I

$\vec{k} = (k_1, \ldots, k_n)$ - sequence of natural numbers

$w(\vec{k}) = k_1 + \ldots + k_n$

$d(\vec{k}) = n$

convergent (admissible) if $k_1 > 1$.

MZV:

$$\zeta(\vec{k}) = \sum_{\substack{m_1 > m_2 > \ldots > m_n \\ m_i \in N}} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}}$$

← converges, if $\vec{k}$ is convergent

Iterated integral:

$$I_r = \int_{1 > t_1 > t_2 > \ldots > t_w > 0} \omega_1(t_1) \wedge \ldots \wedge \omega_w(t_w) \qquad \omega_i = \begin{cases} \frac{dt}{1-t} & y \\ \frac{dt}{t} & x \end{cases}$$

converges if

word of x's & y's

element of $\underbrace{k\langle x, y\rangle}_{A}$

Drinfeld associator:

$$\int_0^1 \left(\frac{x\,dt}{t} + \frac{y\,dt}{1-t} \right) = \sum I_r r$$

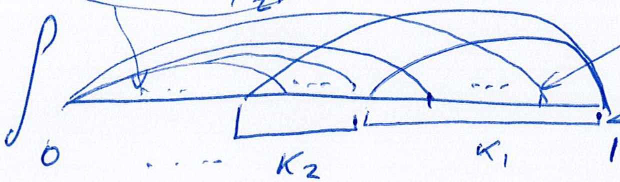
words

(we ignore regularization)

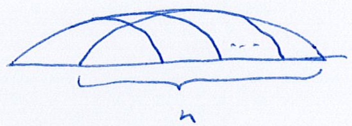
$x \mathbf{A} y \ni$

Iterated integrals are MZVs! $(k_1, k_2, \dots k_n)$!

first is $\frac{dt}{1-t}$ ← Convergence → last is $\frac{dt}{t}$

$\zeta(\vec{k}) = \int$  "length"

$\omega_i(t) = \begin{cases} \frac{dt}{1-t} & i \in \{k_1, k_1+k_2, \dots\} \\ \frac{dt}{t} & \text{otherwise.} \end{cases}$

<u>Proof</u>: 1. Example: $\zeta(n) = \int$  $=$

$$= \int_{1>t_1>\dots>t_n>0} \frac{dt_1}{t_1}\cdots\frac{dt_{n-1}}{t_{n-1}}\frac{dt_n}{1-t_n} = \int_{1>t_1>\dots t_{n-1}>0} \sum \frac{t_{n-1}^k}{k}\cdot\frac{dt_1}{t_1}\cdots\frac{dt_{n-1}}{t} = \dots \int_{1>t_1>0} \frac{t_1^k}{k^{n-1}}\,dt = \sum \frac{1}{n^k}$$

2. Introduce cubical coordinates:

$$x_1 = t_1 \quad x_2 = t_2/t_1 \quad \dots \quad x_n = t_n/t_{n-1}$$ termwise integration

Then $$= \int_{\substack{0<x_1<1 \\ 0<x_2<1 \\ \vdots}} \frac{1}{1-x_1\cdots x_n}\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n} \underset{\text{taylor}}{=} \int_\square \sum \frac{1}{(x_1\cdots x_n)^k} = \sum \frac{1}{n^k}.$$

More generally, $$\int \frac{x_1\cdots x_w}{1-x_1\cdots x_w}\cdot \dots \frac{x_1\cdots x_{k_1}}{1-x_1\cdots x_{k_1}}\frac{dx_1}{x_1}\cdots\frac{dx_w}{x_w}$$

MZV's is an algebra over $\mathbb{Q}$

E.g. $$\zeta(2)\zeta(2) = \sum \frac{1}{n^2} \cdot \sum \frac{1}{m^2} = \left(\sum_{n>m} + \sum_{m>n} + \sum_{m=n} \right) \frac{1}{n^2 m^2} = 2\zeta(2,2) + \zeta(4)$$

More generally, define stuffle (or harmonic) product of two $\vec{k}$ & $\vec{\ell}$ as a <u>multiset</u> of sequences given by the recursive rule

$$st(\vec{k}\cdot x, \vec{\ell}\cdot y) = st(\vec{k}\cdot x, \vec{\ell})\cdot y \cup st(\vec{k}, \vec{\ell}\cdot y) \cup st(\vec{k}, \vec{\ell})\cdot(x+y)$$

where $(k_1, \dots, k_{n-1})\cdot k_n \equiv (k_1, \dots k_n)$

Stuffle relations:
$$\zeta(\vec{k})\,\zeta(\vec{\ell}) = \sum_{\vec{s}\in st(\vec{k},\vec{\ell})} \zeta(\vec{s}) \equiv \zeta(\vec{k} * \vec{\ell})$$

Consider algebra of <u>QSymm</u> : it's lin. generated by monomial quasisym. functions.

$k[\hat{x}_1, x_2 \dots]$

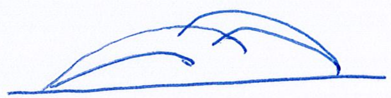
$$M_\alpha = \sum_{i_1 < \dots i_m} x_{i_1}^{\alpha_1} \cdots x_{i_m}^{\alpha_m}$$

For $M_\alpha(1, \frac{1}{2}, \frac{1}{3} \dots)$ is MZV and this is map of algebr.

