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Foundations of Data Science

Exercise sheet 6

Exercise 1

E1)

Goal: $r > \frac{l}{2}$

distance origin → vertex:

$$\sqrt{\sum_d \left(\frac{l}{2}\right)^2} = \sqrt{d}\,\frac{l}{2}$$

therefore:

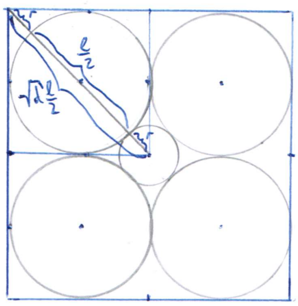
$$2r + \frac{l}{2} = \sqrt{d}\,\frac{l}{2}$$

$$\Leftrightarrow 2r = \sqrt{d}\,\frac{l}{2} - \frac{l}{2}$$

$$\Leftrightarrow r = \frac{1}{2}\left(\sqrt{d}\,\frac{l}{2} - \frac{l}{2}\right) \overset{!}{>} \frac{l}{2} \Leftrightarrow d > 3^2 = 9$$

since the radius of the inner ~~ball~~ hyperball is at most the distance from the surface of an outer hyperball to its corresponding vertex.

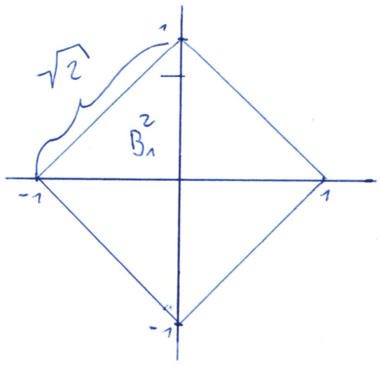
d=2:



Exercise 2

E2)

a)



$vol(B_1^2) = (\sqrt{2})^2 = 2$, since B_1^2 forms a square with side length $\sqrt{2}$.

$vol(B_1^3) = \frac{(\sqrt{2})^3}{3}\sqrt{2}$, since B_1^3 forms a Octahedron with side length $\sqrt{2}$.

b) For $B_2^\ell = \{x \in \mathbb{R}^\ell \mid \|x\|_2 \leq 1\}$ holds $\lim_{\ell \to \infty} vol(B_2^\ell) = 0$.

Let $x \in B_1^\ell = \{x \in \mathbb{R}^\ell \mid \|x\|_1 \leq 1\}$ it follows

$\|x\|_1 \leq 1$

$\Rightarrow \sum_i |x_i| \leq 1$

$\overset{|x_i| \leq 1}{\Rightarrow} \sum_i x_i^2 \leq 1$

$\Rightarrow \sqrt{\sum_i x_i^2} \leq 1$

$\Rightarrow \|x\|_2 \leq 1$

$\Rightarrow x \in B_2^\ell$

$\Rightarrow B_1^\ell \subseteq B_2^\ell$

therefore

$vol(B_1^\ell) \leq vol(B_2^\ell)$

$\Rightarrow \lim_{\ell \to \infty} vol(B_1^\ell) = 0$

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Exercise 3

a)

$$M^TM = \begin{pmatrix} 1 & 3 & 5 & 0 & 1 \\ 2 & 4 & 4 & 2 & 3 \\ 3 & 5 & 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 4 & 3 \\ 0 & 2 & 4 \\ 1 & 3 & 5 \end{pmatrix} = \begin{pmatrix} 36 & 37 & 38 \\ 37 & 49 & 61 \\ 38 & 61 & 84 \end{pmatrix}$$

$$MM^T = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 4 & 3 \\ 0 & 2 & 4 \\ 1 & 3 & 5 \end{pmatrix} \begin{pmatrix} 1 & 3 & 5 & 0 & 1 \\ 2 & 4 & 4 & 2 & 3 \\ 3 & 5 & 3 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 14 & 26 & 22 & 16 & 22 \\ 26 & 50 & 46 & 28 & 40 \\ 22 & 46 & 50 & 20 & 32 \\ 16 & 28 & 20 & 20 & 26 \\ 22 & 40 & 32 & 26 & 35 \end{pmatrix}$$

b)

