HABILITATION À DIRIGER DES RECHERCHES

HOMOLOGIE DE FACTORISATION, OPÉRADES ET STRUCTURES DE POISSON

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Contents

Introduction générale

Voici un résumé de mon parcours de recherche, qui s'est plongé dans les connexions profondes entre les structures algébriques et la géométrie. Voici de brèves descriptions des principaux sujets de mes recherches. Une description plus détaillée peut être trouvée dans les sections suivantes.

Commencant par le travail fondateur de M. Kapranov en 1997, une analogie entre la classe d'Atiyah du fibré tangent d'une variété complexe et les constantes de structure d'une algèbre de Lie a été établie. Dans [Mar09], j'ai approfondi cette observation. J'ai construit un cadre rigoureux décrivant l'interaction entre la cohomologie et l'homologie de Hochschild, la classe d'Atiyah jouant un rôle central. Ce formalisme a ouvert la voie à l'établissement d'un dictionnaire profond entre les algèbres de Lie et les variétés lisses, éclairant leurs connexions intrinsèques. Par la suite, mes recherches se sont étendues à l'application de ces concepts dans une démonstration alternative du théorème de Riemann-Roch.

Un autre sujet assez différent qui m'intéresse est celui des structures de Poisson de Feigin-Odesski. Une structure de Poisson sur une variété est un objet intéressant,

Date: December 23, 2024.

décrivant une déformation non commutative du premier ordre de l'algèbre des fonctions. La structure de Poisson de Feigin-Odesski est une classe intéressante de telles structures sur les espaces projectifs, les grassmanniennes, etc. Elles sont associées à une courbe elliptique, et leur structure reflète la géométrie algébrique de cette courbe elliptique et son plongement dans les espaces mentionnés. Ainsi, l'étude de la structure de Poisson de Feigin-Odesski est un exercice intéressant en géométrie algébrique classique. De plus, on peut supposer que presque toutes les structures de Poisson jouant un rôle important en physique mathématique sont une sorte de dégénérescence de celles de Feigin-Odesski, en particulier, elles sont associées à (une dégénérescence d') une courbe elliptique, ou du moins à une catégorie 1-Calabi-Yau. L'article [Fin+99] en est un exemple de cette approche.

Mes contributions à ce sujet sont diverses. Dans [GM24], nous avons étudié les structures générales des structures de Poisson de Feigin-Odesski. Sur cette base, dans [MP23; MP24], nous avons examiné une nouvelle caractéristique intrigante de ces structures : la présence de grandes familles de structures compatibles. Je crois que cette étude ouvrira de nouvelles perspectives sur ce sujet.

Le thème qui me préoccupe beaucoup est le lien entre les opérades, la topologie des variétés et la géométrie algébrique. La factorisation homologique est un sujet vaste qui réunit la physique mathématique, les catégories supérieures, la topologie algébrique, etc. Je les ai appliqués pour comprendre et étudier les invariants d'ordre fini des variétés de basse dimension et des nœuds. Dans [AS92], une formule explicite de l'invariant de Chern–Simons perturbatif est donnée. Ma première percée dans ce domaine a été un lien direct entre cette formule et la factorisation homologique de certaines e_n-algèbres, que j'ai appelées n-algèbres de Weyl. Ensuite, je les ai appliquées pour étudier les invariants des nœuds et le morphisme de formalité de M. Kontsevich. Je crois que ce sujet est loin d'être épuisé.

Mes intérêts plus récents se situent de l'autre côté de ce thème. On sait que l'opérade des petits disques est équipée d'une structure algébrique supplémentaire. Il s'ensuit que le groupe de Grothendieck–Teichmüller, qui est essentiellement le groupe des automorphismes de cette opérade, contient le groupe de Galois motivique, un objet d'une grande importance pour la géométrie algébrique moderne. Bien que ce sujet soit intensément étudié, il reste assez mystérieux. J'ai un projet pour étudier ces structures sur l'opérade des petits disques. Le premier pas dans cette direction est [Mar23].

Ci-dessous se trouve un aperçu plus détaillé de mon travail. La liste de mes articles sur lesquels cette revue est basée peut être trouvée ci-dessous, après le texte.

General introduction

Here is a summary of my research journey, which has delved into the profound connections between algebraic structures and geometry. Here are short outlines of the main subjects of my research. A more detailed description may be found in the subsequent sections.

Beginning with the seminal work of M. Kapranov in 1997, an analogy between the Atiyah class of the tangent bundle of a complex manifold and the structure constants of a Lie algebra was established. In [Mar09], I further developed this observation. I constructed a rigorous framework describing the interplay between Hochschild cohomology and homology, with the Atiyah class playing a central role.

This formalism paved the way for establishing a profound dictionary between Lie algebras and smooth manifolds, shedding light on their intrinsic connections. Moving forward my research extended to the application of these concepts in an alternative proving the Riemann-Roch theorem.

Another pretty different subject of my interest is Feigin–Odesski Poisson structures. A Poisson structure on a manifold is an interesting gadget on a manifold, describing a first-order non-commutative deformation of the algebra of functions. Feigin–Odesski Poisson structure is an interesting class of such structures on projective spaces, Grassmannians, and so on. They are associated with an elliptic curve, and their structure reflects the algebraic geometry of this elliptic curve and its embedding to the mentioned spaces. Thus, the study of Feigin–Odesski Poisson structure is an interesting exercise in classical algebraic geometry. Besides, one may suppose that nearly all Poisson structures playing important in mathematical physics are some sort of degeneration of Feigin–Odesski ones, in particular, they are associated with (a degeneration of) an elliptic curve, or at least with a 1-Calabi-Yau category. Paper [Fin+99] is an example of this approach.

My contributions to the subject are diverse. In [GM24] we studied the general structures of Feigin-Odesski Poisson structures. Based on it in [MP23; MP24] we investigate an intriguing new feature of these structures: the presence of big families of compatible ones. I believe that this study will lead to new insights into this subject.

The theme that concerns me a lot is the connection between the operads, the topology of manifolds, and algebraic geometry. Factorization homology is a wide subject uniting mathematical physics, higher categories, algebraic topology, and others. I applied them to understand and study so-called finite-order invariants of low-dimensional manifolds and knots. In [AS92], an explicit formula of the perturbative Chern–Simons invariant is given. My first breakthrough in this subject was a direct connection between this formula and the factorization homology of certain e_n-algebras, which I called Weyl n-algebras. Then, I applied them to study invariants of knots and the formality morphism of M. Kontsevich. I believe that this topic is far from being exhausted.

My more recent interests lie on the other side of this theme. It is known that the operad of little 2-disks is equipped with some additional algebraic structure. It follows, that the Grothendieck–Teichmüller group, which is essentially the group of automorphisms of this operad, contains the motivic Galois group, an object of great importance for modern algebraic geometry. Being intensively studied this topic is still rather mysterious. I have a project to investigate these structures on the little 2-disks operad. The first step in this direction is [Mar23].

Below is a more detailed overview of my job. The list of my papers on which this review is based may be found below, after the text.

1. Atiyah class and Riemann–Roch theorem

As far as I know, an analogy between the Atiyah class of the tangent bundle of a complex manifold and the structure constants of a Lie algebra first appeared in the pioneering paper of M. Kapranov [Kap97]. I developed this observation in [Mar09] keeping in mind the question posed to me by B. Feigin: "Why does the Todd class look like the invariant volume form on a Lie group?" The answer is, in a sense, contained in the proof of Proposition 1.3 below. On this way, I was able to develop a dictionary between Lie algebras and smooth manifolds. In modern terms, it is explainable in the paradigm of derived algebraic geometry. Indeed, the Chevalley complex of Lie algebra is a function ring of a supermanifold.

Firstly, I developed a formalism describing the Hochschild cohomology and homology and the relations between them. Essentially, I introduce a global analog of Hochschild–Kostant–Rosenberg isomorphism from [HKR62]. In this development, the Atiyah class plays a crucial role. One may consider the Atiyah class as a morphism from the identity functor to tensoring by the cotangent bundle functor shifted by one on the derived category $\mathrm{D}(X)$ of coherent sheaves on a smooth manifold:

$$\mathrm{at}\colon \mathrm{id} \to \cdot \otimes \Omega^1[1] \tag{1}$$

One may think about (1) as an action of an object $T[-1]$ dual to Ω^1 on $\mathrm{D}(X)$. Iterating this action one can make the tensor power of $T[-1]$ (in fact, the symmetric power) act on $\mathrm{D}(X)$.

Then, I proved the Riemann–Roch theorem as an application of the apparatus developed in the first part combined with the Serre duality. The Riemann–Roch theorem in the form of Grothendieck describes how the Chern character behaves under taking the direct image. There are different forms of the theorem, depending on which definition of the Chern character one uses. If one works with an algebraic manifold and the Chern character takes value in the Chow group, then one should use the intersection theory ([BGI71]); if one works on a complex–analytic manifold and the Chern character is the topological one, then the Atiyah–Singer theorem is appropriate ([Hir66]). We use the Chern character taking value in the Hodge cohomology (see [Ill71]). Of course, one could prove the Riemann–Roch theorem in this case, using some comparison theorems between cohomology theories, but it is more plausible to have independent proof.

Essentially we follow [OTT81]. But instead of explicit calculations with the Čech cocycles we work in the derived category and use our algebraic–differential calculus. The proof consists of two parts: the first says what one needs to calculate (the dual class of the diagonal in the framework of [OTT81]) and the second one is the calculation.

In [Mar01], I applied these methods to prove also the holomorphic Lefschetz formula. This approach was generalized and developed in terms of 2-categories [KP20] following the pioneering ideas suggested in [BN].

Here are some details of my approach.

1.1. **Atiyah class.** We begin with the Atiyah class. Usually, one considers this class for vector bundle as an obstruction to the existence of a connection ([Ati57a]): with any vector bundle one may associate the filtered vector (bundle of first jets) of its sections at the first infinitesimal neighborhood of a point. A splitting of this filtration gives a connection on the vector bundle. The extension obstructing such a splitting is the Atiyah class. I showed that it is very instructive to consider this class for coherent sheaves and complexes of coherent sheaves. And importantly, this class defines a morphism of functors (or natural transformation) from the derived category of complexes on the manifold under consideration to itself.

Let X be a smooth algebraic variety over a field k of characteristic 0 or bigger than $\dim X$. Everything works for the complex analytic case as well. Let $\Delta \subset X \times X$ be the diagonal and let I denote the ideal sheaf of Δ. Then, by definition, $\mathcal{O}_\Delta = \mathcal{O}_{X\times X}/I$ and $\Omega^1_\Delta = I/I^2$. The two-step filtration on $\mathcal{O}_{X\times X}/I^2$ by powers of I gives rise to the exact sequence

$$0 \longrightarrow \Omega^1_\Delta \longrightarrow \mathcal{O}_{X\times X}/I^2 \longrightarrow \mathcal{O}_\Delta \longrightarrow 0 \tag{2}$$

Since the terms of the sequence (2) are supported on the diagonal, one may consider (2) as a sequence of sheaves of $\mathcal{O}_X$-$\mathcal{O}_X$-bimodules on X. Let E be a sheaf of $\mathcal{O}$-modules or a complex of such sheaves on X. Take its tensor product with (2) with respect to the left $\mathcal{O}$ module structure, and consider it as a right $\mathcal{O}$ module. In other words, tensor (2) by p_1^*E and take the direct image p_{2*}. Because all terms in (2) are locally free left $\mathcal{O}$-modules, this operation is exact, and one gets an exact sequence

$$0 \longrightarrow E\otimes\Omega^1 \longrightarrow \mathrm{J}^1(E) \longrightarrow E \longrightarrow 0. \tag{3}$$

Here $\mathrm{J}^1(E)$ denotes $E\otimes\mathcal{O}/I^2$ with the right $\mathcal{O}$-module structure and is called the *sheaf of the first jets*.

Definition 1.1 ([Ati57a; Ill71])**.** For a sheaf of $\mathcal{O}$-modules or a complex of such sheaves E on X the class of extensions represented by (3) is called the *Atiyah class* $\mathrm{at}(E) \in \mathcal{E}xt^1(E, E\otimes\Omega^1)$ of E.

Sheaves on $X\times X$ may be thought as kernel of endofunctors of the derived category: the functor given by such a sheaf is given by pullback, tensor product with a sheaf, and the pushforward, as above. The extension (2) may be considered as an extension of such kernels of functors on the derived category. It follows that the Atiyah class is natural, that is a morphism of functors.

Example 1.1. Consider the simplest example of on locally free sheaf. Let $\mathcal{O}_0$ be the structure sheaf of the origin point of $\mathbb{A}^1$. It has a free resolution:

$$\mathcal{O}_{\mathbb{A}^1} \xrightarrow{x} \mathcal{O}_{\mathbb{A}^1} \longrightarrow \mathcal{O}_0 \longrightarrow 0$$

Let us identify $\Omega^1_{\mathbb{A}^1}$ with $\mathcal{O}_{\mathbb{A}^1}$ by means of section dx of the former. Then one may see, that the Atiyah class is given by the following map of complexes:

$$\begin{tikzcd} & \mathcal{O}_{\mathbb{A}^1} \arrow[r, "x"] \arrow[d, "1"'] & \mathcal{O}_{\mathbb{A}^1} \\ \mathcal{O}_{\mathbb{A}^1} \arrow[r, "x"] & \mathcal{O}_{\mathbb{A}^1} & \end{tikzcd}$$

Note that the Atiyah class in the example above is the derivative of the differential of the complex. More generally, one may prove the following useful lemma.

Proposition 1.1. *Let $E = (E^i, d^i : E^i \to E^{i+1})$ be a complex of sheaves of $\mathcal{O}$-modules. Assume given a connection ∇^i on E^i. Then,* $\mathrm{at}(E)$ *is represented by*

$$(\nabla d)^i \overset{def}{=} (d^i\circ\nabla^i - \nabla^{i+1}\circ d^i)\colon E^i \to E^{i+1}\otimes\Omega^1.$$

Example 1.2. Consider a smooth affine variety $\mathrm{Spec}(A)$. The structure sheaf of the diagonal $\mathcal{O}_\Delta$ has the standard resolution $B = (B_i, d_i)$, $i \geqslant 0$ defined as follows: B_n is a free $A\otimes A$-module generated by tensor power $A^{\otimes n}$ over the base field and the differential is given by

$$\begin{multline} d(a_1\otimes a_2\otimes\cdots\otimes a_{n-1}\otimes a_n) = \\ a_1(a_2\otimes\cdots\otimes a_n) - (a_1a_2\otimes\cdots\otimes a_n) + \cdots + (-1)^n(a_1\otimes a_2\otimes\cdots\otimes a_{n-1})a_n. \end{multline} \tag{4}$$

The terms of the standard resolution are free modules, hence they are equipped with canonical connections. Applying Proposition 1.1 to the standard resolution one obtains the following expression for the Atiyah class of $\mathcal{O}_\Delta$:

$$\begin{aligned}\operatorname{at}(\mathcal{O}_\Delta)\colon (a_1 \otimes a_2 \otimes \cdots \otimes a_{n-1} \otimes a_n) &\mapsto \\ da_1(a_2 \otimes \cdots \otimes a_{n-1} \otimes a_n) &+ (-1)^n (a_1 \otimes a_2 \otimes \cdots \otimes a_{n-1}) da_n,\end{aligned}$$

where d is the exterior differential. This formula is an explicit form of action (7) below.

Consider the Atiyah class of the cotangent bundle

$$\operatorname{at}(\Omega^1)\colon \Omega^1 \to \Omega^1 \otimes \Omega^1[1] \tag{5}$$

Proposition 1.2. (1) $\operatorname{at}(\Omega^1)$ *is symmetric, i. e. invariant under the permutation of factors in* $\Omega^1 \otimes \Omega^1$.
(2) $\operatorname{at}(\Omega^1)$ *obeys the* Jacobi identity, *i. e. the projection of* $\operatorname{at}(\Omega^1) \otimes \operatorname{id} \circ \operatorname{at}(\Omega^1)$ *onto the part of* $\Omega^1 \otimes \Omega^1 \otimes \Omega^1$ *invariant under permutations is equal to zero.*

It follows that there is a structure of a Lie (super)algebra in the derived category $\mathrm{D}(X)$ on the shifted tangent bundle $T[-1]$, as it was observed in [Kap97]. The map dual to (5) is the bracket. Denote this Lie algebra by $\mathcal{T}$:

$$[\ ,\ ]\colon \mathcal{T} \otimes^L \mathcal{T} \to \mathcal{T}.$$

Formally following the analogous construction for Lie algebras one may define the enveloping algebra and the Hopf-dual object, the ring of functions on the formal Lie group. They turn to be *Hochschild cochain complex* by

$$\mathcal{U} = Rp_{1*}\, \underline{RHom}^\bullet{}_{X \times X}(\mathcal{O}_\Delta, \mathcal{O}_\Delta),$$

and *Hochschild chain complex* defined by

$$\mathcal{F} = Rp_{1*}(\mathcal{O}_\Delta \otimes^L \mathcal{O}_\Delta), \tag{6}$$

where Δ is the diagonal in $X \times X$.

Still following this analogy, one expect an analog of action of the universal enveloping algebra, which is the algebra of left-invariant differential operators, on the ring of functions. This canonical action of $\mathcal{U}$ on $\mathcal{F}$

$$\mathcal{D}\colon \mathcal{U} \otimes^L \mathcal{F} \to \mathcal{F} \tag{7}$$

is given by

$$\mathcal{U} = \operatorname{Ext}(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \xrightarrow{-\otimes \mathrm{id}} \operatorname{Ext}(\mathcal{O}_\Delta \otimes^L \mathcal{O}_\Delta, \mathcal{O}_\Delta \otimes^L \mathcal{O}_\Delta) = \operatorname{Ext}(\mathcal{F}, \mathcal{F}).$$

An analog of the Poincaré–Birkhoff–Witt isomorphism is the Hochschil-d-Kostant–Rosenberg isomorphism ([HKR62]), which establishes an isomorphism between $\mathcal{U}$ and the sum of shifted polyvector fields $\bigoplus_i \Lambda^i T[-i]$, and between $\mathcal{F}$ and the sum of shifted differential forms $\bigoplus_i \Omega^i[i]$. In the same way, as for Lie algebras, these isomorphisms neither respect the product on $\mathcal{U}$, nor the action of $\mathcal{U}$ on $\mathcal{F}$.

The problem for Lie algebras is the following. Poincaré–Birkhoff–Witt isomorphism identifies an element of the symmetric power of a Lie algebra with an element of the universal enveloping algebra by taking an average of products of Lie algebra elements forming the former element in all possible orders. Product of two such symmetric elements is not symmetric in an obvious way. The product in terms of Poincaré–Birkhoff–Witt isomorphism is given by the Baker–Campbell–Hausdorff formula. But if one of the factors is linear, the formula simplifies, the formula is known as the one for left-(or right-)invariant vector fields in exponential coordinates.

The analog of the mentioned formula for left-invariant vector fields on a Lie group in exponential coordinates is the following. Let

$$\mathbf{L}\colon \quad \bigoplus_i \Omega^i[i] \to (\bigoplus_i \Omega^i[i]) \otimes \Omega^1[1]$$

denote the morphism defined by the formula

$$\mathbf{L} = \sum l_n L^n \tag{8}$$

where $\sum l_n z^n = z/(e^z - 1)$.

The following theorem provides a description of the action of $\mathcal{T} \subset \mathcal{U}$ on $\mathcal{F}$ which allows us to obtain the action of all of $\mathcal{U}$.

Theorem 1.1. *The diagram*

$$\begin{tikzcd} \mathcal{T} \otimes \mathcal{F} \arrow[r, "\mathcal{D}"] \arrow[d, "\mathrm{id}\otimes\mathbf{E}"'] & \mathcal{F} \arrow[d, "\mathbf{E}"'] \\ T \otimes \bigoplus_i \Omega^i[i][-1] \arrow[r, "\mathbf{L}"] & \bigoplus_i \Omega^i[i] \end{tikzcd}$$

is commutative.

1.2. **Riemnn-Roch theorem.** For X a proper algebraic variety (or a compact analytical one), there is a map $\int : H^{\dim X}(\omega) \to k$, where ω is the canonical sheaf, such that for any $E \in \mathrm{D}(X)$ the composition of maps

$$H^i(E) \otimes H^{\dim X - i}(E^\vee \otimes^L \omega) \xrightarrow{\mathrm{Tr}} H^{\dim X}(\omega) \xrightarrow{\int} k \tag{9}$$

gives a perfect (super)symmetric pairing, where Tr is the trace map. This statement is called the Serre duality, see [Har66], [Con00]. The Serre duality obeys numerous naturality conditions.

As we have seen above, the Atiyah class allows us to produce morphisms of endofunctors of $\mathrm{D}(X)$. Taking a particular composition of these morphisms one may produce a morphism from id to $\otimes^L \omega[\dim X]$ functor. Given an object $E \in \mathrm{D}(X)$, it gives a morphism $\mathrm{Ext}^{\dim X}(E, E \otimes^L \omega)$. Taking trace and applying $\int$ we get a number. Thus, we get a function on the set of objects of $\mathrm{D}(X)$. This function is the Euler characteristic of the cohomology of the object. It is our version of the Riemann–Roch theorem in a nutshell. Here are some details.

For $E \in \mathrm{D}(X)$ let $\mathbf{K}$ denote the composition

$$\mathcal{O}_{X \times X} \xrightarrow{\mathbb{1}} E \boxtimes E^\vee \otimes^L \mathcal{O}_\Delta \overset{can \otimes E \boxtimes E^\vee}{\longrightarrow} E \boxtimes (E^\vee \otimes^L \omega)[\dim X], \tag{10}$$

where the first arrow is given by the identity operator $\mathcal{O} \to E \otimes E^\vee$. By the second statement of the theorem, $\mathbf{K} \in H^*(E) \otimes H^*(E^\vee \otimes \omega) = H^*(E) \otimes H^*(E)^\vee = End\, H^*(E)$ is equal to the identity operator.

By the first statement of the theorem, the trace of the restriction of $\mathbf{K}$ to the diagonal $\Delta^* \mathbf{K} \in H^*(E \otimes E^\vee \otimes \omega)$ followed by $\int$ is equal to the supertrace of the identity operator on $H^*(E)$, that is, to the *Euler characteristic* :

$$\chi(E) \overset{def}{=} \sum_i (-1)^i \dim H^i(E).$$

To state the Riemann-Roch theorem, we need to factorize the morphism $\int \Delta^*\mathbf{K} \in H^*(E \otimes E^\vee \otimes \omega)$. Restricting (10) to the diagonal and taking the trace, we obtain

$$\mathcal{O}_X \xrightarrow{\mathbb{1}\otimes\mathcal{O}_\Delta} E \otimes^L E^\vee \otimes^L \mathcal{F} \xrightarrow{\mathrm{id}\otimes(can\otimes\mathcal{O}_\Delta)} E \otimes^L E^\vee \otimes^L \omega[\dim X] \xrightarrow{\mathrm{Tr}} \omega[\dim X], \tag{11}$$

where $\mathcal{F}$ is defined by (6). Interchanging the last two arrows, we obtain

$$\mathcal{O}_X \xrightarrow{\mathbb{1}\otimes\mathcal{O}_\Delta} E \otimes^L E^\vee \otimes^L \mathcal{F} \xrightarrow{\mathrm{Tr}\otimes\mathrm{id}} \mathcal{F} \xrightarrow{can\otimes\mathcal{O}_\Delta} \omega[\dim X].$$

We introduce the following notations: let

$$\mathrm{Ch}(E)\colon \mathcal{O}_X \xrightarrow{\mathbb{1}\otimes\mathcal{O}_\Delta} E \otimes^L E^\vee \otimes^L \mathcal{F} \xrightarrow{\mathrm{Tr}\otimes\mathrm{id}} \mathcal{F} \tag{12}$$

and let

$$\mathrm{Td}\colon \mathcal{F} \xrightarrow{can\otimes\mathcal{O}_\Delta} \omega[\dim X]. \tag{13}$$

Theorem 1.2 (Riemann-Roch theorem)**.** *For $E \in \mathrm{D}(X)$*

$$\chi(E) = \int \mathrm{Td} \circ \mathrm{Ch}(E).$$

The classes Ch and Td may be calculated explicitly using the formalism described in the previous Subsection.

