Первое задание.

① $\xi \sim R[0, \Theta]$ $\Theta > 0$ вер. модель

$\overrightarrow{x_n}$ – выборка и $p(x) = \frac{1}{\Theta}\{(0; \Theta)\}$

$$\tilde{\Theta}_1 = 2\bar{x} = 2 \cdot \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\tilde{\Theta}_2 = x_{min}; \quad \tilde{\Theta}_3 = x_{max};$$

$$\tilde{\Theta}_4 = x_1 + \frac{1}{n-1} \cdot \sum_{i=2}^{n} x_i$$

1) $\tilde{\Theta}_1$: несмещенность и сост-ть

$$M[2 \cdot \frac{1}{n}\sum x_i] = \frac{2}{n} \cdot M[\sum x_i] = \frac{2}{n} \cdot$$

все одинак

$$\cdot \sum M x_i = 2 M\xi = \Theta \Rightarrow \text{несмещ.}$$

$$\left\{ M\xi = \int_0^{\Theta} x \cdot \frac{1}{\Theta} dx = \frac{\Theta}{2} \right\}$$

$$D[2 \cdot \frac{1}{n}\sum x_i] = \frac{4}{n^2} D[\sum x_i] = \frac{4}{n^2} \sum \underbrace{D x_i}_{\text{все одинак}} =$$

$$= \frac{4}{n} D\xi = \frac{1}{3n}\Theta^2 \xrightarrow[n \to \infty]{} 0 \Rightarrow \text{состоят.}$$

(по дост. усл. сост.)

$$\left\{ D\xi = M\xi^2 - M^2\xi \right.$$

$$M\xi^2 = \int_0^{\Theta} x^2 \cdot \frac{1}{\Theta} dx = \frac{\Theta^2}{3}$$

$$\left. D\xi = \frac{\Theta^2}{3} - \frac{\Theta^2}{4} = \frac{\Theta^2}{12} \right\}$$

2) $\tilde{\Theta}_2 = x_{min} = x_{(1)}$

$$M[\tilde{\Theta}_2] = \int_{-\infty}^{\infty} y\, q(y)\, dy \;=$$

$x_i \sim R(0, \Theta) \quad x_i \sim F(x)$

$x_{(1)} \sim 1 - (1 - F(y))^n$

$$q(y) = n \cdot (1 - F(y))^{n-1} \cdot \frac{1}{\Theta} \{(0, \Theta)\}$$

$$= \int_0^{\Theta} y \cdot n \left(1 - \frac{y}{\Theta}\right)^{n-1} \cdot \frac{1}{\Theta}\, dy = \{t = 1 - y/\Theta\} =$$

$$= -\int_1^0 \Theta(1-t)\, n\, t^{n-1} \cdot \frac{1}{\Theta}\, \Theta\, dt = n\Theta\left(\int_0^1 t^{n-1} dt - \int_0^1 t^n dt\right) =$$

$$= n\Theta \cdot \frac{1}{n} - n\Theta \cdot \frac{1}{n+1} = \frac{\Theta}{n+1}$$ — смещ., но можно исправить.

$\tilde{\Theta}_2' = (n+1)\tilde{\Theta}_2 = (n+1) x_{min}$ — теперь несмещ.

$$D[\tilde{\Theta}_2] = M[\tilde{\Theta}_2^2] - M^2[\tilde{\Theta}_2] \;=$$

$$\Big\{ M\tilde{\Theta}_2^2 = \int_0^{\Theta} y^2 n \left(1 - \frac{y}{\Theta}\right)^{n-1} \cdot \frac{1}{\Theta}\, dy = \Big\{ t = 1 - \frac{y}{\Theta} \Big\} =$$

$$= -\int_1^0 \Theta^2 (1-t)^2 n\, t^{n-1} \frac{1}{\Theta}\, \Theta\, dt = n\Theta^2 \left(\int_0^1 t^{n-1} dt - \right.$$

$$\left. -2\int_0^1 t^n dt + \int_0^1 t^{n+1} dt\right) = n\Theta^2 \left(\frac{1}{n} - 2 \cdot \frac{1}{n+1} + \frac{1}{n+2}\right) =$$

$$= \cancel{n}\Theta^2 \left(\frac{(n+1)(n+2) - 2n(n+2) + n(n+1)}{\cancel{n}(n+1)(n+2)}\right) =$$

$$= \Theta^2 \left(\frac{\cancel{n^2} + \cancel{3n} + 2 - \cancel{2n^2} - \cancel{4n} + \cancel{n^2} + \cancel{n}}{(n+1)(n+2)}\right) = \frac{2\Theta^2}{(n+1)(n+2)}$$

$$= \frac{2\Theta^2}{(n+1)(n+2)} - \frac{\Theta^2}{(n+1)^2} = \frac{2\Theta^2 n + \cancel{2\Theta^2} - \Theta^2 n - \cancel{2\Theta^2}}{(n+1)^2(n+2)} =$$

