Gnomes and Glomes

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The Problem

Given 3-dimensional Euclidean space, we can think of different ways of breaking it up into non-intersecting lines. For instance, pick a plane in $\mathbb{R}^3$ and take all the lines that are perpendicular to it. Then any two such lines wont intersect and their union will be all of Euclidean space.

Now consider a harder version of this problem:

> Is there a set of lines in $\mathbb{R}^3$ whose union is $\mathbb{R}^3$, such that no two lines intersect and **no two lines are parallel**?

It seems easy at first—there is so much space to place lines and so many ways for a pair of lines to be skew (non-intersecting and non-parallel). But at the same time the lines have to fill up all of Euclidean space. Maybe at some point we'll run out of room and some lines will be forced to be parallel.

Perhaps, we can take our original example and modify it by keeping the intersection of the lines with our plane fixed and wiggling them around. Imagine a bunch of needles poked through a sheet of paper. Could they somehow be set at different angles with the paper to avoid being parallel? This seems like it should work but again there is no way to be certain and no particular arrangement comes to mind.

The problem with giving a concrete answer is that we would have to picture an arrangement of an infinite amount of lines, all densely packed together.

To spoil the answer to the question, such a set of lines does exists and in fact we can concretely visualize their placement. The solution to the problem involves many mathematical concepts: quaternions, higher-dimensional spheres, group theory, and projections used in map-making.

The Gnomonic Projection

A map projection is a way of taking part of a sphere, usually Earth, and capturing the points of that surface on a plane to create a map. One simple example, involves going to space and taking a picture of Earth. This will result in a 2-dimensional picture of half of the Earth and is called an orthographic projection, seen in figure 1.

One of way of interpretting this figure is to picture Earth's surface as being transparent. Then we can send rays of light upward through this surface and track where they hit a fixed plane that is being projected onto. Now let's consider a different map projection. Instead of having a bunch of parallel rays of light, position a tiny light source at the center of the Earth with rays travelling in all directions outward. The resulting projection we get is called the **gnomonic projection**, seen in figure 2.

This projection yields a horrific-looking map and to make matters worse, the map stretches out infinitely (we only see a piece of the map in figure 2). This is because points just above the equator are mapped very

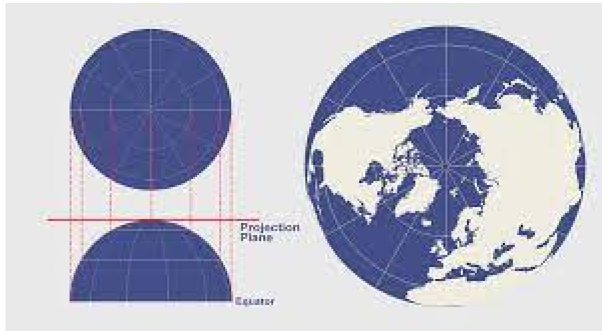

Figure 1: Orthographic projection of Earth. Source.

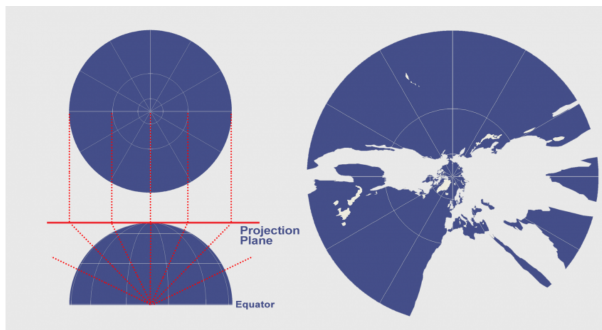

Figure 2: Gnomonic projection of Earth. Source.

far away from the north pole on the map.

However, this projection has one redeeming quality that will be useful for our problem. If you want to find the shortest path between two points on the gnomonic map, just draw a line connecting them along the map. This can be applied to finding flight paths for planes trying to fly to their destinations in minimal distance. In fact, no other map projection has this property and this is why flight paths end up looking curved on more standard maps (see figure 3).

