(5)

$$p(x) = \begin{cases} \frac{\theta - 1}{x^{\theta}}, & x \geq 1 \\ 0, & x < 1 \end{cases}, \quad \theta > 1$$

a) <u>ОМП:</u>

$$L(x, \theta) = \sum_{i=1}^{n} \frac{\theta - 1}{x_i^{\theta}}$$

$$\ln L(x, \theta) = \sum_{i=1}^{n} \ln(\theta - 1) - \sum_{i=1}^{n} \ln(x_i^{\theta}) =$$

$$= n \cdot \ln(\theta - 1) - \theta \sum_{i=1}^{n} \ln x_i$$

$$\frac{\partial \ln L}{\partial \Theta} = \frac{n}{\Theta - 1} - \sum_{i=1}^{n} \ln x_i = 0$$

$$\frac{n}{\Theta - 1} = \sum_{i=1}^{n} \ln x_i \; ; \quad \Theta - 1 = \frac{n}{\sum_{i=1}^{n} \ln x_i}$$

значит $\tilde{\Theta} = \frac{n}{\sum_{i=1}^{n} \ln x_i} + 1$

$$\frac{\partial^2 \ln L}{\partial \Theta^2} = -\frac{n}{(\Theta-1)^2} < 0 \Rightarrow \max$$

в) Докажем сильную регулярность модели:

$p(x, \theta)$ - непр. дифф. по θ.

$$\frac{\partial}{\partial \Theta} \int_{-\infty}^{\infty} \frac{\Theta - 1}{x^{\Theta}} dx = \frac{\partial}{\partial \Theta} (\Theta - 1) \cdot \frac{x^{-\Theta+1}}{-(-\Theta+1)} \Big|_{-\infty}^{+\infty} =$$

$$= \frac{\partial}{\partial \Theta} \left(x^{-\Theta+1} \right) = x^{1-\Theta} \ln x$$

$$\int_{-\infty}^{+\infty} \frac{\partial}{\partial \Theta} (\Theta - 1)(x^{-\Theta}) dx = \int_{-\infty}^{\infty} x^{-\Theta} - (\Theta - 1) x^{-\Theta} \ln x \, dx =$$

$$= \int_{-\infty}^{\infty} x^{-\Theta} (1 - \Theta \ln x + \ln x) dx =$$

$$= x^{1-\Theta} \ln x$$

Значит, интеграл перестановочный.

$$I(\theta) = M\left[\left(\frac{\partial \ln p}{\partial \theta}\right)^2\right]$$

$$\ln p = \ln(\theta-1) - \theta \ln x$$

$$\frac{\partial \ln p}{\partial \theta} = \frac{1}{\theta-1} - \ln x$$

$$I(\theta) = \int_1^{\infty} \left(\left(\frac{1}{\theta-1} - \ln x\right)^2 \cdot \frac{\theta-1}{x^\theta}\right) dx =$$

$$= \int_1^{+\infty} \left(\frac{1}{(\theta-1)^2} \cdot \frac{\theta-1}{x^\theta} - \frac{2\ln x}{\theta-1} \cdot \frac{\theta-1}{x^\theta} + \ln^2 x \cdot \frac{\theta-1}{x^\theta}\right) dx$$

$$= \int_1^{+\infty} \left(\frac{1}{(\theta-1)x^\theta} - \frac{2\ln x}{x^\theta} + \ln^2 x \cdot \frac{\theta-1}{x^\theta}\right) dx =$$

$$= \dots = \frac{1}{(\theta-1)^2} \quad \text{непр. и} > 0$$

$$\frac{\partial^2}{\partial \theta^2} \int_{-\infty}^{+\infty} p(x,\theta)\,dx = \frac{\partial^2}{\partial \theta^2} \int_{-\infty}^{\infty} \frac{\theta-1}{x^\theta}\,dx =$$

$$= \frac{\partial^2}{\partial \theta^2}\left(x^{-\theta+1}\right) = \frac{\partial}{\partial \theta}\left(x^{1-\theta}\ln x\right) = -x^{1-\theta}\ln^2 x$$

$$\int_{-\infty}^{+\infty} \frac{\partial^2}{\partial \theta^2} p(x,\theta)\,dx = \int_{-\infty}^{+\infty} \frac{\partial^2}{\partial \theta^2}\left(\frac{\theta-1}{x^\theta}\right) dx =$$

$$= \int_{-\infty}^{\infty} \frac{\partial}{\partial \theta}\left(x^{-\theta}(1 - \theta\ln x + \ln x)\right) dx =$$

$$= \int_{-\infty}^{+\infty} \left((1-\theta \ln x + \ln x) \cdot (-x^{-\theta} \ln x) - x^{-\theta} \ln x \right) dx =$$

$$= \int_{-\infty}^{\infty} \left(x^{-\theta} (\theta - 1) \ln^2 x - 2x^{-\theta} \ln x \right) dx =$$

$$= -x^{-\theta+1} \cdot \ln^2 x \Rightarrow$$ одинаково

$p(x, \theta)$ беск. дифф. при $\theta > 1$, модель регул. а частные производные перестановочны $\Rightarrow$ модель сильно регулярна