The key statement here is the following Proposition.

Proposition 1.3. *Let*

$$\mathrm{td} = \exp(\sum t_i \,\mathrm{ch}(\Omega^1)) \in \bigoplus_i H^i(\Omega^i), \tag{14}$$

where $\sum t_i z^i = \log(z/(e^z - 1))$. Then, the class Td *from* (13) *may be expressed as the composition*

$$\mathrm{Td}\colon \mathcal{F} \xrightarrow{\mathbf{E}} \bigoplus_i \Omega^i[i] \xrightarrow{\wedge\,\mathrm{td}} \bigoplus_i \Omega^i[i] \twoheadrightarrow \omega[\dim X],$$

where the last arrow is the projection onto the differential forms of the top degree.

The proof is based on the Theorem 1.1 above.

2. Feigin–Odesski Poisson structures

In the late 80s B. Feigin and A. Odesskii invented elliptic algebras, which are generalizations of Sklyanin algebras. These are algebras with quadratic relations, which are flat deformations of the polynomial algebra. The construction of algebra depends on a choice of an elliptic curve, and the deformation parameter is a shift on this elliptic curve. Generators of such an algebra form a space naturally isomorphic to sections of a simple vector bundle over the elliptic curve. By [Ati57b], such bundles up to a twist of a line bundle of degree 0, are classified by pairs of relatively prime natural numbers (n, r), where n is degree of the bundle and r is its rank.

In [FO95] Feigin and Odesskii studied the quadratic Poisson structures associated with these deformations. Such a homogeneous Poisson structure on vector space gives a Poisson structure on the associated projective space. The latter Poisson structure is called Feigin–Odesski (FO) Poisson structure. Thus, this is a Poisson structure on $\mathbb{P}^{n-1}H^1(C, \mathcal{E}_{n,r}^\vee)$, where C is an elliptic curve, $\mathcal{E}_{n,r}$ is a simple vector bundle of rank r and degree n, and we identify $H^1(C, \mathcal{E}_{n,r}^\vee)$ with the space dual to $H^0(C, \mathcal{E}_{n,r})$ by means of the Serre duality. Denote this Poisson structure by $q_{n,r}$.

I learned about FO Poisson structures in the 90s from the authors. But at that time, the only result of my interest to this subject was the joint paper with M. Finkelberg, A. Kuznetsov, and I. Mirković [Fin+99]. In this paper, we build a symplectic structure on the space of G-monopoles, that is the moduli space of maps from $\mathbb{P}^1$ to a semisimple complex Lie group G, which send infinity to the Borel subgroup. If one treat $\mathbb{P}^1$ as an additive degeneration of an elliptic curve, this construction is similar to the construction of the FO bracket, as I describe below.

There are many equivalent ways to define FO Poisson structures. Essentially, they are given by Massey products in the derived category of complexes of coherent sheaves on the elliptic curve, see [Pol98]. Below, I give a more practical definition in terms of a spectral sequence associated with a filtered object on the elliptic curve. The crucial property discovered already by Feigin and Odesski is the following. Points of the projective space classify extensions of $\mathcal{E}^\vee_{r,n}$ by $\mathcal{O}$ on the elliptic curve. Subsets of points with isomorphic middle terms of the extension are algebraic. One may show that such subset is a union of symplectic leaves of the FO Poisson structure. In particular, these symplectic leaves are algebraic. One can not expect such a property for a general Poisson structure. It makes the FO Poisson structure so special and interesting algebrogeometric object.

Example 2.1. Consider an example of FO Poisson structure; it corresponds to 4-dimensional Sklyanin algebra introduced in [Skl83]. The vector bundle involved is $\mathcal{E}_{4,1}$, that is a line bundle of degree 4.

This Poisson structure is on the 3-dimensional projective space $\mathbb{P}^3$, which is the projectivization of the space dual to the space of sections of our line bundle.

There is a canonical embedding of the elliptic curve $C \hookrightarrow \mathbb{P}^3$ to this vector space, this is a normal curve of degree 4. This curve defines a pencil of quadrics passing through it. There are four singular quadrics in this pencil.

Let us describe symplectic leaves of this Poisson structure. Leaves of rank zero are points of the embedded elliptic curve plus 4 singular points of singular quadrics in the pencil. Each quadric is the closure of a leaf of rank 2.

As we mentioned above, leaves of the FO Poisson structure correspond to isomorphism classes of the middle terms of non-trivial extensions given by points of the projective spaces, that is non-zero elements of $H^1(C, \mathcal{E}_{4,1})$ up to a scalar. Let us describe these classes. Points on the embedded elliptic curves correspond to bundles of type $\mathcal{E}_{3,1} \oplus \mathcal{E}_{1,1}$, points on non-singular quadrics out of $C \hookrightarrow \mathbb{P}^3$ correspond to $\mathcal{E}_{2,1} \oplus \mathcal{E}'_{2,1}$ (sum of two *non-isomorphic* line bundles), points on singular quadrics out of $\mathcal{E}_{3,1} \oplus \mathcal{E}_{1,1}$ and singular points correspond to the non-trivial extension of $\mathcal{E}_{2,1}$ with itself, and singular points of singular quadrics correspond to $\mathcal{E}_{2,1} \oplus \mathcal{E}_{2,1}$ (sum of two *isomorphic* line bundles).

One may notice that the FO Poisson structure around a singular point of a quadric resembles the Kirillov–Kostant–Souriau Poisson structure, corresponding to Lie algebra sl_2. This fact has an explanation: by Theorem 2.2 below, the linearization at a point of rank 0 of the FO Poisson structure is the Kirillov–Kostant–Souriau Poisson structure of the Lie algebra of traceless endomorphisms of the corresponding vector bundle. In our case, the vector bundle is $\mathcal{E}_{2,1} \oplus \mathcal{E}_{2,1}$ and the algebra of its endomorphisms is 2×2 matrices.

Note that each quadric in the pencil is a variety formed by lines passing through two points of C with a fixed sum in the sense of the group law on the elliptic curve.

It works for all FO Poisson structures of type $q_{n,1}$ as well: leaves of rank 0 are given by points of the normal embedded elliptic curve $C \hookrightarrow \mathbb{P}^{n-1}$ of degree n, closures of leaves of rank two are formed by lines passing through two pairs of C with a fixed sum, closures of leaves of rank four are formed by planes passing through triples of points of C with a fixed sum, and so on, plus additional leaves corresponding to singularities of these chord varieties. Also, one may easily describe vector bundles corresponding to symplectic leaves. But description of geometry of FO Poisson structures of type $q_{n,r}$ for $r > 1$ is less clear.

The FO Poisson structures satisfy even stronger algebraic conditions. As the underlying space of the FO Poisson structure is a moduli space, there is a groupoid associated with this moduli problem. This algebraic groupoid is the symplectic groupoid of the Poisson structure (see [Wei87]); this is a subject of a future project. It implies, in particular, an isomorphism between conormal Lie algebra and the Lie algebra of traceless endomorphisms of the vector bundle corresponding to the point of the projective space. In the joint preprint with L. Gorodetsky [GM24] we give a direct proof of this result. Details are below

In the work [OW13] Odesskii and Wolf gave a construction of 9-dimensional subspaces of Poisson structures on projective spaces whose general member is a FO Poisson structure for some elliptic curve and a linear bundle on it. In [HP07], Hua and Polishchuk interpreted this construction geometrically in terms of families of anticanonical divisors on Hirzebruch surfaces and extended it to give new examples of compatible Poisson structures.

In our joint work with A. Polishchuk [MP23], we show that these constructions are essentially the only way to produce compatible FO Poisson structure given by an elliptic curve and a line bundle. In [MP24] we do the same for the first non-trivial FO Poisson structure of rank bigger than 1, for $q_{5,2}$. Some details are below.

2.1. **Conormal Lie algebra.** Let $\mathcal{E}_{n,r}$ be a simple vector bundle of rank $r > 0$ and degree $n > 0$ on an elliptic curve C, as this is described in [Ati57b]. As in [FO95], consider the following moduli space of filtered vector bundles. The $(n-1)$-dimensional projective space $P = \mathbb{P}\operatorname{Ext}^1(\mathcal{E}_{n,r}, \mathcal{O})$ is the moduli space of filtered vector bundles $E \supset L \supset 0$ with fixed associated quotients

$$E/L \simeq \mathcal{E}_{n,r}, \quad L \simeq \mathcal{O} \tag{15}$$

Isomorphisms (15) are not specified, i. e. they are not a part of the data of a filtered vector bundle. Moreover, we throw away the trivial filtered vector bundle $\mathcal{E}_{n,r} \oplus \mathcal{O} \supset \mathcal{O} \supset 0$.

We will define the Feigin–Odeskii Poisson structure π on P as a morphism of vector bundles

$$\pi \colon T^*P \to TP.$$

Since P is a projective space, for any non-zero $\phi \in \operatorname{Ext}^1(\mathcal{E}_{n,r}, \mathcal{O})$ we have the identification

$$T_\phi P = \operatorname{Ext}^1(\mathcal{E}_{n,r}, \mathcal{O})/\langle \phi \rangle,$$

of degree 1 in ϕ, and using the Serre duality pairing

$$\langle -, - \rangle \colon \operatorname{Hom}(\mathcal{O}, \mathcal{E}_{n,r}) \otimes \operatorname{Ext}^1(\mathcal{E}_{n,r}, \mathcal{O}) \to k$$

we can write

$$T^*_\phi P = \langle \phi \rangle^\perp \subset \operatorname{Hom}(\mathcal{O}, \mathcal{E}_{n,r}).$$

Fix a non-zero element $\phi \in \operatorname{Ext}^1(\mathcal{E}_{n,r}, \mathcal{O})$ corresponding to an extension

$$0 \longrightarrow \mathcal{O} \longrightarrow E \longrightarrow \mathcal{E}_{n,r} \longrightarrow 0 \tag{16}$$

and consider the corresponding filtered vector bundle $E \supset L \supset 0$. For this filtered vector bundle, consider the vector bundle $\mathcal{E}nd(E)$ with the induced three-term filtration on it, and this filtration gives a spectral sequence computing $H^*(C, \mathcal{E}nd(E)) = \operatorname{Ext}^*(E, E)$. Moreover, one can use the vector bundle $\mathcal{E}nd(E)_0$ instead of $\mathcal{E}nd(E)$ to get the reduced spectral sequence computing $\operatorname{Ext}^\bullet(E, E)_0$. Since $\mathcal{E}nd(E) = \mathcal{E}nd(E)_0 \oplus \mathcal{O}$, the reduced spectral sequence is the traceless part of the initial spectral sequence, and it is a direct summand in it.

The first page of the reduced spectral sequence looks as follows.

$$\begin{array}{c|ccc}
1 & \operatorname{Hom}(\mathcal{O}, \mathcal{E}_{n,r}) \xrightarrow{\langle \phi, - \rangle} & k & \\
0 & & k \xrightarrow{\cdot \phi} & \operatorname{Ext}^1(\mathcal{E}_{n,r}, \mathcal{O}) \\
\hline
 & -1 & 0 & 1
\end{array} \tag{17}$$

The only non-trivial differential on the second page is a map

$$d_2 \colon \langle \phi \rangle^\perp \to \operatorname{Ext}^1(\mathcal{E}_{n,r}, \mathcal{O})/\langle \phi \rangle,$$

As we mentioned, the source of this map is the cotangent space, and the target is the tangent space to our projective space. This map defines the FO Poisson structure. One may show this definition coincides with both given in [FO95] and [HP23].

Because d_2 is the last non-zero differential in the spectral sequence, this immediately implies the following theorem, stated in [FO95].

Theorem 2.1. *Connected components of isomorphism classes of E are symplectic leaves of the Feigin–Odesskii Poisson structure π on P.*

It is known (see [Wei83]) that locally any Poisson structure looks like a direct product of a symplectic structure and a Poisson structure vanishing at the origin. The linear part of the latter gives a linear Poisson structure on the normal space to the symplectic leave of the initial one, that is a Lie algebra structure on the conormal space to the symplectic leaf.

As mentioned above, a leaf of FO Poisson structure corresponds to a vector bundle E on the elliptic curve, which is the middle term of the corresponding extension. One may see that the conormal space to the leaf is naturally isomorphic to the space of traceless endomorphisms of this vector bundle. In [GM24], we prove the following theorem by analyzing the very definitions of both terms.

Theorem 2.2. *Let $\langle \phi \rangle$ be a point of P corresponding to a filtered vector bundle $E \supset L \supset 0$. Then the conormal Lie algebra of the Feigin–Odesskii Poisson structure π at the point $\langle \phi \rangle$ is isomorphic to the Lie algebra $\operatorname{End}(E)_0$ of traceless endomorphisms of E.*

2.2. **Compatible FO Poisson structures.** Recall that two Poisson structures Π_1, Π_2 on the same space X are called compatible if every linear combination $\lambda_1\Pi_1+\lambda_2\Pi_2$ is still a Poisson structure. This is equivalent to the identity $[\Pi_1, \Pi_2] = 0$, where we use the Schouten bracket of bivectors. More generally, one can consider larger linear subspaces of Poisson bivectors. In the work [OW13] Odesskii and Wolf gave a construction of 9-dimensional subspaces of Poisson structures on $\mathbb{P}^n$ whose general member is a bracket Π_C for some normal elliptic curve $C \subset \mathbb{P}^n$. In [HP23] this construction was interpreted geometrically in terms of families of anticanonical divisors on Hirzebruch surfaces, and extended to give new examples of compatible Poisson structures.

In a joint paper with A. Polishchuk [MP23] we show that these constructions are the only way to produce compatible FO Poisson structures of type $q_{n+1,1}(C)$, with one exception occurring for $n = 3$. The following theorem from there describes all compatible FO Poisson brackets of type $q_{n,1}$.

Let us fix $n \geq 4$ (case $n = 3$ is simple, but requires a separate treatment, see our text for details). With every normal elliptic curve C in $\mathbb{P}^n$ we can associate a 1-dimensional family $\mathcal{S}_C$ of rational normal scrolls $S(r,r) \subset \mathbb{P}^n$ if $n = 2r+1$, with several exceptional members of type $S(r-1, r+1)$ (resp., $S(r-1, r)$ if $n = 2r$), which are parametrized by points of $\mathrm{Pic}_2(C)$. These scrolls are symplectic leaves of rank 2 of $q_{n,1}$. In the case of $n = 5$, each normal elliptic curve C in $\mathbb{P}^5$ is contained in four Veronese surfaces, corresponding to choices of a square root of the line bundle $\mathcal{O}(1)|_C$ of degree 6.

Theorem 2.3. *For a collection of normal elliptic curves $(C_i)_{i\in I}$ in $\mathbb{P}^n$ the Poisson structures (Π_{C_i}) are compatible if and only if*

- *either the corresponding families $(\mathcal{S}_{C_i})$ have an element in common,*
- *or $n = 5$ and all C_i are contained in a Veronese surface $\mathbb{P}^2 \subset \mathbb{P}^5$.*

The idea of the proof is based on Theorem 2.2 about the conormal Lie algebra. Suppose Π_1 and Π_2 are two compatible FO Poisson structures of type $q_{n,1}$. Let us look at the point, where Π_1 vanishes. Then by Theorem 2.2 we know its linear at this point. The compatibility condition implies strong restrictions on Π_2 at this point. In particular, it must be of rank 2 there. Analysis of the geometry of symplectic leaves of $q_{n,1}$ gives the result.

This idea may be realized for higher ranks as well. So far we only managed to do it of the simplest case of $q_{5,2}$ in [MP24]. In the process, we discovered a lot of interesting geometry associated with this Poisson structure, such as an explicit formula for $q_{5,2}$, see Theorem 2.4 below. In the process, we discovered a lot of interesting geometry associated with this structure

Let $\mathcal{C}_{n,r}$ be a 5-dimensional vector space. Consider the Plucker embedding

$$G(2,V) \to \mathbb{P}(\bigwedge^2 V).$$

It is well known that for a generic 5-dimensional subspace $W \subset \bigwedge^2 V$ the corresponding linear section

$$C_W := G(2,V) \cap \mathbb{P}W$$

is an elliptic curve. Furthermore, if $\mathcal{U} \subset V \otimes \mathcal{O}$ is the unversal subbundle on $G(2,V)$, then one can check that the restriction

$$V_W := \mathcal{U}^\vee|_{C_W}$$

is a simple vector bundle of rank 2 and degree 5 on C_W. Thus, we have the corresponding Feigin-Odesskii Poisson structure of type $q_{5,2}$ on $\mathbb{P}H^0(C_W, V_W)^*$.

Furthermore, one can check that the restriction map

$$V^* = H^0(G(2,V), \mathcal{U}^\vee) \to H^0(C_W, V_W)$$

is an isomorphism. Thus, we get a Poisson structure Π_W on $\mathbb{P}V$ (defined up to a rescaling).

On the other hand, we have a natural $\mathrm{GL}(V)$-invariant map

$$\pi_{5,2} : \bigwedge^5 (\bigwedge^2 V) \to H^0(\mathbb{P}V, \bigwedge^2 T) \otimes \det^2(V)$$

constructed as follows.

Note that we have a natural isomorphism $V \simeq H^0(\mathbb{P}V, T(-1))$, hence we get a natural map $V \otimes \mathcal{O}(1) \to T$, and hence, the composed map

$$\phi : W \otimes \mathcal{O}(2) \to \bigwedge^2 V \otimes \mathcal{O}(2) \to \bigwedge^2 T$$

on $\mathbb{P}V$. Taking the 5th exterior power of this map we get a map

$$\bigwedge^5(\phi) : \det(W) \otimes \mathcal{O}(10) \to \bigwedge^5 (\bigwedge^2 T) \simeq (\bigwedge^2 T)^\vee \otimes \det^3(T),$$

where we used the identification $\det(\bigwedge^2 T) \simeq \det^3(T)$. Note that we have a nondegenerate pairing given by the exterior product,

$$\bigwedge^2 T \otimes \bigwedge^2 T \to \det(T),$$

hence, we have an isomorphism $\bigwedge^2 T \simeq (\bigwedge^2 T)^\vee \otimes \det(T)$, and we can rewrite the above map as

$$\det(W) \to \bigwedge^2 T \otimes \det^2(T)(-10) \simeq \bigwedge^2 T \otimes \det^2(V).$$

Theorem 2.4. *For every* 5*-dimensional subspace* $W \subset \bigwedge^2 V$*, such that* $C_W := G(2,V) \cap \mathbb{P}W$ *is an elliptic curve, one has an equality*

$$\pi_{5,2}(\lambda_W) = \Pi_W \otimes \delta,$$

for some trivializations $\lambda_W \in \bigwedge^5 W$ *and* $\delta \in \det^2(V)$.

Theorem 2.5. *(1) For* 5*-dimensional subspaces* $W, W' \subset \bigwedge^2 V$ *such that* C_W *and* $C_{W'}$ *are elliptic curves, the Poisson brackets* Π_W *and* $\Pi_{W'}$ *are compatible if and only if* $\dim W \cap W' \geq 4$.
(2)For any collection (W_i) *of* 5*-dimensional subspaces in* $\bigwedge^2 V$*, the brackets* (Π_{W_i}) *are pairwise compatible if and only if either there exists a* 6*-dimensional subspace* $U \subset \bigwedge^2 V$ *such that each* W_i *is contained in* U*, or there exists a* 4*-dimensional subspace* $K \subset \bigwedge^2 V$ *such that each* W_i *contains* K*.*

The proof is analogous to the one for $q_{n,1}$, but the geometry is more complicated: conditions of compatibility are translated to some constrains on the Grassmannian, rather than the projective space.

3. Factorization homology

The definition of factorization homology goes back to the late 70s when loop spaces were extensively studied by algebraic topologists. But only relatively recently they were applied to problems initiated by mathematical physicists such as Chern–Simons theory, see e. g. [CG]. I discovered in [Mar17], that such old invariants as perturbative Chern–Simons invariants may be naturally defined in terms of factorization homologies. In [Mar16] inspired by [AM00] I apply this definition to calculate the Kontsevich integral of unknot. Surprisingly, methods from my old paper [Mar09] were very helpful in this research. The approach developed in these two papers seems to be very prominent: practically any result about invariants of knots and manifolds, which are defined in terms of chord diagrams and similar objects, may be formulated and interpreted in terms of factorization homologies. A big opened question here is how (and is it possible?) to make it with quantum Chern–Simons, Donaldson and other "non-perturbative" invariants of manifolds. In [Mar21] these methods surprisingly leads to a construction of a rational Kontsevich formality isomorphism.

The main character of this story is n-Weyl algebra (or Weyl e_n-algebra that is algebras over the operad of chains of the little n-discs operad), which is a generalization of usual Weyl algebras, that is algebras of differential operators on a vector space. They are, in the same way as Weyl algebras are, in a sense, the simplest deformations of the polynomial algebras.

We are interested not in n-Weyl algebras themselves, but in factorization homologies of them. There are many ways to define them, see [Lur; Gin]. We use the most explicit way via Fulton–MacPherson compactification. Being the simplest deformations of polynomial algebras, factorization homologies of n-Weyl algebras are not very interesting. But the situation becomes more interesting if we take into account the adjoint action of a Weyl e_n-algebra, being considered as a Lie algebra, on itself. In other words, if we consider inner automorphisms (or deformations) of these e_n-algebras. The Lie algebra of inner automorphism is essentially the Lie algebra of Hamiltonian vector fields of a shifted symplectic structure. Its cohomology is calculated by the graph complex. Thus, n-Weyl algebras explain the appearance of the graph complex in the formulas for perturbative Chern–Simons invariants from [AS92; BC98] and others. It was one of the initial motivations to introduce and study n-Weyl algebras.

Nowadays, there are a lot of places where graph complex appears e. g. [CGP21; Wil10] and others. An ambitious plan for the future is to interpret them in terms of the factorization complex of Weyl n-algebras.

An important property of Weyl n-algebras is that their factorization homology on a closed manifold is one-dimensional (Theorem 3.1 below). It would be plausible to find some conceptual proof of this statement, perhaps by using some kind of Morita invariance of factorization homology. As far as I know, such arguments are unknown even in the classical situation, when $n = 1$. Besides, as I learned from O. Gwilliam, in [Gwi12] it is shown, that the factorization algebra of any "free" BV theory has one-dimensional factorization homology over a closed manifold, which implies the result for the Weyl case.

In [Mar17] I built perturbative Chern–Simons invariants by means of the factorization complex of Weyl n-algebras. In [Mar16] I continue this line and introduce the Wilson loop invariant. This invariant is supposed to be equal to the Bott–Taubes invariant and the Kontsevich integral. In fact, we are only interested in one question here: calculating the Wilson loop invariant of unknot in S^3. This problem appears to be connected with the Duflo isomorphism.