Figure 3: Of the two flight paths seen on the Mercator projection (left), the curved one is actually a shorter distance and is seen again on the right. Source.

To understand why this works, think about the shortest path on a sphere's surface. The points of that path lie on a circle centered at the center of the Earth. Imagine walking between two points on the equator, for instance. Such circles that have the same center and radius as the Earth are called **great circles**, of which the equator is one example. The key observation is that great circles on the sphere are mapped to straight lines by the gnomonic projection, which is demonstrated in the geometric diagram of figure 4.

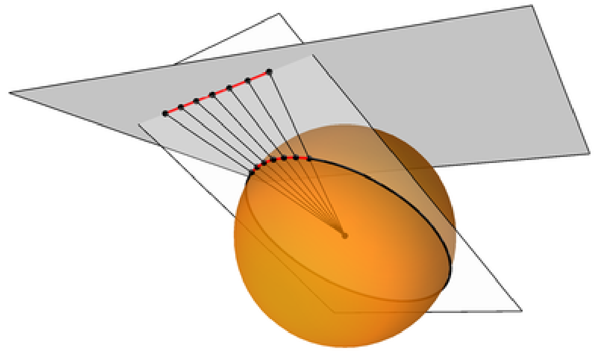

Figure 4: Since points of a great circle lie on the same plane, they remain on the plane throughout the projection process and end up mapping onto the same line. Source.

The diagram is a sort of proof without words. We position the grey plane to be projected onto above the sphere. Now take any great circle that isn't the equator. There is a plane that contains this circle and the center of the sphere; it is seen as a transparent plane in figure 4. The rays of light emitted from the center of the sphere travel along this plane until they hit points on the great circle and continue travelling outward along this plane. Where these light rays hit the grey projection plane is the intersection of the grey plane and the transparent plane—a line!

To get a better idea of how this relates to our problem, take two great circles and project them as in figure 5 to yield two lines. Now these two great circles must intersect at a pair of points that are at opposite ends of the sphere. If this intersection doesn't occur on the equator it will show up as an intersection of lines on the plane (we only see where the great circles intersect **above** the equator). Otherwise both intersection points of the great circles show up on the equator and there is no way to see a corresponding intersection on the map (the equator doesn't appear on the map). Thus the lines must be parallel. To summarize:

> A pair of (distinct) great circles on a sphere intersect on the equator of the sphere, if and only if, they show up as parallel lines after the gnomonic projection is applied.

This shows how a property of lines on the plane (being parallel or not) is turned into a property of great circles (where they intersect on the sphere). However, we aren't concerned with lines in a 2-dimensional plane, but rather lines in 3-dimensional space. For this, we will have to push all the concepts we just saw up one dimension.

The Glome

The unit sphere can be thought of as the points in space, (x, y, z), satisfying the equation $x^2 + y^2 + z^2 = 1$ (the equation tells us the points are a distance of 1 unit away from the origin). The next higher dimensional analog of this is called the **glome** or hypersphere: the set of points, (x, y, z, w), satisfying $x^2+y^2+z^2+w^2 = 1$. This is sometimes written as

$$\mathbb{S}^3 = \left\{(x, y, z, w) \in \mathbb{R}^4 : x^2 + y^2 + z^2 + w^2 = 1\right\}.$$

Remark: For those of you wondering why there is a superscript of 3, it's because this is a 3-dimensional analog of our sphere. The sphere we're used to is often thought of as 2-dimensional despite existing in 3-dimensional space. This is because the surface of the sphere is infinitely thin and plane-like in some ways. So you can sometimes see a unit sphere called $\mathbb{S}^2$ and a unit circle called $\mathbb{S}^1$.

