just large enough to hold a different glass paperweight.
   a. What is the volume of the new box?
   b. What is a reasonable estimate for the volume of this glass paperweight?
   c. How many times bigger do you think the volume of Tyler's paperweight is than the volume of Mai's paperweight? Explain your thinking.

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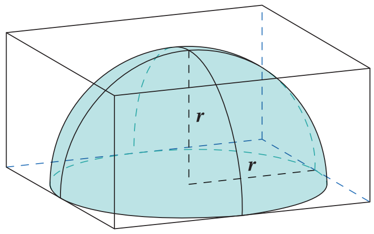
Tell students that in this activity, they will think about building a box (or rectangular prism) around a hemisphere. Arrange students in groups of 2. Ask students to quietly work through the first question and then share their reasoning with their partner. Pause for a whole-class discussion, and invite students to share their responses. The purpose of this problem is to see how the measurements of a hemisphere determine the dimensions of the prism. Ensure everyone understands the box must have edge lengths 6, 6, and 3 inches and that the hemisphere must have a volume that is less than 108 cubic inches (since $6^2 \cdot 3 = 108$). Discuss how much less students think the volume of the sphere is along with their reasoning, then ask students to move onto the second problem with their partner.

Student Task Statement



1. Mai has a dome paperweight that she can use as a magnifier. The paperweight is shaped like a hemisphere and made of solid glass, so she wants to design a box to keep it in so it won't get broken. Her paperweight has a radius of 3 cm.

   a. What should the dimensions of the inside of the box be so that the box is as small as possible?

   6 cm by 6 cm by 3 cm

   The radius of the hemispherical paperweight is only half of the box's side length. $3 \cdot 2$ gives us the entire length of the side. The height is the same as the radius.

   b. What is the volume of the box?

   108 cubic centimeters. $6^2 \cdot 3 = 108$.

   c. What is a reasonable estimate for the volume of the paperweight?

   Correct responses are less than 108 cubic centimeters.

   The box's volume is $6^2 \cdot 3$, and since there is space in the box that the hemisphere does not take up, the volume of the hemisphere has to be less than the volume of the box.

2. Tyler has a different box with side lengths that are twice as long as the sides of Mai's box. Tyler's box is just large enough to hold a different glass paperweight.

a. What is the volume of the new box?

864 cubic centimeters

If every side length is doubled then the volume gets 8 times larger, and $108 \cdot 8 = 864$.

b. What is a reasonable estimate for the volume of this glass paperweight?

Correct responses are less than 864 cubic centimeters.

There is still space in the box that the hemisphere does not take up, so the volume of the hemisphere has to be less than the volume of the box.

c. How many times bigger do you think the volume of Tyler's paperweight is than the volume of Mai's paperweight? Explain your thinking.

Sample response: 8 times bigger

If the volume of a box gets 8 times bigger and the hemisphere has to fit inside the new box, then the hemisphere's volume must also get 8 times bigger.

Activity Synthesis

The purpose of this discussion is for students to see that since a rectangular prism's volume gets larger by a factor of 2^3 when edge lengths are doubled, it would make sense for a hemisphere's volume to do the same.

Invite students to share their answers to the second question. Ensure that students understand the volume calculated for the box holding the hemisphere is greater than the actual volume of the hemisphere because of the space left around the paperweight in the box. This will be an upper bound of the volume of a sphere—that is, a value we know the volume of the hemisphere must be less than.

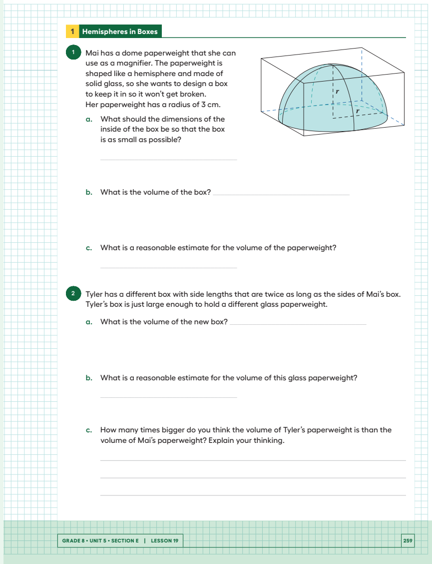
Tell students that in the next activity they are going to investigate a better way to estimate the volume of a hemisphere. If time allows, ask students to suggest shapes they are already familiar with that they could use to find the volume of a hemisphere.