I consider the Duflo isomorphism for Lie algebras with a scalar product, which is much simpler to prove than the general statement from [Duf77]. There are (at least) two proofs of the Duflo isomorphism for a Lie algebra with a scalar product. In [AM00] the authors use a quantization of the Weil algebra. In [BLT03] the Kontsevich integral of knots and link is used. Our sketch of a proof is related to the both. The work [Kri11] also connects these two approaches and it would be very interesting to compare it with our arguments.

My next paper on this subject [Mar21] continues studies of Weyl n-algebras began above. We describe how these ideas can be applied to prove formality theorems, which are isomorphisms between higher Hochschild cohomology of polynomial algebras and Weyl n-algebras. The substantial part of this paper is rephrasing and generalization of the pioneer paper [Kon03], where the formality for usual Hochschild cohomology was firstly proved, in terms of Weyl n-algebras.

The construction from [Kon03] depends on choice of a propagator. There is another approach to formality via the factorization homology of Weyl n-algebras, which was implicitly stated and used in [Mar16]. We show, that for the usual Hochschild cohomology this formality is equivalent to the one introduced in [Kon03] but with a different propagator. Due to the geometric nature of this approach, all coefficients of this morphism are rational. It leads us to a surprising conjecture that a family of propagators we define gives formalities with rational coefficients.

Two approaches to the formality described in the present paper resemble two approaches to the Kontsevich integral of a knot. The first one using iterated integrals (see e. g. [CDM12, Part 3]) is similar to the approach via propagator. The second partly conjectural approach corresponds to the one via the factorization complex we discussed above.

Below are some details.

3.1. **Factorization complex.**

3.1.1. *Fulton-MacPherson operad.* Let $\mathbb{R}^n$ be an affine space. For a finite set S let denote by $(\mathbb{R}^n)^S$ the set of ordered S-tuples in $\mathbb{R}^n$. Let $\mathscr{C}^0(\mathbb{R}^n)(S) \subset (\mathbb{R}^n)^S$ be the configuration space of distinct ordered points in $\mathbb{R}^n$ labeled by S. In [GJ] (see also [Sal01] and [AS92]) the Fulton–MacPherson compactification $\mathscr{C}(\mathbb{R}^n)(S)$ of $\mathscr{C}^0(\mathbb{R}^n)(S)$ is introduced. This is a manifold with corners and a boundary with interior $\imath\colon \mathscr{C}^0(\mathbb{R}^n)(S) \hookrightarrow \mathscr{C}(\mathbb{R}^n)(S)$. There is a projection $\pi\colon \mathscr{C}(\mathbb{R}^n)(S) \to (\mathbb{R}^n)^S$ such that $\pi \circ \imath\colon \mathscr{C}^0(\mathbb{R}^n)(S) \to (\mathbb{R}^n)^S$ is the natural embedding.

For any $S' \subset S$ there is the projection map

$$\mathscr{C}(\mathbb{R}^n)(S) \to \mathscr{C}(\mathbb{R}^n)(S'), \tag{18}$$

compatible with the same maps $\mathscr{C}^0(\mathbb{R}^n)(S) \to \mathscr{C}^0(\mathbb{R}^n)(S')$ and $(\mathbb{R}^n)^S \to (\mathbb{R}^n)^{S'}$.

The natural action of the group of affine transformations on $\mathscr{C}^0(\mathbb{R}^n)(S)$ is lifted to $\mathscr{C}(\mathbb{R}^n)(S)$. Denote by $\mathrm{Dil}(n)$ its subgroup consisting of dilatations and shifts. Group $\mathrm{Dil}(n)$ acts freely on $\mathscr{C}(\mathbb{R}^n)(S)$ and the quotient is isomorphic to the fiber $\pi^{-1}(\vec{0})$, where $\vec{0} \in (\mathbb{R}^n)^S$ is the S-tuple sitting at the origin. To build this isomorphism consider dilatations with positive coefficients with the center at the origin:

$\mathbb{R}_{>0} \times \mathscr{C}^0(\mathbb{R}^n)(S) \to \mathscr{C}^0(\mathbb{R}^n)(S)$. By the construction of the compactification their action is lifted to $r\colon \mathbb{R}_{\geq 0}\times\mathscr{C}(\mathbb{R}^n)(S) \to \mathscr{C}(\mathbb{R}^n)(S)$, which is a fiber bundle. The map $r(0\times -)$ factors through the quotient by $\mathrm{Dil}(n)$ and its image lies in $\pi^{-1}(\vec{0})$. This gives the required isomorphism. It follows that $\pi^{-1}(\vec{0})$ is a retract of $\mathscr{C}(\mathbb{R}^n)(S)$.

As it is just mentioned, manifolds with corners $\mathscr{C}(\mathbb{R}^n)(S)/\mathrm{Dil}(n)$ and $\pi^{-1}(\vec{0})$ are isomorphic. Denote any of these manifolds by $\mathbf{FM}_n^S$. The sequence of manifolds $\mathbf{FM}_n^S$ is a contravariant functor from $\mathbf{Set}_{\hookrightarrow}$ to topological spaces: the map corresponding to an embedding of sets forgets points that are not in its image. The sequence $\mathbf{FM}_n^S$ may be equipped with a structure of a unital operad in the category of topological spaces. This operad is a free as an operad of sets and as such is generated by quotients of $\mathscr{C}^0(\mathbb{R}^n)(S) \hookrightarrow \mathscr{C}(\mathbb{R}^n)(S)$ by $\mathrm{Dil}(n)$. The action of k-ary operations $\mathscr{C}^0(\mathbb{R}^n)(\underline{\mathbf{k}})/\mathrm{Dil}(n)$ on $\mathscr{C}(\mathbb{R}^n)(S)$ looks as follows. Consider the submanifold of $\mathscr{C}(\mathbb{R}^n)(S)$ for which the image of $\pi\colon \mathscr{C}(\mathbb{R}^n)(S) \to (\mathbb{R}^n)^S$ consists exactly of k different points. This submanifold is isomorphic to $\mathscr{C}^0(\mathbb{R}^n)(\underline{\mathbf{k}})\times\pi^{-1}(\vec{0})$ because fibers of π over any point are isomorphic due to parallel translations. The embedding of this submanifold to $\mathscr{C}(\mathbb{R}^n)(S)$ in composition with the quotient by $\mathrm{Dil}(n)$ gives a map

$$\mathscr{C}(\mathbb{R}^n)(\underline{\mathbf{k}})/\mathrm{Dil}(n) \times (\mathbf{FM}_n)^{\times k} \to \mathscr{C}(\mathbb{R}^n)(\bullet)/\mathrm{Dil}(n) = \mathbf{FM}_n,$$

which is the desired action, where $\underline{\mathbf{k}}$ is the set of k elements.

Definition 3.1. The sequence of topological spaces $\mathbf{FM}_n^S$ with the unital operad structure as above is called the *Fulton–MacPherson operad*.

3.1.2. *Chains of Fulton-MacPherson operad.* Given a topological operad, one may produce a dg-operad by taking complexes of chains of its components.

Definition 3.2. Denote by $\mathfrak{fm}_n$ the operad of $\mathbb{R}$-chains of $\mathbf{FM}_n$.

Real numbers appear here are to simplify things, in fact all object and morphism we shall use may be defined over rationals, see remark before Example 3.2 below.

By chains we mean the complex of de Rham currents, that is why we need real chains. Alternatively, one may think about the cooperad of de Rham cochains of $\mathbf{FM}_n$.

Proposition 3.1. *Operad $\mathfrak{fm}_n$ is weakly homotopy equivalent to e_n, the operad of chains of the little discs operad.*

Spaces $\mathbf{FM}_n^S$ are acted on by the general linear group, and, in particular, by its maximal compact subgroup $SO(n)$, we suppose that a scalar product on the space is chosen. The semidirect product $\mathbf{FM}_n \rtimes SO(n)$ is called the operad of framed disks $f\mathbf{FM}_n$. Any operad is equipped with a natural structure of an operad colored over the classifying space of its invertible 1-ary elements. In this way, we will consider $f\mathbf{FM}_n$ as an operad colored by the classifying space $BSO(n)$.

Definition 3.3. Denote by $f\mathfrak{fm}_n$ the operad of $\mathbb{R}$-chains of $f\mathbf{FM}_n$.

The closely connected, but not identical object is the operad of framed disks from [Get94]. And much like with Definition 3.2, real numbers may be replaced with rational for our purposes.

Operations of arity s of $f\mathfrak{fm}_n$ form complexes over $BSO(n)^{s+1}$. An algebra over $f\mathfrak{fm}_n$ is given by a family of complexes over appropriate powers of $BSO(n)$. Below we will need only the following restrictive, but a simpler class of such algebras.

Definition 3.4. We say that a dg-algebra A over $\mathfrak{fm}_n$ is *invariant*, if all structure maps of complexes

$$\mathfrak{fm}_n \otimes A \otimes \cdots \otimes A \to A$$

are invariant under the action of group $SO(n)$ on complexes of operations of $\mathfrak{fm}_n$. An invariant algebra over $\mathfrak{fm}_n$ is naturally an algebra over $f\mathfrak{fm}_n$.

Note, that we mean invariance on the level of complexes, not up to homotopy. An important class (and the only class we need, in fact) of invariant e_n-algebras is universal enveloping e_n-algebras, see the end of the next Subsection.

3.1.3. L_∞ *operad.* A *tree* is an oriented connected graph with three type of vertices: the *root* has one incoming edge and no outgoing ones, *leaves* have one outgoing edge and no incoming ones and *internal vertexes* have one outgoing edge and more than one incoming ones. Edges incident to leaves will be called *inputs*, the edge incident to the root will be called the *output* and all other edges will be called *internal edges*. The degenerate tree has one edge and no internal vertexes. Denote by $T_k(S)$ the set of non-degenerate trees with k internal edges and leaves labeled by a finite set S.

For two trees $t_1 \in T_{k_1}(S_1)$ and $t_2 \in T_{k_2}(S_2)$ and an element $s \in S_1$ the composition of trees $t_1 \circ_s t_2 \in T_{k_1+k_2+1}$ is obtained by identification of the input of t_1 corresponding to s and the output of t_2. Composition of trees is associative and the degenerate tree is the unit. The set of trees with respect to the composition forms an operad.

We call a tree with only one internal vertex the *star*. Any non-degenerate tree with k internal edges may be uniquely presented as a composition of $k+1$ stars.

The operation of *edge splitting* is the following: take a non-degenerate tree, present it as a composition of stars and replace one star with a tree that is a product of two stars and has the same set of inputs. The operation of an edge splitting depends on an internal vertex and a proper subset of incoming edges with more than one element.

For a non-degenerate tree t denote by $\mathrm{Det}(t)$ the one-dimensional $\mathbb{Q}$-vector space that is the determinant of the vector space generated by internal edges. For $s > 1$ consider the complex

$$L(s)\colon \bigoplus_{t \in T_0(\underline{\mathbf{s}})} \mathrm{Det}(t) \to \bigoplus_{t \in T_1(\underline{\mathbf{s}})} \mathrm{Det}(t) \to \bigoplus_{t \in T_2(\underline{\mathbf{s}})} \mathrm{Det}(t) \to \cdots, \tag{19}$$

where $\underline{\mathbf{s}}$ is the set of s elements, the cohomological degree of a tree $t \in T_k(\underline{\mathbf{s}})$ is $2 - s + k$ and the differential is given by all possible splittings of an edge (see e. g. [SGA]). The composition of trees equips the sequence $L(i) \otimes \mathrm{sgn}$ with the structure of a non-unital *dg*-operad, here sgn is the sign representation of the symmetric group.

This operad is called the L_∞ *operad.* For simplicity denote by the same symbol the operad $L_\infty \otimes_{\mathbb{Q}} \mathbb{R}$, it will be clear from the context which one is meant. Denote by $L_\infty[n]$ the *dg*-operad given by the complex $L(s)[n(s-1)] \otimes (\mathrm{sgn})^n$ and refer to it as n-shifted L_∞ operad.

As $\mathbf{FM}_n$ is freely generated by $\mathscr{C}^0(\mathbb{R}^n)(S)/\mathrm{Dil}(n)$ as the operad of sets, there is a map μ from it to the free operad with one generator in each arity, which sends generators to generators. Elements of the latter operad are enumerated by rooted trees. The map above sends $\mathscr{C}^0_{\underline{\mathbf{k}}}(\mathbb{R}^n)/\mathrm{Dil}(n)$ to the star tree with k leaves. For a tree $t \in T(S)$ denote by $[\mu^{-1}(t)] \in C_*(F_n(S))$ the chain presented by its preimage under μ.

Proposition 3.2. *Map $[\mu^{-1}(\cdot)]$ as above gives a morphism from shifted L_∞ operad $L(s)[s(1-n)]$ to the dg-operad $\mathfrak{fm}_n$ of chains of the Fulton–MacPherson operad. The last operad here is treated as a non-unital one.*

Proof. To see that the map commutes with the differential, note, that two strata given by μ with dimensions differing by 1 are incident if and only if one of the corresponding trees is obtained from another by edge splitting. In this way, we get a basis in the conormal bundle to a stratum labeled by the internal edges, it follows that orientations on the chains of the boundary of a stratum match signs in the complex (19). □

It follows that there is a morphism of *dg*-operads

$$L_\infty[1-n] \to \mathfrak{fm}_n \tag{20}$$

Definition 3.5. For a $\mathfrak{fm}_n$-algebra A call its pull-back under (20) the *associated L_∞-algebra* and denote it by $L(A)$.

Since the operad $\mathfrak{fm}_n$ is weakly homotopy equivalent to e_n (Proposition 3.1), it gives a homotopy morphism of operads $L_\infty[1-n] \to e_n$.

This morphism of operads produces a functor from the category of e_n-algebras to that of L_∞-algebras. This functor has a left adjoint, which is called the universal enveloping e_n-algebra. The important example of the latter is the complex of rational chains of an iterated loop space $\Omega^n X$, which is a universal enveloping e_n-algebra of the homotopy groups Lie algebra $\pi_{*-1}(X)$, for more details see e. g. [Fra, Section 5]. Note, that $\Omega^n X$ is equipped with a natural $SO(n)$ action. This is in good agreement with the fact that any universal enveloping e_n-algebra is invariant.

3.1.4. *Factorization complex.* Let M be a n-dimensional oriented topological manifold. In the same way, as for $\mathbb{R}^n$ there is the Fulton–MacPherson compactification $\mathscr{C}(M)(S)$ of the space $\mathscr{C}^0(M)(S)$ of ordered pairwise distinct points in M labeled by S. Locally it is the same thing. Inclusion $\mathscr{C}^0(M)(S) \hookrightarrow \mathscr{C}(M)(S)$ is a homotopy equivalence, there is a projection $\mathscr{C}(M)(S) \xrightarrow{\pi} M^S$.

Recall that a point in the Fulton–MacPherson compactification $\mathscr{C}(\mathbb{R}^n)(S)$ of the configuration space of $\mathbb{R}^n$ looks like a configuration from the configuration space $\mathscr{C}^0(\mathbb{R}^n)(S')$ with elements of $\mathbf{FM}_n$ sitting at each point of the configuration. It follows that spaces $\mathscr{C}(\mathbb{R}^n)(\bullet)$ form a right $\mathbf{FM}_n$-module, which is freely generated by $\mathscr{C}^0(\mathbb{R}^n)(\bullet)$ as a set. The same is nearly true for the Fulton–MacPherson compactification of any oriented manifold M. But to define such an action one needs to choose coordinates at the tangent space of any point of a configuration of $\mathscr{C}(M)(S)$. To fix it one has to consider either only framed manifolds or introduce framed configuration space.

Definition 3.6. The framed Fulton–MacPherson compactification $f\mathscr{C}(M)(S)$ is the principal $SO(n)^S$ bundle over $\mathscr{C}(M)(S)$, which is the pullback of product of principal bundles associated with the tangent bundles to each point under the projection map $\pi\colon \mathscr{C}(M)(S) \to M^S$.

The chain complex $C_*(f\mathscr{C}(M)(S))$ over $BSO(n)^S$ for various S make up a right $f\mathfrak{fm}_n$-module (see Definition 3.3).

Definition 3.7. For an algebra A over $f\mathfrak{fm}_n$ and an oriented manifold M the *factorization complex* $\int_M A$ is the tensor product of the left $f\mathfrak{fm}_n$-module $A^{\otimes}$ and the right $f\mathfrak{fm}_n$-module $C_*(f\mathscr{C}(M)(S))$.

The homology of $\int_M A$ is called the *factorization homology* of A on M.

For an invariant $\mathfrak{fm}_n$-algebra (Definition 3.4) the definition of the factorization complex may be rephrased as follows.

Proposition 3.3. *For an invariant unital $\mathfrak{fm}_n$-algebra A and an oriented manifold M the factorization complex $\int_M A$ is the complex given by the colimit of the diagram*

$$\begin{array}{c} \bigoplus_{S'} C_*(\mathscr{C}(M)(S')) \underset{Aut(S')}{\otimes} A^{\otimes S'} \\ \uparrow \\ \bigoplus_{i\colon S'\to S} C_*(f\mathscr{C}^0(M)(S)) \underset{SO(n)^S \rtimes Aut(S)}{\otimes} \bigotimes_{s\in S} (\mathfrak{fm}_n(i^{-1}s) \underset{Aut(i^{-1}s)}{\otimes} A^{\otimes(i^{-1}s)}) \\ \downarrow \\ \bigoplus_{S} C_*(\mathscr{C}^0(M)(S)) \underset{Aut(S)}{\otimes} A^{\otimes S} \end{array} \tag{21}$$

where the summation in the middle runs over maps between finite sets, the downwards arrow is given by the left action of $\mathfrak{fm}_n$ on A for $\operatorname{Im} i$ and the unit for $S\setminus \operatorname{Im} i$ and the upwards arrow is given by the right action of $\mathfrak{fm}_n$ on $C_*(f\mathscr{C}(M)(\bullet))$.

The formula (21) is a direct interpretation of Definition 3.7.

Note that relations (21) include in particular colimits

$$\begin{array}{c} \bigoplus_{S'} C_*(\mathscr{C}(M)(S')) \underset{Aut(S')}{\otimes} A^{\otimes S'} \\ \uparrow \\ \bigoplus_{i\colon S'\hookrightarrow S} C_*(\mathscr{C}(M)(S)) \underset{Aut(S')}{\otimes} A^{\otimes S'} \\ \Big\downarrow{\scriptstyle \otimes 1^{(S\setminus S')}} \\ \bigoplus_{S} C_*(\mathscr{C}(M)(S)) \underset{Aut(S)}{\otimes} A^{\otimes S} \end{array} \tag{22}$$

where the upward arrow is the projection, which forgets points labeled by $S \setminus S'$.

If the manifold is framed, that is its tangent bundle is trivialized, the definition may be simplified: one should substitute $\mathscr{C}^0(M)(S)$ instead of $f\mathscr{C}^0(M)(S)$ and remove $SO(n)$ from the tensor product.

Since the upwards arrow in (21) is an isomorphism of underlying vector spaces, for any class of the colimit above there is a unique chain downstairs, which is in the interior of the Fulton–MacPherson compactification, that is in a configuration space of distinct points. Thus on the complex (21) (that calculates the factorization homology) there is an increasing filtration by the number of points of the configuration space and the associated graded object is $\bigoplus_S C_*(\mathscr{C}^0(M)(S)) \otimes A^{\otimes S}$.

Note, that this filtration splits as a filtration of vector spaces. Thus, any morphism from or to the factorization complex may be presented as the one for all graded pieces of the filtration consistent in a proper way.

The definition above may be again rephrased as follows. Denote by $Ran(M)$ the *Ran space* of M, that is the set of finite subsets of M with the natural topology. There is a natural map $M^{\times i} \to Ran(M)$, which sends a set of points to its support. Denote the composite map $\mathscr{C}(M)(\underline{\mathbf{i}}) \to M^{\times i} \to Ran(M)$ by ϖ_i. The fiber of this map is the product of some copies of the Fulton–MacPherson operad. Take a $\mathfrak{fm}_n$-algebra A and consider chains $\bigoplus_i C_*(\mathscr{C}(M)(\underline{\mathbf{i}})\otimes_{\Sigma_i} A^{\otimes i})$ modulo relations (21). As all relations respect ϖ_*, for any open subset of the complex of these chains modulo relations is defined; being restricted $\mathscr{C}^0(M)(\underline{\mathbf{i}}) \hookrightarrow Ran(M)$ this complex equals to $C_*(\mathscr{C}^0(M)(\underline{\mathbf{i}})) \otimes_{\Sigma_i} \otimes A^{\otimes i}$. The way these complexes are glued together defines a cosheaf (see e. g. [Cur]) on the Ran space. The factorization homology is homology of this cosheaf, for details see [Lur].

Example 3.1. Any commutative algebra canonically is an algebra over chains of any topological operad, because it is the operad of chains of the terminal object in the category of topological operad. In particular, any commutative algebra is an $\mathfrak{fm}_n$-algebra over and it is invariant.

Let A be the polynomial algebra $k[V]$ generated by a $\mathbb{Z}$-graded vector space V over the base field k of characteristic zero containing $\mathbb{R}$. Its factorization complex $\int_M A$ is a commutative algebra because any commutative algebra is a commutative algebra in the category of commutative algebras. The multiplication in $\int_M A$ is given by taking unions of points in M and multiplication of labels for coinciding points.

One may see that $\int_M A = k[H_*(M) \otimes V]$, where $H_*(M)$ is the integer homology groups of M negatively graded.

To show it, choose a homogeneous basis of V enumerated by a set B. The action of $\mathfrak{fm}_n$ on a commutative algebra factorizes through the augmentation map $\mathfrak{fm}_n(\bullet) \to k$. It means, that the complex $\oplus A^{\otimes i} \otimes C_*(\mathscr{C}(M)(\underline{\mathbf{i}}))$ modulo relations (21) equals to $\oplus A^{\otimes i} \otimes \overline{C}_*(\mathscr{C}(M)(\underline{\mathbf{i}}))/\sim$, where $\sim$ are relations given by the unit and $\overline{C}_*(\mathscr{C}(M)(\underline{\mathbf{i}}))$ is the chain complex of the Fulton–MacPherson compactification with all border components shrunk to points. The latter space is simply the power $M^{\times i}$. Thus taking into account relations $\sim$ we see that $\int_M A$ is the homology of space of finite subsets of M labeled by B, that is the direct sum of homology of $M^{\times i_1} \times \cdots \times M^{\times i_{|B|}}$ modulo the action of product of symmetric groups $\Sigma_{i_1} \times \cdots \times \Sigma_{i_{|B|}}$, which is given by permutations for components that corresponds to elements of the basis of even degree and by permutation multiplied by the sign representation for odd degrees. The multiplication on this space is obviously defined.

Proposition 3.4 ([GTZ14, Section 5]).

(1) *The factorization complex $\int_{M^k} A$ of an invariant $\mathfrak{fm}_n$-algebra on a closed compact oriented k-manifold M^k is naturally equipped with a structure of $\mathfrak{fm}_{n-k}$-algebra.*

(2) *For a fiber bundle $E^n \xrightarrow{F^k} B^{n-k}$ with closed compact oriented base and fiber and an invariant $\mathfrak{fm}_n$-algebra A*
$$\int_{B^{n-k}} (\int_{F^k} A) = \int_{E^n} A,$$
where $\int_{F^k} A$ is a $\mathfrak{fm}_{n-k}$-algebra by the previous item.

This theorem may be formulated for maps more general than projections of fiber bundles. To define push-forward in a more general situation one needs to introduce factorization sheaves, see [AFT; Gin] for details. The construction from Subsection 3.4.3 below is an example of such a push-forward.