Visualising what the glome looks like is no easy task. We can try looking at sections of it a time. Just like the unit sphere has circles of radius 1 as subsets (these are just the great circles), the glome has both circles and spheres of radius 1 as subsets (which we can call great circles and **great spheres**). We might not be able to see all of $\mathbb{S}^3$ at once but we could just picture a "slice" of it a time in the form of a great sphere.

Alternatively, we could use a map projection. Just as regular map projections create 2-dimensional maps of spheres living in 3-dimensional space, we can try creating a 3-dimensional map of an object living in 4-dimensional space. In fact, we can generalize the gnomonic projection.

To generalize the gnomonic projection, let's revisit it and try to right down how it works more formally. We can picture the Earth as a unit sphere with $(1, 0, 0)$ as the north pole. Then the gnomonic projection is a function, $g(x, y, z)$, which takes points on the upper half of the sphere and outputs points on a plane.

As a ray of light hits a point on the sphere, (x, y, z), it carries it outward to further and further points. In terms of vectors, all these points are just scalar multiples of the vector (x, y, z) (points of the form (ax, ay, az) for bigger and bigger values a). Now there is a plane balanced on the north pole at $(1, 0, 0)$ waiting to be hit by this ray. This plane consists of all points of the form $(1, y', z')$. Thus we are looking for a value a such that (ax, ay, az) is on this plane. This yields an equation

$$(ax, ay, az) = (1, y', z'),$$

which just boils down to solving for a in $ax = 1$. So $a = 1/x$. Observe that if $x = 0$ then $ax = 1$ has no solution and this occurs precisely when the point (x, y, z) is on the equator, which is "sent off to infinity" and thus can't show up on our map.

To summarize, the point (x, y, z) is mapped to the point

$$\frac{1}{x}(x, y, z) = \left(1, \frac{y}{x}, \frac{z}{x}\right)$$

on the plane and since all points on the plane have the same x-coordinate, we can drop it to consider the plane as a 2-dimensional object. Thus, we obtain a concrete way of writing the gnomonic projection as a function:

$$g(x, y, z) = \left(\frac{y}{x}, \frac{z}{x}\right),$$

where $x^2 + y^2 + z^2 = 1$ and $x > 0$ (this condition tells us that (x, y, z) is on the upper half of the sphere).

Can you now guess what the 4-dimensional gnomonic projection looks like? Just as (x, y, z) was sent out until it hit a point with x-coordinate 1, the point, (x, y, z, w), on the glome is sent out until it hits a point with x-coordinate 1. Thus, the gnomonic projection for the glome can be written as

$$g(x, y, z, w) = \left(\frac{y}{x}, \frac{z}{x}, \frac{w}{x}\right),$$

where $x^2 + y^2 + z^2 + w^2 = 1$ and $x > 0$. Following this train of thought we can generalize the gnomonic projection to higher and higher dimensions.

Similar to how the regular gnomonic projection maps great circles to lines in $\mathbb{R}^2$, this glome-gnomonic projection maps great circles to lines in $\mathbb{R}^3$ and maps great spheres to planes in $\mathbb{R}^3$.

Now think back to how the condition of parallel lines translated to intersections of great circles. If two great circles intersect in the glome, then we can consider the great sphere that contains both of them. Since the great sphere maps to a plane, the resulting pair of lines must both lie on that plane. Then they either intersect or are parallel (since they are lines on the same plane). This can be seen in figure 5.