Building on Student Thinking

Students may believe that since this box is not a cube, but a rectangular prism, the volume will not increase by 8 when the side lengths are doubled. Consider asking:

"Tell me how you calculated the volume of the box."
"What happens to the volume if you replace the length, width, and height with values that are doubled?"

Student Workbook


1 Hemispheres in Boxes

1 Mai has a dome paperweight that she can use as a magnifier. The paperweight is shaped like a hemisphere and made of solid glass, so she wants to design a box to keep it in so it won't get broken. Her paperweight has a radius of 3 cm.

a. What should the dimensions of the inside of the box be so that the box is as small as possible?

b. What is the volume of the box?

c. What is a reasonable estimate for the volume of the paperweight?

2 Tyler has a different box with side lengths that are twice as long as the sides of Mai's box. Tyler's box is just large enough to hold a different glass paperweight.

a. What is the volume of the new box?

b. What is a reasonable estimate for the volume of this glass paperweight?

c. How many times bigger do you think the volume of Tyler's paperweight is than the volume of Mai's paperweight? Explain your thinking.

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Access for Students with Diverse Abilities (Activity 2, Launch)

Representation: Develop Language and Symbols.
Use virtual or concrete manipulatives to connect symbols to concrete objects or values. Provide examples of actual three-dimensional models of cylinders, hemispheres, and cones for students to view or manipulate. Ask students to use the three-dimensional models and volumes calculated in each question to estimate the volume of the hemisphere.

Supports accessibility for: Visual-Spatial Processing, Conceptual Processing

Activity 2

Estimating Hemispheres

15 min

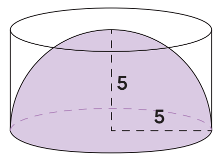
Activity Narrative

In this activity, students use different solid figures to estimate an upper and lower bound for the volume of a hemisphere. For the upper bound, the hemisphere fits snugly inside a cylinder whose height and radius are equal to the radius of the hemisphere. For the lower bound, the cone fits snugly inside the hemisphere, and its radius and height also equal the radius and height of the hemisphere.

The *Activity Synthesis* concludes by inviting students to compare the formulas for the volume of a cylinder and cone where $h = r$ and consider what the formula for the volume of a hemisphere might look like, which helps prepare them for making sense of upcoming work determining the formula for the volume of a sphere.

Launch

Keep students in the same groups. Display the image of a hemisphere of radius 5 that fits snugly inside a cylinder of the same radius and height.



Ask,

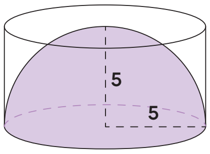
"How does the hemisphere affect the height of the cylinder?"

and then give students one minute of quiet think time, then one minute to discuss their response with a partner. Ask partners to share their responses. If not mentioned by students, point out that the height of the cylinder is equal to the radius of the hemisphere.

Give students work time for the activity, and follow with a whole-class discussion. As students work, select groups who use the reasoning from the first problem to assist them in answering the second problem to share during the *Activity Synthesis*. For example, since the cylinder and cone have the same dimensions, the volume of the cone must be $\frac{1}{3}$ that of the cylinder.

Student Task Statement

1. A hemisphere with radius 5 units fits snugly into a cylinder of the same radius and height.



   a. Calculate the volume of the cylinder.

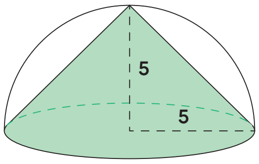
   125π

   The cylinder's volume can be calculated using $V = \pi(5)^2(5)$.

   b. Estimate the volume of the hemisphere. Explain your reasoning.

   Correct responses are less than the volume of the cylinder.