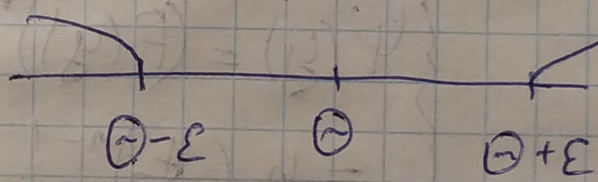
$$= \frac{\Theta^2 n}{(n+1)^2(n+2)} \xrightarrow[n\to\infty]{} 0$$, но смещ. / не можем польз. дост. усл.)

Проверяем по опр. сост-ти.

$\tilde{\Theta}_2 \xrightarrow{P} \Theta \quad \forall \Theta > 0$

$\forall \varepsilon > 0 \quad P(|\tilde{\Theta}_2 - \Theta| \geq \varepsilon) \xrightarrow[n\to\infty]{} 0$

$$\underset{x_{min}}{\underset{\parallel}{\tilde{\Theta}_2}} \geq \Theta + \varepsilon \text{ или } \tilde{\Theta}_2 \leq \Theta - \varepsilon$$



$x_i \sim R(0, \Theta) \to P(x_{min} \geq \Theta + \varepsilon) = 0.$

$$P(x_{min} \leq \Theta - \varepsilon) \underset{\text{в силу непр.}}{=} P(x_{min} < \Theta - \varepsilon) = \Phi(\Theta - \varepsilon).$$

$$\Phi(y) = 1 - (1 - F(y))^n = 1 - (1 - F(\Theta - \varepsilon))^n =$$

$$= 1 - \left(1 - \frac{\Theta - \varepsilon}{\Theta}\right)^n = 1 - \left(\frac{\varepsilon}{\Theta}\right)^n \xrightarrow[n\to\infty]{} 1 \qquad \begin{matrix} 0 < \Theta - \varepsilon < \Theta \\ 0 < \varepsilon < \Theta \end{matrix}$$

т.е. $\exists \varepsilon$, что $\not\to 0 \Rightarrow$ оценка не явл. сост.

Для $\tilde{\Theta}_2'$ по опр. сост.

$\tilde{\Theta}_2' \xrightarrow{P} \Theta, \ \forall \Theta > 0.$

$\forall \varepsilon > 0 \quad P(|\tilde{\Theta}_2' - \Theta| \geq \varepsilon) \xrightarrow[n\to\infty]{} 0$

$$P(|(n+1)x_{min} - \theta| \geq \varepsilon) \geq P(x_{min}(n+1) \geq \theta + \varepsilon) =$$

$$= P\left(x_{min} \geq \frac{\theta+\varepsilon}{n+1}\right) = 1 - P\left(x_{min} < \frac{\theta+\varepsilon}{n+1}\right) =$$

$$= 1 - \Phi\left(\frac{\theta+\varepsilon}{n+1}\right) = \left(1 - F\left(\frac{\theta+\varepsilon}{n+1}\right)\right)^n \underset{n \geq n_0}{=\!=}$$

$$= \left(1 - \frac{\theta+\varepsilon}{\theta(n+1)}\right)^n \xrightarrow[n \to \infty]{} e^{-\frac{\theta+\varepsilon}{\theta}} > 0.$$

$\Rightarrow$ <u>не явл. сост.</u>

3) $\tilde{\theta}_3 = x_{max}$

$$M[\tilde{\theta}_3] = \int_{-\infty}^{\infty} y\, q(y)\, dy \;=$$

$$\begin{cases} \Psi(y) = (F(y))^n;\; q(y) = \Psi'(y) = n(F(y))^{n-1} \cdot F'(y) = \\ = n\left(\frac{y}{\theta}\right)^{n-1} \cdot \frac{1}{\theta} \{(0;\theta)\}. \end{cases}$$

$$= \int_0^{\theta} y^n \frac{n}{\theta^n}\, dy = \frac{n}{\theta^n} \cdot \frac{\theta^{n+1}}{n+1} = \frac{\theta n}{n+1} \text{ — смещ.}$$