Conversely, if the great circles don't intersect then there can't be a great sphere that contains both of them (since any two great circles on a sphere intersect). Thus, there can't be a plane containing both of

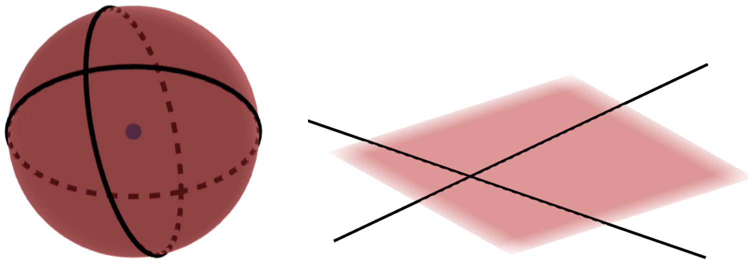

Figure 5: A pair of great circles on one of the great spheres of the glome being projected to a pair of intersecting lines on a plane

the resulting lines and the lines are skew. This shows that two great circles of a glome don't intersect, if and only if, the corresponding lines in $\mathbb{R}^3$ are skew. We see again how the gnomonic projection translates a property of great circles (not intersecting) into a property of corresponding lines (skewness).

We have thus formulated an equivalent problem to our original one:

Is there a set of great circles in $\mathbb{S}^3$, whose union is $\mathbb{S}^3$, such that no two great circles intersect?

This question seems harder than our original problem since it require visualizing 4-dimensions. But the solution to this problem turns out to be an interesting mathematical result known as the Hopf fibration. To answer this question we will need to introduce a 4-dimensional analog of the complex numbers, the quaternions.

Quaternions

For a great introduction to quaternions, I refer the reader to this video by 3Blue1Brown. The quaternions are a generalization of the complex numbers and are numbers of the form $a + bi + cj + dk$, where a, b, c, d are real numbers. So some examples of quaternions include $1 + i + 3j$, 2, $j - k$. The set of all quaternions is denoted as $\mathbb{H}$ in honor of their discoverer, William Hamilton. Like the imaginary unit i, the imaginary units of the quaternions, i,j,k, satisfy $i^2 = j^2 = k^2 = -1$. To understand how to multiply quaternions, we also have define the values of products like ij and jk. They are defined as

$$ij = k,\ ji = -k,\ jk = i,\ kj = -i\ , ki = j,\ ik = -j.$$

Notice that this multiplication operation isn't commutative since $ij \neq ji$.

The quaternions naturally resemble points in 4-dimensional space, $\mathbb{R}^4$. In this sense, we can define the norm or absolute value of a quaternion as being the distance of the corresponding point to the origin of $\mathbb{R}^4$:

$$|a + bi + cj + dk| = \sqrt{a^2 + b^2 + c^2 + d^2},$$

so the norm of $1 + i + 3j$ is $\sqrt{11}$ since the point $(1, 1, 3, 0)$ is that distance from the origin.

One can work out by hand that any two quaternions $u, v \in \mathbb{H}$ satisfy

$$|uv| = |u| \cdot |v|.$$

You may recall that the norms of the complex and real numbers also satisfy the above equation. If we interpret the points on the glome, $\mathbb{S}^3$, as quaternions, then it will just consist of the quaternions u with $|u| = 1$. Then above equation tells us that multiplying two points on the glome gives another point on the glome (if $|u| = |v| = 1$, then $|uv|$ is also 1). For those of you that have seen some group theory, it turns out that $\mathbb{S}^3$ forms a group with quaternion multiplication as the operation.

One last important fact we will need about quaternions is that multiplying by them represents 4-dimensional rotation (a concept discussed in greater detail in the 3Blue1Brown video I linked earlier). In particular, for a quaternion on the glome, $u \in \mathbb{S}^3$, we can define a function of quaternions

$$f(w) = wu.$$

This function will be a rotation of 4-dimensional space around the origin. For those of you who want to prove this, you will need to show that f satisfies the defining properties of such a rotation. In particular, you will have to show f is a linear transformation that preserves distances ($|f(w) - f(v)| = |w - v|$) and preserves orientation (has determinant 1).