3.1.5. *Factorization homology and Lie algebra homology.* Following the definition of a tree from the beginning of Subsection 3.1.3, we say that a *bush* is an oriented connected graph with three type of vertices: *root* has no outgoing ones, *leaves* have one outgoing edge and no incoming ones and *internal vertexes* have one outgoing edge and more than one incoming ones. That is the only difference is that the root may have many incoming edges. The composition of bushes is not defined, but one may compose a tree and a bush by identification of an input of the bush and the output of the tree. Thus, bushes form a right module over the operad of trees. Denote by $B_k(S)$ the set of bushes with k edges not incident to leaves and leaves labeled by a set S.

Continuing on the same lines, define the operation of *edge splitting* in the same way as for trees: we choose a vertex and a subset of incoming edges with more than one element, then we cut off trees that grow from the chosen edges, then glue an incoming edge to the vertex we choose and then glue trees we cut to the input of the glued edge. Note that an edge splitting for a bush may be done not only for an internal vertex, but for a root as well. But for an internal edge, the subset of edges must be proper and for the root it may be the whole set.

For a bush b denote by $\mathrm{Det}(b)$ the one-dimensional $\mathbb{Q}$-vector space that is the determinant of the vector space generated by internal edges. For $s > 0$ consider the complex

$$B(s)\colon \bigoplus_{b\in B_0(\underline{\mathrm{s}})} \mathrm{Det}(b) \to \bigoplus_{b\in B_1(\underline{\mathrm{s}})} \mathrm{Det}(b) \to \bigoplus_{b\in B_2(\underline{\mathrm{s}})} \mathrm{Det}(b) \to \cdots, \tag{23}$$

where $\underline{\mathrm{s}}$ is the set of s elements, the cohomological degree of a bush $B \in B_k(\underline{\mathrm{s}})$ is $k-s$ and the differential is given by all possible splitting of an edge. The composition of a tree and a bush is compatible with differentials on complexes (19) and (23) and thus equips the complex with a structure of right L_∞-module.

Given a L_∞-algebra $\mathfrak{g}$ its homology (with trivial coefficients) may be calculated by means of the homological Chevalley–Eilenberg complex. Its n-th term is the symmetric power $S^n(\mathfrak{g}[1])$ and the differential is the coderivation defined by the operations $l_i\colon S^i(\mathfrak{g}[1]) \to \mathfrak{g}[1]$ corresponding to star trees (for the definition of the latter see Subsection 3.1.3).

This definition may be nicely formulated in terms of modules over operads as follows.

Proposition 3.5. *For a L_∞-algebra $\mathfrak{g}$ the product $\mathfrak{g}^{\otimes} \otimes_{L_\infty} B(\bullet)$ is isomorphic to the Chevalley–Eilenberg complex calculating homology of $\mathfrak{g}$ with trivial coefficients modulo the zero-degree component.*

The proof is straightforward. For a more conceptual treatment see [Bal98].

The homology of a L_∞-algebra with coefficients in the adjoint module is calculated by the complex with n-th term $S^n(\mathfrak{g}[1]) \otimes \mathfrak{g}$. The differential is a sum of the Chevalley–Eilenberg differential and the coderivation $d_{ad}\colon \mathfrak{g} \otimes S^n(\mathfrak{g}[1]) \to \bigoplus_i S^i(\mathfrak{g}[1])$ given by the adjoint action. A light modification of the foregoing allows us to define it in terms of modules over operads.

A *marked bush* is a bush with one of the edges incoming to root marked. Denote by $B'_k(S)$ the set of marked bushes with k non-marked edges not incidental to leaves and leaves labeled by a set S. The edge splitting for marked bushes is defined in the same way, if the root vertex is chosen then the inserted edge is marked if the chosen subset of edges contains the marked edge and is not marked otherwise.

As before, for a bush b denote by $\mathrm{Det}(b)$ the one-dimensional $\mathbb{Q}$-vector space that is the determinant of the vector space generated by not marked edges. For $s > 0$ consider the complex

$$B'(s)\colon \bigoplus_{b\in B'_0(\underline{\mathbf{s}})} \mathrm{Det}(b) \to \bigoplus_{b\in B'_1(\underline{\mathbf{s}})} \mathrm{Det}(b) \to \bigoplus_{b\in B'_2(\underline{\mathbf{s}})} \mathrm{Det}(b) \to \cdots, \tag{24}$$

where $\underline{\mathbf{s}}$ is the set of s elements, the cohomological degree of a bush $B \in B_k(\underline{\mathbf{s}})$ is $k-s$ and the differential is given by all possible splitting of an edge. The composition of a tree and a bush again equips the complex with a structure of right L_∞-module.

On the analogy of Proposition 3.5 we have the following.

Proposition 3.6. *For a L_∞-algebra $\mathfrak{g}$ the product $\mathfrak{g}^\otimes \otimes_{L_\infty} B'(\bullet)$ is isomorphic to the Chevalley–Eilenberg complex calculating homology of $\mathfrak{g}$ in the adjoint module.*

The proof is straightforward. For a more conceptual treatment see [Bal98].

In Subsection 3.1.3 we have defined a morphism from operad L_∞ to $\mathfrak{fm}_n$. Applying this morphism to the right $\mathfrak{fm}_n$-module $C_*(f\mathscr{C}(M)(S))$ introduced in Subsection 3.1.4 we get the right action of L_∞ on $C_*(f\mathscr{C}(M)(S))$. A morphism from the right L_∞-module given by complexes (23) and (24) generated by bushes to this right L_∞-module produces morphisms from Chevalley–Eilenberg complexes to the factorization complex. It may be formulated as follows.

Proposition 3.7. *Let A be an invariant $\mathfrak{fm}_n$-algebra. Let $C_{Ch} = (S^*(L(A)[1]), d_{Ch})$ $C^{ad}_{Ch} = (S^*(L(A)[1]) \otimes L(A), d_{Ch} + d_{ad})$ be the Chevalley–Eilenberg complexes calculating the homology of L_∞-algebra $L(A)$ with trivial coefficients and in the adjoint module correspondingly. Let M be a closed manifold and $p \in M$ is a point. Then morphisms*

$$\begin{aligned} a_1 \otimes \cdots \otimes a_i &\mapsto [\mathscr{C}^0(M)(\underline{\mathbf{i}})] \otimes_{\Sigma_i} (a_1 \otimes \cdots \otimes a_i) \\ a_1 \otimes \cdots \otimes a_i \otimes a_0 &\mapsto [\mathscr{C}^0(M \setminus p)(\underline{\mathbf{i}})] \otimes_{\Sigma_i} (a_1 \otimes \cdots \otimes a_i) \otimes a_0 \end{aligned}$$

define maps from complexes $C_{Ch}(L(A))$ and $C^{ad}_{Ch}(L(A))$ respectively to the factorization complex $\int_M A$, where $[\mathscr{C}^0(M)(S)]$, $[\mathscr{C}^0(M \setminus p)(S)]$ and $[p]$ are cycles in $C_*(\mathscr{C}(M)(S))$ presented by the configuration space of distinct points, distinct points different from p and the point p.

3.2. **Weyl n-algebras.**

3.2.1. *Definition.* The usual Weyl algebra is a deformation of the polynomial algebra. We have seen that a commutative algebra is an algebra over operad $\mathfrak{fm}_n$ for any n. The analogous deformation of a commutative algebra in the category of $\mathfrak{fm}_n$-algebras gives us what we call the Weyl n-algebra.

Let V be a $\mathbb{Z}$-graded finite-dimensional vector space over the base field k of characteristic zero containing $\mathbb{R}$ equipped with a non-degenerate skew-symmetric pairing $\omega\colon V \otimes V \to k$ of degree $1-n$. Let $k[V]$ be the polynomial algebra generated by V and $k[[h]]$ be the ring of formal series and $k[V][[h]]$ is the polynomial algebra over it. Denote by

$$\partial_\omega\colon k[V] \otimes k[V] \to k[V] \otimes k[V] \tag{25}$$

the differential operator that is a derivation in each factor and acts on generators as ω.

Consider $\mathbf{FM}_n(\underline{\mathbf{2}})$, the space of 2-ary operations of the Fulton–MacPherson operad. This is homeomorphic to the $(n-1)$-dimensional sphere. Denote by v the standard $SO(n)$-invariant $(n-1)$-differential form on it. For any two-element subset $\{i,j\} \subset S$ denote by $p_{ij} \colon \mathbf{FM}_n(S) \to \mathbf{FM}_n(\underline{\mathbf{2}})$ the map that forgets all points except ones marked by i and by j. Denote by v_{ij} the pullback of v under projection p_{ij}. Let α be an element of endomorphisms of $k[V]^{\otimes S} \underset{Aut(S)}{\otimes} C^*(\mathbf{FM}_n(S))$ (where $C^*(-)$ is the de Rham complex) given by

$$\alpha = \sum_{i,j \in S} \partial_\omega^{ij} \wedge v_{ij},$$

where ∂_ω^{ij} is the operator ∂_ω applied to the i-th and j-th factors.

Proposition 3.8. *The composition*

$$k[V]^{\otimes S} \xrightarrow{\exp(h\alpha)} k[V][[h]]^{\otimes S} \otimes C^*(\mathbf{FM}_n(S)) \xrightarrow{\mu} k[V][[h]] \otimes C^*(\mathbf{FM}_n(S)),$$

where μ is the product in the polynomial algebra, defines a $k[[h]]$-algebra over the operad $\mathfrak{fm}_n$ with the underlying space $k[V][[h]]$.

This is a simple check.

The algebra defined in this way is obviously invariant under the action of $SO(n)$, thus it is invariant (see Definition 3.4).

Definition 3.8. For a pair (V,ω) as above the invariant $\mathfrak{fm}_n$-algebra given by Proposition 3.8 is called the *Weyl $\mathfrak{fm}_n$-algebra*. Denote it by $\mathcal{W}^n{}_h(V)$.

Note that Proposition 3.1 provides us with the *Weyl e_n-algebra*.

One may give an alternative definition of the Weyl algebra as the universal enveloping of the Heisenberg Lie algebra, compare with [BD04, p. 3.8.1]. It allows us to define the rational version of the Weyl algebra, which is an algebra over rational chains of the Fulton–MacPheson operad.

Let obl_n^m denotes the functor from $\mathfrak{fm}_n$-algebras to $\mathfrak{fm}_m$-algebras induces by the natural map of operads $\mathfrak{fm}_m \to \mathfrak{fm}_n$ for $m < n$.

Proposition 3.9. *For $m < n$ the $\mathfrak{fm}_m$-algebra $\mathrm{obl}_n^m \mathcal{W}^n(V)$ is isomorphic to the commutative polynomial algbebra $\mathbb{k}[V]$.*

Example 3.2. For $n = 1$ and a vector space of degree 0 one gets the Moyal product.

Denote by $\mathcal{W}^n(V)$ the algebra over Laurent formal series, which is the localization $\mathcal{W}^n{}_h(V)\hat{\otimes}_{k[h]}k[h^{-1},h]]$. Both of algebras $\mathcal{W}^n{}_h(V)$ and $\mathcal{W}^n(V)$ are equipped with increasing filtration, which is multiplicative with respect to the commutative product on $k[V][[h]]$, and h and elements of V lie in the component of degree 1.

Consider the L_∞-algebra $L(\mathcal{W}^n{}_h(V))$ associated with the Weyl algebra . By the very definition, all operations on it are given by integration of closed forms by chains of the Fulton–MacPherson operad. But one may see, that chains representing higher operations (that is operations, which are not composition of Lie brackets) in L_∞ are all homologous to zero, because L_∞ is a resolution of the Lie operad. Thus $L(\mathcal{W}^n{}_h(V))$ is a $\mathbb{Z}$-graded Lie algebra, all higher operations vanish. This Lie algebra $L(\mathcal{W}^n{}_h(V))$ is a deformation of the Abelian one. The first order deformation gives the *Poisson Lie algebra*: the underlying space is the $\mathbb{Z}$-graded commutative algebra $k[V][[h]]$, the bracket is defined by $h\omega\colon V\otimes V\to k[[h]]$ on generators and satisfies the Leibniz rule. For the classical one-dimensional Weyl algebra it is known, that higher terms of the deformation are non-trivial: $L(\mathcal{W}^1_h(V))$ differs from the Poisson Lie algebra ([Vey75]). But for $n>1$ the situation is simpler.

Proposition 3.10. *For $n>1$ Lie algebra $L(\mathcal{W}^n)$ is isomorphic to the Poisson Lie algebra of $(V\hat{\otimes}k[h^{-1},h]],\omega)$ over $k[h^{-1},h]]$, the definition of the latter is as above.*

3.2.2. *Factorization homology of $\mathcal{W}^n$.* Weyl n-algebra is a deformation of a commutative algebra. From Subsection 3.1 we know factorization homology of a commutative algebra. Below we use deformation arguments to calculate factorization homology of the Weyl algebra on a closed manifold M.

Theorem 3.1. *Let V be a $\mathbb{Z}$-graded finite-dimensional vector space with a skew-symmetric pairing of degree $1-n$ and $V=\oplus_i V_i$ is its decomposition by degrees. Let M be a n-dimensional closed oriented manifold and b_i its Betti numbers. Then factorization homology $H_*(\int_M \mathcal{W}^n(V))$ is a one-dimensional $k[h^{-1},h]]$-module of total degree*

$$\sum_{\substack{\{i,j\}\\ i+j\ odd}} (-i+j)b_i \dim V_j$$

The idea of proof is to consider n-Weyl algebra as a deformation of the polynomial algebra.

Example 3.3. Let $n=1$ and V is concentrated in degree 0. Then by Example 3.2, $\mathcal{W}^n(V)$ is the usual Weyl algebra. For $M=S^1$ the factorization homology is the Hochschild homology and Theorem 3.1 matches with the well-known fact about Weyl algebra:

$$\dim HH_i(\mathcal{W}^1(V))=\begin{cases}1, & i=\dim V,\\ 0, & \text{otherwise},\end{cases}$$

see e. g. [FT89].

The proof of Theorem 3.1 allows to produce an explicit cycle presenting the only non-trivial class in factorization homology of the Weyl algebra on a closed manifold similarly to the example. Below we consider the simplest case, leaving the general one to the reader.

Proposition 3.11. *Let M be an odd-dimensional rational homology sphere and the $\mathbb{Z}$-graded vector space V has only odd-degree components. Then the only non-trivial cycle in the homology of $\int_M \mathcal{W}^n(V)$ is presented by a cycle in $C_0(M)$ given by a point marked by an element $S^{\mathrm{top}}V$ of the top degree in the symmetric power of V, since V lies in the odd degree the latter makes sense.*

As it was mentioned after Proposition 3.3, framing on a manifold simplifies the definition of the factorization complex. For Weyl n-algebra, a weaker structure, which I call the Euler structure, is sufficient. I do not know whether this is just a technical point or it has some deep relations with [Tur89], where the term is taken from.

For a manifold M and a map of finite sets $S' \to S$ denote by $\mathscr{C}(M)(S' \to S)$ the fiber product

$$\begin{array}{ccc} & & \mathscr{C}(M)(S') \\ & & \downarrow \\ \mathscr{C}^0(M)(S) & \longrightarrow & M^{S'} \end{array} \tag{26}$$

where the horizontal map is composition of the embedding $\mathscr{C}^0(M)(S) \hookrightarrow M^S$ and the map $M^S \to M^{S'}$ induced by the map $S' \to S$, and the vertical map is the projection. Space $\mathscr{C}(M)(S' \to S)$ is equipped with the projection

$$\pi\colon \mathscr{C}(M)(S' \to S) \to \mathscr{C}^0(M)(S).$$

For the only map from $\underline{\mathbf{2}}$ to $\underline{\mathbf{1}}$ the space $\mathscr{C}(M)(\underline{\mathbf{2}} \to \underline{\mathbf{1}})$ is the total space of the sphere bundle associated with the tangent bundle.

Definition 3.9. An *Euler structure* on a n-manifold M is a closed differential form $\mathfrak{v}$ on $\mathscr{C}(M)(\underline{\mathbf{2}} \to \underline{\mathbf{1}})$ such that its restriction on any fiber of the projection $\mathscr{C}(M)(\underline{\mathbf{2}} \to \underline{\mathbf{1}}) \to M$ is the standard volume form on the sphere.

The only obstruction to the existence of the Euler structure is the rational Euler class. In particular, on odd-dimensional manifolds an Euler structure always exists.

Fix an Euler structure on M given by a form $\mathfrak{v}$ on $\mathscr{C}(M)(\underline{\mathbf{2}} \to \underline{\mathbf{1}})$. For any morphism of arrows from $\underline{\mathbf{2}} \to \underline{\mathbf{1}}$ to $S' \to S$ the natural map

$$\mathscr{C}(M)(S' \to S) \to \mathscr{C}(M)(\underline{\mathbf{2}} \to \underline{\mathbf{1}})$$

is defined. Denote by $\mathfrak{v}_{ij}$ the pull back of $\mathfrak{v}$ under this map.

Let V be a $\mathbb{Z}$-graded finite-dimensional vector space equipped with a non-degenerate skew-symmetric pairing $\omega\colon V \otimes V \to k$ of degree $1-n$. Let $k[V]$ be the polynomial algebra generated by V. As before let A be an element of endomorphisms of $k[V]^{\otimes S} \underset{Aut(S')}{\otimes} C^*(\mathscr{C}(M)(S' \to S))$ given by

$$\mathrm{A} = \sum_{\{i,j\}} \partial_\omega^{ij} \wedge \mathfrak{v}_{ij},$$

where the sum is taken by all morphisms of arrows from $\underline{\mathbf{2}} \to \underline{\mathbf{1}}$ to $S' \to S$ and ∂_ω^{ij} is the operator ∂_ω applied to the i-th and j-th factors, where ∂_ω is defined by (25). The exponent of $h\mathrm{A}$ in composition with the cup product gives endomorphism of $k[V][[h]]^{\otimes S'} \underset{Aut(S')}{\otimes} C_*(\mathscr{C}(M)(S' \to S))$. Consider the composite map

$$\begin{array}{c} k[V][[h]]^{\otimes S'} \underset{Aut(S)}{\otimes} C_*(\mathscr{C}(M)(S' \to S)) \\ \Big\downarrow{\scriptstyle \exp(h\mathrm{A})} \\ k[V][[h]]^{\otimes S'} \underset{Aut(S')}{\otimes} C_*(\mathscr{C}(M)(S' \to S)) \\ \Big\downarrow{\scriptstyle \mu\otimes\pi_*} \\ k[V][[h]]^{\otimes S} \underset{Aut(S)}{\otimes} C_*(\mathscr{C}^0(M)(S)), \end{array} \tag{27}$$

where μ is action of morphism in the category $Comm^{\otimes}$.

Proposition 3.12. *Let V be a $\mathbb{Z}$-graded finite-dimensional vector space equipped with a non-degenerate skew-symmetric pairing $\omega\colon V\otimes V\to k$ of degree $1-n$, $A=\mathcal{W}^n{}_h(V)$ be the corresponding Weyl algebra and M be a closed manifold with an Euler structure. Then the factorization complex $\int_M \mathcal{W}^n{}_h(V)$ is the colimit of the diagram*

$$\begin{tikzcd} \bigoplus_{i\colon S'\to S}(C_*(\mathscr{C}(M)(S'\to S)))\underset{Aut(S')}{\otimes} A^{\otimes S'} \\ \bigoplus_{S'} C_*(\mathscr{C}(M)(S'))\underset{Aut(S')}{\otimes} A^{\otimes S'} \arrow[u] \arrow[d] \\ \bigoplus_{S} C_*(\mathscr{C}^0(M)(S))\underset{Aut(S)}{\otimes} A^{\otimes S} \end{tikzcd} \tag{28}$$

where the downwards arrow is the composite map (27) and the upwards arrow is induced by the natural embedding.

3.3. **Perturbative invariants.**

3.3.1. *Propagator.* Let M be a rational homological sphere of dimension n. Let us denote by $\tilde{M}$ the complement in M to the interior of a little ball around a point $p\in M$.

Below we will need the Fulton–MacPherson compactification of manifolds with boundary. Let X be such a manifold and $X\hookrightarrow X'$ be its closed embedding in a manifold of the same dimension, for example, X' is obtained from X by gluing a collar. Then denote by $\mathscr{C}(\tilde{X})(S)$ the fiber product

$$\begin{tikzcd} & \mathscr{C}(X')(S) \arrow[d] \\ \tilde{X}^S \arrow[r] & X'^S \end{tikzcd}$$

where the upwards arrow is the embedding and the vertical one is the projection.

Consider the differential $(n-1)$-form on $\mathscr{C}^0(\mathbb{R}^n)(\underline{\mathbf{2}})$ which is the pullback of the standard form on the sphere under the map $(x,y)\mapsto (x-y)/|x-y|$ and continue it on $\mathscr{C}(\mathbb{R}^n)(\underline{\mathbf{2}})$ straightforwardly (in Subsection 3.2.1 it was denoted by v). Consider the subset of $\mathscr{C}(\mathbb{R}^n)(\underline{\mathbf{2}})$ where both points lie on the unit sphere and restrict the form as above to it. Call the result the *standard form*.

The following proposition stays, that on the 2-point Fulton–MacPherson configuration space of the "fake disk" $\tilde{M}$ there is a differential $(n-1)$-form similar to the standard form on the configuration space of the real disk.

Proposition 3.13. *For a rational homological sphere M choose a point O in the interior of its complement $\tilde{M}$ to a little disk. Then on manifold with corners $\mathscr{C}(\tilde{M})(\underline{\mathbf{2}})$ as above there exists a differential $(n-1)$-form such that*

1. *it is smooth and closed;*
2. *its restriction to any fiber of $\pi\colon \mathscr{C}(\tilde{M})(\underline{\mathbf{2}})\to\tilde{M}^2$ over any point on the diagonal, which is a sphere, is equal to the standard form on the sphere;*
3. *its restriction to the subset where both points of the configuration lie on the boundary equals to the standard form;*

(4) *its restriction to $O \times \partial\tilde{M}$ and $\partial\tilde{M} \times O$ equals to the standard form on the sphere.*

Definition 3.10. We call the $(n-1)$-form as above on $\mathscr{C}(\tilde{M})(\underline{\mathbf{2}})$ a *propagator* and denote it by ν.

Note, that our definition of propagator differs slightly from the one given in [AS92], [BC98].

3.3.2. *Collapse.* Let M and M' be any closed n-manifolds. Choose a point in each manifold and cut off small open balls around them. We get two manifolds $\tilde{M}$ and $\tilde{M}'$ with boundaries S^{n-1}. Denote their interiors by M_0 and M'_0. The connected sum $M\#M'$ is a result of gluing together of these two manifolds by their boundaries. Call the continuous map $\mathfrak{Col}\colon M\#M' \to M'$ that shrinks M to a point $p \in M'$ by the *collapse map*.

In general, the collapse map does not produce any map between factorization homologies of $M\#M'$ and M. There are two cases when it obviously does.

The first case is when the algebra is commutative. The factorization homology is given by homology of the powers of the space and the morphism is given by the direct image on homology of the powers.

The second case is when $M = S^n$. Then $M\#M' = M'$. To build the morphism one need loosely speaking to take everything sitting in M, multiply it and put the result to the point p. One may see that this is an isomorphism.

There is another case when such morphism exists: when M is an odd-dimensional homology sphere and the algebra at hand is the Weyl algebra. Its construction occupies the rest of this Subsection.