$\tilde{\theta}_3' = \frac{n+1}{n} x_{max}$ — <u>несмещ.</u>

$$D[\tilde{\theta}_3] = M[\tilde{\theta}_3^2] - M^2[\tilde{\theta}_3] \;=$$

$$\begin{cases} M[\tilde{\theta}_3^2] = \int_0^{\theta} y^2 q(y)\, dy = \int_0^{\theta} \frac{n}{\theta^n} y^{n+1}\, dy = \\ = \frac{n}{\theta^n} \cdot \frac{\theta^{n+2}}{n+2} \end{cases}$$

$$= \theta^2 \cdot \frac{n}{n+2} - \frac{\theta^2 n^2}{\underset{n^2+2n+1}{(n+1)^2}} = \cancel{\frac{\theta^2 n^2 + \theta^2 n - \theta^2 n^3 - \theta^2 n^2}{(n+2)(n+1)}}$$

$$= \Theta^2 \left(\frac{\cancel{n^3} + \cancel{2n^2} + n - \cancel{n^3} - \cancel{2n^2}}{(n+1)^2 (n+2)} \right) = \frac{\Theta^2 n}{(n+1)^2 (n+2)} \xrightarrow[n \to \infty]{} 0$$

по дост. усл. несмещ. (смещена)

Проверяем по опр. сост-ти:

$$\tilde{\Theta}_3 \xrightarrow{P} \Theta \qquad \forall \Theta > 0$$

$$\forall \varepsilon > 0 \quad P(|\tilde{\Theta}_3 - \Theta| \geq \varepsilon) \xrightarrow[n \to \infty]{} 0$$

$$P(|x_{max} - \Theta| \geq \varepsilon) = P(x_{max} \leq \Theta - \varepsilon) = 0$$

$\Rightarrow$ <u>сост-на</u>

$$D[\tilde{\Theta}_3] = D\left[\frac{n+1}{n} x_{max}\right] = \frac{\cancel{(n+1)^2}}{n^2} \frac{\Theta^2 n}{\cancel{(n+1)^2}(n+2)} \xrightarrow[n \to \infty]{} 0$$

(и несмещ.)

$\Rightarrow$ <u>сост-на</u> (по дост. усл.)

4) $$\tilde{\Theta}_4 = \overbrace{x_1}^{\xi_n} + \overbrace{\frac{1}{(n-1)} \sum_{i=2}^{n} x_i}^{\eta_n}$$

$$M[\tilde{\Theta}_4] = M\left[x_1 + \frac{1}{(n-1)} \sum x_i\right] = M[x_i] +$$

$$+ \frac{1}{n-1} \sum M[x_i] = M\xi + M\xi = 2M\xi = \Theta$$

$$D[\tilde{\Theta}_4] = D[x_1] + \frac{1}{(n-1)^2} \sum_{i=2}^{n} Dx_i = \frac{\Theta^2}{12} +$$

$$+ \frac{1}{n-1} \cdot \frac{\Theta^2}{12} \xrightarrow[n \to \infty]{} \frac{\Theta^2}{12}$$

(дост. усл. не работ.)

Проверяем по опр.:

$\tilde{\Theta}_4 \xrightarrow{P} \Theta$

① $\xi_n \xrightarrow{P} \xi$ $\quad \eta_n \xrightarrow{P} \eta$

$\left.\begin{array}{l} \\ \xi_n + \eta_n \xrightarrow{P} \xi + \eta \end{array}\right\}$ св-ва

② ЗБЧ Хинчина: ξ_i незав. и одинак. распр. и $M\xi_i < \infty$, тогда $\frac{1}{n}\sum_1^n \xi_i \xrightarrow{P} M\xi_i$

$\xi_n \xrightarrow{P} x_1$

$\eta_n \xrightarrow{P} M\xi = \frac{\Theta}{2}$ т.е. $\tilde{\Theta}_4 \xrightarrow{P} \xi + \frac{\Theta}{2}$

(опр не вып.) $\Rightarrow$

<u>не явл. сост.</u>

Сравнение оценок:

$D\tilde{\Theta}_1 = \frac{\Theta^2}{3n}$

$D\tilde{\Theta}_3' = \frac{\Theta^2}{(n+2)n}$ $\quad \frac{1}{3n} \geq \frac{1}{(n+2)n}$

$n+2 \geq 3 \Rightarrow n > 1$

т.е. $\tilde{\Theta}_3'$ эффективнее $\tilde{\Theta}_1$.