2. A cone fits snugly inside a hemisphere, and they share a radius of 5.



   a. What is the volume of the cone?

   $\frac{125}{3}\pi$

   The cone's volume can be calculated using $V = \frac{1}{3}\pi(5)^2(5)$.

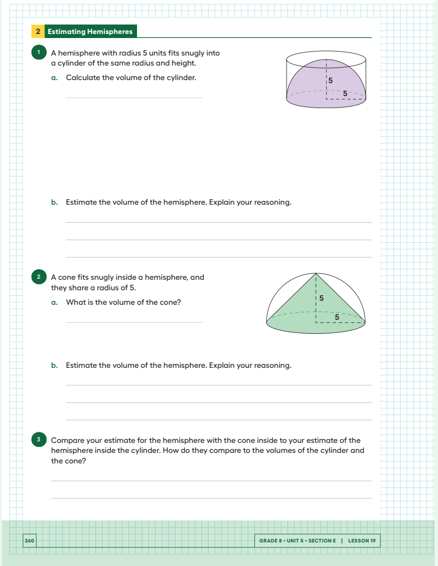
   b. Estimate the volume of the hemisphere. Explain your reasoning.

   Correct responses are greater than the volume of the cone and less than the volume of the cylinder from the previous question.

3. Compare your estimate for the hemisphere with the cone inside to your estimate of the hemisphere inside the cylinder. How do they compare to the volumes of the cylinder and the cone?

   Sample response: The volume of the hemisphere will be greater than the volume of the cone but less than the volume of the cylinder.

Student Workbook


2 Estimating Hemispheres

1 A hemisphere with radius 5 units fits snugly into a cylinder of the same radius and height.

a. Calculate the volume of the cylinder.

b. Estimate the volume of the hemisphere. Explain your reasoning.

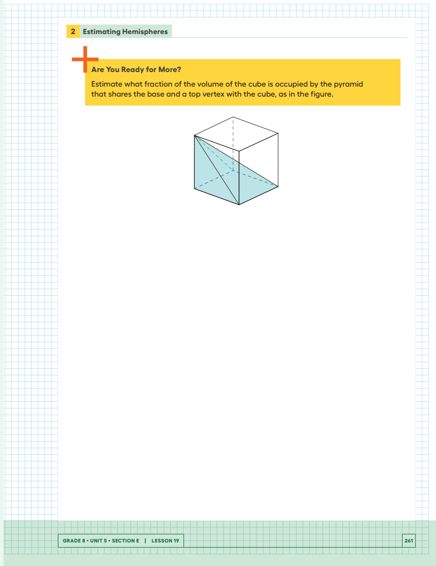
2 A cone fits snugly inside a hemisphere, and they share a radius of 5.

a. What is the volume of the cone?

b. Estimate the volume of the hemisphere. Explain your reasoning.

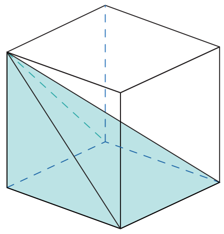
3 Compare your estimate for the hemisphere with the cone inside to your estimate of the hemisphere inside the cylinder. How do they compare to the volumes of the cylinder and the cone?

Student Workbook

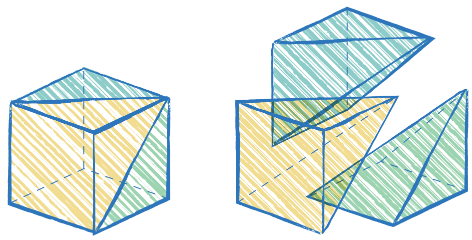


Are You Ready for More?

Estimate what fraction of the volume of the cube is occupied by the pyramid that shares the base and a top vertex with the cube, as in the figure.



In fact, the pyramid is precisely $\frac{1}{3}$ of the volume of the cube. One way to see this is by decomposing the cube into 3 pyramids each congruent to the original.



Activity Synthesis

The purpose of this discussion is for students to recognize how the upper and lower bounds for the volume of a hemisphere are established by the cylinder and cone. Begin the discussion by asking previously selected groups to share their responses to the first two problems. Draw attention to any connections made between the two problems.