$\nearrow$ QSymm $\to$ $\mathbb{Q}[MZV]$

Hopf algebra dual to NSymm $= k\langle x_1, x_2, \dots \rangle \Rightarrow$ free comm. algebra.

MZVs is algebra in another way!

$\zeta(2)\cdot\zeta(2) =$  $\cdot$  $=\sum$  $= \underbrace{4\zeta(3,1)+2\zeta(2,2)}$

$\int_{\Delta_1}$ $\qquad$ $\int_{\Delta_2}$ $\qquad$ all shuffles $\qquad$ $\int_{\Delta_1\times\Delta_2}$

respects depth!

More generally, define shuffle product of two words in x, y as a multiset by

$$Sh(v\cdot z_1, u\cdot z_2) = v\cdot Sh(z, u\cdot z_2) \cup u\cdot Sh(v\cdot z_1, z_2)$$

$u, v \in \langle x, y\rangle$.

<u>Shuffle relation:</u> $\zeta(\vec{k})\,\zeta(\vec{\ell}) = \sum_{s\in Sh(\vec{k},\vec{\ell})} \zeta(\vec{s}) \equiv \zeta(\vec{k} ш \vec{\ell})$

In other words, Drinfeld associator is group-like.

Shuffle product equip formal MZV's by product, which is Hopf dual to $x\Delta y$. This is free comm. algebra.

Shuffle + Stuffle relations $\rightsquigarrow$ $\underbrace{\text{Double Shuffle relations}}$

plethora of relations, incl linear ones.

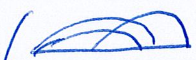
e.g. $\boxed{2\zeta(2,2)+\zeta(4) = 2\zeta(2,2)+4\zeta(3,1)}$

<u>Main problem</u>: describe all (geometric! motivic!) relations.

E.g. (pentagon) relations on Drinfeld ass-r give some (are supposed to be the strongest).

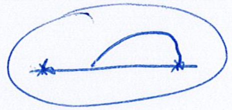
All this relations should respect <u>Weight</u>!

though,

DSR is not full. Euler relation $\zeta(2,1) = \zeta(3)$ ( = )

$\text{DSR} \not\Rightarrow$ <u>duality</u>

But it is implied by <u>Regularized</u> DSR.

Here is the idea: add $\zeta(1)$. Then   $\zeta(2)$

$$\zeta(1)\zeta(2) = \cancel{\zeta(1,2)} + \zeta(2,1) + \zeta(3) \quad — \quad \zeta(1 * 2)$$

$$\| \quad 2\zeta(2,1) + \cancel{\zeta(1,2)} \quad — \quad \zeta(1 ш 2)$$

$$\Downarrow$$

$$\zeta(2,1) = \zeta(3).$$

Let $\underset{\text{divergent series}}{R_*} \supset \underset{\text{convergent}}{R_*^\circ}$ be space of sequences with $*$-product

divergent integrals $\to R_ш \supset R_ш^\circ$ —— " —— with ш product.

<u>Key fact:</u> $R_* = R_*^\circ[(1)]$ $\quad R_ш = R_ш^*[(1)]$

That is (1) is "the only" problem. Thus one may expect

maps $(1) \mapsto T \Rightarrow \underline{\rho(T) = T}$

$$\begin{tikzcd}
R_* \arrow[r, "\text{“}\Sigma\text{”}", "\text{map}"'] \arrow[d, "\wr"', "\text{linear isomorph.}"] & R[T] \arrow[d, "\rho\text{- } \cancel{R}\text{ linear map}"] \\
R_ш \arrow[r, "\text{“}\int\text{”}", "\text{of algebras}"'] & R[T]
\end{tikzcd}$$

$(1)^{*2} = \sum \frac{1}{n} \sum \frac{1}{m} = 2(1,1) + (2) = (1)^{ш2} + (2)$

$\Downarrow$

$\underline{\rho(T^2) = T^2 + \zeta(2)}$

$\rho(T^3) = T^3 + 3\zeta(2)T - 2\zeta(3)$

To get the Euler identity, consider divergent $(1,2)$,

$\sum(1,2) = \zeta(2)T - \zeta(2,1) - \zeta(3)$ $\quad$ "$\int$"$(1,2) = \zeta(2)T - 2\zeta(2,1)$

This is <u>r. DSR</u>. $\quad$ Associator relations

$\Longleftarrow$ is known, many proofs

All relations? $\Longrightarrow$? is an interesting question.

If yes

Proof is straightforward: consider divergent sums and integrals

Excplicit formula for ρ:

$$\rho(\underbrace{e^{Tu}}_{\substack{\text{generating}\\ \text{series in } u}}) = A(u)\, e^{Tu}$$

$$A(u) = \exp\left(\sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n) u^n\right)$$

In other words,

$$\rho = \exp\left(\sum_{2}^{\infty} \frac{(-1)^n}{n} \zeta(n) \left(\frac{\partial}{\partial T}\right)^n\right)$$

The analytic proof gives a formula

$$A(u) = e^{\gamma u}\, \Gamma(1+u) \qquad (|u|<1)$$

II Let us express stuffle relations in terms of integrals.