$det(M^TM - \lambda I) = -\lambda^3 + 169\lambda^2 - 2370 = -\lambda * (\lambda^2 - 169\lambda + 2370) = 0$
$\lambda_1 = 0 \;\; \lambda_{2,3} = \frac{169}{2} \pm \sqrt{\frac{19081}{4}}$

$det(MM^T - \lambda I) = -\lambda^5 + 169\lambda^4 - 2370\lambda^3 = -\lambda^3 * (\lambda^2 - 169\lambda + 2370) = 0$
$\lambda_{1,2,3} = 0 \;\; \lambda_{4,5} = \frac{169}{2} \pm \sqrt{\frac{19081}{4}}$

c)

$(M^TM - \lambda I) * x = 0$
$\lambda_1 = 0$

$$\left(\begin{array}{ccc|c} 36 & 37 & 38 & 0 \\ 37 & 49 & 61 & 0 \\ 38 & 61 & 84 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 36 & 37 & 38 & 0 \\ 0 & 395 & 790 & 0 \\ 0 & 395 & 790 & 0 \end{array}\right)$$

$x_3 = -2x_2 \Rightarrow x_1 = x_3 \Rightarrow x = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$\lambda_2 = \frac{169}{2} - \sqrt{\frac{19081}{4}}$
$x = \begin{pmatrix} -1.45 \\ -0.22 \\ 1 \end{pmatrix}$
$\lambda_3 = \frac{169}{2} + \sqrt{\frac{19081}{4}}$
$x = \begin{pmatrix} 0.57 \\ 0.79 \\ 1 \end{pmatrix}$

$(MM^T - \lambda I) * x = 0$
$\lambda_{1,2,3} = 0$

$$\left(\begin{array}{ccccc|c} 14 & 26 & 22 & 16 & 22 & 0 \\ 26 & 50 & 46 & 28 & 40 & 0 \\ 22 & 46 & 50 & 20 & 32 & 0 \\ 16 & 28 & 20 & 20 & 26 & 0 \\ 22 & 40 & 32 & 26 & 35 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccccc|c} 14 & 26 & 22 & 16 & 22 & 0 \\ 0 & 12 & 36 & -12 & -6 & 0 \\ 0 & 36 & 108 & -36 & -18 & 0 \\ 0 & -12 & -36 & 12 & 6 & 0 \\ 0 & -6 & -18 & 6 & 3 & 0 \end{array}\right)$$

$x_2 = r, x_3 = s, x_4 = t \Rightarrow x_5 = 2r + 6s - 2t \Rightarrow x_1 = 2t - 11s - 5r$

$$x = \begin{pmatrix} 2t - 11s - 5r \\ r \\ s \\ t \\ 2r + 6s - 2t \end{pmatrix}$$

three eigenvectors for eigenvalue 0:

$$\begin{pmatrix} -5 \\ 1 \\ 0 \\ 0 \\ 2 \end{pmatrix} \quad \begin{pmatrix} -11 \\ 0 \\ 1 \\ 0 \\ 6 \end{pmatrix} \quad \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \\ -2 \end{pmatrix}$$

$\lambda_4 = \frac{169}{2} - \sqrt{\frac{19081}{4}}$

$x_4 = \begin{pmatrix} 0.38 \\ -0.08 \\ -1.78 \\ 1.23 \\ 1 \end{pmatrix}$

$\lambda_5 = \frac{169}{2} + \sqrt{\frac{19081}{4}}$

$x_5 = \begin{pmatrix} 0.65 \\ 1.24 \\ 1.13 \\ 0.70 \\ 1 \end{pmatrix}$

d)

$\lambda_1 = \frac{169}{2} + \sqrt{\frac{19081}{4}}, \lambda_2 = \frac{169}{2} - \sqrt{\frac{19081}{4}}$

$\sigma_1 = \sqrt{\lambda_1} = 12.39$

$\sigma_2 = \sqrt{\lambda_2} = 3.93$

$u_1 = \begin{pmatrix} 0.57 \\ 0.79 \\ 1 \end{pmatrix}$

$u_2 = \begin{pmatrix} -1.45 \\ -0.22 \\ 1 \end{pmatrix}$

Exercise 4