Dissecting the glome

Recall that we are trying to break the glome up into disjoint great circles. An example of one such circle is

$$S = \{\cos(\theta) + \sin(\theta)i : \theta \in \mathbb{R}\},$$

which are the points of the complex unit circle if we were dealing with just complex numbers. We can also write these points down using Euler's formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

Now because the function from earlier, $f(w) = wu$, is a rotation, we can apply it to the points of S to get another unit circle of $\mathbb{S}^3$. For short hand we'll write this new circle as

$$Su = \left\{e^{i\theta}u : \theta \in \mathbb{R}\right\}.$$

For those familiar with group theory, S is actually a subgroup of $\mathbb{S}^3$ and Su is a right coset of this subgroup.

By using different u, we can obtain different unit circles of $\mathbb{S}^3$ this way; although we may get the same circle more than once (one can work out that $S1 = Si$ for example). Let the set of all circles we get this way be called

$$\mathcal{C} = \{Su : u \in \mathbb{S}^3\}.$$

We claim that this is precisely the set of disjoint circles that we want.

Claim: No two great circles in $\{Su : u \in \mathbb{S}^3\}$ intersect and the union of all the circles is $\mathbb{S}^3$. We can say that the circles **partition** $\mathbb{S}^3$.

Proof: To see that the union of the circles is $\mathbb{S}^3$, take any point $u \in \mathbb{S}^3$. We need to show that this point appears on one of the great circles. Observe that it appears in the circle Su, since 1 is an element of S and so $1u = u$ is an element of Su.

Now recall that we may get the same circle using different unit quaternions, u, v, and it's ok that $Su = Sv$ since they will show up as a single element of $\mathcal{C}$. The problem of intersection only occurs if the circles, Su, Sv, intersect but don't completely overlap. So suppose they intersect, sharing a common point $e^{i\theta_1}u = e^{i\theta_2}v$. We must show that $Su = Sv$.

To do this take any other point in Su; call it $e^{i\varphi}u$. We can manipulate $e^{i\theta_1}u = e^{i\theta_2}v$ by multiplying both sides by $e^{-i\theta_1}$ to get

$$e^{-i\theta_1}e^{i\theta_2}v = e^{-i\theta_1}e^{i\theta_1}u = e^0u = u.$$

Now

$$e^{i\varphi}u = e^{i\varphi}\left[e^{-i\theta_1}e^{i\theta_2}v\right] = e^{i(\varphi-\theta_1+\theta_2)}v,$$

which is an element of Sv since it is something in S times v. Using a similar argument we can show that any point of Sv lies in Su and so we must have that the two circles are equal: $Su = Sv$. $\square$

The proof above is actually a special case of a fact in group theory: a group, G (in this case $\mathbb{S}^3$), is partitioned by the cosets of a subgroup of G (in this case S). To learn more about these concepts, I refer the reader to two great videos on what groups are and what cosets are.

The Hopf fibration map

In 1931, Heinz Hopf had discovered an even more fascinating fact about the set of disjoint circles $\mathcal{C} = \{Su : u \in \mathbb{S}^3\}$; each circle corresponded to a unique point on a regular sphere. This allows one to quickly and efficiently compute circles in our set $\mathcal{C}$, but the details of this are a bit technical and you can visualize the solution to this article's problem without these details.

Instead of quaternions, Hopf thought of the points in $\mathbb{R}^4$ as a pairs of complex numbers, denoted $\mathbb{C}^2$. So the point (a, b, c, d) or $a + bi + cj + dk$ would be written as $(a + bi, c + di)$. In a similar way, he thought of points $(a, b, c) \in \mathbb{R}^3$ as a complex number and a real number $(a + bi, c)$.

We can write the condition for a point $(z, w) \in \mathbb{C}^2$ to lie on $\mathbb{S}^3$, as $|z|^2 + |w|^2 = 1$ since this means the point is a unit distance away from the origin. Now Hopf defined a function on these points, (z, w),

$$h(z, w) = (2z\overline{w}, |z|^2 - |w|^2),$$

where $\overline{a + bi} = a - bi$ is complex conjugation. The function outputs a complex number and a real number, which as mentioned earlier can be thought of as a point in $\mathbb{R}^3$. It can be verified by hand that the outputed points by $h(z, w)$ have unit distance to the origin of $\mathbb{R}^3$ and thus lie on the regular sphere, $\mathbb{S}^2$.