(Cartier? Goncharov Brown)

$t_2 \quad t_1$

$0 \quad x_2 \quad x_1 \quad 1$

$\ldots \quad t_2 = x_2 \quad t_1 = x_1$

t_1

E.g $\quad \zeta(2) = \int_{\square} \frac{dx_1\, dx_2}{1 - x_1 x_2}$

$$\zeta(2) \cdot \zeta(2) = \int_{\square} \underbrace{\frac{dx_1\, dx_2}{1-x_1x_2} \cdot \frac{dx_3\, dx_4}{1-x_3x_4}} \left(= \int_{\Delta} \frac{\cdots}{(1-t_2)\left(1-\frac{t_4}{t_2}\right)} = \int_{\Delta} \frac{\frac{dt_1}{t_1}\, dt_2 \cdots}{(1-t_2)(t_2-t_4)} \right)$$

reorder this series.

non-iterated!

Relation: $\frac{1}{(1-\alpha)(1-\beta)} = \frac{\alpha}{(1-\alpha)(1-\alpha\beta)} + \frac{\beta}{(1-\beta)(1-\beta\alpha)} + \frac{1}{1-\alpha\beta}$

Proof: eg. consider Taylor series

Substituting, we get

$$= \int_{\square} \left(\frac{x_1 x_2}{(1-x_1x_2)(1-x_1x_2x_3x_4)} + \frac{x_3x_4}{(1-x_3x_4)(1-x_3x_4x_1x_2)} + \frac{1}{1-x_1x_2x_3x_4} \right) d\vec{x}$$

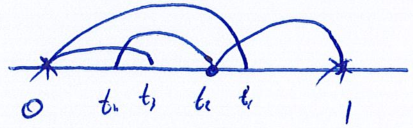
$\downarrow$ $\zeta(2,2)$

$\downarrow$ after permutations $x_i \to x_{\sigma(i)}$ $\|$ $\zeta(2,2)$

$\downarrow$ $\zeta(4)$.

Find geometric meaning of these manipulations ②

1. $\zeta(2)\cdot\zeta(2) = \int\limits_{..<t_i<1} \frac{dt_1}{t_1}\,\frac{dt_2}{1-t_2}\,\frac{d(t_2-t_4)}{t_2-t_4}\,\frac{dt_3}{t_3}$



$\| \ \boxed{\text{Fubini}}$

$$\int\limits_{0<t_2<t_1<1}\frac{dt_1}{t_1}\,\frac{dt_2}{1-t_2}\cdot\int\limits_{0<t_4<t_3<t_2}\frac{d(t_2-t_4)}{t_2-t_4}\,\frac{dt_3}{t_3}$$

$\rightsquigarrow$ change of var. $t_3 \to t_3/t_2$, $t_4 \to t_4/t_2$ $\to \int \frac{dt_4}{1-t_4}\,\frac{dt_3}{t_3}$

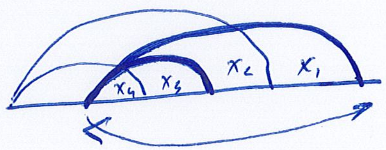
2. Relation $\frac{1}{(1-\alpha)(1-\beta)}$ -- is Arnold's relation:

$$\omega_{ij} = \frac{d(t_i-t_j)}{t_i-t_j} \Rightarrow \omega_{ij}\wedge\omega_{jk}+\omega_{jk}\wedge\omega_{ik}+\omega_{ik}\wedge\omega_{ij}=0$$

substituting $\alpha = t_j/t_i$ $\beta = t_k/t_j$ $t=t_i$, term with $d\alpha\wedge d\beta$ is

Arnold relations gives (up to signs


=

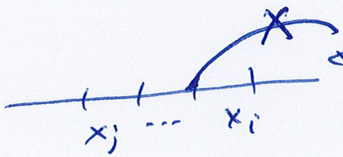
+


← what is it?!

iterated $= \zeta(2,2)$

Permutation of x_i - birational automorfism - flop

More generally, we need flop



← such that no arcs like this.

Integral $\int_\Delta$  is an example of cell zeta values.

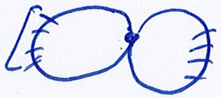
Plan: find relations between CZVs and use them to prove stuffle.

$$\mathcal{M}_{0,n+3} = \{ z_i \in \mathbb{P}^1,\ z_i \neq z_j,\ \underbrace{i \in [n+3]}_{S} \} / PGL_2$$

$\mathcal{M}_{0,n+3} \subset \overline{\mathcal{M}}_{0,n+3} \leftarrow$ stable curves

$\overline{\mathcal{M}}_{0,n+3} \setminus \mathcal{M}_{0,n+3}$ – $\cup$ of divisors, corresponding to subdivisions

$S = S_1 \cup S_2$

$S_1 \ni$  $\in S_2$.

$S_1 \cap S_2 = \emptyset$

In every orbit there is a uniqe point with $z_1 = 0$ $z_2 = \infty$ $z_3 = 1$.

$$(0, \infty, 1, \underbrace{t_1 \dots t_n})$$

$\nwarrow$ simplicial coordinates on $\mathcal{M}_{0,n+3}$

$$\Delta_n = \{ 1 > t_1 > \dots > t_n > 0 \} \quad t_i \in \mathbb{R}$$

$\uparrow$ standard simplex in $\mathcal{M}_{0,n+3}(\mathbb{C})$

Fix a cyclic order on S. Standard simplex depends on order on S, but in fact only on cyclic order (it is a component of $M_{0,n+3}(\mathbb{R})$)

Fact: Algebra of regular diff forms on $M_{0,n+3}$ is generated by $\frac{d(x_i - x_j)}{x_i - x_j}$

" forms with log. singularities at ∞

Def CZV is an integral of convergent reg. diff form on $M_{0,n+3}(\mathbb{Q})$ by the standard simplex.