The morphism factorizes through an intermediate object we are going to define.

Let M be a rational homology odd-dimensional sphere and M' be any closed n-manifold of the same dimension. Choose Euler structures on M and M', this is possible because they are odd-dimensional. These Euler structures naturally define an Euler structure on the connected sum $M\#M'$ due to the following trick, which works for any pair of odd-dimensional manifolds. Choose as above small embedded open balls $D \hookrightarrow M$ and $D \hookrightarrow M'$ and suppose, that the sphere bundle associated with the tangent bundle is trivialized over D and the Euler structure is constant there. To build the connected sum $M\#M'$ one need to glue the complements of D in M and M' by some orientation-reversing linear automorphism of the sphere $S = \partial D$. Let us choose the antipodal map. One may see, that under the natural isomorphism over S of sphere bundles associated with tangent bundles over M and M', the Euler structures on M and M' fit together.

For a surjective morphism of manifolds $f\colon X' \to X$ and a map of sets $S' \to S$ define space $\mathscr{C}(X'/X)(S' \to S)$ as the fiber product

$$\begin{tikzcd} & \mathscr{C}(X')(S') \arrow[d] \\ \mathscr{C}^0(X)(S) \arrow[r] & X^{S'} \end{tikzcd} \tag{29}$$

where the vertical arrow is the composition of projection $\mathscr{C}(X')(S') \to {X'}^{S'}$ with $f^{S'}$ and the lower arrow is composition of the embedding $\mathscr{C}^0(M)(S) \hookrightarrow X^S$ and the map $X^S \to X^{S'}$ induced by the map $S' \to S$. Space $\mathscr{C}(X'/X)(S' \to S)$ is equipped with the projection

$$\pi\colon \mathscr{C}(X'/X)(S'\to S)\to \mathscr{C}^0(X)(S).$$

For the collapse map $M\#M'\to M'$ consider space $\mathscr{C}(M\#M'/M')(\underline{\mathbf{2}}\to\underline{\mathbf{1}})$. This space contains $\mathscr{C}(M')(\underline{\mathbf{2}}\to\underline{\mathbf{1}})$ and M_0^2 as subspaces. On the first one the Euler structure gives a differential $(n-1)$-form and on the second one choose a propagator (Definition 3.10). Property 3 of propagator (Proposition 3.13) allows to glue it in a global $(n-1)$-cocycle in the cochain complex of $\mathscr{C}(M\#M'/M')(\underline{\mathbf{2}}\to\underline{\mathbf{1}})$. Denote it by $\mathcal{V}$. Note that the space is not manifold, but $\mathcal{V}$ is a well-defined cochain of the corresponding relative complex.

Let V be a $\mathbb{Z}$-graded finite-dimensional vector space equipped with a non-degenerate skew-symmetric pairing $\omega\colon V\otimes V\to k$ of degree $1-n$. Let $k[V]$ be the polynomial algebra generated by V. Mimicking construction from Preposition 3.12, let $\mathcal{A}$ be an element of endomorphisms of $k[V]^{\otimes S}\underset{Aut(S')}{\otimes} C^*(\mathscr{C}(M\#M'/M')(S'\to S))$ given by

$$\mathrm{A}=\sum_{\{i,j\}}\partial_\omega^{ij}\wedge\mathcal{V}_{ij},$$

where the sum is taken by all morphisms of arrows from $\underline{\mathbf{2}}\to\underline{\mathbf{1}}$ to $S'\to S$ and ∂_ω^{ij} is the operator ∂_ω applied to the i-th and j-th factors, where ∂_ω is defined by (25). The exponent of $h\mathcal{A}$ in composition with cup product gives endomorphism of $k[V][[h]]^{\otimes S'}\underset{Aut(S')}{\otimes} C_*(\mathscr{C}(M\#M'/M')(S'\to S))$. Consider the composite map

$$\begin{array}{c} k[V][[h]]^{\otimes S'}\underset{Aut(S)}{\otimes} C_*(\mathscr{C}(M\#M'/M')(S'\to S)) \\ \Big\downarrow{\scriptstyle \exp(h\mathcal{A})} \\ k[V][[h]]^{\otimes S'}\underset{Aut(S')}{\otimes} C_*(\mathscr{C}(M\#M'/M')(S'\to S)) \\ \Big\downarrow{\scriptstyle \mu\otimes\pi_*} \\ k[V][[h]]^{\otimes S}\underset{Aut(S)}{\otimes} C_*(\mathscr{C}^0(M')(S)), \end{array}\tag{30}$$

where μ is the morphism in the category $Comm^{\otimes}$.

By analogy with (28) consider the diagram

$$\begin{array}{c} \bigoplus_{S'} C_*(\mathscr{C}(M\#M')(S'))\underset{Aut(S')}{\otimes} A^{\otimes S'} \\ \Big\uparrow \\ \bigoplus_{i\colon S'\to S}(C_*(\mathscr{C}(M\#M'/M')(S'\to S)))\underset{Aut(S')}{\otimes} A^{\otimes S'} \\ \Big\downarrow \\ \bigoplus_{S} C_*(\mathscr{C}^0(M')(S))\underset{Aut(S)}{\otimes} A^{\otimes S} \end{array}\tag{31}$$

where the downwards arrow is the composite map (30) and the upwards arrow is induced by the natural embedding.

The desired intermediate object is the colimit of diagram (31). Property 2 of propagator (Proposition 3.13) supplies us with a natural map from the diagram presenting $\int_{M\#M'} \mathcal{W}^n{}_h(V)$ by Proposition 3.12 to (31), thus with a map from $\int_{M\#M'} \mathcal{W}^n{}_h(V)$ to the colimit of (31).

The following Proposition completes the construction.

Theorem 3.2. *The colimit of (31) is isomorphic to* $\int_{M'} \mathcal{W}^n{}_h(V)$.

Call the morphism $\mathfrak{col}\colon \int_{M\#M'} \mathcal{W}^n{}_h(V) \to \int_{M'} \mathcal{W}^n{}_h(V)$ just constructed the *collapse morphism.*

One may prove this Proposition by means of cosheaves in the spirit of the discussion at the end of Subsection 3.1.4. Indeed, the colimit of Diagram (31) gives a cosheaf on the Ran space of M'. Theorem 3.2 states that it is isomorphic to the one given by the Weyl algebra.

Note finally, that Theorem 3.2 may be reformulated as follows: for a homological sphere M the factorization complex $\int_{\tilde{M}} \mathcal{W}^n{}_h(V)$ is isomorphic to $\mathcal{W}^n{}_h(V)$ as an $\int_{[0,1]\times S^{n-1}} \mathcal{W}^n{}_h(V)$-module (about the module structure on the factorization complex of a manifold with boundary see e. g. [Gin] and references therein).

3.3.3. *Invariants.* Factorization homology of Weyl n-algebras may be used to construct invariants of manifolds. Let M be a closed n-manifold and V be a $\mathbb{Z}$-graded finite-dimensional vector space with a non-degenerate pairing of degree $1-n$. By Theorem 3.1 the factorization homology of $\mathcal{W}^n{}_h(V)$ on M is one-dimensional. The idea of the invariant we are going to build is to produce in some manner a cycle in $\int_M \mathcal{W}^n{}_h(V)$ and calculate the class represented by it. As the homology group is one-dimensional, this class is a multiple of a standard one. The series we get this way is the invariant of the manifold.

Let us restrict ourselves with the following conditions: M is a rational homology sphere of odd dimension n and V be a $\mathbb{Z}$-graded finite-dimensional vector space, which has only odd-dimensional components. Under these conditions due to Proposition 3.11 the only class in the factorization homology is presented by an especially simple cycle, just an element of the top degree power of V sitting at a point, call this cycle the standard one.

To produce a different cycle we shall resort to the morphism given by Proposition 3.7. It sends the Chevalley–Eilenberg complex of the Lie algebra $L(\mathcal{W}^n(V))$ associated with the Weyl algebra $\mathcal{W}^n(V)$ to the factorization complex of $\mathcal{W}^n(V)$.

By Proposition 3.10, for $n > 1$ $L(\mathcal{W}^n(V))$ is $\mathbb{Z}$-graded Poisson Lie algebra. Suppose that $\dim V \geq 3$ and denote by $L(\mathcal{W}^n(V)^{\geq 3})$ the Lie subalgebra of polynomials of degree not less than 3. One may see that a generator of $S^{top}V$ is in the center of $L(\mathcal{W}^n(V)^{\geq 3})$. Thus the map

$$k \to L(\mathcal{W}^n(V)^{\geq 3}),$$

which sends the generator to a non-zero element from $S^{top}V$ is a morphism from the trivial $L(\mathcal{W}^n(V)^{\geq 3})$-module to the adjoint one. Consider the induced map

$$C_{Ch}(L(\mathcal{W}^n(V)^{\geq 3})) \to C^{ad}_{Ch}(L(\mathcal{W}^n(V)^{\geq 3}))$$

and combine it with map

$$C^{ad}_{Ch}(L(\mathcal{W}^n(V)^{\geq 3})) \to \int_M \mathcal{W}^n(V)$$

given by Proposition 3.7. The composite map

$$C_{Ch}(L(\mathcal{W}^n(V)^{\geq 3})) \to \int_M \mathcal{W}^n(V) \xrightarrow{\sim} k[[h^{-1}, h]] \tag{32}$$

is the desired invariant. In other words, the invariant is a cohomology class of total degree zero of $\mathbb{Z}$-graded Lie algebra $L(\mathcal{W}^n(V)^{\geq 3})$ with coefficients in $k[[h^{-1},h]]$. To get just an element of $k[[h^{-1},h]]$ one may substitute a homology class of this Lie algebra in it. Note, that the coefficients of this series are rational due to the remark preceding Example 3.2.

As it was already mentioned in the Introduction, a cocycle of the complex linear dual to the factorization complex $\int_M \mathcal{W}^n(V)$ which does not vanish on the standard cycle would make this invariant more explicit. As such a cocycle is unavailable, we shall make use of the collapse morphism from the previous Subsection.

Proposition 3.14. *If M and M' are both rational homology odd-dimensional spheres and V has only odd-degree components then the collapse morphism*

$$\mathsf{col}\colon \int_{M\#M'} \mathcal{W}^n(V) \to \int_{M'} \mathcal{W}^n(V)$$

induces isomorphism on homologies.

Assuming $M' = S^n$ in the Proposition above we get an isomorphism $\int_M \mathcal{W}^n(V) \to \int_{S^n} \mathcal{W}^n(V)$. In composition with (32) we get a morphism

$$C_{Ch}(L(\mathcal{W}^n(V)^{\geq 3})) \to \int_{S^n} \mathcal{W}^n(V),$$

which is better than (32), because the target does not depend on M.

Unwinding the definition of the collapse morphism one may see that this cocycle of $L(\mathcal{W}^n(V)^{\geq 3})$ taking values in $\int_{S^n} \mathcal{W}^n(V)$ is a sort of cocycle given by the graph complex, see [Kon93; Kon94; QZ11]. It is known ([AS92; AS94]), that perturbative Chern–Simons invariants also give classes in the graph complex in the same way, by integration of the powers of the propagator. It makes us believe that our invariants coincide with the perturbative Chern–Simons ones. Perhaps, some good choice of the propagator will lead to a more explicit formula.

Finally, let $n = 3$, V be a $\mathbb{Z}$-graded finite-dimensional vector space of dimension more than 2 concentrated in degree 1 with skew-symmetric pairing of degree -2, that is $V[1]$ is equipped with a symmetric pairing. In this case for dimensional reasons the cocycle is given by trivalent graphs. If $V[1]$ is the underlying space of a Lie algebra with non-degenerate pairing, then the element in $S^3V[1]$, which is the composition of the Lie bracket and the pairing, is a Maurer–Cartan element in $L(\mathcal{W}^n(V)^{\geq 3})$. Its power gives a homology class. Values of the cocycle on it must be the perturbative invariants associated with given Lie algebra. More about this case the reader may find in the following subsection.

3.3.4. *Oscillatory integrals.* The physical definition of perturbative Chern–Simons invariant is based on the asymptotic series of the oscillatory integral $\int e^{iS}$ taken over the space of all G-connection A on M, where $S = \frac{\kappa}{4\pi}\int_M \mathrm{tr}(A\wedge dA + \frac{2}{3}A\wedge A\wedge A)$ is the Chern–Simons functional, M is a 3-manifold and G is a semi-simple Lie group. The aim of this appendix is to demonstrate speculatively how to interpret the calculation of such an integral in terms of the factorization complex.

Thus, we have the infinite-dimensional space of connections, a function S on it and we want to calculate the asymptotic series in $1/\kappa$ of the oscillatory integral. If M is a homology sphere, then function S has a non-degenerate critical point at the origin. Thus, the free term of the series in hand is the Gaussian integral by an infinite-dimensional space and is unapproachable by algebraic methods. But after dividing by this term the series may be calculated by means of the method of Feynman diagrams.

To explain how this method works consider an abstract situation, the reader can find more at [Joh]. Let V be a Euclidean vector space and f be a smooth function on it such that its Taylor series at the origin start with terms of degree, at least, three. Choose a volume form on V and consider the integral $\int e^{(-|x|^2+tf)}$. Consider the twisted de Rham complex of polynomial forms Ω^*_t given by differential forms on V with differential $d_{dR} - 2(\mathbf{x}, d\mathbf{x}) + t\, df$, where d_{dR} is the de Rham differential. One may see that complex $\Omega^*_t \otimes \mathbb{R}[[t]]$ has only top degree cohomology, which is one-dimensional over $\mathbb{R}[[t]]$. This one-dimensional vector bundle over the t-line has the Gauß–Manin connection and a section given by the chosen volume form on V. Their quotient is a series in t up to a constant factor and one may show that this is the asymptotic expansion of the oscillatory integral up to a constant.

We are now going to construct a $\mathfrak{fm}_3$-algebra (or equivalently, by Proposition 3.1, an e_3-algebra) the factorization complex of which on a homology 3-sphere M resembles the twisted de Rham complex as above. Let g be a Lie algebra with a non-degenerate invariant bilinear form. The desired $\mathfrak{fm}_3$-algebra is a deformation of the Chevalley–Eilenberg commutative dg-algebra $C^\bullet_{Ch}(g)$ in the class of $\mathfrak{fm}_3$-algebras. The deformation may be described as follows: forget about the differential on the Chevalley–Eilenberg complex and deform the underlying polynomial algebra as in the definition of the Weyl algebra, that is apply Definition 3.8 to the space $g^\vee$ and the pairing given by the invariant bilinear form. It is easy to check that this deformation respects the differential. Note, that this e_3-algebra is the algebra of Ext's from the unit to itself in e_2-category of representations of a quantum group. Denote it by $Ch^\bullet_h(g)$.

Alternatively, this $\mathfrak{fm}_3$-algebra may be defined as follows. Start with $\mathbb{Z}$-graded finite-dimensional vector space $g^\vee[1]$ with the pairing of degree -2 given by the invariant scalar product and build the Weyl algebra $\mathcal{W}^3_h(g^\vee[1])$. Then define differential on it as $\frac{1}{h}\{\cdot, q\}$, where $\{,\}$ is image of the Lie bracket under (20) and $q \in S^3(g^\vee[1])$ is the composition of the Lie bracket on g and the scalar product. One may show, that similarly to Hochschild homology (see e. g. [Lod98, Propositon 1.3.3]), factorization homology on a closed manifold is invariant under inner deformations. It follows by Theorem 3.1 that the homology of $\int_M Ch^\bullet_h \otimes k[h^{-1}, h]]$ is free $k[h^{-1}, h]]$-module of rank 1. And moreover, this homology is equipped with a connection along the formal deleted h-line.

To fulfill the analogy (note, that t corresponds to $1/h$) we have to present a section of this one-dimensional vector bundle and compare it with a horizontal one. Formula (32) produces elements in the factorization complex of $Ch^\bullet_h(g)$. One may see, that 1 goes to a cycle (in fact, this is the cycle given by Proposition 3.11) and this is an analog of the section given by the volume form on V in the example above. On the other hand, one may see that a cycle horizontal with respect to the connection is the image under (3.11) of the cycle $\sum_i \frac{h^{-i}}{i!} \underbrace{q \wedge \cdots \wedge q}_{i}$. The quotient of these two sections is an analog of the asymptotic series and is given by the invariants as in Subsection 3.3.3.

3.4. **Weyl n-algebras and the Kontsevich integral.**

3.4.1. *Quantization of the Chevalley–Eilenberg complex.* Let us start with some calculations similar to the ones we made with the Atiyah class in section 1.

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. The Chevalley–Eilenberg algebra $\mathrm{Ch}^\bullet(\mathfrak{g})$ is a super-commutative dg-algebra $S^*(\mathfrak{g}^\vee[1])$ generated by the dual space $\mathfrak{g}^\vee$ placed in degree 1. The differential is a derivation of this free super-commutative algebra defined on the generators by the tensor $\mathfrak{g}^\vee \to \mathfrak{g}^\vee \wedge \mathfrak{g}^\vee$ dual to the bracket. The Jacobi identity guarantees that this is indeed, a differential. In terms of [Ale+97] the Chevalley–Eilenberg algebra may be thought of as the function ring of a Q-manifold.

With any $\mathfrak{g}$-module E one may associate the module $\mathrm{Ch}^\bullet(\mathfrak{g}, E)$ over $\mathrm{Ch}^\bullet(\mathfrak{g})$ as follows. As a $S^*(\mathfrak{g}^\vee[1])$-module it is freely generated by E and the differential is defined by its value on $E \otimes 1$ given by the tensor $E \to E \otimes \mathfrak{g}^\vee$ of the $\mathfrak{g}$-action. As a complex, $\mathrm{Ch}^\bullet(\mathfrak{g}, E)$ calculates the cohomology of $\mathfrak{g}$ with coefficients in E.

The $\mathrm{Ch}^\bullet(\mathfrak{g})$-module $\mathrm{Ch}^\bullet(\mathfrak{g}, \mathfrak{g}^\vee_{ad})$ corresponding to the adjoint $\mathfrak{g}$-module may be thought of as a cotangent complex of $\mathrm{Ch}^\bullet(\mathfrak{g})$. The de Rham differential $d_{dR}\colon \mathrm{Ch}^\bullet(\mathfrak{g}) \to \mathrm{Ch}^\bullet(\mathfrak{g}, \mathfrak{g}^\vee_{ad})$, which is a derivation of $\mathrm{Ch}^\bullet(\mathfrak{g})$-modules, is tautologically defined on the generators. Define the $\mathrm{Ch}^\bullet(\mathfrak{g})$-module of differential forms of $\mathrm{Ch}^\bullet(\mathfrak{g})$ as $\mathrm{Ch}^\bullet(\mathfrak{g}, \mathbf{k}[[\mathfrak{g}^\vee]]^{ad})$. It is a super-commutative algebra and the de Rham differential acts on it in the usual way, it is a derivation.

For a unital dg-algebra A define the reduced (or normalized) Hochschild complex $C_*(A)$ (see e. g. [Lod98, Ch 1.1]) as the total complex of the bi-complex with the $(-i)$-th term

$$\prod_{i\geq 0}(A \otimes \underbrace{A/\mathbf{k} \otimes \cdots \otimes A/\mathbf{k}}_{i}), \tag{33}$$

the first differential coming from A and the second differential given by

$$\begin{aligned} &a_0 \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_i \mapsto \\ &a_0a_1 \otimes a_2 \otimes \cdots \otimes a_i - a_0 \otimes a_1a_2 \otimes \cdots \otimes a_i + \dots \\ &+ (-1)^{i+\deg a_i(\deg a_0 + \dots \deg a_{i-1})} a_ia_0 \otimes a_1 \otimes \cdots \otimes a_{i-1}. \end{aligned} \tag{34}$$

Here one have to choose representatives of quotients $A/\mathbf{k}$, then apply formula and take quotients again, the result does not depend on choices. Note, that the usual definition uses direct sums instead of products, but we need the one we gave. In other words, we shall consider unbounded chains, that is the graded completion ([CDM12, Definition A.25]) of $\sum_{i\geq 0}(A \otimes \underbrace{A/\mathbf{k} \otimes \cdots \otimes A/\mathbf{k}}_{i})$ with respect to the grading given by the grading on A. For an ungraded algebra the reduced Hochschild complex calculates $\mathrm{Tor}_*^{A\otimes A^o}(A, A)$.

The following proposition is a variant of the Hochschild–Kostant–Rosenberg isomorphism.

Proposition 3.15. *The formula*

$$a_0 \otimes a_1 \otimes \cdots \otimes a_i \mapsto a_0\, d_{dR}a_1 \cdots\, d_{dR}a_i \tag{35}$$

defines a morphism from the reduced Hochschild complex $C_*(\mathrm{Ch}^\bullet(\mathfrak{g}))$ of the Chevalley–Eilenberg algebra to its differential forms $\mathrm{Ch}^\bullet(\mathfrak{g}, \mathbf{k}[[\mathfrak{g}^\vee]]^{ad})$. This morphism is a quasi-isomorphism.

Equip $C_*(\mathrm{Ch}^\bullet(\mathfrak{g}))$ with a descending filtration F: the subcomplex $F_kC_*(\mathrm{Ch}^\bullet(\mathfrak{g}))$ is spanned by chains $a_0 \otimes a_1 \otimes \cdots \otimes a_i$ such that $\deg a_0 \geq k$.

Proposition 3.16. *The spectral sequence associated with the filtration F on $C_*(\mathrm{Ch}^\bullet(\mathfrak{g}))$ degenerates at the second sheet. The complex $E_1^{p,0}$ is isomorphic to $\mathrm{Ch}^\bullet(\mathfrak{g}, \mathbf{k}[[\mathfrak{g}^\vee]]^{ad})$ and $E_1^{p,>0} = 0$.*

Note, that $F_iC_i(\mathrm{Ch}^\bullet(\mathfrak{g}))$ is spanned by chains $a_0 \otimes a_1 \otimes \cdots \otimes a_i$ such that $\deg a_{>0} = 1$. Taking into account Proposition 3.16 we get the following.

Corollary 3.1. *Every cycle in $C_*(\mathrm{Ch}^\bullet(\mathfrak{g}))$ may be presented by a sum of chains $a_0 \otimes a_1 \otimes \cdots \otimes a_i$ with $\deg a_{>0} = 1$.*

Finding an explicit formula for these cycles seems to be an interesting question.

Along with the Hochschild complex as above one may consider the Hochschild complex $C_*(A, M)$ of a dg-algebra A with coefficients in a A-bimodule M (see e. g. [Lod98, Ch 1.1]). It is given by the same formulas (33) and (34), but a_0 now is an element of M. For a ungraded algebra the reduced Hochschild complex calculates $\mathrm{Tor}_*^{A\otimes A^o}(A, M)$.

The $\mathrm{Ch}^\bullet(\mathfrak{g})$-module of 1-forms $\mathrm{Ch}^\bullet(\mathfrak{g}, \mathfrak{g}^\vee)$ is a bimodule as well, because the algebra is supercommutative. Introduce the Hochschild complex of $\mathrm{Ch}^\bullet(\mathfrak{g})$ with coefficients in this bimodule $C_*(\mathrm{Ch}^\bullet(\mathfrak{g}), \mathrm{Ch}^\bullet(\mathfrak{g}, \mathfrak{g}^\vee))$.

the following statement is an analog of Example 1.2.

