Course Project

Deadline: Sunday, Nov 15th, at 11:59pm.

Submission: You need to submit your code files and a PDF report on Microsoft Teams.

DO NOT submit zipped files.

Late Submission: 50% of the marks will be deducted for any submission beyond the deadline. No submissions will be accepted after two days past the deadline.

Collaboration: Collaboration in a pair of maximum two students is allowed.

1 Linear Elastic Constants Identification of 2D Linear Elastic Plate

1.1 A little background on linear elastic materials

Linear elastic materials are those that exhibit a linear relationship between stress and strain within their elastic limit. This means that the deformation is proportional to the applied load, and the material returns to its original shape when the load is removed. Many engineering materials, such as metals, ceramics, and certain types of plastics, exhibit linear elastic behavior within a reasonable range of stresses.

Linear elastic material model is typically used when a structural element experiences small deformations. This allows us to simplify the analysis by neglecting the effects of deformation on the geometry of the body, meaning that it would not matter whether you work with the undeformed configuration or with the deformed configuration (as both of them are taken to be approximately same under the assumption of small deformation).

1.2 Stress Measure

For linear elastic materials, the most common stress measure is the **Cauchy's stress tensor** ($\underline{\sigma}$). This tensor represents the stress at a point in the *deformed configuration*. It has the units of force per unit area in the deformed configuration.

1.3 Defining the boundary value problem (BVP) for a square plate

In this project, we will consider a boundary value problem (BVP) involving a two-dimensional square plate with an elliptic hole made of homogeneous isotropic linear elastic material. The internal domain of the plate is encompassed by the set $\Omega \in \mathbb{R}^2$ in its undeformed configuration $\underline{x}$, as shown in Fig. 1. The boundary of the plate is denoted by $\partial\Omega$ and each point on the boundary can be defined by its outward normal vector $\underline{n}(\underline{x})$. The plate is only subjected to Dirichlet boundary conditions – that specify the displacement at points on the boundary $\partial\Omega_u$, and there are no Neumann boundary conditions involved.

1.4 Quasi-Static Loading and Experimental Setup

An experimentation under *displacement-controlled loading* is assumed to be undertaken with quasi-static loading conditions. Under quasi-static loading, the displacement at the Dirichlet boundary (in this case, the top and right edges of the plate) is gradually increased in small steps; this sequence is called load steps. At each load steps, we measure the displacements of the 2D plate at various locations using full-field digital image correlation (DIC). Additionally, we measure the corresponding net reaction forces on the left (R_3) and bottom (R_1) edges of the plate (which form portions of $\partial\Omega_u$) using load cells.

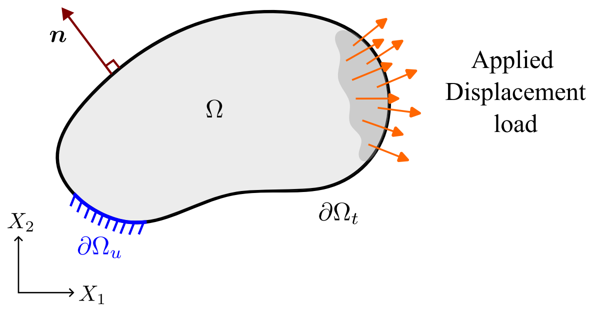

Figure 1: A generic 2D body Ω in undeformed configuration.

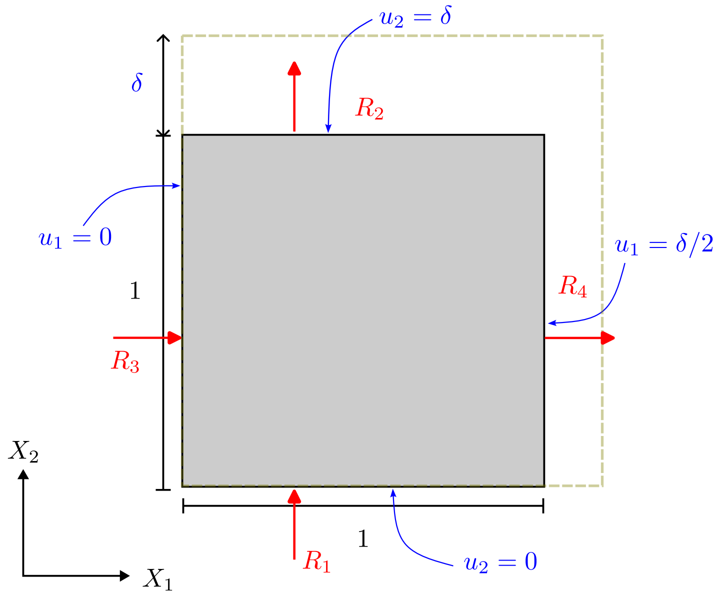

Figure 2: A 2D square hyperelastic plate Ω in deformed configuration. The left and bottom edges are placed on roller support, while displacement-controlled loading is applied to the top and right edges. As such, all material points in the entire boundary $\partial\Omega$ in this case constitutes the Dirichlet boundary condition (i.e., $\partial\Omega_u = \partial\Omega$).