Reg. form may be (not uniquely!) presented by arc diagram: 

due to Arnold's rel

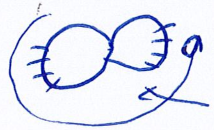
Form is convergent if in some presentation it does not contain  $\frac{d(x_i - x_{i+1})}{x_i - x_{i+1}}$.

In other words, $M_{0,n+3} \subset M^{\delta}_{0,n+3} \subset \overline{M}_{0,n+3}$

smooth affine, introduced by Brown.

"planar" Hypersurf = $M_{0,n+3}$ + divisors

Form is convergent if it comes from $M^{\delta}_{0,n+3}$.



subdivision respects cyclic order

Th (Brown) CZV = MZV $(/\mathbb{Q})$

inclusion, because iterated integrals are particular types of cell.

1. Generalized shuffle relations b/w CZVs

$\Downarrow$ (Brown, Carr, Schneps 10, Brown's thesis)

CZV form an algebra.

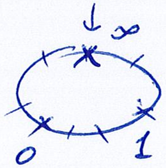
with cyclic order $\longrightarrow \mathcal{M}_{S,0}$

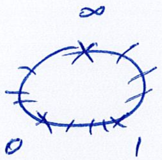
Choose $\{0,\infty,1\} \subset S$

$\Downarrow$

$\Delta_{S_1} \simeq \Delta^1_{S_1} \times \Delta^2_{S_1} \times \Delta^3_{S_1}$

Standard simplex





$\omega_{1,2} \in \Omega^{top}(\mathcal{M}_{S_{1,2},0})$

$p_{1,2} : \mathcal{M}_{S_1 \sqcup_{\{0,1,\infty\}} S_2} \to \mathcal{M}_{S_{1,2}}$

$$\int_{\Delta_{S_1} \times \Delta_{S_2}} p_1^* \omega_1 \wedge p_2^* \omega_2 = \int_{\Delta_{S_1}} \omega_1 \int_{\Delta_{S_2}} \omega_2$$

$\Delta_{S_1} \times \Delta_{S_2} = (\Delta^1_{S_1} \times \Delta^1_{S_2}) \times (\Delta^2_{S_1} \times \Delta^2_{S_2}) \times (\Delta^3_{S_1} \times \Delta^3_{S_2})$

Fubini th.

Gen. shuffle relations!

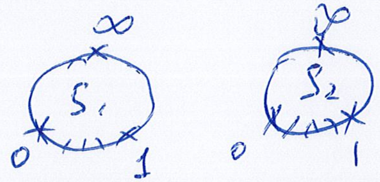
Given cyclic orders S_1 & S_2 + choice of $\{0,\infty,1\} \subset S_i$

$$\Rightarrow \int_{\Delta_{S_1}} \omega_1 \cdot \int_{\Delta_{S_2}} \omega_2 = \sum_{\text{gen. shuffles}} \int_{\Delta_S} p_1^* \omega_1 \wedge p_2^* \omega_2$$

Whitney–Alexander

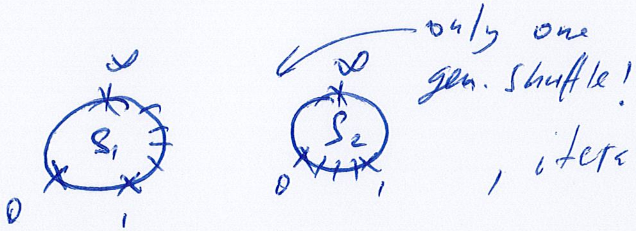
gen. shuffles, cyclic orders on $S = S_1 \sqcup_{\{0,\infty,1\}} S_2$, compatible with ones on S_i.

Examples! 1.



, iterated forms $\Rightarrow$ shuffle relations for MZVs

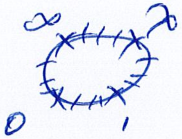
2.



, iterated forms $\Rightarrow$ relation we use at the beginning

(observed by Brown in his thesis)

In Brown, Carr, Schneps all relations was gen. shuffle + duality rels.
We extend last ones.

Let $S_{1,2} =$  – cyclic sets with a cyclic subset $\underline{4} \subset S$ chosen.

Let $\nu \in V_4$ acts on $\underline{4}$. Denote by S^ν the twisted pair $\underline{4} \to S$

Klein 4-group

cyclic order is reversed, if needed

or vice versa

Let $\omega_1 \in \Omega^{top} \mathcal{M}_{S_1,0}$ $\quad \omega_2 \in \Omega^{top-1} \mathcal{M}_{S_2,0}$ $\quad p_{1,2} : \mathcal{M}_{S_1 \cup_{\underline{4}} S_2} \to \mathcal{M}_{1,2}$

2. Generalized shuffle relations:

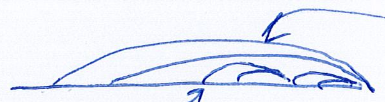
$$\sum_{\substack{\text{all cyclic orders} \\ \text{on } S = S_1 \cup_{\underline{4}} S_2, \text{ compatible} \\ \text{with ones on } S_i}} \int_{\Delta_S} p_1^* \omega_1 \wedge p_2 \omega_2 = \pm \left(\text{the same, but } S_1 \to S_1^\nu\right)$$

linear relations b/w CZVs.