Here are some more questions for discussion:

- *"What do the volumes of the cone and cylinder tell us about the volume of the hemisphere?"*

  The volume of the hemisphere has to be between the two volumes.

- *"Did anyone revise their original estimate for the hemisphere based on the calculation of the volume of the cone?"*

  If students estimated low values after calculating the volume of a cylinder, some may have needed to adjust their estimates.

- *"Compare the equations for volume of a cylinder and volume of a cone in which the radius and height of each solid are equal. If the volume of the hemisphere has to be between these two, what might an equation for the volume of a hemisphere look like?"*

  Cylinder volume: $V = \pi r^3$; Cone volume: $V = \frac{1}{3}\pi r^3$. A possible hemisphere volume could be the average of these two, or $V = \frac{2}{3}\pi r^3$.

Lesson Synthesis

To prompt students to describe some of the important understandings of the lesson, ask:

- *"How did we use a cylinder to overestimate the volume of a hemisphere?"*

  We found the volume of a cylinder that the hemisphere could fit inside without extra space.

- *"How did we use a cone to underestimate the volume of a hemisphere?"*

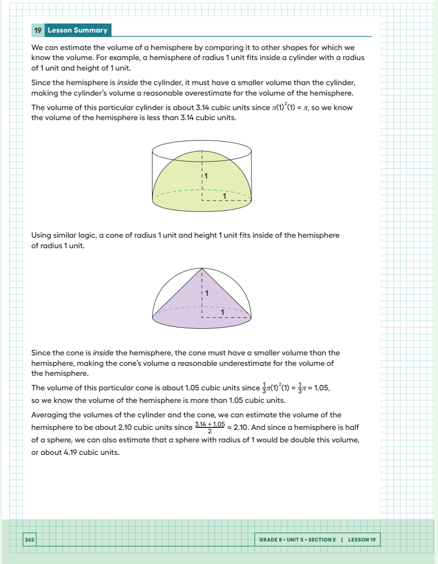
  We found the volume of the largest cone that could fit inside of the hemisphere.

- *"How did we get a closer estimate for the volume of a hemisphere?"*

  Earlier we figured out an upper bound using a prism. Here, we have a smaller upper bound using a cylinder and a lower bound using a cone.

Explain to students that we used figures that we know how to find the volume of (a cone and cylinder) to try and estimate the volume of a figure we do not know how to find the volume of (a sphere). Next, we will use similar reasoning to find the volume of a sphere.

Student Workbook


19 Lesson Summary

We can estimate the volume of a hemisphere by comparing it to other shapes for which we know the volume. For example, a hemisphere of radius 1 unit fits inside a cylinder with a radius of 1 unit and height of 1 unit.

Since the hemisphere is *inside* the cylinder, it must have a smaller volume than the cylinder, making the cylinder's volume a reasonable overestimate for the volume of the hemisphere.

The volume of this particular cylinder is about 3.14 cubic units since $\pi(1)^2(1) = \pi$, so we know the volume of the hemisphere is less than 3.14 cubic units.

Using similar logic, a cone of radius 1 unit and height 1 unit fits inside of the hemisphere of radius 1 unit.

Since the cone is *inside* the hemisphere, the cone must have a smaller volume than the hemisphere, making the cone's volume a reasonable underestimate for the volume of the hemisphere.

The volume of this particular cone is about 1.05 cubic units since $\frac{1}{3}\pi(1)^2(1) = \frac{1}{3}\pi \approx 1.05$, so we know the volume of the hemisphere is more than 1.05 cubic units.

Averaging the volumes of the cylinder and the cone, we can estimate the volume of the hemisphere to be about 2.10 cubic units since $\frac{3.14 + 1.05}{2} \approx 2.10$. And since a hemisphere is half of a sphere, we can also estimate that a sphere with radius of 1 would be double this volume, or about 4.19 cubic units.