Applying the principle of balance of linear momentum to a generic body Ω in a state of static equilibrium yields:

$$\nabla_{\underline{x}} \cdot \underline{\underline{\sigma}}(\underline{x}) = \underline{0} \quad \text{in } \Omega, \tag{1a}$$

$$\underline{u}(\underline{x}) = \underline{\tilde{u}} \quad \text{on } \partial\Omega_u, \tag{1b}$$

$$\underline{t}(\underline{x}) = \underline{\underline{P}}(\underline{x})\underline{n}(\underline{x}) = \underline{0} \quad \text{on } \partial\Omega_t, \tag{1c}$$

Here, $\underline{\nabla} \cdot (\circ)$ represents the divergence operator, with $\underline{\nabla}_{\underline{x}}$ denoting the gradient operator with respect to the deformed coordinate $\underline{x}$. The displacement field is represented by $\underline{u}(\underline{x}) \in \mathbb{R}^2$ while the prescribed displacement at $\partial\Omega_u$ is $\underline{\tilde{u}} \in \mathbb{R}^2$. Note that the body force is considered negligible and hence omitted in Eq. (1a). Finally, by incorporating a constitutive model that establishes the relationship between $\underline{u}(\underline{x})$ and $\underline{\underline{\sigma}}(\underline{x})$, the boundary value problem is fully defined and can be solved.

1.5 Constitutive Relation for Linear Elastic Isotropic Materials

The constitutive relation for a linear elastic isotropic material is given by Hooke's law:

$$\underline{\underline{\sigma}}(\underline{x}) = \underline{\underline{\underline{\underline{C}}}}\, \underline{\underline{\epsilon}}(\underline{x}) \tag{2}$$

where $\underline{\underline{\sigma}}(\underline{x})$ is the Cauchy stress tensor, $\underline{\underline{\underline{\underline{C}}}}$ is the fourth-order elasticity tensor, and $\underline{\underline{\epsilon}}(\underline{x})$ is the infinitesimal strain tensor.

For isotropic materials, the elasticity tensor $\underline{\underline{\underline{\underline{C}}}}$ can be expressed in terms of two material constants: **Lame's first constant** (λ) and **Lame's second constant** (μ). These parameters are related to Young's modulus (E) and Poisson's ratio (ν) as follows:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \tag{3}$$

$$\mu = \frac{E}{2(1+\nu)} \tag{4}$$

Using Lame's constants, the constitutive relation for isotropic linear elastic material can be written in component form as:

$$\sigma_{ij}(\underline{x}) = \lambda \text{ trace}\left(\underline{\underline{\epsilon}}(\underline{x})\right) \delta_{ij} + 2\mu\, \epsilon_{ij}(\underline{x}) \tag{5}$$

where δ_{ij} is the Kronecker delta and $\epsilon_{ij}(\underline{x})$ are the components of the infinitesimal strain tensor.

This Cauchy's stress tensor $\underline{\underline{\sigma}}(\underline{x})$ is defined for each material point $\underline{x}$, and it should satisfy the equilibrium equation Eq. (1a) and the zero traction boundary condition Eq. (1c). Additionally, at any material point $\underline{x}$ (on $\partial\Omega_u$) associated with the Dirichlet (or displacement) boundary constraints, the internal traction becomes equal to the reaction force:

$$\underline{t}_{int}(\underline{x}) = \underline{\underline{\sigma}}(\underline{x})\, \underline{n}(\underline{x}), \quad \underline{x} \in \partial\Omega_u \tag{6}$$

However, in an experimental setup, the reaction forces are not measured at the individual material points on the Dirichlet boundary Ω_u. Instead, one typically measures the total reaction force, say $\underline{R} \in \mathbb{R}^2$, using load transducers.

$$\underline{r} = \int_{\underline{x} \in \partial\Omega_u} \underline{t}_{int}(\underline{x}) d\underline{x} \tag{7}$$

1.6 Measurements

During the experiment with the square plate, multiple full-field displacement datasets are available at different load steps. We apply the quasi-static displacement loading on the upper and the right edges of the square plate in a sequence of five loading steps. At each displacement-loading step (ℓ), $\ell = 1, \ldots, 5$, we have M displacements measurements $\left\{\underline{u}_j^{(\ell)}\right\}_{j=1}^{M}$ at M locations $\left\{\underline{x}_j\right\}_{j=1}^{M}$ of the square plate.