It turns out that a pair of inputs (z, w) and (z', w') are mapped by h to the same point in $\mathbb{S}^2$ precisely when they are both on the same circle in $\mathcal{C}$. In other words, h "collapses" different circles in $\mathcal{C}$ onto different points of $\mathbb{S}^2$.

To better understand this, it helps to observe that the pair of complex numbers

$$(z, w) = (a + bi, c + di)$$

corresponds to the quaternion

$$u = a + bi + cj + dk = (a + bi) + (c + di)j = z + wj$$

and so when we are rotating a point e^{it} of the circle S to get a point $e^{it}u$ of Su, we are just multiplying the z and w by e^{it}. In other words, point of the circle Su in $\mathcal{C}$ have the form

$$\left(e^{it}z, e^{it}w\right)$$

for changing the value of t gives different point on the circle.

Remark: In a sense to get around the different points of a circle Su, we need to take scalar multiples of the vector $(z, w) \in \mathbb{C}^2$ by multiplying it by complex numbers on the unit circle, e^{it}.

So to prove the special property of Hopf's function, $h(z, w)$, (the fact that all the points of Su are mapped to the same place) one needs to show that $h(z, w) = h(z', w')$, if and only if, there exists an $e^{it} \in \mathbb{C}$ such that $z' = e^{it}z$ and $w' = e^{it}w$. Let's prove part of this statement. If $z' = e^{it}z$ and $w' = e^{it}w$, then

$$\begin{aligned} h(z', w') &= h\left(e^{it}z, e^{it}w\right) \\ &= \left(2\left[e^{it}z\right]\left[\overline{e^{it}w}\right], \quad |e^{it}z|^2 - |e^{it}w|^2\right) \\ &= \left(2e^{it}ze^{-it}\overline{w}, \quad \left|e^{it}\right|^2 |z|^2 - \left|e^{it}\right|^2 |w|^2\right) \\ &= (2z\overline{w}, |z|^2 - |w|^2) = h(z, w). \end{aligned}$$

In the next section, we will try to better visualize this correspondence between great circles, Su, and points of a sphere, $\mathbb{S}^2$.

Graphing the projected circles

The standard way of visualizing the circles of the Hopf fibration is by using a stereographic projection, instead of gnomonic one. This projection is achieved by placing the light source at the south pole of instead of the center of the sphere. More about the different dimensional stereographic projections can be found in 3Blue1Brown's quaternion video.

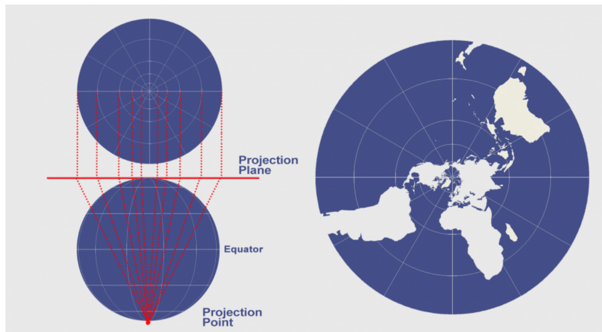

Figure 6: Stereographic projeciton of Earth. Source.

Unlike the gnomonic projection, which projects great circles to lines and great spheres to planes, the stereographic projection projects great circles to circles and great sphere to spheres (although it can greatly vary their radii). So in a sense, the stereographic projection preserves shape but not size.

Thus, one can visualize the Hopf fibration by picking points on a sphere and seeing the resulting stereographically projected circles pop up in 3-dimensional space. Since these circles don't intersect on the glome, they won't intersect in 3-dimensional space either. Philogb and Samuelj have provided two great interactive tools for this. Try picking different circles on the sphere and seeing how they interlink but don't intersect.

