③ $\xi \sim F(x)$

$$p(x) = \begin{cases} e^{-x/\theta}/\theta, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$n = 3$

$$\tilde{\theta}_1 = \bar{x} = \frac{1}{n}\sum_{1}^{n=3} x_i ; \quad \tilde{\theta}_2 = x_{(2)}$$

а) несмещенность:

$\tilde{\theta}_1$: $M[\tilde{\theta}_1] = M[\frac{1}{n}\sum x_i] =$

$= \frac{1}{n} M\sum x_i = \frac{1}{n}\sum M\xi = M\xi \ominus$

$$M\xi = \int_0^\infty x e^{-x/\theta}/\theta \, dx = \int_0^\infty \frac{x}{\theta} \cdot e^{-x/\theta} dx =$$

$$= \left\{ \begin{matrix} t = \frac{x}{\theta} \\ dt = \frac{dx}{\theta} \end{matrix} \right\} = \theta \int_0^\infty t \cdot e^{-t} dt = \frac{1}{\theta}\left(-\int_0^\infty t \cdot de^{-t}\right) =$$

$$= \theta\left(-t e^{-t}\Big|_0^\infty - \int_0^\infty e^{-t} d(-t)\right) =$$

$$= \theta\left(0 - e^{-t}\Big|_0^\infty\right) = \theta$$

$\ominus \theta$ $\Rightarrow$ несмещ.

$\tilde{\theta}_2$: $M[\tilde{\theta}_2] = \int_0^\infty y\, q(y) dy \ominus$

$x_i \sim F(x)$

$$x_{(2)} \sim n \cdot p(y) \cdot C_{n-1}^{k-1} \cdot (F(y))^{k-1} \cdot (1 - F(y))^{n-k} =$$

$$= 6 p(y) \cdot F(y) \cdot (1 - F(y))$$

$$F(y) = -\int_0^y e^{-t/\theta}\, d\left(-\frac{t}{\theta}\right) = -e^{-x/\theta} + 1 = 1 - e^{-x/\theta}$$

$$q(y) = \frac{6e^{-2y/\theta}}{\theta}\left(1 - e^{-y/\theta}\right)$$

$$= \int_0^\infty y\, \frac{6e^{-2y/\theta}}{\theta}\left(1 - e^{-y/\theta}\right) dy =$$

$$= \frac{6}{\theta}\int_0^\infty y e^{-2y/\theta}\, dy - \frac{6}{\theta}\int_0^\infty y e^{-3y/\theta}\, dy =$$

$$= \left\{\begin{array}{l} t = y/\theta \\ dt = \frac{dy}{\theta} \end{array}\right\} = 6\left(\int_0^\infty t e^{-2t}\,\theta\, dt - \int_0^\infty t e^{-3t}\,\theta\, dt\right) =$$

$$= 6\theta\left(-\frac{1}{2}\int_0^\infty t\, de^{-2t} + \frac{1}{3}\int_0^\infty t\, de^{-3t}\right) =$$

$$= 6\theta\left(\cancel{-\frac{1}{2} t e^{-2t}\Big|_0^\infty}^{\to 0} + \frac{1}{2}\int_0^\infty e^{-2t}\, dt + \right.$$

$$\left. + \cancel{\frac{1}{3} t e^{-3t}\Big|_0^\infty}^{\to 0} - \frac{1}{3}\int_0^\infty e^{-3t}\, dt\right) =$$

$$= 6\theta\left(-\frac{1}{4} e^{-2t}\Big|_0^\infty + \frac{1}{9} e^{-3t}\Big|_0^\infty\right) =$$

$$= 6\theta \cdot \frac{5}{36} = \frac{5}{6}\theta \Rightarrow \text{смещ.}$$

$$\tilde{\theta}_2' = \frac{6}{5} x_{(2)} - \text{несмещ.}$$

в) Рассмотр. несмещ. оценки:

$$D[\tilde{\Theta}_1] = D[\frac{1}{n}\Sigma x_i] = D[\Sigma x_i]\cdot\frac{1}{n^2} =$$

$$= \frac{1}{n^2} D[\Sigma \xi_i] = \frac{1}{n} D\xi \ (=)$$

$$M\xi^2 = \int_0^\infty \frac{x^2 e^{-x/\theta}}{\Theta} dx = -\int_0^\infty x^2 de^{-x/\Theta} =$$

$$= \underbrace{-x^2 e^{-x/\Theta}\Big|_0^\infty}_{0} + 2\cdot\int_0^\infty \frac{xe^{-x/\theta}}{\theta} dx = 2\Theta\cdot\Theta =$$

$$= 2\Theta^2 \quad ; \quad D\xi = 2\Theta^2 - \Theta^2 = \Theta^2$$

$$(=) \quad \frac{\Theta^2}{3}$$

$$D[\tilde{\Theta}_2'] = M\tilde{\Theta}_2{}^2 - M^2\tilde{\Theta}_2' \ (=)$$

$$M[\tilde{\Theta}_2'^2] = \int_0^\infty y^2\cdot\frac{36}{25}\cdot 6\,\frac{e^{-2y/\theta}}{\Theta}(1-e^{-y/\theta})dy =$$

$$= \ldots = \frac{38}{25}\Theta^2$$

$$(=) \quad \frac{38}{25}\Theta^2 - \Theta^2 = \frac{13}{25}\Theta^2$$

$\Longrightarrow$ $\tilde{\Theta}_1$ – более эффективная.

