Artificial Intelligence 2 – Assignment #2

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1. Generative Models, Naïve Bayes Classifier [20 points]

a) $\rho(Water\ =\ warm|Play\ =\ yes), \rho(Water\ =\ warm|Play\ =\ no)$

$$\rho(Water = warm|Play = yes) = \frac{2}{3}$$

$$\rho(Water = warm|Play = no) = 1$$

b) $\rho(Play = yes|Water = warm), \rho(Play = no|Water = warm)$

$$\rho(Play\ =\ yes|Water\ =\ warm) = \frac{2}{3}$$

$$\rho(Play\ =\ no|Water\ =\ warm) = \frac{1}{3}$$

c) $\rho(Play\ =\ yes|Forecast\ =\ same), \rho(Play\ =\ yes|Forecast\ =\ change)$

$$\rho(Play\ =\ yes|Forecast\ =\ same) = \frac{1}{1}$$

$$\rho(Play\ =\ yes|Forecast\ =\ change) = \frac{1}{2}$$

d) $\rho(Water\ =\ warm|Play\ =\ yes), \rho(Water\ =\ warm|Play\ =\ no)$ $*\ \boldsymbol{with\ Laplace\ smoothing}\ *$

$$\rho(Water = warm|Play = yes) = \frac{2+1}{3+2} = \frac{3}{5}$$

$$\rho(Water = warm|Play = no) = \frac{1+1}{1+2} = \frac{1}{3}$$

2. Kernels [30 points]

a) $\boldsymbol{k(x,z)\ =\ a_1 k_1(x,z)\ -\ a_2 k_2(x,z)\ where\ a_{1\,,} a_2\ >\ 0\ are\ real\ numbers}$ Not Valid

I first prove that $k(x,z)\ =\ \alpha k(x,z)$ is a kernel and then I prove that subtraction of these two kernels is not a valid kernel.

Proof: K (x, z) =$\sum_{i=1}^{m} \alpha_i k_i\,(z,x)$ is a valid kernel. Because each k_i is a valid kernel function, it is symmetric so $k_i(x,z)\ =\ k_i(z,x)$; So:

Symmetry: K (z, x) = $\sum_{i=1}^{m} \alpha_i k_i\,(z,x)\ =\ \sum_{i=1}^{m} \alpha_i k_i\,(x,z)$ = K (x, z) => K (z, x) = K (x, z)

Positive Semi-definite: Let u ∈ R^n be arbitrary. The Gram matrix of K, denoted by G has the property,

$$G_{i,j} = K(x_{i,}z_j) = \sum_{i=1}^{m} \alpha_i k_i\,(z,x) \to G = \alpha_1 G_1 + \cdots + \alpha_m G_m$$

Now $u^T Gu = u^T(\alpha_1 G_1 + \cdots + \alpha_m G_m)u = u^T\alpha_1 G_1 u + \cdots + u^T \alpha_m G_m u = \sum_{i=1}^{m} \alpha_i u^T k_i\,(z,x)u$

$u^T G_i u > 0$ *and* $\alpha_i >= 0$, *So*: $\alpha_i u^T G_i u >= 0$

Now I prove K (x, z) = $k_1(x,z) - k_2(x,z)$ is not a valid kernel:

This equation is symmetric however it is not Positive-Semidefinite; the counter example is:

Suppose Gram matrix associated with k_1 is $\mathrm{I_p}$ - the identity matrix and Gram matrix associated with k_2 is $2\mathrm{I_p}$ - twice the identity matrix, then k_1-k_2 = -I which is negative.