Proposition 3.17. *The formulas*

$$\begin{aligned} &a_0 \otimes a_1 \otimes \cdots \otimes a_i \mapsto a_0\, d_{dR} a_1 \otimes a_2 \otimes \cdots \otimes a_i \\ &a_0 \otimes a_1 \otimes \cdots \otimes a_i \mapsto \pm a_0\, d_{dR} a_i \otimes a_1 \otimes \cdots \otimes a_{i-1}, \end{aligned} \tag{36}$$

where the sign is defined by the Koszul rule, define morphisms from the Hochschild complex $C_*(\mathfrak{g})$ to the Hochschild complex with coefficients $C_*(\mathrm{Ch}^\bullet(\mathfrak{g}), \mathrm{Ch}^\bullet(\mathfrak{g}, \mathfrak{g}^\vee))$ of degree 1.

The following proposition describes these morphisms in terms of the quasi-isomorphism (3.15).

Recall some basic facts from Lie group theory. For a finite-dimensional Lie algebra $\mathfrak{g}$ denote by $U_\mathfrak{g}$ its enveloping algebra. This is a Hopf algebra which is dual to the Hopf algebra of formal functions $F(G)$ on the formal group associated with $\mathfrak{g}$. The Poincaré–Birkhoff–Witt map from the symmetric power of $\mathfrak{g}$ to its universal enveloping $i_{PBW}\colon S^*\mathfrak{g} \to U_\mathfrak{g}$ provides an isomorphism between them as adjoint $\mathfrak{g}$-modules. It is dual to the exponential coordinate map $exp^*\colon F(G) \to \mathbf{k}[[\mathfrak{g}^\vee]]$. Maps

$$F_L\colon F(G) \to F(G) \otimes \mathfrak{g}^\vee \quad \text{and} \quad F_R\colon F(G) \to F(G) \otimes \mathfrak{g}^\vee \tag{37}$$

dual to the multiplications

$$U_\mathfrak{g} \otimes \mathfrak{g} \to U_\mathfrak{g} \quad \text{and} \quad \mathfrak{g} \otimes U_\mathfrak{g} \to U_\mathfrak{g}$$

respectively. After identifying G and $\mathfrak{g}$ by the exponential map, the maps (37) are given by elements of $Vect(\mathfrak{g}) \otimes \mathfrak{g}^\vee$. Corresponding maps from $\mathfrak{g}$ to $Vect(\mathfrak{g})$ are given by left and right invariant vector fields on G. Applying the constant trivialization of the tangent bundle to $\mathfrak{g}$ one may identify such a tensor with a section of the trivial vector bundle with fiber $\mathrm{End}(\mathfrak{g})$ over $\mathfrak{g}$. In other words, this section is the transformation matrix between the constant basis of the tangent bundle and the one given by left (right) invariant vector fields. By e. g. [Reu93, Ch. 3.4] they are given by formulas

$$\mathrm{id} \pm \frac{1}{2}\mathrm{Ad} + \sum_{n\geq 1} \frac{B_{2n}}{(2n)!}\mathrm{Ad}^{2n} \tag{38}$$

("$+$" for the first and "$-$" for the second tensor), where Ad is the structure tensor of the $\mathfrak{g}$ considered as linear function on $\mathfrak{g}$ taking values in $\mathrm{End}(\mathfrak{g})$ and B_n are Bernoulli numbers:

$$\sum_{n\geq 0} \frac{B_n}{n!} z^n = \frac{z}{e^z - 1}. \tag{39}$$

Recall that Proposition 3.15 identifies $C_*(\mathrm{Ch}^\bullet(\mathfrak{g}))$ with the complex $\mathrm{Ch}^\bullet(\mathfrak{g}, \mathbf{k}[[\mathfrak{g}^\vee]]^{ad})$. In the same way, one can build a quasi-isomorphism between $C_*(\mathrm{Ch}^\bullet(\mathfrak{g}), \mathrm{Ch}^\bullet(\mathfrak{g}, \mathfrak{g}^\vee))$ and $\mathrm{Ch}^\bullet(\mathfrak{g}, \mathfrak{g}^\vee \otimes \mathbf{k}[[\mathfrak{g}^\vee]]^{ad})$.

Proposition 3.18. *Under the quasi-isomorphism as above, maps (36)*

$$\mathrm{Ch}^\bullet(\mathfrak{g}, \mathbf{k}[[\mathfrak{g}^\vee]]^{ad}) \to \mathrm{Ch}^\bullet(\mathfrak{g}, \mathfrak{g}^\vee \otimes \mathbf{k}[[\mathfrak{g}^\vee]]^{ad})$$

are induced by (37), where $\mathbf{k}[[\mathfrak{g}^\vee]]$ *is identified with* $F(G)$ *by the exponential map; that is, (36) are given by formulas (38).*

Let now $\mathfrak{g}$ be an finite-dimensional Lie algebra with a non-degenerate invariant scalar product $\langle \cdot, \cdot \rangle$ as in the Subsection 3.3.4. The scalar product may be thought of as a constant symplectic structure of degree -2 on the *dg*-manifold (or Q-manifold), which is the spectrum of $\mathrm{Ch}^\bullet(\mathfrak{g})$. That is, we define a Poisson bracket on $\mathrm{Ch}^\bullet(\mathfrak{g})$ on the generators by $\{x, y\} = \langle x, y\rangle$ and extend it to the whole algebra by the Leibnitz rule. In terms of [Ale+97] we get a QP-manifold.

A symplectic structure gives a first order deformation of the product of functions on a manifold and thus deforms the Hochschild complex. Our aim is to calculate it in our case.

More precisely, consider the ring $\mathbf{k}[\varepsilon]$, where $\deg \varepsilon = 2$ and $\varepsilon^2 = 0$ and the Chevalley–Eilenberg complex $\mathrm{Ch}^\bullet(\mathfrak{g}) \otimes \mathbf{k}[\varepsilon]$ over $\mathbf{k}[\varepsilon]$ with the differential as before, with the product given by $x \cdot y = x \wedge y + \frac{1}{2}\varepsilon\langle x, y\rangle$. Take the Hochschild complex of $\mathbf{k}[\varepsilon]$-algebra $\mathrm{Ch}^\bullet(\mathfrak{g}) \otimes \mathbf{k}[\varepsilon]$, that is, all tensor products are taken over $\mathbf{k}[\varepsilon]$. It is a module over $\mathbf{k}[\varepsilon]$. Multiplication by ε defines a 2-step filtration on it. Consider the spectral sequence associated with this filtration. The 0-th sheet is $C_*(\mathrm{Ch}^\bullet(\mathfrak{g})) \otimes \mathbf{k}[\varepsilon]$. The following proposition describes d_0 of this spectral sequence, which is the first order deformation of the differential in the Hochschild complex.

Proposition 3.19. *Contract tensors (37) from* $Vect(\mathfrak{g}) \otimes \mathfrak{g}^\vee$ *with the pairing* $\langle \cdot, \cdot \rangle$ *and consider the resulting element of* $Vect(\mathfrak{g}) \otimes \mathfrak{g}$ *as a differential operator on* $\mathrm{Ch}^\bullet(\mathfrak{g}, \mathbf{k}[[\mathfrak{g}^\vee]]^{ad})$ *of the second order, where term* $\cdot \otimes \mathfrak{g}$ *differentiates* $\mathrm{Ch}^\bullet(\mathfrak{g})$ *and term* $Vect(\mathfrak{g}) \otimes \cdot$ *differentiates* $\mathbf{k}[[\mathfrak{g}^\vee]]$*. Under quasi-isomorphism (35) differential* d_0 *of the above-mentioned spectral sequence is given by half-sum of these operators on the complex* $\mathrm{Ch}^\bullet(\mathfrak{g}, \mathbf{k}[[\mathfrak{g}^\vee]]^{ad})$*. By (38), the matrix of this differential operator is given by*

$$\mathrm{id} + \sum_{n\geq 1} \frac{B_{2n}}{(2n)!}\mathrm{Ad}^{2n}, \tag{40}$$

B_n *are Bernoulli numbers,* Ad *is the structure tensor of the* $\mathfrak{g}$*, being considered as linear function on* $\mathfrak{g}$ *taking values in* $\mathrm{End}(\mathfrak{g})$*.*

Proposition 3.19 defines, therefore, on the algebra $\mathrm{Ch}^\bullet(\mathfrak{g}, \mathbf{k}[[\mathfrak{g}^\vee]]^{ad})$ a differential operator of order 2 and of cohomological degree -1. On this algebra another differential operator of the same order and degree is defined, in terms of the above proposition it is given by the unit matrix. Call it the Brylinski differential after [Bry88] and denote it by d_{Br} They are not chain homotopic, but by the following proposition they become such after conjugation by an automorphism of complex $\mathrm{Ch}^\bullet(\mathfrak{g}, \mathbf{k}[[\mathfrak{g}^\vee]]^{ad})$. This automorphism equals to multiplication by the Duflo character.

Given a Lie group G, equip it with the left invariant volume form (which is the right invariant as well, due to the invariant scalar product). Equip its Lie algebra $\mathfrak{g}$ with the constant volume form and denote by $j \in \mathbf{k}[[\mathfrak{g}^\vee]]$ the Jacobian of the exponential map. The Duflo character is the power series on $\mathfrak{g}$ which is the square root of the Jacobian and is given by

$$j^{\frac{1}{2}} = \exp \sum_{n=1}^{\infty} \frac{B_{2n}}{4n(2n)!} \operatorname{Tr}(\mathrm{Ad}^{2n}), \tag{41}$$

where B_n are the Bernoulli numbers from (39) and Ad is the linear function on $\mathfrak{g}$ taking values in $\mathrm{End}(\mathfrak{g})$ as above.

Proposition 3.20. *Under the quasi-isomorphism (35), the differential d_0 on $\mathrm{Ch}^\bullet(\mathfrak{g}, \mathbf{k}[[\mathfrak{g}^\vee]]^{ad})$ is chain homotopic to $j^{-\frac{1}{2}} \circ d_{Br} \circ j^{\frac{1}{2}}$, where $j^{\frac{1}{2}}$ is the operator of the multiplication of $\mathbf{k}[[\mathfrak{g}^\vee]]$ by the Duflo character and $j^{-\frac{1}{2}}$ is the inverse operator.*

Remark 1. The above proposition can be stated and proved in a coordinate-free manner for any QP-manifold in terms of [Ale+97]. In the setting of Section 1 it describes the differential on the differential forms on a complex symplectic manifold, that is, on the Hochschild homology of the structure sheaf, coming from the first order deformation of the structure sheaf along the symplectic structure. It seems that when applied to the cotangent bundle of a complex manifold, it gives an alternative way of calculating the Todd class of this manifold.

Remark 2. Proposition 3.20 was inspired by the proof of the Duflo isomorphism for a Lie algebra with an invariant scalar product from [AM00]. As we will see below, the calculation above is connected with another proof of the Duflo isomorphism, the one from [BLT03].

3.4.2. *Factorization homology.* Above, we introduced e_n-algebras and their factorization homology. To define the latter for a non-parallelized manifold, one needs to modify the notion of e_n-algebra, incorporating action of $SO(n)$. The way we used is to deal with equivariant e_n-algebras. Another way to take this action into account is to consider $SO(n)$ as an operad with 1-ary only operations and take the semi-direct product of this operad and the little discs operad. The result is called framed little discs operad, see [SW03]. We denote the dg-operad of chains of this operad by fe_n.

In general, the category of such algebras is not the same as the one of fe_n-algebras. However, for $n = 2$, the commutativity of the group simplifies things, and these categories are essentially the same. Consider the latter case in some detail. The cohomology of fe_2 is known as the Batalin–Vilkovisky (BV) operad, see e. g. [SW03]. It is generated by the product $\cdot$ and the bracket $\{\,,\}$ obeying the same relations as those in e_2 and an additional 1-ary operation Δ of degree -1 obeying the relations

$$\Delta^2 = 0, \qquad \{a, b\} = (-1)^{|a|}\Delta(ab) - (-1)^{|a|}\Delta(a)b - a\Delta(b).$$

One important property of the factorization complex is its behavior with respect to gluing, see e. g. [Gin] and references therein. Let M_1 and M_2 be two manifolds with isomorphic boundaries B. Then for a e_n-algebra A there is a map of complexes

$$\int_{M_1} A \otimes \int_{M_2} A \to \int_{M_1 \cup_B M_2} A.$$

It follows that for $k < n$, a k-manifold M^k and a e_n-algebra A, the complex $\int_{M^k \times I^{n-k}} A$ is a e_k-algebra, and it is equivariant, if A is. In particular, for an n-manifold M with boundary B the complex $\int_{B\times I} A$ is a (homotopy) algebra, and the map above equips $\int_M A$ with a module structure over it. In terms of this action, the gluing rule may be written as

$$\int_{M_1 \cup_B M_2} A = \int_{M_1} A \underset{\int_{B\times I} A}{\otimes} \int_{M_2} A. \tag{42}$$

Another important property of the factorization complex is a kind of homotopy invariance:

$$\int_{M^k \times I^{n-k}} A = \int_{M^k} \operatorname{obl}_k^n A.$$

Below we will make no difference between the two sides of this equality and will denote them simply by $\int_{M^k} A$. In particular, the factorization complex on a disk is quasi-isomorphic, as a complex, to the algebra itself.

Example 3.4. Let A be an equivariant e_2-algebra. Then its factorization complex on the disc $\int_{D^2} A$, which is A itself, is a module over $\int_{S^1\times I^1} A = \int_{S^1} \operatorname{obl}_1^2 A$, which is the Hochschild homology complex of $\operatorname{obl}_1^2 A$. The equivariance of A is essential here: without it, the Hochschild complex of e_2-algebra A does not act on A, and, if an equivariance structure is chosen, the action depends on this choice. In order to see it, note that $S^1 \times I^1$ is a framed manifold, that is why we do not need equivariance to take its factorization complex for any, not only equivariant algebra. However, this framing, which comes from the constant framing on the square after gluing together two opposite edges, can not be extended to the whole disc obtained from the annulus $S^1 \times I^1$ by gluing one of its boundary circles with the disc. Hence, in order to construct the desired action by gluing the annulus with the disc one need to identify factorization complexes with different framings, and here one needs the equivariance.

The type of equivariant e_n-algebras we need are the Weyl n-algebras introduced in Subsection 3.2. In order to build such an algebra one needs a super-vector space V with a super-skew-symmetric non-degenerate bilinear form on it. The e_n-algebra associated with such data is denoted by $\mathcal{W}^n(V)$. In analogy with the usual Weyl algebra, it is the deformation of the polynomial algebra generated by V in the direction given by pairing. In fact, this is an algebra over the field of Laurent formal series in the quantization parameter h; this, however, must be ignored, assuming, loosely speaking, that $h = 1$.

There are some important properties we need. Firstly, considered as an e_k-algebra, where $k < n$, it is commutative. In other words, $\mathrm{obl}_k^n \mathcal{W}^n(V) = \mathrm{obl}_k^\infty \mathbf{k}[V]$ for any $k < n$, where $\mathbf{k}[V]$ is the polynomial algebra.

The following property is crucial for our construction of the perturbative invariants: for any n-manifold M the complex $\int_M \mathcal{W}^n(V)$ has one dimensional cohomology (Theorem 3.1). I conjecture that, for any $k < n$, the factorization complex $\int_{N^k \times I^{n-k}} \mathcal{W}^n(V)$ is again a Weyl algebra for any k-dimensional manifold N^k.

Example 3.5. Let V be a vector space. Equip $V \oplus V^\vee[-1]$ with the standard form of degree -1. Then $\mathcal{W}^2(V \oplus V^\vee[-1])$ is the space of polyvector fields on $V^\vee$ and standard operations on it — the Gerstenhaber bracket and the cup product — are the operations of the cohomology of e_2.

As any Weyl algebra, $\mathcal{W}^2(V \oplus V^\vee[-1])$ is equivariant. Thus it is acted on by the operad fe_2 and by its cohomology, which is the BV operad. The operation Δ is equal to the de Rham differential, where the polyvector fields are identified with the differential forms by means of the constant volume form. Another choice of the volume form leads to another fe_2-structure with the same underlying e_2-structure.

3.4.3. *The action.* For associative (or e_1-) algebras the notion of modules plays the central role. The higher generalization of this notion is a e_n-algebra acting on a e_{n-1}-algebra, for the definition and the discussion see e. g. [Gin] and references therein. Constructively, it may be defined by means of the Swiss cheese operad, which is especially convenient for algebras over the operad of chains of the Fulton–MacPherson operad. In the same way as the operations of the little discs operad are given by the configuration spaces of $\mathbb{R}^n$, the operations of the Swiss cheese operad are given by the spaces of distinct points in $\mathbb{R}^{\geq 0} \times \mathbb{R}^{n-1}$. There are points of two types: those on the boundary and those in the interior. This gives a colored operad with two colors. If an e_n-algebra B acts on an e_{n-1}-algebra A, then elements of B sit on the interior points and elements of A — on boundary points. For further details we refer the reader to [Vor99].

Note that the action of the Swiss cheese operad may be formulated in terms of factorization sheaves; for the definition of the latter see e. g. [Gin] and references therein. Namely, such an action is equivalent to a factorization sheaf on the half-space such that its restriction to the boundary and to the interior are constant factorization sheaves, corresponding to the e_{n-1}-algebra A and the e_n-algebra B.

It is known that for any e_n-algebra there exists a universal e_{n+1} algebra $\mathrm{End}(A)$ acting on it ([Lur]). In other words, an action of an e_{n+1} algebra B on A is the same as a morphism of e_{n+1}-algebras $B \to \mathrm{End}(A)$. For an associative (or e_1-) algebra the End-object is its Hochschild cohomology complex.

Let V be a vector space. Equip $V \oplus V^\vee[1-n]$ with the standard form of degree $(1-n)$. Then $\mathcal{W}^n(V \oplus V^\vee[1-n])$ is $\mathrm{End}(\mathbf{k}[V])$, where $\mathbf{k}[V]$ is the polynomial algebra. In order to see it, one may construct an action of $\mathcal{W}^n(V \oplus V^\vee[1-n])$ on $\mathbf{k}[V]$ directly by using the Swiss cheese operad and the Fulton–MacPherson compactification. Then one need to check that the resulting map $\mathcal{W}^n(V \oplus V^\vee[1-n]) \to \mathrm{End}(\mathbf{k}[V])$ is a quasi-isomorphism.

This action commutes with taking the factorization complex. That is, if an equivariant e_{n+1}-algebra B acts on an equivariant e_n-algebra A, then for a k-manifold N

the e_{n-k+1}-algebra $\int_{N^k\times I^{n-k+1}} B$ acts on e_{n-k}-algebra $\int_{N^k\times I^{n-k}} A$. It follows immediately from definitions of the Swiss cheese operad and of the factorization complex. It seems plausible that under appropriate conditions $\int_{N^k\times I^{n-k+1}} \mathrm{End}(A) = \mathrm{End}(\int_{N^k\times I^{n-k}} A)$.

Example 3.6. Consider the polynomial algebra $A = \mathbf{k}[V]$ as an associative algebra. Its Hochschild cohomology complex $C^*(A, A)$ (which, as it was mentioned above, is $\mathcal{W}^2(V \oplus V^\vee[-1])$) acts on it. It follows, that $\int_{S^1} C^*(A, A)$, which is a e_1-algebra, acts on $\int_{S^1} A$. The latter complex is the Hochschild homology complex of A, which is known to be quasi-isomorphic to the direct sum of shifted differential forms (see e. g. [Lod98]). It is shown in [NT99] that the first complex is quasi-isomorphic to the differential operators on differential forms, and this is in good agreement with the speculation preceding the present example.

Recall, that in Example 3.4 for any an equivariant e_2-algebra A we construct action of e_1-algebra $\int_{S^1} A$ on the underlying complex of A. In the same way for any equivariant e_n-algebra A the e_1-algebra $\int_{S^{n-1}} A$ acts on the underlying complex of A: the action is given by gluing a n-ball and $S^{n-1} \times I$. It may be generalized even further. The factorization complex $\int_{S^k} A$, which is a e_{n-k}-algebra, analogously acts on e_{n-k-1}-algebra $\mathrm{obl}^n_{n-k-1} A$. As this action plays a crucial role in the next Section, let us phrase it below as the construction.

Construction 3.1. Let A be an equivariant e_n-algebra. Then, for any $k < n$, the e_{n-k}-algebra $\int_{S^k} A$ naturally acts on $\mathrm{obl}^n_{n-k-1} A$. The corresponding action of the Swiss cheese operad is defined as follows. Embed $\mathbb{R}^{\geq 0} \times \mathbb{R}^{n-k-1}$ linearly into $\mathbb{R}^n$. Put at any point of this half-space the factorization complex of A on the k-sphere lying into the $k+1$ space perpendicular to the half-space, with its center on $0\times\mathbb{R}^{n-k-1}$ and passing through this point. In particular, for points on $0\times\mathbb{R}^{n-k-1}$ we get the sphere of zero diameter, that is a point and the factorization complex is A itself.

In other words, consider a map $\mathbb{R}^n \to \mathbb{R}^{\geq 0} \times \mathbb{R}^{n-k-1}$ which sends a point to the pair which consists of the distance from the point to the subspace $\{0\}\times\mathbb{R}^{n-k-1}$ and the orthogonal projection on $\mathbb{R}^{n-k-1}$. Then the direct image of the factorization sheaf on $\mathbb{R}^n$ corresponding to A is the desired factorization sheaf on $\mathbb{R}^{\geq 0}\times\mathbb{R}^{n-k-1}$.

3.4.4. *Wilson loop.* Given a Lie algebra $\mathfrak{g}$ with an invariant scalar product, in Chapter Subsection 3.3.4 a e_3-*dg*-algebra $\mathrm{Ch}^\bullet_h(\mathfrak{g})$ is defined as follows. Take the Weyl 3-algebra given by the space $\mathfrak{g}^\vee[1]$ with the scalar product and equip it with a differential $\frac{1}{h}\{\cdot, q\}$, where $\{\ ,\ \}$ is the image of the Lie bracket under the map $L_\infty \to e_3$ (see e. g. Proposition 3.2) and q is the degree 3 element, which is the composition of the Lie bracket on g and the scalar product. Call this e_3-algebra the quantum Chevalley–Eilenberg algebra.

Consider the Hochschild complex $C_*(\mathrm{Ch}^\bullet_h(\mathfrak{g}))$. Here and in what follows we will consider unbounded Hochschild chains, that is, the Hochschild complex which is the direct product of its terms.

This Hochschild complex is the factorization complex $\int_{S^1} \mathrm{Ch}^\bullet_h(\mathfrak{g})$. As $\mathrm{Ch}^\bullet_h(\mathfrak{g})$ is e_3-algebra, the Hochschild complex is an e_2-algebra. Consider it as an e_1-algebra, that is take $\mathrm{obl}^2_1 \int_{S^1} \mathrm{Ch}^\bullet_h(\mathfrak{g})$. By the very definition it is equal to $\int_{S^1} \mathrm{obl}^3_2 \mathrm{Ch}^\bullet_h(\mathfrak{g})$. We mentioned above an important property of Weyl algebras: $\mathrm{obl}^n_k \mathcal{W}^n(V) = \mathrm{obl}^\infty_k \mathbf{k}[V]$

for any $k < n$. It follows, that $\mathrm{obl}_2^3 \mathrm{Ch}_h^\bullet(\mathfrak{g}) = \mathrm{obl}_2^\infty \mathrm{Ch}^\bullet(\mathfrak{g})$. Thus $\mathrm{obl}_2^3 \mathrm{Ch}_h^\bullet(\mathfrak{g})$ is just the super-commutative Chevalley–Eilenberg algebra. Its Hochschild complex is again a super-commutative algebra quasi-isomorphic to $\mathrm{Ch}^\bullet(\mathfrak{g}, \mathbf{k}[[\mathfrak{g}^\vee]]^{ad})$ by Proposition 3.15.To recap, $\int_{S^1} \mathrm{Ch}_h^\bullet(\mathfrak{g})$ as e_1-algebra, that is $\mathrm{obl}_1^2 \int_{S^1} \mathrm{Ch}_h^\bullet(\mathfrak{g})$ is isomorphic to $\mathrm{Ch}^\bullet(\mathfrak{g}, \mathbf{k}[[\mathfrak{g}^\vee]]^{ad})$.