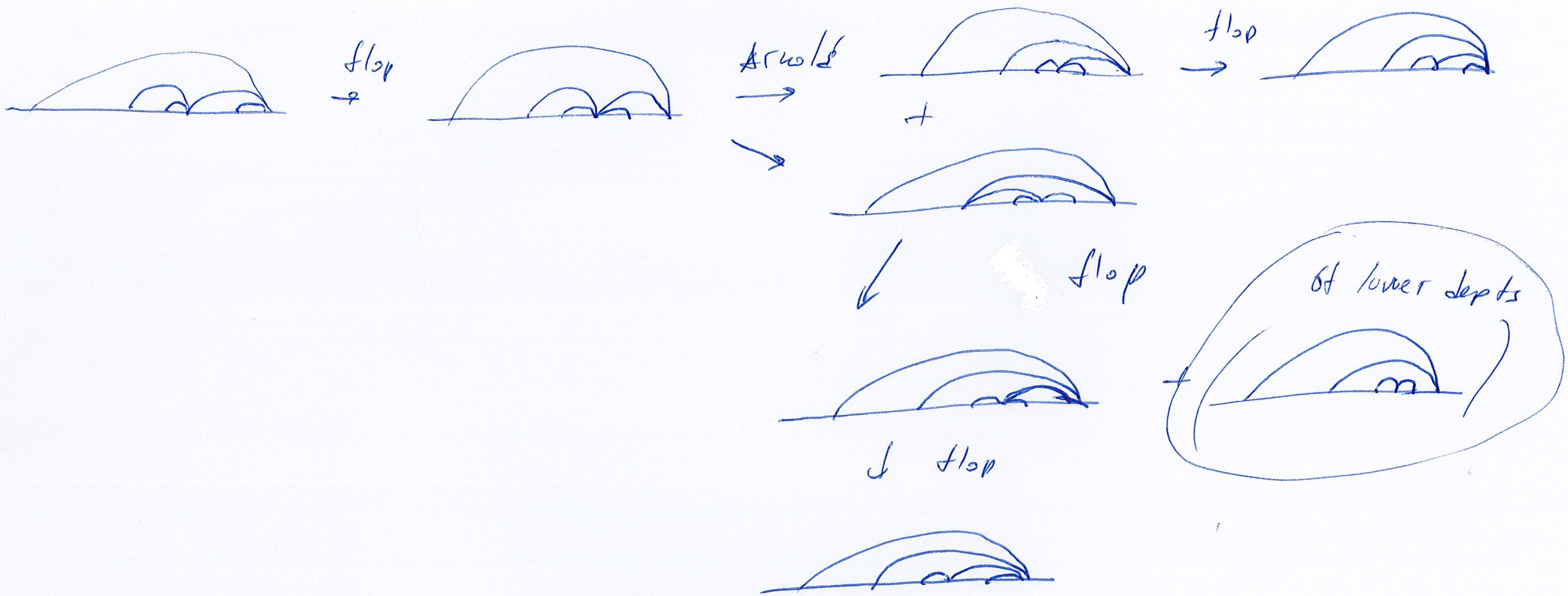
<u>Proof</u> Consider projection $\pi_{\underline{4}} : \mathcal{M}_{S_1 \cup_{\underline{4}} S_2, 0} \to \mathcal{M}_{\underline{4},0}$. Then both sides equal to $\int_{\Delta_1} \pi^* \omega_1 \wedge \pi^* \omega_2$, <u>because</u> V_4 acting on $\underline{4}$ acts trivially on $\mathcal{M}_{\underline{4},0}$.

relative shuffle relations

Theorem: Gen. shuffle + stuffle relations $\Rightarrow$ DSR for MZVs.

General procedure: given form 

we can decrease number of arcs of this type and increase of this

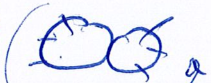


Previously:

smooth, affine

$$\mathcal{M}_{0,n+3} \subset \mathcal{M}^{\delta}_{0,n+3} \subset \overline{\mathcal{M}}_{0,n+3}$$

depends on cyclic order on points

contains only cyclic diagonals ,

$\Delta_n = \{(0,\infty,1,t_1,\dots,t_n),\ 1>t_1>\dots t_n>0,\ t_i\in\mathbb{R}\}$ – standard simplex $\in \mathcal{M}_{0,n+3}$

$\bar{\Delta}_n \subset \mathcal{M}^{\delta}_{0,n+3}$ – Stasheff polytope.

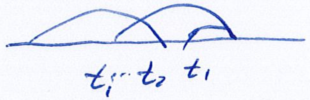
Let ω – reg. differential form on $\mathcal{M}^{\delta}_{0,n+3}$

with simple poles on ∞

(in other words, form on $\mathcal{M}_{0,n+3}$ without poles on cyclic diagonals)

$$\int_{\Delta_n} \omega \quad - \quad CZV$$

Th (Brown): Any CZV is a rational combination of MZVs.

$\int$ 

To prove, integrate by t_1, then by t_2 ....