Additionally, we have the reactions measured along the normal (perpendicular) directions of all edges $R_1^{(\ell)}$ (bottom), $R_2^{(\ell)}$ (top), $R_3^{(\ell)}$ (left), and $R_4^{(\ell)}$ (right), as shown in Fig. 2, all of which are assembled into an array $\underline{R}^{(\ell)}$ of size 4×1. You will find that $R_1^{(\ell)} = -R_2^{(\ell)}$ and $R_3^{(\ell)} = -R_4^{(\ell)}$. For the provided loading condition, the reactions along the edges are zero, and hence not measured.

The experimental data we measure is as follows and the link for downloading it is provided HERE.

$$
\begin{aligned}
\text{Known locations: } & \underline{x}_1, \ldots, \underline{x}_M \\
\text{Loading step: } & \ell = 1 \quad \mathcal{D}^{(1)} := \left\{ \underline{u}^{(1)}(\underline{x}_1), \ldots \underline{u}^{(1)}(\underline{x}_M), \underline{R}^{(1)} \right\} \\
& \ell = 2 \quad \mathcal{D}^{(2)} := \left\{ \underline{u}^{(2)}(\underline{x}_1), \ldots \underline{u}^{(2)}(\underline{x}_M), \underline{R}^{(2)} \right\} \\
& \ldots \\
& \ell = 5 \quad \mathcal{D}^{(5)} := \left\{ \underline{u}^{(5)}(\underline{x}_1), \ldots \underline{u}^{(5)}(\underline{x}_M), \underline{R}^{(5)} \right\}
\end{aligned}
\tag{8}
$$

2 Assessment

You will be assessed based on a short 2-7 pages report that is to be submitted in addition to appropriate codes (preferably in Python). Answer the following in the report:

- **[2 marks]** Instead of 15 equations of elasticity, derive a set of three governing equations involving only the three displacements for a linear elastic isotropic material. Use Lame's constants and substitute the displacements into the stress-strain relations and then into the stress-equilibrium relations. Include the derivation to this question in your report under a section named *Derivation of governing equation*.

- **[1+1 marks]** Formulate an optimization problem to maximize the fit between your model predictions (based on Lame's constants) and the experimental dataset. Define the constraints for this optimization problem.

  Also, write out a clear algorithm explaining every significant step of the parameter identification process, similar to that shown in Fig. 3. No need to write out the optimization algorithm steps.

Algorithm 1 An algorithm with caption

Require: $n \geq 0$
Ensure: $y = x^n$

```
  y ← 1
  X ← x
  N ← n
  while N ≠ 0 do
      if N is even then
          X ← X × X
          N ← N/2                      ▷ This is a comment
      else if N is odd then
          y ← y × X
          N ← N − 1
      end if
  end while
```

Figure 3: Algorithm format to be followed.

Include the answer to this question in your report under a section named *Parameter identification methodology.*

- **[6 marks]** Report the values of the Lame's constants obtained from your parameter identification optimization in your report under a section named *Results*. Upload the code file (do not zip it) on Teams.

  Provide figures illustrating the stress distributions of σ_{xx}, σ_{yy}, and τ_{xy} throughout the square plate for each load step ℓ. These figures should be presented in a similar format to the example provided below. Make sure that the color bar limits are set the same for all plots so as to be able to compare between plots.

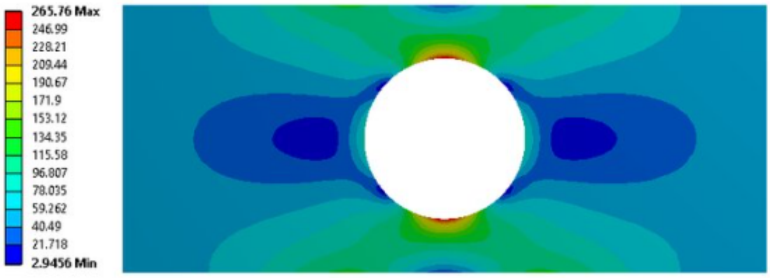

Figure 4: Example figures for stress and strain distribution plots.

A Appendix on Displacement-controlled loading

Displacement-controlled loading is a method used in material testing where the applied load is adjusted to maintain a specified rate of deformation or displacement in the specimen. In other words, instead of controlling the force applied to the specimen, the displacement (or deformation) experienced by the specimen is controlled. Here is how displacement-controlled loading typically works:

1. **Setup**: The specimen is mounted in the testing machine, and the desired testing parameters such as displacement rate and maximum displacement are set.

2. **Testing Procedure**: The testing machine applies a continuous displacement to the specimen at the specified rate. As the displacement increases, the load applied to the specimen may vary depending on its mechanical properties and the testing conditions.

3. **Feedback Mechanism**: The testing machine continuously monitors the displacement of the specimen and adjusts the applied load to maintain the desired displacement rate. This feedback mechanism ensures that the specimen experiences a controlled deformation throughout the test.

4. **Data Collection**: During the test, data such as applied load (measured in the form of generated reaction forces at the specimen's boundary constraints) and the corresponding displacements are typically recorded.