Now, let us apply the construction from the previous section to $A = \mathrm{Ch}_h^\bullet(\mathfrak{g})$, $n = 3$ and $k = 1$. It gives an action of the e_2-algebra $\int_{S^1} \mathrm{Ch}_h^\bullet(\mathfrak{g})$ on $\mathrm{obl}_1^3 \mathrm{Ch}_h^\bullet(\mathfrak{g})$, which is $\mathrm{obl}_1^\infty \mathrm{Ch}^\bullet(\mathfrak{g})$. That is we get a map from the e_2-algebra $\mathrm{Ch}^\bullet(\mathfrak{g}, \mathbf{k}[[\mathfrak{g}^\vee]]^{ad})$ to the Hochschild cohomology complex of $\mathrm{Ch}^\bullet(\mathfrak{g})$ by the universal property, which is easily seen to be a quasi-isomorphism. The Hochschild cohomology complex of $\mathrm{Ch}^\bullet(\mathfrak{g})$ is known to be equal to $\mathrm{Ch}^\bullet(\mathfrak{g}, U_{\mathfrak{g}}^{ad})$, where $U_{\mathfrak{g}}$ is the universal enveloping algebra of $\mathfrak{g}$.

To be more precise, in this way we get a map from $\mathrm{Ch}^\bullet(\mathfrak{g}, \mathbf{k}[[\mathfrak{g}^\vee]]^{ad})$ to $\mathrm{Ch}^\bullet(\mathfrak{g}, U_{\mathfrak{g}}^{ad}) \otimes \mathbf{k}[[h]]$. The e_1-structure on this complex comes from the one on the universal enveloping algebra. On the other hand, as it is shown in the previous paragraph, $\int_{S^1} \mathrm{Ch}_h^\bullet(\mathfrak{g})$ as e_1-algebra isomorphic to $\mathrm{Ch}^\bullet(\mathfrak{g}, \mathbf{k}[[\mathfrak{g}^\vee]]^{ad})$. Thus, an explicit form of this map, which is supplied by the proposition below, implies the Duflo isomorphism.

Proposition 3.21. *The map of complexes*

$$\mathrm{Ch}^\bullet(\mathfrak{g}, \mathbf{k}[[\mathfrak{g}^\vee]]^{ad}) = \int_{S^1} \mathrm{Ch}_h^\bullet(\mathfrak{g}) \to \mathrm{Ch}^\bullet(\mathfrak{g}, U_{\mathfrak{g}}^{ad}) \otimes \mathbf{k}[[h]] \tag{43}$$

as above is chain homotopic to the map induced by the composition

$$\mathbf{k}[[\mathfrak{g}^\vee]] \xrightarrow{\exp(h(\cdot,\cdot))} S^*\mathfrak{g} \otimes \mathbf{k}[[h]] \xrightarrow{j^{\frac{1}{2}}} S^*\mathfrak{g} \otimes \mathbf{k}[[h]] \xrightarrow{PBW} U_{\mathfrak{g}} \otimes \mathbf{k}[[h]], \tag{44}$$

where the first arrow is given by the scalar product multiplied by h, the second is the contraction with the Duflo character (41) and the third one is the PBW map.

Sketch of proof. As it was mentioned above, $\mathrm{Ch}_h^\bullet(\mathfrak{g})$ as an e_2-algebra is isomorphic to the commutative algebra $\mathrm{Ch}^\bullet(\mathfrak{g})$. It follows that the map induced by the unit embedding $\mathrm{Ch}^\bullet(\mathfrak{g}) \to \mathrm{Ch}^\bullet(\mathfrak{g}, \mathbf{k}[[\mathfrak{g}^\vee]]^{ad})$ is a morphism of e_2-algebras and in composition with (43) it gives the standard map $\mathrm{Ch}^\bullet(\mathfrak{g}) \to \mathrm{Ch}^\bullet(\mathfrak{g}, U_{\mathfrak{g}}^{ad})$. Thus we know the image of the subalgebra $\mathrm{Ch}^\bullet(\mathfrak{g})$ under (43). One may see that the whole map (43) may be uniquely determined from it as the unique extension compatible with the Lie bracket coming from the e_2-structure. To see this one may use the faithful action of $\int_{S^1\times S^1} \mathrm{Ch}_h^\bullet(\mathfrak{g})$ on $\int_{S^1} \mathrm{Ch}_h^\bullet(\mathfrak{g})$ as in the sketch of the proof of Proposition 3.22.

So our immediate purpose is to calculate the bracket on $\mathrm{Ch}^\bullet(\mathfrak{g}, \mathbf{k}[[\mathfrak{g}^\vee]])$, which is $\int_{S^1} \mathrm{Ch}_h^\bullet(\mathfrak{g})$. As we will see below, it is enough to calculate the bracket with an element which is image of $a \in \mathrm{Ch}^\bullet(\mathfrak{g})$ under the embedding map as above. Given an element $b \in \int_{S^1} \mathrm{Ch}_h^\bullet(\mathfrak{g})$, the bracket $\{a, b\}$ may be interpreted geometrically as follows. Consider the solid torus $D^2 \times S^1$ and two circles in it: $C = (0, S^1)$, call it the big one, and $c = (\{x \in D^2 \mid |x| = 1/2\}, *)$, call it the small one. The cycle in the factorization complex of the solid torus, which is $\int_{S^1} \mathrm{Ch}_h^\bullet(\mathfrak{g})$, representing $\{a, b\}$ equals $C_b \otimes ([c] \otimes a)$, where by C_b we denote the image of b in $\int_{D^2\times S^1} \mathrm{Ch}_h^\bullet(\mathfrak{g})$ under the embedding $C \hookrightarrow D^2 \times S^1$. One may see that cycle $[c] \otimes a$ is equal to $c_{d_{dR}a}$, where d_{dR} is the de Rham differential. If $a = x_1 \wedge \cdots \wedge x_i$, then $d_{dR}x = \sum \pm d_{dR}x_i\, x_1 \wedge \ldots \hat{x}_i \cdots \wedge x_n$.

Let us now start pulling the small circle to unlink it from the big one. That is, consider a family of cycles c^t where c^t is a family of circles in the solid torus such that c^0 is the small circle, c^1 is a circle unlinked with the big circle and only one circle in the family intersects the big one. Until the circles do not intersect, nothing happens and the cycle $C_b \otimes c_a^t$ remains in the same class. But, as soon as they intersect each other, this class is changed by the class which is a derivation of b. The calculation shows that for $b = d_{dR}x_0\, x_1 \wedge \cdots \wedge x_n$ it is given by the sum of maps (36) contracted with x_0 and multiplied by $x_1 \wedge \cdots \wedge x_n$. The reasoning is analogous to Proposition 3.19: unlinking influences only around the intersection point. When the small circle is unlinked from the big one, $C_b \otimes c_a^1$ vanishes, because $c_a^1 = [c^1] \otimes a$ is a boundary.

Note, that the e_2-algebra $\mathrm{Ch}^\bullet(\mathfrak{g}, \mathbf{k}[[\mathfrak{g}^\vee]]^{ad})$ is, in fact, a fe_2-algebra. Thus, instead of the Lie bracket, one may calculate the operator Δ corresponding to the rotation. Given an element $x = \sum a_i b_i \in \mathrm{Ch}^\bullet(\mathfrak{g}, \mathbf{k}[[\mathfrak{g}^\vee]]^{ad})$, where a_i are in the odd part and b_i in the even part, one may show, that

$$\Delta x = \sum \{a_i, b_i\}.$$

Apply the calculations from the previous paragraph to it. Comparing it with Proposition 3.19 we see, that the operator Δ on $\mathrm{Ch}^\bullet(\mathfrak{g}, \mathbf{k}[[\mathfrak{g}^\vee]]^{ad})$ coincides with the operator d_0 from there. Proposition 3.20 implies that the Duflo character gives an isomorphism between this operator and d_{Br}. In order to complete the proof, one has to verify that d_{Br} is the operator Δ for the fe_2-algebra $\mathrm{Ch}^\bullet(\mathfrak{g}, U_{\mathfrak{g}}^{ad})$. □

While proving the proposition we found that the operator Δ on the fe_2-algebra $\int_{S^1} \mathrm{Ch}_h^\bullet(\mathfrak{g})$ is equal to the first order deformation of the Hochschild differential of $\mathrm{Ch}^\bullet(\mathfrak{g})$ that we discussed in the first section. I have no explanation for this coincidence.

3.4.5. *Invariants of knots.* In Subsection 3.3.3 we constructed invariants of manifolds using Weyl n-algebras. Below we develop this idea for manifolds with embedded links. Let us restrict ourselves to a 3-sphere with a knot in it.

As we know, the cohomology of the factorization complex of the Weyl n-algebra $\mathcal{W}^n(V)$ on a closed n-manifold is one-dimensional. If V lies in degree 1 and the manifold is a 3-sphere (or a homology sphere), then the generator of this cohomology is given by the class $[p] \otimes S^{\mathrm{top}}V$, where p is a point in the manifold. As it was explained in Subsection 3.3.4, the factorization complex $\int_{S^3} \mathrm{Ch}_h^\bullet(\mathfrak{g})$ is isomorphic to the complex of the underlying Weyl 3-algebra. Since the Chevalley–Eilenberg differential is inner, one needs to consider here unbounded chains that is, take direct product rather than the direct sum. It is easy to see that the generator in the cohomology of $\int_{S^3} \mathrm{Ch}_h^\bullet(\mathfrak{g})$ is given by $[p] \otimes S^{\mathrm{top}}\mathfrak{g}^\vee$. Call it the standard cycle. The idea of invariants we construct is to produce another cycle and compare it with the standard one.

Given a knot $K\colon S^1 \hookrightarrow S^3$ and a class $f \in \int_{S^1} \mathrm{Ch}^\bullet(\mathfrak{g}) = \mathrm{Ch}^\bullet(\mathfrak{g}, \mathbf{k}[[\mathfrak{g}^\vee]]^{ad})$, denote by K_f the direct image of this class under K. The class we are interested in is $([p] \otimes S^{\mathrm{top}}\mathfrak{g}^\vee) \otimes K_f$. For dimensional reasons, only f of degree 0 are interesting, in fact, $f \in \mathbf{k}[[\mathfrak{g}^\vee]]^{inv}$. Thus we get the following definition.

Definition 3.11. For a knot K in $\mathbb{R}^3$ the Wilson loop invariant is the function on $\mathbf{k}[[\mathfrak{g}^\vee]]^{inv}$ given by

$$f \mapsto ([\infty] \otimes S^{\mathrm{top}}(\mathfrak{g}^\vee[1])) \otimes K_f \in \int_{S^3} \mathrm{Ch}_h^\bullet(\mathfrak{g}),$$

where we identify $\int_{S^3} \mathrm{Ch}_h^\bullet(\mathfrak{g})$ with $\mathbf{k}[[h]]$ using the standard cycle as the generator.

In Subsection 3.3.3 it is showed that invariants constructed there are described by formulas similar to formulas for the Axelrod–Singer invariants. Following the same line, we see that the Wilson loop invariants are connected with Bott–Taubes invariants; for a survey of the latter see e. g. [Vol07]. There is another invariant of knots — the Kontsevich integral, see [CDM12, Part 3]. In principle, it should coincide with the Bott–Taubes invariants, see [Kon94]. As far as I know, this point is not clear, for discussion see [Les02]. One may hope that the definition above will help to elucidate this.

Our construction of the Wilson loop invariant depends on the choice of a Lie algebra with a scalar product. One may give a more complicated, but universal definition of these invariants with values in the graph complex, which is the Chevalley–Eilenberg complex of Hamiltonian vector fields, in the same way as it is outlined in Subsection 3.3.4.

An interesting property of the Kontsevich integral is its value on the unknot: it is equal to the Duflo character and this allows to prove the Duflo isomorphism, see [BLT03] and [CDM12, Ch. 11]. The following proposition states that the Wilson loop invariant shares this property.

Proposition 3.22. *The Wilson loop invariant of the unknot is equal to the composition*

$$\mathbf{k}[[\mathfrak{g}^\vee]]^{inv} \hookrightarrow \mathbf{k}[[\mathfrak{g}^\vee]] \to U_{\mathfrak{g}} \otimes \mathbf{k}[[h]] \to \mathbf{k}[[h]],$$

where the second arrow is given by (44) and the third one is the standard augmentation.

This Proposition confirms that our definition of the Kontsevich integral coincides with the standard one.

There is another application of Construction 3.1 above. As it was mentioned above in a particular case, integration of a n-Weyl algebra $\mathcal{W}$ on the pair (k-disk, $k-1$-sphere) gives an action of $\int_{S^{k-1}} \mathcal{W}$ on $\mathcal{W}$ itself as on e_{n-k-1}-algebra. By the universal property of the higher Hochschild cohomology (see [Tho16]), it gives a map of e_{n-k}-algebras from $\int_{S^{k-1}} \mathcal{W}$ to the higher Hochschild cohomology of e_{n-k-1}-algebra $\mathcal{W}$. One may show that this is quasi-isomorphism.

In [Mar21] we show that this map may be used to build a formality morphism, that is a quasi-isomorphism between the Lie algebra of higher Hochschild cohomology of polynomial algebra and the Lie algebra of appropriate polyvector fields.

The classical paper [Kon03] describes a way to build the formality (only for usual Hochschild cohomology, but the construction may be generalized to higher ones), which depends on a choice of a propagator. Moreover, there a specific propagator is chosen and the corresponding formality morphism is written down. Our formality is also given by a propagator, but this propagator differs from the one chosen in [Kon03]. A surprising consequence of the geometric nature of this new propagator is the rationality of coefficients of this formality morphism.

4. Operads and MZV

My scientific interests are largely centered around the concept of operads. Specifically, I focus on operads of little disks and their modifications. Factorization homologies are intricately linked to it.

My joint paper with A. Khoroshkin and S. Shadrin [KMS13] is devoted to the study of the hypercommutative operad, formed by homologies of the Deligne–Mumford compactification of the moduli space of stable marked curves of genus 0. We give an explicit formula for a quasi-isomorphism between the operads Hycomm (the homology of the moduli space of stable genus 0 curves) and BV/Δ (the homotopy quotient of Batalin-Vilkovisky operad by the BV-operator). In other words, we derive an equivalence of Hycomm-algebras and BV-algebras enhanced with a homotopy that trivializes the BV-operator.

Because the operad of little 2-disks can be realized in the category of Tate motives, it possesses additional structure. Understanding this structure has been occupying my attention in recent years. The only result of these efforts so far is my paper [Mar23] about multiple zeta values. It essentially investigates the Hodge structure on the operad of little 2-disks.

Multiple zeta values is an important series of numbers. It plays fundamental role in number theory, algebraic geometry, mathematical physics. These numbers are defined as follows.

Call a finite sequence of natural numbers $(k_1, \dots, k_n)$ convergent if $k_1 \geq 2$. For a convergent sequence $\mathbf{k} = (k_1, \dots, k_n)$ the multiple zeta value is defined by the integral (see e. g. [IKZ06])

$$\zeta(\mathbf{k}) = \int_{\Delta_{w(\mathbf{k})}} \omega_1(t_1) \wedge \cdots \wedge \omega_{w(\mathbf{k})}(t_{w(\mathbf{k})}), \tag{45}$$

where $\Delta_{w(\mathbf{k})} = \{1 > t_1 > \cdots > t_{w(\mathbf{k})} > 0\}$, $w(\mathbf{k}) = k_1 + \cdots + k_n$ is weight of the sequence and

$$\omega_i(t) = \begin{cases} dt/(1-t) & \text{if } i \in \{k_1, k_1+k_2, \dots, k_1+\cdots+k_n\} \\ dt/t & \text{otherwise} \end{cases}$$

Thus, multiple zeta values are values of iterated integrals.

A more conventional way to define multiple zeta values is by series representation ascending to Euler. For a convergent $\mathbf{k} = (k_1, \dots, k_n)$ as above,

$$\zeta(\mathbf{k}) = \sum_{\substack{l_1 > l_2 > \cdots > l_n > 0 \\ l_i \in \mathbb{N}}} \frac{1}{l_1^{k_1} l_2^{k_2} \cdots l_n^{k_n}} \tag{46}$$

To ensure that formulas (45) and (46) are consistent one may calculate integral (45) iteratively by all variables t_i. There is another way to do it. Define *cubical coordinates* on the standard simplex $\Delta_k = \{1 > t_1 > \cdots > t_k > 0\}$ by

$$x_1 = t_1 \quad x_2 = t_2/t_1 \quad \dots \quad x_k = t_k/t_{k-1}.$$

Coordinates t_i are called *simplicial coordinates*. In cubical coordinates definition (45) looks as

$$\zeta(\mathbf{k}) = \int_{\square} \frac{x_1 \cdots x_{k_1}}{1 - x_1 \cdots x_{k_1}} \frac{x_1 \cdots x_{k_1+k_2}}{1 - x_1 \cdots x_{k_1+k_2}} \cdots \frac{x_1 \cdots x_{w(\mathbf{k})}}{1 - x_1 \cdots x_{w(\mathbf{k})}} \, dV, \tag{47}$$

where $dV = dx_1/x_1 \wedge dx_2/x_2 \dots$ is the standard volume form on the torus and symbol $\square$ here and below means the unit cube $\{0 < x_i < 1\}$. Expanding the integrand of (47) into a series and integrating term-wise we obtain (46).

Multiple zeta values form an algebra over rational numbers. A product of two of them may be presented as a linear combination of multiple zeta values with integer coefficients by means of each representation: as integrals and as number series. It gives two systems of relations which multiple zeta values obey. We are interested only in geometric relations between MZVs az periods of integrals. One may say, that we are dealing with motivic MZVs.

The first set of relations is called shuffle relations. They immediately follow from Fubini's theorem applied to the integral representation (46). They are defined as follows.

With a finite sequence of natural numbers $\mathbf{k} = (k_1, \dots, k_n)$ associate the word $z_{\mathbf{k}} = x^{k_1-1}yx^{k_2-1}\dots x^{k_{n-1}}y$ of two letters x and y. It establishes a bijection between sequences and words ending in y.

A finite multiset is an unordered finite list with possible repetitions. For a multiset M, denote by $x \cdot M$ the result of the action of operation $x\cdot$ on M elementwise.

Define shuffle product $\mathrm{sh}(\,\cdot\,,\,\cdot\,)$ of two words in letters x and y as a multiset of words given by the recursive rule

$$\mathrm{sh}(v \cdot z_1, u \cdot z_2) = v \cdot \mathrm{sh}(z_1, u \cdot z_2) \cup u \cdot \mathrm{sh}(v \cdot z_1, z_2), \tag{48}$$

where $u, v \in \{x, y\}$ and $\mathrm{sh}(1, z) = \mathrm{sh}(z, 1) = \{z\}$.

One may see that the shuffle product of two words ending in y consists of words ending in y. It defines the shuffle product of sequences of natural numbers, which we denote likewise by $\mathrm{sh}(\,\cdot\,,\,\cdot\,)$.

Shuffle relations:

Let $\mathbf{k}$ and $\mathbf{l}$ be convergent sequences. Then

$$\zeta(\mathbf{k})\,\zeta(\mathbf{l}) = \sum_{\mathbf{s}\in \mathrm{sh}(\mathbf{k},\mathbf{l})} \zeta(\mathbf{s}) \tag{49}$$

Example 4.1.

$$\zeta(2)\,\zeta(2) = \left(\int_{1>t_1>t_2>0} \frac{dt_1}{t_1}\frac{dt_2}{1-t_2}\right)\left(\int_{1>t_1>t_2>0} \frac{dt_1}{t_1}\frac{dt_2}{1-t_2}\right) =$$
$$\int_{1>t_1>t_2>t_3>t_4>0} \left(4\frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dt_3}{1-t_3}\frac{dt_4}{1-t_4} + 2\frac{dt_1}{t_1}\frac{dt_2}{1-t_2}\frac{dt_3}{t_3}\frac{dt_4}{1-t_4}\right) =$$
$$4\,\zeta(3,1) + 2\,\zeta(2,2)$$

To prove the second equality we divide a product of two simplices in simplices and apply the Fubini theorem.

The second family of relations is called stuffle relations and is given by rearrangement of summands in the product of number series.

For a finite sequence of natural numbers $\mathbf{k} = (k_1, \dots, k_{n-1})$, denote the sequence $(k_1, \dots, k_{n-1}, k_n)$ by $\mathbf{k} \cdot k_n$. Introduce the empty sequence $()$ such that $() \cdot k = (k)$.

Define stuffle product $\mathrm{st}(\,\cdot\,,\,\cdot\,)$ of two sequences as a multiset of sequences given by the recursive rule

$$\mathrm{st}(\mathbf{k}\cdot x, \mathbf{l}\cdot y) = \mathrm{st}(\mathbf{k}\cdot x, \mathbf{l})\cdot y \cup \mathrm{st}(\mathbf{k}, \mathbf{l}\cdot y)\cdot x \cup \mathrm{st}(\mathbf{k}, \mathbf{l})\cdot (x+y) \tag{50}$$

and by $\mathrm{st}((),\mathbf{k}) = \mathrm{st}(\mathbf{k},()) = \{\mathbf{k}\}$. Note that $\mathrm{st}(\mathbf{k},\mathbf{l}) = \mathrm{st}(\mathbf{l},\mathbf{k})$.

Stuffle relations:

Let $\mathbf{k}$ and $\mathbf{l}$ be convergent sequences. Then

$$\zeta(\mathbf{k})\,\zeta(\mathbf{l}) = \sum_{\mathbf{s}\in\mathrm{st}(\mathbf{k},\mathbf{l})} \zeta(\mathbf{s}) \tag{51}$$

Stuffle relations follow easily from the series representation of multiple zeta values by rearranging terms of the product. The idea is clear from the following example.

Example 4.2.

$$\zeta(2)\,\zeta(2) = \sum_l \frac{1}{l^2}\cdot\sum_m \frac{1}{m^2} = \left(\sum_{l>m} + \sum_{l<m} + \sum_{l=m}\right)\frac{1}{l^2m^2} = 2\,\zeta(2,2) + \zeta(4) \tag{52}$$

Stuffle relations are not so evident in the integral presentation, see [Gol86; Bro09; Sou10].

Example 4.3. Let us prove (52) in the integral presentation.