The technical details

In this section, we will lay out the specifics of how to graph the gnomonically projected circles Su, as lines in $\mathbb{R}^3$. Feel free to skim this part to get to visualization part.

Since each point on a sphere yields a great circle we need to somehow go about picking different point on a sphere. We can use a special coordinate system for locating these point, called spherical coordinates. Every point on the unit sphere can be described by two angles, a polar angle, θ, and an azimuthal angle, φ, which can be seen in the diagram of figure 7. The polar angle tells you which line of longitude you are on, while the azimuthal angle tells you which line of latitude you are on based on how far away you are from the north pole.

A point, P, with polar and azimuthal angles θ, φ, has Cartesian coordinates

$$(\cos(\theta)\sin(\varphi), \sin(\theta)\sin(\varphi), \cos(\varphi)).$$

Now recall that Hopf thought of points on the sphere as a complex number and a real number. In this sense, the point P could be simply written as

$$\left(e^{i\theta}\sin(\varphi), \cos(\varphi)\right).$$

Now you can check that

$$h\left(e^{i\theta/2}\cos(\varphi/2), e^{-i\theta/2}\sin(\varphi/2)\right) = \left(e^{i\theta}\sin(\varphi), \cos(\varphi)\right).$$

So the points of the great circle corresponding to P have the form

$$e^{it}\left(e^{i\theta/2}\cos(\varphi/2), e^{-i\theta/2}\sin(\varphi/2)\right)$$

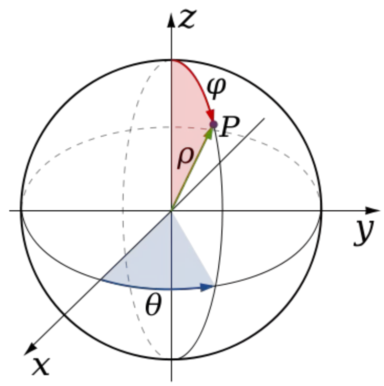

Figure 7: We see a sphere with radius ρ, with polar and azimuthal angles θ and φ shown for point P.

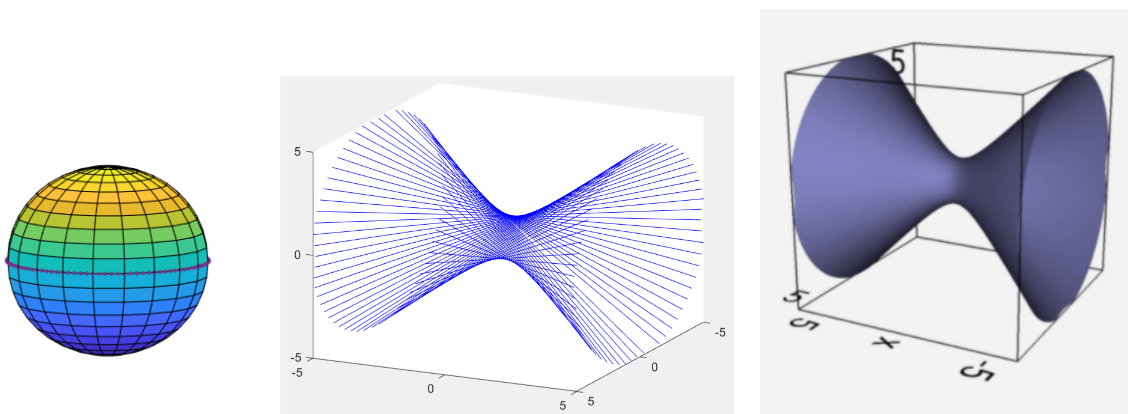

Figure 8: Letting $\varphi = \frac{\pi}{2}$, we get lines corresponding to points on the equator of the sphere (seen as purple points). These lines all lie on the surface on the far left, which has equation $y^2 + z^2 - x^2 = 1$.