c) исследование на эффективность (нер-во Крамера-Рао)

$\tilde{\Theta}_1$: $\rho(x, \tilde{\Theta}_1) = \dfrac{e^{-x/\theta}}{\Theta}$ – непр. дифф. по Θ на $(0; +\infty)$

$\Theta > 0.$

$$\frac{\partial}{\partial\theta}\int_0^{\infty} \rho(x,\theta)\,dx = \underbrace{\frac{\partial}{\partial\theta}\int_0^{\infty} \frac{e^{-x/\theta}}{\theta}\,dx}_{=0} \overset{?}{=}$$

$$\overset{?}{=} \int_0^{\infty} \frac{\partial}{\partial\theta}\left(\frac{e^{-x/\theta}}{\theta}\right)dx = \int_0^{\infty} \frac{e^{-x/\theta}}{\theta}\cdot\frac{x}{\theta^2}\,dx -$$

$$- \int_0^{\infty} \frac{e^{-x/\theta}}{\theta^2}\,dx = \int_0^{\infty} \frac{e^{-x/\theta}}{\theta^3}(x-\theta)\,dx = 0$$

Инф-я Фишера:

$$I(\tilde{\theta}_1) = M\left[\left(\frac{\partial \ln\left(\frac{e^{-x/\theta}}{\theta}\right)}{\partial\theta}\right)^2\right] =$$

$$= \int_0^{+\infty} \left(\frac{\partial \ln\left(\frac{e^{-x/\theta}}{\theta}\right)}{\partial\theta}\right)^2 \cdot \frac{e^{-x/\theta}}{\theta}\,dx =$$

$$= \int_0^{+\infty} \left(\frac{e^{-x/\theta}}{\theta^3}(x-\theta)\right)^2 \cdot \frac{\theta^{\cancel{2}}}{e^{-2x/\theta}} \cdot \frac{e^{-x/\theta}}{\cancel{\theta}}\,dx =$$

$$= \int_0^{+\infty} \frac{e^{-x/\theta}}{\theta}\cdot\left(\frac{x^2}{\theta^4} - \frac{2\theta x}{\theta^4} + \frac{\theta^2}{\theta^4}\right)dx =$$

$$= \int_0^{+\infty} \ldots = \frac{1}{\theta^4}\cdot 2\theta^2 - \frac{2}{\theta^3}\cdot\theta + \frac{1}{\theta^2} = \frac{1}{\theta^2}$$

$I(\tilde{\theta}_1) = \frac{1}{\theta^2}$ – непр. и >0 на $(0;+\infty)$

• $D = \frac{\theta^2}{3}$ – огр. $\Leftarrow$ оценка регулярна $\Rightarrow$ модель регулярна.

• несмещ. и • модель рег.)

$$D[\tilde{\theta}] \geq \frac{1}{n I(\theta)}; \quad \frac{\theta^2}{3} = \frac{1}{3 \cdot \frac{1}{\theta^2}}$$

(достигли).

$\Downarrow$

оценка <u>эффективна</u>

$\tilde{\theta}_2'$: $\underbrace{\frac{\partial}{\partial \theta} \int \frac{6e^{-2y/\theta}}{\theta}\left(1-e^{-y/\theta}\right)}_{=0} \overset{?}{=}$

$$\overset{?}{=} \int_0^{\infty} \frac{\partial}{\partial \theta}\left(\frac{6e^{-2y/\theta}}{\theta}\left(1-e^{-y/\theta}\right)\right) =$$

$$= \int_0^{\infty} \frac{\partial}{\partial \theta} \cdot \frac{6e^{-2y/\theta}}{\theta} - \int_0^{\infty} \frac{\partial}{\partial \theta} \frac{6e^{-3y/\theta}}{\theta} =$$

$$= \int_0^{\infty} \frac{6e^{-2y/\theta}}{\theta^3} \cdot 2y - \int_0^{\infty} \frac{6e^{-2y/\theta}}{\theta^2} - \int_0^{\infty} \frac{6e^{-3y/\theta}}{\theta^3} \cdot 3y +$$

$$+ \int_0^{\infty} \frac{6e^{-3y/\theta}}{\theta^2} = \frac{3}{\theta} - \frac{3}{\theta} + \frac{2}{\theta} - \frac{2}{\theta} = 0$$

Инф-я Фишера:

$$I(\tilde{\theta}_2') = M\left[\left(\frac{\partial \ln\left(\frac{6e^{-2y/\theta}}{\theta}\left(1-e^{-y/\theta}\right)\right)}{\partial \theta}\right)^2\right] =$$

$$= \int_0^{+\infty} \left(\frac{\partial \ln(*)}{\partial \theta}\right)^2 \cdot (*)\, dy =$$

$$= \int_0^{+\infty} \frac{\theta}{6e^{-2y/\theta}(1-e^{-y/\theta})} \cdot \frac{\partial}{\partial \theta}(*)\, dy =$$ интеграл не берется

$\tilde{\theta}_3$ – ничего нельзя сказать, т.к. не смогли проверить регулярность модели.