In cubical coordiantes we have

$$\zeta(2) = \int_\square \frac{dx_1dx_2}{1-x_1x_2}$$

$$\zeta(4) = \int_\square \frac{dx_1dx_2dx_3dx_4}{1-x_1x_2x_3x_4}$$

$$\zeta(2,2) = \int_\square \frac{x_1x_2\,dx_1dx_2dx_3dx_4}{(1-x_1x_2)(1-x_1x_2x_3x_4)}$$

$$\zeta(2)\,\zeta(2) = \int_\square \frac{dx_1dx_2}{(1-x_1x_2)}\,\frac{dx_3dx_4}{(1-x_3x_4)}$$

For any variables α and β we have the equality:

$$\frac{1}{(1-\alpha)(1-\beta)} = \frac{\alpha}{(1-\alpha)(1-\alpha\beta)} + \frac{\beta}{(1-\beta)(1-\beta\alpha)} + \frac{1}{1-\alpha\beta} \tag{53}$$

Substituting $\alpha = x_1x_2$ and $\beta = x_3x_4$ we get the stuffle relation:

$$\begin{aligned}\zeta(2)\zeta(2) = \\ \int_\square\Bigg(\frac{x_1x_2}{(1-x_1x_2)(1-x_1x_2x_3x_4)} &+ \frac{x_3x_4}{(1-x_3x_4)(1-x_3x_4x_1x_2)} \\ &+\frac{1}{1-x_1x_2x_3x_4}\Bigg)\,dx_1dx_2dx_3dx_4 = \\ \zeta(2,2)+\zeta(2,2)+\zeta(4)\end{aligned} \tag{54}$$

Analyzing this example, we see that there are three techniques used in the proof. The first is a variant of Fubini's theorem, which allows to rewrite the product of integrals as a single integral. This, as it was observed in [Bro09], follows from the generalized shuffle relations we will discuss below. Then, we used the formula (53), which is essentially a form of Arnold's relations between differential forms on $\mathcal{M}_{0,n+3}$. Finally, if we look at the first two summands under the integral in (54) we can see that they differ by the order of the coordinates. Permutation of the cubical coordinates obviously does not affect the result of the integration. However, in simplicial coordinates, this permutation gives a birational transformation. My main observation is that the invariance of integrals with respect to permutations of cubical coordinates of a certain type ("flips") implies certain relations on the integrals, which I call generalized stuffle relations, see below.

Stuffle relations may be extended by regularizations of some equalities with divergent series, see [IKZ06; Rac02]. This extended system of relations is called regularized double shuffle relations.

Multiple zeta values of weight n being values of integrals are periods of the pair $(\mathcal{M}^{\delta}_{0,n+3}, \mathcal{M}^{\delta}_{0,n+3} \setminus \mathcal{M}_{0,n+3})$ of a special kind ([GM04]), where $\mathcal{M}^{\delta}_{0,n+3}$ is a partial compactification of $\mathcal{M}_{0,n+3}$ introduced in [Bro09]. In [Bro09; BCS10] all periods of such pairs were studied. In [BCS10] they were called *cell-zeta values*.

As well as multiple zeta values, cell-zeta values obey a lot of relations over rational numbers. By the one of the main results of [Bro09], all cell-zeta values are rational combinations of multiple zeta values. In light of this, it is natural to try to find a set of relations on cell-zeta values, which allows to express any cell-zeta value in terms of multiple zeta values and implies all known relation on multiple zeta values. In [Mar23] I suggest a candidate for this, which consists of two families of relations. In [BCS10] another system of relations on cell-zeta values is written down, which is a subset of my relations.

The first family containing quadratic-linear relations was introduced in [Bro09]. It is analogous to shuffle relations and follows from Fubini's theorem. In [BCS10] these relations called product map relations. We suggest the term "generalized shuffle relations" to emphasize the similarity between our pair of families with the pair of shuffle and stuffle relations.

Let $\underline{\mathbf{3}}$ be a cyclically ordered set with three elements. Define a 3-pointed cyclically ordered set $\mathcal{T}$ as a pair of a cyclically ordered set T and a monotonic embedding $\imath\colon \underline{\mathbf{3}} \hookrightarrow T$.

Let $\mathcal{T}_{1,2}$ be a pair of 3-pointed cyclically ordered sets and $\imath_{1,2}\colon \underline{\mathbf{3}} \hookrightarrow T_{1,2}$ are corresponding embeddings. Let $T_1 \coprod_{\underline{\mathbf{3}}} T_2$ be the colimit of the diagram in the category of sets given by these embeddings. Denote by $\mathfrak{sh}(\mathcal{T}_1, \mathcal{T}_2)$ the set of cyclically ordered sets given by all cyclic orders on $T_1 \coprod_{\underline{\mathbf{3}}} T_2$ for which projections on T_1 and T_2 are monotonic.

For any $C \in \mathfrak{sh}(\mathcal{T}_1, \mathcal{T}_2)$ consider the map

$$\beta_C\colon \mathcal{M}^{\delta}_{0,C} \to \mathcal{M}^{\delta}_{0,T_1} \times \mathcal{M}^{\delta}_{0,T_2}, \tag{55}$$

which is the forgetful map on each factor. In [Bro09, p. 2.7] this map is called the product map.

The following proposition is taken from [Bro09; BCS10], where it is called product map relations.

Generalized shuffle relations:

Using notations as above let ϕ and ψ be regular top-degree differential forms on $\mathcal{M}^{\delta}_{0,T_1}$ and $\mathcal{M}^{\delta}_{0,T_2}$ correspondingly. Then

$$\left(\int_{\Delta(T_1)} \phi\right) \cdot \left(\int_{\Delta(T_2)} \psi\right) = \sum_{C \in \mathfrak{sh}(\mathcal{T}_1, \mathcal{T}_2)} \int_{\Delta(C)} \beta_C^{*}(\phi \boxtimes \psi) \tag{56}$$

The second family is new. I call it generalized stuffle relations. This is a family of linear relations following from the relative version of Fubini's theorem.

Let $\underline{\mathbf{4}}$ be a cyclically ordered set with four elements. Define a 4-pointed cyclically ordered set $\mathcal{T}$ as a pair of a cyclically ordered set T and a monotonic embedding $\imath\colon \underline{\mathbf{4}} \hookrightarrow T$.

The Klein four-group V acts on $\underline{\mathbf{4}}$. Half of this group respects the cyclic order and the other half reverses it. For $\nu \in V$ and a 4-pointed cyclically ordered set $\mathcal{T} = (T, \imath)$ denote by $\mathcal{T}^\nu$ the 4-pointed cyclically ordered set with the embedding equal to $\imath$ composed with ν and with the cyclic ordered set equal to T or to T^{op} depending on whether ν respects cyclic order on $\underline{\mathbf{4}}$ or not, where $\cdot^{op}$ means the same set with the opposite order. Denote the latter cyclically ordered set by T^ν.

Let $\mathcal{T}_{1,2}$ be a pair of 4-pointed cyclically ordered sets and $\imath_{1,2}\colon \underline{\mathbf{4}} \hookrightarrow T_{1,2}$ are corresponding embeddings. Let $T_1 \coprod_{\underline{\mathbf{4}}} T_2$ be the colimit of the diagram in the category of sets given by these embeddings. Denote by $\mathfrak{st}(\mathcal{T}_1, \mathcal{T}_2)$ the set of cyclically ordered sets given by all cyclic orders on $T_1 \coprod_{\underline{\mathbf{4}}} T_2$ for which projections on T_1 and T_2 are monotonic.

For any $\nu \in V$, $C \in \mathfrak{st}(\mathcal{T}_1, \mathcal{T}_2)$ and $C_\nu \in \mathfrak{st}(\mathcal{T}_1, \mathcal{T}_2^\nu)$ consider maps

$$\begin{tikzcd}[column sep=small] \gamma_C\colon \mathcal{M}^\delta_{0,C} \arrow[dr] & \\ & \mathcal{M}^\delta_{0,T_1} \times \mathcal{M}^\delta_{0,T_2} \\ \gamma_{C_\nu}\colon \mathcal{M}^\delta_{0,C_\nu} \arrow[ur] & \end{tikzcd} \tag{57}$$

which are forgetful map on each factor.

Generalized stuffle relations:

Using notations as above let ν be a non-trivial element of the Klein four-group and ϕ and ψ be regular differential forms on $\mathcal{M}^\delta_{0,T_1}$ and $\mathcal{M}^\delta_{0,T_2}$ correspondingly such that

$$\deg\phi + \deg\psi = |T_1| + |T_2| - 7$$

Then

$$\sum_{C \in \mathfrak{st}(\mathcal{T}_1, \mathcal{T}_2)} \int_{\Delta(C)} \gamma_C^*(\phi \boxtimes \psi) = \epsilon \cdot \sum_{C_\nu \in \mathfrak{st}(\mathcal{T}_1, \mathcal{T}_2^\nu)} \int_{\Delta(C_\nu)} \gamma_{C_\nu}^*(\phi \boxtimes \psi), \tag{58}$$

where $\epsilon = (-1)^{\frac{(|T_2|-3)(|T_2|-2)}{2}}$ if ν reverses the cyclic order on $\underline{\mathbf{4}}$ and $\epsilon = 1$ if not.

These relations generalize above-mentioned manipulations with integrals in cubical coordinates. It allows to prove the following theorem, which is the main result of the paper.

Theorem 4.1. *Generalized shuffle relations (56) and generalized stuffle relations (58) jointly imply shuffle relations (49) and stuffle relations (51).*

My papers

[Fin+99] Michael Finkelberg et al. "A note on a symplectic structure on the space of G-monopoles". *Comm. Math. Phys.* 201.2 (1999), pp. 411–421.

[GM24] Leonid Gorodetsky and Nikita Markarian. *On Conormal Lie Algebras of Feigin–Odesskii Poisson Structures.* arXiv:2403.02805. 2024.

[KMS13] A. Khoroshkin, N. Markarian, and S. Shadrin. "Hypercommutative Operad as a Homotopy Quotient of BV". *Communications in Mathematical Physics* 322.3 (Sept. 2013), pp. 697–729.

[Mar01] Nikita Markarian. *Poincaré–Birkoff–Witt isomorphism, Hochschild homology and Riemann–Roch theorem.* Preprints MPI Mathematik Bonn 2001-52. 2001.

[Mar09] Nikita Markarian. "The Atiyah class, Hochschild cohomology and the Riemann-Roch theorem". *J. Lond. Math. Soc. (2)* 79.1 (2009), pp. 129–143.

[Mar16] Nikita Markarian. "Weyl n-algebras and the Kontsevich integral of the unknot". *J. Knot Theory Ramifications* 25.12 (2016), 1642008, 18p.

[Mar17] Nikita Markarian. "Weyl n-algebras". *Comm. Math. Phys.* 350.2 (2017), pp. 421–442.

[Mar21] Nikita Markarian. "Weyl n-algebras and the Swiss cheese operad". *Forum Mathematicum* 32.2 (2021), pp. 531–545.

[Mar23] Nikita Markarian. "On generalized stuffle relations between cell-zeta values". *Communications in Number Theory and Physics* 17.1 (2023), pp. 63–76.

[MP23] Nikita Markarian and Alexander Polishchuk. "Compatible Feigin–Odesskii Poisson brackets". *Manuscripta Mathemtica* (May 2023), pp. 1–17.

[MP24] Nikita Markarian and Alexander Polishchuk. "Compatible Poisson brackets associated with elliptic curves in G(2,5)". *SIGMA* (2024). Forthcoming.

References

[AFT] David Ayala, John Francis, and Hiro Lee Tanaka. *Factorization homology of stratified spaces.* arXiv:1409.0848 [math.AT].

[Ale+97] Mikhail Alexandrov et al. "The geometry of the master equation and topological quantum field theory". *Internat. J. Modern Phys. A* 12.7 (1997), pp. 1405–1429.

[AM00] Anton Alekseev and Eckhard Meinrenken. "The non-commutative Weil algebra". *Invent. Math.* 139.1 (2000), pp. 135–172.

[AS92] Scott Axelrod and I. M. Singer. "Chern-Simons perturbation theory". *Proceedings of the XXth International Conference on Differential Geometric Methods in Theoretical Physics, Vol. 1, 2 (New York, 1991).* World Sci. Publ., River Edge, NJ, 1992, pp. 3–45.

[AS94] Scott Axelrod and I. M. Singer. "Chern-Simons perturbation theory. II". *J. Differential Geom.* 39.1 (1994), pp. 173–213.

[Ati57a] M. F. Atiyah. "Complex analytic connections in fibre bundles". *Trans. Amer. Math. Soc.* 85 (1957), pp. 181–207.

[Ati57b] M. F. Atiyah. "Vector Bundles over an Elliptic Curve". *Proc. London Math. Soc.* 7 (1957), pp. 414–452.

[Bal98] David Balavoine. "Homology and cohomology with coefficients, of an algebra over a quadratic operad". *J. Pure Appl. Algebra* 132.3 (1998), pp. 221–258.

[BC98] Raoul Bott and Alberto S. Cattaneo. "Integral invariants of 3-manifolds". *J. Differential Geom.* 48.1 (1998), pp. 91–133.

[BCS10] Francis Brown, Sarah Carr, and Leila Schneps. "The algebra of cell-zeta values". *Compositio Mathematica* 146.3 (2010), pp. 731–771.

[BD04] Alexander Beilinson and Vladimir Drinfeld. *Chiral algebras*. Vol. 51. American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 2004, pp. vi+375.

[BGI71] Pierre Berthelot, Alexander Grothendieck, and Luc Illusie. *Théorie des Intersections et Théorème de Riemann-Roch*. Vol. 225. Lecture notes in mathematics. Springer-Verlag, 1971.

[BLT03] Dror Bar-Natan, Thang T. Q. Le, and Dylan P. Thurston. "Two applications of elementary knot theory to Lie algebras and Vassiliev invariants". *Geom. Topol.* 7 (2003), 1–31 (electronic).

[BN] David Ben-Zvi and David Nadler. *Nonlinear Traces*. arXiv:1305.7175 [math.AG].

[Bro09] Francis C. S. Brown. "Multiple zeta values and periods of moduli spaces $\overline{\mathfrak{M}}_{0,n}(\mathbb{R})$". en. *Annales scientifiques de l'École Normale Supérieure* Ser. 4, 42.3 (2009), pp. 371–489.

[Bry88] Jean-Luc Brylinski. "A differential complex for Poisson manifolds". *J. Differential Geom.* 28.1 (1988), pp. 93–114.

[CDM12] Sergei Chmutov, Sergei Duzhin, and Jacob Mostovoy. *Introduction to Vassiliev knot invariants*. Cambridge University Press, Cambridge, 2012, pp. xvi+504.

[CG] Kevin Costello and Owen Gwilliam. *Factorization algebras in perturbative quantum field theory*. `http://math.northwestern.edu/~costello/factorization_public.html`.

[CGP21] Melody Chan, Søren Galatius, and Sam Payne. "Tropical curves, graph complexes, and top weight cohomology of $\mathcal{M}_g$". *J. Amer. Math. Soc.* 34 (2021), pp. 565–594.

[Con00] Brian Conrad. *Grothendieck duality and base change*. Vol. 1750. Lecture Notes in Mathematics. Berlin: Springer-Verlag, 2000, pp. vi+296.

[Cur] Justin Curry. *Sheaves, Cosheaves and Applications*. arXiv:1303.3255 [math.AT].

[Duf77] Michel Duflo. "Opérateurs différentiels bi-invariants sur un groupe de Lie". *Ann. Sci. École Norm. Sup. (4)* 10.2 (1977), pp. 265–288.

[FO95] B. Feigin and A. Odesskii. *Vector bundles on elliptic curve and Sklyanin algebras*. 1995.

[Fra] John Francis. *Factorization homology of topological manifolds*. arXiv:1206.5522 [math.AT].

[FT89] B. L. Feigin and B. L. Tsygan. "Riemann-Roch theorem and Lie algebra cohomology. I". *Proceedings of the Winter School on Geometry and Physics (Srni, 1988)*. 21. 1989, pp. 15–52.

[Get94] E. Getzler. "Batalin-Vilkovisky algebras and two-dimensional topological field theories". *Comm. Math. Phys.* 159.2 (1994), pp. 265–285.

[Gin] Grégory Ginot. *Notes on factorization algebras, factorization homology and applications*. arXiv:1307.5213 [math.AT].

[GJ] Ezra Getzler and J. D. S. Jones. *Operads, homotopy algebra and iterated integrals for double loop spaces*. arXiv:hep-th/9403055v1.

[GM04] A. B. Goncharov and Yu. I. Manin. "Multiple ζ-motives and moduli spaces $\overline{\mathcal{M}}_{0,n}$". *Compositio Mathematica* 140.1 (2004), pp. 1–14.

[Gol86] William M. Goldman. "Invariant functions on Lie groups and Hamiltonian flows of surface group representations". *Invent. Math.* 85.2 (1986), pp. 263–302.

[GTZ14] Grégory Ginot, Thomas Tradler, and Mahmoud Zeinalian. "Higher Hochschild homology, topological chiral homology and factorization algebras". *Comm. Math. Phys.* 326.3 (2014), pp. 635–686.

[Gwi12] Owen Gwilliam. *Factorization Algebras and Free Field Theories*. Thesis (Ph.D.)–Northwestern University. ProQuest LLC, Ann Arbor, MI, 2012, p. 282.

[Har66] Robin Hartshorne. *Residues and duality*. Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20. Berlin: Springer-Verlag, 1966, pp. vii+423.

[Hir66] F. Hirzebruch. *Topological methods in algebraic geometry*. Third enlarged edition. New appendix and translation from the second German edition by R. L. E. Schwarzenberger, with an additional section by A. Borel. Die Grundlehren der Mathematischen Wissenschaften, Band 131. Springer-Verlag New York, Inc., New York, 1966, pp. x+232.

[HKR62] G. Hochschild, Bertram Kostant, and Alex Rosenberg. "Differential forms on regular affine algebras". *Trans. Amer. Math. Soc.* 102 (1962), pp. 383–408.

[HP07] Zheng Hua and Alexander Polishchuk. *Elliptic bihamiltonian structures from relative shifted Poisson structures*. arXiv:2007.12351 [math.AG]. 2007.

[HP23] Zheng Hua and Alexander Polishchuk. "Elliptic bihamiltonian structures from relative shifted Poisson structures". *Journal of Topology* 16.4 (2023), pp. 1389–1422.

[IKZ06] Kentaro Ihara, Masanobu Kaneko, and Don Zagier. "Derivation and double shuffle relations for multiple zeta values". *Compositio Mathematica* 142.2 (2006), pp. 307–338.

[Ill71] Luc Illusie. *Complexe cotangent et déformations. I*. Berlin: Springer-Verlag, 1971, pp. xv+355.

[Joh] Theo Johnson-Freyd. *Homological perturbation theory for nonperturbative integrals*. arXiv:1206.5319 [math-ph].

[Kap97] Mikhail Kapranov. "Rozansky–Witten invariants via Atiyah classes". *Compositio Mathematica* 115 (1997), pp. 71–113.

[Kon03] Maxim Kontsevich. "Deformation quantization of Poisson manifolds". *Lett. Math. Phys.* 66 (2003), pp. 157–216.

[Kon93] Maxim Kontsevich. "Formal (non)commutative symplectic geometry". *The Gel'fand Mathematical Seminars, 1990–1992*. Birkhäuser Boston, Boston, MA, 1993, pp. 173–187.

[Kon94] Maxim Kontsevich. "Feynman diagrams and low-dimensional topology". *First European Congress of Mathematics, Vol. II (Paris, 1992)*. Vol. 120. Progr. Math. Birkhäuser, Basel, 1994, pp. 97–121.

[KP20] Grigory Kondyrev and Artem Prikhodko. "Categorical proof of holomorphic Atiyah-[Bott formula". *Journal of the Institute of Mathematics of Jussieu* 19.5 (2020), pp. 1739–1763.

[Kri11] Andrew Kricker. "Noncommutative Chern-Weil theory and the combinatorics of wheeling". *Duke Math. J.* 157.2 (2011), pp. 223–281.

[Les02] Christine Lescop. "About the uniqueness of the Kontsevich integral". *J. Knot Theory Ramifications* 11.5 (2002), pp. 759–780.

[Lod98] Jean-Louis Loday. *Cyclic homology.* Second. Vol. 301. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Berlin: Springer-Verlag, 1998, pp. xx+513.

[Lur] Jacob Lurie. *Higher Algebra.* `http://www.math.harvard.edu/~lurie/papers/HigherAlgebra.pdf`.

[NT99] Ryszard Nest and Boris Tsygan. "On the cohomology ring of an algebra". *Advances in geometry.* Vol. 172. Progr. Math. Birkhäuser Boston, Boston, MA, 1999, pp. 337–370.

[OTT81] Nigel R. O'Brian, Domingo Toledo, and Yue Lin L. Tong. "The trace map and characteristic classes for coherent sheaves". *Amer. J. Math.* 103.2 (1981), pp. 225–252.

[OW13] Alexander Odesskii and Thomas Wolf. "Compatible quadratic Poisson brackets related to a family of elliptic curves". *Journal of Geometry and Physics* 63.0 (2013), pp. 107–117.

[Pol98] A. Polishchuk. "Poisson structures and birational morphisms associated with bundles on elliptic curves". *IMRN* 13 (1998), pp. 683–703.

[QZ11] Jian Qiu and Maxim Zabzine. "Introduction to graded geometry, Batalin-Vilkovisky formalism and their applications". *Arch. Math. (Brno)* 47.5 (2011), pp. 415–471.

[Rac02] Georges Racinet. "Doubles mélanges des polylogarithmes multiples aux racines de l'unité". fr. *Publications Mathématiques de l'IHÉS* 95 (2002), pp. 185–231.

[Reu93] Christophe Reutenauer. *Free Lie algebras.* Vol. 7. London Mathematical Society Monographs. New Series. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993, pp. xviii+269.

[Sal01] Paolo Salvatore. "Configuration spaces with summable labels". *Cohomological methods in homotopy theory (Bellaterra, 1998).* Vol. 196. Progr. Math. Basel: Birkhäuser, 2001, pp. 375–395.

[SGA] Victor Ginzburg and Mikhail Kapranov. "Koszul duality for operads". *Duke Math. J.* 76.1 (1994), pp. 203–272.

[Skl83] E. K. Sklyanin. "Some algebraic structures connected with the Yang-Baxter equation. Representations of quantum algebras". English. *Funct. Anal. Appl.* 17 (1983), pp. 273–284.

[Sou10] Ismael Soudères. "Motivic double shuffle". *International Journal of Number Theory* 06.02 (2010), pp. 339–370.

[SW03] Paolo Salvatore and Nathalie Wahl. "Framed discs operads and Batalin-Vilkovisky algebras". *Q. J. Math.* 54.2 (2003), pp. 213–231.

[Tho16] Justin Thomas. "Kontsevich's Swiss cheese conjecture". *Geom. Topol.* 20.1 (2016), pp. 1–48.

[Tur89] V. G. Turaev. "Euler structures, nonsingular vector fields, and Reidemeister-type torsions". *Izv. Akad. Nauk SSSR Ser. Mat.* 53.3 (1989), pp. 607–643, 672.

[Vey75] Jacques Vey. "Déformation du crochet de Poisson sur une variété symplectique". *Comment. Math. Helv.* 50.4 (1975), pp. 421–454.

[Vol07] Ismar Volić. "A survey of Bott-Taubes integration". *J. Knot Theory Ramifications* 16.1 (2007), pp. 1–42.

[Vor99] Alexander A. Voronov. "The Swiss-cheese operad". *Homotopy invariant algebraic structures (Baltimore, MD, 1998)*. Vol. 239. Contemp. Math. Amer. Math. Soc., Providence, RI, 1999, pp. 365–373.

[Wei83] A. Weinstein. "The local structure of Poisson manifolds". *J. Differential Geom.* 18 (1983), pp. 523–557.

[Wei87] Alan J. Weinstein. "Symplectic groupoids and Poisson manifolds". *Bulletin of the American Mathematical Society* 16 (1987), pp. 101–104.

[Wil10] Thomas Willwacher. "M. Kontsevich's graph complex and the Grothendieck–Teichmüller Lie algebra". *Inventiones mathematicae* 200 (2010), pp. 671–760.

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