for various $t \in \mathbb{R}$. We can write these points explicitly as points of $\mathbb{R}^4$:

$$(\cos(t+\theta/2)\cos(\varphi/2),\ \ \sin(t+\theta/2)\cos(\varphi/2),\ \ \cos(t-\theta/2)\sin(\varphi/2),\ \ \sin(t-\theta/2)\sin(\varphi/2))\,.$$

Then applying the gnomonic projection formula

$$g(x,y,z,w) = \frac{1}{x}(y,z,w)$$

to such a point gives a point

$$\left(\tan(t+\theta/2),\ \frac{\cos(t-\theta/2)}{\cos(t+\theta/2)}\tan(\varphi/2),\ \frac{\sin(t-\theta/2)}{\cos(t+\theta/2)}\tan(\varphi/2)\right).$$

So to graph a line in our magical set of non-skew lines, fix values θ, φ and plot the above point for all possible values $t \in \mathbb{R}$. Now the above expression doesn't look like it would produce a line with all those trigonometric functions, but miraculously it does.

Visualizing the results

To better understand the arrangement of the resulting lines and better answer our original problem, let's fix a value of φ and graph a bunch of lines for different $\theta \in [0, 2\pi]$. This would correspond to great circles represents by points on a line of latitude of $\mathbb{S}^2$.

Letting $\varphi = \pi/2$ for example, we get the lines seen in figure 8. It looks like what you would get if you held some dry spaghetti and let it fan out. In fact, all the lines lie on a surface called a **hyperboloid**, which is the shape used for designing power-plant towers. The hyperboloid is an example of a **ruled surface**, which is a surface that that can be written as a union of lines.

The equation of this particular hyperboloid is $y^2 + z^2 - x^2 = 1$. The general equation (if we used a different φ instead of $\varphi = \pi/2$) would be

$$\frac{1}{\tan^2(\varphi/2)}(y^2 + z^2) - x^2 = 1.$$

As φ increases towards its maximum value, π, these hyperboloids spread farther and farther away from the origin. Also note that if $\varphi = 0$, then the great circle corresponding to the north pole is the complex unit circle from earlier, S. This circle maps to a line along the x-axis.

So we can imagine nesting these hyperboloids within each and drawing lines on the surface of each hyperboloid, and this is the solution to our problem:

> Consider the set of hyperboloids satisfying equations of the form $\alpha(y^2 + z^2) - x^2 = 1$ for $\alpha > 0$. Then write each hyperboloid as a union of lines fanning out clockwise around the hyperboloid. This set of lines combined with the x-axis is a set of lines whose union is $\mathbb{R}^3$, such that no two lines are skew.

With a little more work we can work out a formula for pairs of points on these lines (specifically when the x-coordinate of a point on the line is 1 or -1). It turns out if you pick a pick a point $(-1, a, b)$, you can rotate 90 degrees around the x-axis and translate the point along the x-axis to $(1, -b, a)$ to get another point on the line containing $(-1, a, b)$. This allows us to further simplify the presentation of these lines:

> For various $(a, b) \in \mathbb{R}^2$, consider the line through the points $(-1, a, b)$ and $(1, -b, a)$. The set of these lines have union $\mathbb{R}^3$, and no two lines are skew.

Conclusion

In this article, we learned about spheres of different dimension and how map projections help us visualize them. We also saw how different projections affect these spheres differently (the gnomonic projection maps circles to lines, while the stereographic projection maps them to circles).

The main trick to solving the geometry problem of this article, breaking $\mathbb{R}^3$ into a set of skew lines, was to turn the problem into an algebraic one. Namely, the algebraic problem involved quaternion multiplication and the concept of groups. This is why quaternions (and complex numbers) are important: they aren't just points in space, but points that can be multiplied together in some meaningful geometric way.

A lot of higher level math is about finding these kinds of connections between different fields of math (like geometry and algebra) and using them to solve problems.