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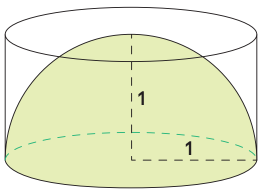
Lesson Summary

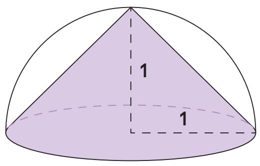
We can estimate the volume of a hemisphere by comparing it to other shapes for which we know the volume. For example, a hemisphere of radius 1 unit fits inside a cylinder with a radius of 1 unit and height of 1 unit.

Since the hemisphere is *inside* the cylinder, it must have a smaller volume than the cylinder, making the cylinder's volume a reasonable overestimate for the volume of the hemisphere.

The volume of this particular cylinder is about 3.14 cubic units since $\pi(1)^2(1) = \pi$, so we know the volume of the hemisphere is less than 3.14 cubic units.



Using similar logic, a cone of radius 1 unit and height 1 unit fits inside of the hemisphere of radius 1 unit.



Since the cone is *inside* the hemisphere, the cone must have a smaller volume than the hemisphere, making the cone's volume a reasonable underestimate for the volume of the hemisphere.

Responding To Student Thinking

More Chances
Students will have more opportunities to understand the mathematical ideas addressed here. There is no need to slow down or add additional work to the next lessons.

The volume of this particular cone is about 1.05 cubic units since $\frac{1}{3}\pi(1)^2(1) = \frac{1}{3}\pi \approx 1.05$, so we know the volume of the hemisphere is more than 1.05 cubic units.

Averaging the volumes of the cylinder and the cone, we can estimate the volume of the hemisphere to be about 2.10 cubic units since $\frac{3.14 + 1.05}{2} \approx 2.10$. And since a hemisphere is half of a sphere, we can also estimate that a sphere with radius of 1 would be double this volume, or about 4.19 cubic units.

Cool-down

A Reasonable Estimate

5 min

Student Task Statement

A hemisphere fits exactly inside a rectangular prism box with a square base that has edge length 10 inches. What is a reasonable estimate for the volume of the hemisphere?

Sample responses:

- Less than 500 cubic inches

  The volume of the box that the hemisphere fits in is $10^2 \cdot 5$, and the hemisphere does not take up all the space in the box.

- Less than 125π cubic inches

  The volume of the cylinder that the hemisphere fits in is $\pi(5)^2 \cdot 5$, and the hemisphere does not take up all the space in the cylinder.

- More than $\frac{125}{3}\pi$ cubic inches

  The volume of the cone that fits in the hemisphere is $\frac{1}{3}\pi(5)^2 \cdot 5$, and the hemisphere is larger than the cone.

Lesson 19

Practice Problems

4 Problems

Problem 1

A baseball fits snugly inside a transparent display cube. The length of an edge of the cube is 2.9 inches.

Is the baseball's volume greater than, less than, or equal to 2.9^3 cubic inches? Explain how you know.

Less than

Sample reasoning: The baseball fits inside the cube, and the cube's volume is 2.9^3 cubic inches. Therefore, the baseball's volume is less than 2.9^3 cubic inches.

Problem 2

A hemisphere fits snugly inside a cylinder with a radius of 6 cm. A cone fits snugly inside the same hemisphere.

a. What is the volume of the cylinder?

216π cm^3

b. What is the volume of the cone?

72π cm^3

c. Estimate the volume of the hemisphere by calculating the average of the volumes of the cylinder and cone.

144π cm^3

Problem 3

a. Find the hemisphere's diameter if its radius is 6 cm.

12 cm

b. Find the hemisphere's diameter if its radius is $\frac{1,000}{3}$ m.

$\frac{2,000}{3}$ m (or equivalent)

c. Find the hemisphere's diameter if its radius is 9.008 ft.

18.016 ft

d. Find the hemisphere's radius if its diameter is 6 cm.