Тогда:

$$\sqrt{n} \cdot \frac{f(\tilde{\theta}) - f(\theta)}{\sigma(\theta)} \rightsquigarrow N(0, 1)$$

$$\sigma(\theta) = \sqrt{\nabla^T f(\theta) I^{-1}(\theta) \nabla f(\theta)}$$

$$\int_1^{x_{1/2}} p(x) dx = 0{,}5$$

$$\int_1^{x_{1/2}} \frac{\theta - 1}{x^\theta} dx = (\theta - 1) \int_1^{x_{1/2}} \frac{1}{x^\theta} dx =$$

$$= (\theta - 1) \cdot \frac{x^{-\theta+1}}{-\theta+1} = -x^{-\theta+1} \Big|_1^{x_{1/2}} = 1 - x_{1/2}^{1-\theta} = 0{,}5$$

$$x_{1/2}^{1-\theta} = 0{,}5; \quad x_{1/2} = \sqrt[\theta-1]{2}$$

$$\sqrt{n}\cdot\frac{g(\tilde\Theta)-g(\Theta)}{\sigma(\tilde\Theta)} \rightsquigarrow N(0,1)$$

$$\nabla g = \nabla\left(2^{\frac{1}{\Theta-1}}\right) = -\frac{\ln 2\cdot 2^{\frac{1}{\Theta-1}}}{(\Theta-1)^2}$$

$$\sigma(\bar\Theta) = \sqrt{\frac{\ln 2\cdot 2^{1/(\bar\Theta-1)}}{(\bar\Theta-1)^2}\cdot(\bar\Theta-1)^2\cdot\frac{\ln 2\cdot 2^{1/\bar\Theta-1}}{(\bar\Theta-1)^2}} =$$

$$= \frac{\ln 2\cdot 2^{1/(\bar\Theta-1)}}{(\bar\Theta-1)}$$

$$\sigma(\bar\Theta) \longrightarrow \sigma(\Theta)$$

<u>ОМП:</u>

$$-1{,}96 < \frac{\sqrt{n}\left(2^{1/(\bar\Theta-1)}-g(\Theta)\right)}{\frac{\ln 2}{(\bar\Theta-1)}\cdot 2^{1/(\bar\Theta-1)}} < 1{,}96$$

$$-\frac{1{,}96\cdot\ln 2\cdot 2^{1/(\bar\Theta-1)}}{(\bar\Theta-1)\sqrt{n}}+2^{1/(\bar\Theta-1)} < g(\Theta) < \frac{1{,}96\cdot\ln 2\cdot 2^{1/(\bar\Theta-1)}}{(\bar\Theta-1)(\sqrt{n})}+2^{1/(\bar\Theta-1)}$$

c) $g(\Theta)=\Theta$

$$\sigma(\bar\Theta) = \sqrt{(\bar\Theta-1)^2} = \tilde\Theta-1$$

$$-1{,}96 < \frac{\sqrt{n}(\bar\Theta-\Theta)}{\bar\Theta-1} < 1{,}96$$

$$-\frac{1{,}96(\bar\Theta-1)}{\sqrt{n}}+\tilde\Theta < \Theta < \frac{1{,}96(\bar\Theta-1)}{\sqrt{n}}+\tilde\Theta$$

$$\tilde{\Theta} = \frac{n}{\sum_{i=1}^{n} \ln x_i} + 1$$

d)

$$F(x) = \int_1^x \frac{\Theta - 1}{t^\Theta} dt = (\Theta - 1) \cdot \int_1^x t^{-\Theta} dt =$$

$$= (\Theta - 1) \cdot \frac{t^{-\Theta+1}}{1-\Theta} = -t^{-\Theta+1}\Big|_1^x =$$

$$= 1 - x^{1-\Theta}$$

$$\eta = 1 - \xi^{1-\Theta}$$

$$\xi^{1-\Theta} = 1 - \eta \Rightarrow \xi = \sqrt[1-\Theta]{1-\eta}$$