We have defined relations on CZV, coming from tensor structure. But we missed the most obvious one:

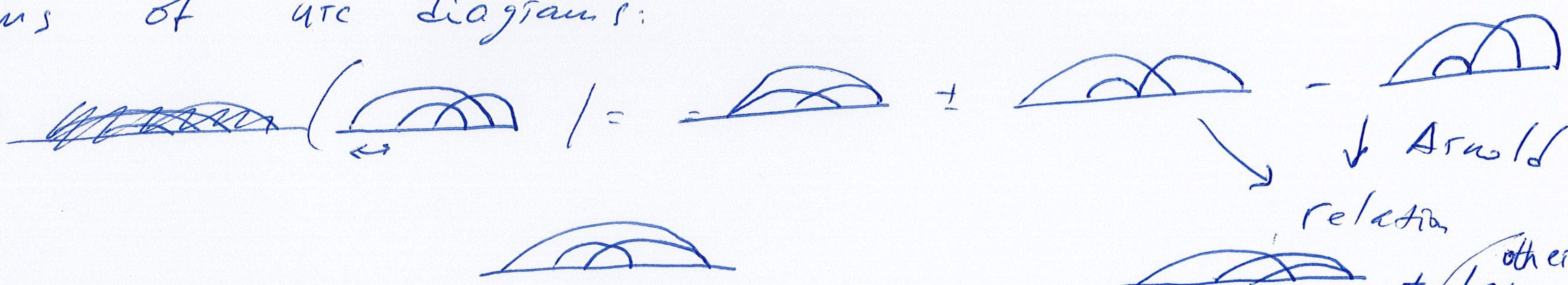
Stokes theorem: $\int_\Delta \underbrace{d\omega}_{\substack{\uparrow \\ \shortparallel \\ 0}} = \int_{\partial\Delta}$

it ω is regular $\Rightarrow$ $\boxed{\int_{\partial\Delta} \omega = 0}$ $\omega \in \mathrm{reg}\, \Omega^{\mathrm{top}-1} \mathcal{M}^{\delta}_{0,n+3}$.

boundary of Stasheff polytope
$\shortparallel$
$\cup$ of products of S.p's.

In terms of arc diagrams:

$0 = \partial$



↓ Arnold relation

+ other terms, products in general

there are some relations without these terms....

Observation: it gives nothing for iterated integrals.

Iterated integrals are a particular type of cell integrals $\Rightarrow$

map $MZV \to CZV$

? this is an embedding, not obvious at once, because

it depends on choice of $0, \infty, 1$.

<u>Naive conjecture</u>: (Cell integrals)//Stokes relations $=$ $\mathbb{Q}$-lin comb of iterated integrals.

If it is not true, what we need?

To understand we need to analyse proof of Brown's theorem.

Gen (stuffle + shuffle) relations $\Rightarrow$ DSR bw MZV.

<u>Conj</u> 1. Gen st & st rels $\not\Rightarrow$ regularised DSR

2. <u>Gen st & st + Stokes</u> $\Rightarrow$ regularized DSR

3. this is the full list of geometric relations among CZV

to prove it I am planning to use <u>Kaneko-Yamamoto relations</u>.

In general how to prove <u>Gen st & st + Stokes</u> ?

Kaneko - Yamamoto ~~zeta~~ integral-series relations (2018)

Let $\vec{k} = (k_1, \dots, k_r)$ $\vec{l} = (l_1, \dots, l_s)$ - sequences of $\mathbb{N}$-numbers (order is opposite to previous lectures!)

Define

1. $$\zeta(\vec{k} \circledast \vec{l}^{\,\star}) = \sum_{0 < m_1 < \dots < m_r = n_s \geq \dots \geq n_1 > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r} n_1^{l_1} \cdots n_s^{l_s}} = \sum (\text{all terms})$$

like stuffle product

MZVs

no convergency conditions!

Eg: $\zeta(\vec{k} \circledast (1)) = \zeta(k_1, \dots, k_{r-1}, k_r + 1)$

order is opposite!

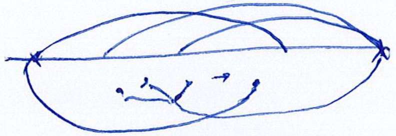
$\zeta((1) * \vec{l}^{\,\star}) = \zeta^\star(l_1, \dots, l_{s-1}, l_s + 1)$

2. $$\zeta(\mu(k,l)) = \int_{0 < t_1 < \dots < t_{k_1 + \dots + k_r + l_s} > t_{k_1 + \dots + k_r + l_s + 1} < \dots \; t_{k_1 + \dots + k_r + l_s + l_{s-1}} > t_{k_1 + \dots + k_r + l_s + l_{s-1} + 1} \dots} \prod \left(\omega_i = \begin{cases} \frac{dt}{1 - t_i}, & i = 1, k_1 + 1, \dots, 1 + k_1 + \dots + k_{r-1}, 1 + k_1 + \dots + k_r + l_s, \dots + l_s + l_{s-1}, \dots \\ \frac{dt}{t_i}, & \text{otherwise.} \end{cases} \right) = \sum (\text{iterated integrals})$$

analog of shuffle product.