3 cm

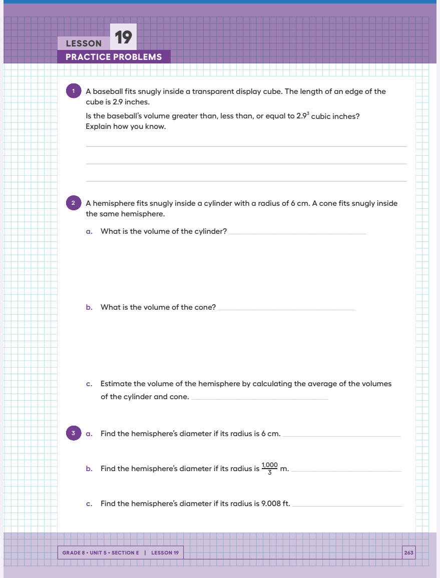
e. Find the hemisphere's radius if its diameter is $\frac{1,000}{3}$ m.

$\frac{500}{3}$ m (or equivalent)

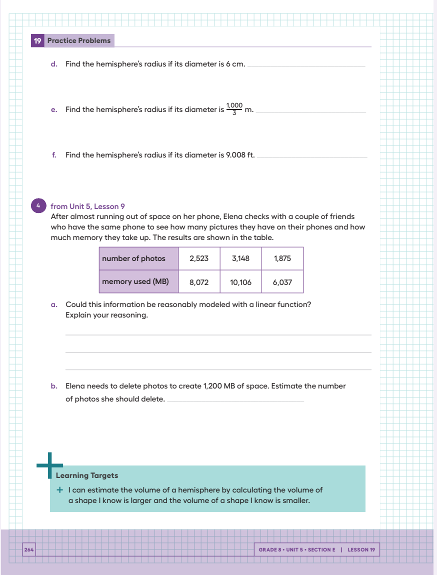
f. Find the hemisphere's radius if its diameter is 9.008 ft.

4.504 ft

Student Workbook


LESSON 19
PRACTICE PROBLEMS
1 A baseball fits snugly inside a transparent display cube. The length of an edge of the cube is 2.9 inches.
Is the baseball's volume greater than, less than, or equal to 2.9^3 cubic inches? Explain how you know.
2 A hemisphere fits snugly inside a cylinder with a radius of 6 cm. A cone fits snugly inside the same hemisphere.
a. What is the volume of the cylinder?
b. What is the volume of the cone?
c. Estimate the volume of the hemisphere by calculating the average of the volumes of the cylinder and cone.
3 a. Find the hemisphere's diameter if its radius is 6 cm.
b. Find the hemisphere's diameter if its radius is $\frac{1000}{3}$ m.
c. Find the hemisphere's diameter if its radius is 9.008 ft.

Student Workbook


19 Practice Problems

d. Find the hemisphere's radius if its diameter is 6 cm.

e. Find the hemisphere's radius if its diameter is $\frac{1,000}{3}$ m.

f. Find the hemisphere's radius if its diameter is 9.008 ft.

4 from Unit 5, Lesson 9

After almost running out of space on her phone, Elena checks with a couple of friends who have the same phone to see how many pictures they have on their phones and how much memory they take up. The results are shown in the table.

number of photos	2,523	3,148	1,875
memory used (MB)	8,072	10,106	6,037

a. Could this information be reasonably modeled with a linear function? Explain your reasoning.

b. Elena needs to delete photos to create 1,200 MB of space. Estimate the number of photos she should delete.

Learning Targets

+ I can estimate the volume of a hemisphere by calculating the volume of a shape I know is larger and the volume of a shape I know is smaller.

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Problem 4

from Unit 5, Lesson 9

After almost running out of space on her phone, Elena checks with a couple of friends who have the same phone to see how many pictures they have on their phones and how much memory they take up. The results are shown in the table.

number of photos	2,523	3,148	1,875
memory used (MB)	8,072	10,106	6,037

a. Could this information be reasonably modeled with a linear function? Explain your reasoning.

Yes

Sample reasoning: All the points are close to $y = 3.2x$, where y represents the memory usage and x represents the number of photos.

b. Elena needs to delete photos to create 1,200 MB of space. Estimate the number of photos she should delete.

about 375 photos, because $1,200 \div 3.2 = 375$