For Example: $k_1 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ $k_2 = \begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}$ $\mathrm{k} = k_1 - k_2 = \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$

Therefore $k(x,z) = a_1 k_1(x,z) - a_2 k_2(x,z)$ *is not a kernel function.*

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b) $\boldsymbol{k(x,z) = k_1(x,z)k_2(x,z)}$ Valid

Symmetry: K (z, x) = $\sum_{i=1}^{m} \mathrm{k}_{i,1}\,(z,x)\mathrm{k}_{i,2}(z,x) = \sum_{i=1}^{m} \mathrm{k}_{i,1}\,(x,z)\mathrm{k}_{i,2}(x,z)$ = K (x, z) => K (z, x) = K (x, z)

Positive Semi-definite: Let u ∈ R^n be arbitrary. The Gram matrix of K, denoted by G has the property,

$$G_{i,j} = K(x_{i,}z_j) = \sum_{i=1}^{m} k_{i,1}\,(z,x)k_{i,2}(z,x) = G_{1,1}G_{2,1} + \cdots + G_{1,\mathrm{m}}G_{2,\mathrm{m}}$$

$$u^T Gu = u^T(G_{1,1}G_{2,1} + \cdots + G_{1,\mathrm{m}}G_{2,\mathrm{m}})u = u^T G_{1,1}G_{2,1}u + \cdots + u^T G_{1,\mathrm{m}}G_{2,\mathrm{m}}u$$
$$= \sum_{i=1}^{m} u^T k_{1,i}\,(z,x)k_{2,i}(z,x)u$$

Since $u^T G_{1,\mathrm{i}}u \geq 0$ *and* $u^T G_{2,\mathrm{i}}u \geq 0 \Rightarrow u^T G_{1,\mathrm{m}}G_{2,\mathrm{m}}u \geq 0$

Therefore, $\boldsymbol{k(x,z) = k_1(x,z)k_2(x,z)}$ is a valid kernel function.

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c) $k(x,z) = f_1(x)f_1(z) + f_2(x)f_2(z)$ Valid

I first prove that adding two valid kernel results in a valid kernel, then I prove $k(x,z) = f(x)f(z)$ is a valid kernel.

Symmetry: K (z, x) = $\mathrm{k}_1(z,x) + \mathrm{k}_2(z,x) = \mathrm{k}_1(x,z) + \mathrm{k}_2(x,z)$ = K (x, z) => K (z, x) = K (x, z)

Positive Semi-definite: Let u ∈ R^n be arbitrary. The Gram matrix of K, denoted by G has the property,

$$G_{i,j} = K(x_{i,}z_j) = \sum_{i=1}^{m} \mathrm{k_1(x_i, z_i)} + \mathrm{k_2(x_j, z_j)}$$

$$u^T Gu = u^T(G_{1,1} + G_{2,1} + \cdots + G_{1,\mathrm{m}} + G_{2,\mathrm{m}})u$$
$$= u^T G_{1,1}\mathrm{u} + u^T G_{2,1}u + \cdots + u^T G_{1,\mathrm{m}} + u^T G_{2,\mathrm{m}}u$$
$$= \sum_{i=1}^{m} u^T k_{1,i}(z,x)\mathrm{u} + u^T k_{2,i}(z,x)u$$

Since $u^T k_{2,i}(z,x)u$ >= 0 and $u^T k_{1,i}(z,x)u$ >= 0 thus: $u^T k_{1,i}(z,x)\mathrm{u} + u^T k_{2,i}(z,x)u >= 0$

Second, I have to prove $k(x,z) = f_1(x)f_1(z)$ is a valid kernel. I do this with finding the φ function such that k (x, z) = $\varphi(\mathrm{x})^{\mathrm{T}}\varphi(\mathrm{x})$

So, if f_1 or f_2 are functions whose transpose is same as themselves, we can consider them a valid φ function for the formula. An example of such function is identity.

$k(x,z) = f_1(x)f_1(z) = \mathrm{I(x)I(z)} = \mathrm{I(x)^T I(z)} = \varphi(\mathrm{x})^{\mathrm{T}}\varphi(\mathrm{x})$

As a result, $k(x,z) = f_1(x)f_1(z) + f_2(x)f_2(z)$ is a valid kernel.

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3. PCA and Eigenface [50 points]

This section is implemented in the Jupiter notebook attached. All answers are embedded within the code.