Th: $\zeta(\vec{k} \circledast \vec{l}) = \zeta(\mu(k, l))$

Example: $\vec{k} = (1,1)$ $\quad \vec{\ell} = (2,1)$



$$\int_{t_1<t_2<t_3>t_4<t_5} \frac{dt_1}{1-t_1}\,\frac{dt_2}{1-t_2}\,\frac{dt_3}{t_3}\,\frac{dt_4}{1-t_4}\,\frac{dt_5}{t_5} =$$

sum of 6 iterated integrals

$$= \int_{t_2<t_3>t_4<t_5} \sum \frac{t_2^{\ell}}{\ell} \dots = \int_{t_3>t_4<t_5} \sum_{\ell,m} \frac{t_3^{\ell+m}}{\ell(\ell+m)} \dots =$$

$$= \int_{t_4<t_5} \sum \frac{1-t_4^{\ell+m}}{\ell(\ell+m)^2}\,\frac{dt_4}{1-t_4}\,\frac{dt_5}{t_5} = \int_{t_4<t_5} \sum_{0<m_1<m_2} \frac{1}{m_1 m_2^2} \sum_{n=1}^{m_2} t_4^{n-1}\,dt_4\,\frac{dt_5}{t_5} =$$

$$= \int_{t_5} \sum_{m_1<m_2} \frac{1}{m_1 m_2^2} \sum_{n=1}^{m_2} \frac{t_5^{n}}{n}\,\frac{dt_5}{t_5} = \sum_{0<m_1<m_2\geq n>0} \frac{1}{m_1 m_2^2 n^2} = \zeta(2,2,1)+\zeta(2,1,2)+\zeta(3,2)+\zeta(4,1)$$

old notations!

$$6\zeta(3,1,1) + 2\zeta(2,2,1) + \zeta(2,1,2)$$

Th (KY) KY + shuffle ⇔ KY + shuffle ⇔ reg. DSR

The proof is based on relations like this: $\vec{k} = (k_1, \dots, k_r)$ $\vec{l} = (l_1, \dots, l_s)$

$$\sum (-1)^i \Big(\sum_{m_1 < \dots < m_r = n_s \geq \dots \geq n_{i+1}} \frac{1}{m_1^{k_1} \cdots m_r^{k_r} n_s^{l_s} \cdots n_{i+1}^{l_{i+1}}} \Big) \cdot \sum_{n_i < \dots < n_1} \frac{1}{n_i^{l_i} \cdots n_1^{l_1}} + (-1)^s \sum_{m_1 < \dots < m_r = n_s < \dots < n_1} \frac{1}{m_1^{k_1} \cdots m_r^{k_r} n_s^{l_s} \cdots n_1^{l_1}} = 0$$

Hop!

$$\sum (-1)^i (\vec{k} \circledast (l_{i+1}, \dots l_s)^*) * (l_i, \dots l_1) + (-1)^s (k_1, \dots, k_r + l_s, l_{s-1}, \dots, l_1) = 0$$

relation in R_*

To prove it divide in two parts: $n_i \leq n_{i+1}$ and $n_i > n_{i+1}$

They cancel each other

There is other, which splits $\vec{k}$. And two relations in $R_ш$.

Substitute here $\vec{l} = (1, \dots, 1)$. We get a relation in R_*, where one term is not convergent, and other are in $R_*^0[(1)]$.

Applying the same relation in $R_ш$ and comparing, we get ⇒ of theorem.

IV

Operad (Gerstenhaber)

↳ $e_2(n) = H_*(\underbrace{M_{0,n+1} \times S^1})$ ← little discs

It is equipped with Hodge, motivic structure. (How to see it?

Resolution : $\overset{\text{bar}}{B}(e_2^!)$ ← Koszul dual $= e_2$ up to shift

Geometrically this is $(\mathbb{C}^{2n}$; diagonals)

Hodge structure on e_2, is very simple, no interesting extensions.

All non-trivial $e_2^!$ structure in differential there, implicit.

$M_{0,n+1} \times S^1 = \overset{\text{class. space}}{B}$(braid group) has Hodge structure, but this is long way.

Want Have something geometric like Aomoto polylogarithms:

$\mathbb{P}^n$

two simplices

Remove one, and shrink another (relative homology).

↓ interesting mixed H.s.

↓ interesting extensions

" periods

If we only remove or shrink ⇒ not interesting

Example: $\mathbb{P}^1$

H^1 is 2-dim, gives extension $\mathrm{Ext}^1(\mathbb{R}, \mathbb{R}(1))$

Beilinson
Goncharov
Schechtman
Varchenko...

→ Period - integral of hol. form by cycle $=$

$\ln$(cross ratio)

Consider planar (non-symmetric) operad.

Example: planar Lie is free.

Consider the bar-dual to e_2^{pl}. It is

$$D(e_2) = H_*\left(\left(\mathcal{M}^{\delta}_{0,n+1}, \mathcal{M}^{\delta}_{0,n+1} \setminus \mathcal{M}_{0,n+1}\right) \times S^1\right)$$

we shrink only cyclic diagonals, thats we remove non-cyclic and shrink cycling

$\Downarrow$

interesting extensions (Goncharov-Manin).

By the way $H_*(\mathcal{M}^{\delta}_{0,n+1})$ form a planar operad, which is Br^{pl}/Δ, dual thing is the same, without δ!

Question: Structure of e_2^{pl}? It is known that $grav^{pl} \subset e_2^{pl}$ is free due to Bergström-Brown-Dupont-Vallette-Alm-Petersen.

also with help of Hodge theory!

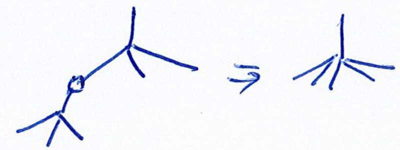
generators and relations?

At least, generators and relations of $\langle Ass^{pl}, Grav^{pl} \rangle$ in e_2^{pl}.

Here $Ass^{pl} \to e_2^{pl}$ is

associative or planar commutative operad,

one operation in each arity



Instead of Hodge structure on e_2^{P1} on may try to study H.S. on the morphism (derived!)

$$Ass^{P1} \to e_2^{P1} \qquad \rightleftarrows \qquad e_1 \to e_2$$

adjunction.

$\downarrow$ resolution freely generated by $\curlywedge$,

d $\curlywedge$ is composition.

Morphism is defined by values on generators. They live in

$$H_{n-2}(M_{0,n+1} \times S^1)/(\mathrm{Im}\ \text{of}\ d).$$

Variant: consider $Ass^{P1} \to bv^{P1}/\Delta \Rightarrow \underbrace{\text{classes in } H_{n-2}(M_{0,n+1}^{\delta})}$

$\underbrace{Gal}$ acts on $Mor(Ass^{P1} \to e_2^{P1}) \longrightarrow$ cell-zeta values!

$\uparrow$
motivic,
Hodge Galois,
GT

Or, $Lie(Gal) \to \underbrace{Def(Ass^{P1}, e_2^{P1})}$.

differential is a linearisation of Stokes relations on CZVs.

Elements here are solutions of MC equation, they obey non-linear Stokes.

(to describe it is an interesting question)

It is known (Wilwacher ...) that for Hopf operad ($\simeq$ spaces)

$$\mathrm{Hom}_{Hopf}(Ass^{pl}, e_2^{pl}) = GT \leftarrow \text{Grothendieck-Teichmuller}.$$

(what is $\mathrm{Hom}_{Hopf}(BVP^{pl}/\Delta, e_2^{pl})$?)

Thus, consider $\mathrm{Def}_{Hopf}(Ass^{pl}, e_2^{pl})$. There are 2 conditions: respect operad structure + Hopf structure $\Rightarrow$ 2 differentials.

We have spectral sequence (Fresse, Wilwacher, Merkulov) $\mathrm{Def}(Ass^{pl}, \mathrm{Res}_{Hopf}(e_2^{pl})) \Rightarrow \mathrm{Def}_{Hopf}(Ass^{pl}, e_2^{pl})$

The first leaf is Def of linear operad.

What is it? If one believe in naive conjectures about Stokes relations, this is $\simeq$ Free Lie $\langle x, x_2 \rangle$.

See the question above

Differentials in this sp. sequence help organize relations b/w CZV. 1st differential – Stokes 2nd – shuffle relations, ...? Besides it may help to prove (CZV relations) $\Leftarrow$ (Associator rel.).

What if we consider dihedral operads? Does 2nd differential gives gen st & st relations on CZV? Does spectral sequence collapsed at 2nd page (at least at degree 0?) Does it mean, that (gen st & sh) = (Assosiator)?

Let us calculate CZV directly, using cubical coordinates
What kind of series we get?

Definition: big zeta value is the sum of series

BZV
$$\sum_{\substack{n_i \in N \\ 1 \le i \le d}} \frac{1}{\prod_j a_{ij} n_i}, \quad \text{where } (a_{ij}) = A$$

$d \times w$ - matrix of rang d

depth, weight ~~with coordinates~~

Such that any row is $(0 \ldots 0 \underbrace{1 \ldots 1}_{\text{at least two, convergence.}} 0 \ldots 0)$

Example: !
$$\begin{pmatrix} \overset{k_d}{\overbrace{1 \; 1 \; 1 \; 1}} & \overset{k_{d-1}}{} & \overset{k_1}{1} \\ & 1 \; 1 & 1 \\ 0 & \ddots & \end{pmatrix}$$ - series is $\zeta(k_1, \ldots, k_d)$

Example 2: $\begin{pmatrix} 1 \ldots \overbrace{1 \ldots 1}^{a+b} & 0 \ldots \\ 0 \ldots 0 & \underbrace{1 \ldots 1 \; 1 \ldots 1}_{b+c} \end{pmatrix}$ $\quad \frac{1}{m^a m^c (m+n)^b}$ ← Tornheim sum

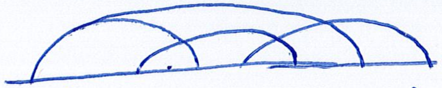
To prove it is $\mathbb{Q}$-comb. of MZV, consider $\frac{1}{mn} = \frac{1}{(m+n)n} + \frac{1}{(m+n)m}$

$$\frac{1}{(m+n)n} = \frac{1}{mn} - \frac{1}{(m+n)m}$$

$\left(\frac{\partial}{\partial m}\right)^{b-1} \left(\frac{\partial}{\partial n}\right)^{a-1}$, multiply by $\frac{1}{m^c}$,

summing get the relation.

CZV

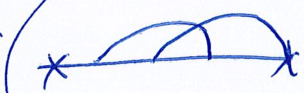
BZV e.g.  $\begin{pmatrix}0&0&0&1&1&1\\0&1&1&1&0&0\end{pmatrix}$ corresponds up to terms of lower degrees.

Two family of relations b/w BZV:

1. Orlik-Solomon: $\sum \alpha_i l_i = 0$ ← linear forms $\Rightarrow \sum \frac{\alpha_i}{l_1 \cdots \hat{l_i} \cdots l_k} = 0$ and their derivatives

2. Regrouping (like shuffle product), harmonic relations.

Analogous to Arnold relations.

Analogous to Stokes relations: $\frac{1}{mn} = \frac{1}{(m+n)n} + \frac{1}{(m+n)m}$ $\partial\Big($  $\Big)$

Th: Using only these relations one may transform BZV to MZV. ← evidence of naive conjecture about Stokes

This is suprising, because, this is not all relations!

We also may permutate columns!

(like in cubical coordinates.)