CHAPTER 16

Simultaneous Equations Models

In the previous chapter, we showed how the method of instrumental variables can solve two kinds of endogeneity problems: omitted variables and measurement error. Conceptually, these problems are straightforward. In the omitted variables case, there is a variable (or more than one) that we would like to hold fixed when estimating the ceteris paribus effect of one or more of the observed explanatory variables. In the measurement error case, we would like to estimate the effect of certain explanatory variables on y, but we have mismeasured one or more variables. In both cases, we could estimate the parameters of interest by OLS if we could collect better data.

Another important form of endogeneity of explanatory variables is **simultaneity**. This arises when one or more of the explanatory variables is *jointly determined* with the dependent variable, typically through an equilibrium mechanism (as we will see later). In this chapter, we study methods for estimating simple simultaneous equations models (SEMs). Although a complete treatment of SEMs is beyond the scope of this text, we are able to cover models that are widely used.

The leading method for estimating simultaneous equations models is the method of instrumental variables. Therefore, the solution to the simultaneity problem is essentially the same as the IV solutions to the omitted variables and measurement error problems. However, crafting and interpreting SEMs is challenging. Therefore, we begin by discussing the nature and scope of simultaneous equations models in Section 16.1. In Section 16.2, we confirm that OLS applied to an equation in a simultaneous system is generally biased and inconsistent.

Section 16.3 provides a general description of identification and estimation in a two-equation system, while Section 16.4 briefly covers models with more than two equations. Simultaneous equations models are used to model aggregate time series, and in Section 16.5 we include a discussion of some special issues that arise in such models. Section 16.6 touches on simultaneous equations models with panel data.

16.1 The Nature of Simultaneous Equations Models

The most important point to remember in using simultaneous equations models is that each equation in the system should have a ceteris paribus, causal interpretation. Because we only observe the outcomes in equilibrium, we are required to use counterfactual reasoning in constructing the equations of a simultaneous equations model. We must think in terms of potential as well as actual outcomes.

The classic example of an SEM is a supply and demand equation for some commodity or input to production (such as labor). For concreteness, let h_s denote the annual labor hours supplied by workers in agriculture, measured at the county level, and let w denote the average hourly wage offered to such workers. A simple labor supply function is

$$h_s = \alpha_1 w + \beta_1 z_1 + u_1, \qquad [16.1]$$

where z_1 is some observed variable affecting labor supply—say, the average manufacturing wage in the county. The error term, u_1, contains other factors that affect labor supply. [Many of these factors are observed and could be included in equation (16.1); to illustrate the basic concepts, we include only one such factor, z_1.] Equation (16.1) is an example of a **structural equation**. This name comes from the fact that the labor supply function is derivable from economic theory and has a causal interpretation. The coefficient α_1 measures how labor supply changes when the wage changes; if h_s and w are in logarithmic form, α_1 is the labor supply elasticity. Typically, we expect α_1 to be positive (although economic theory does not rule out $\alpha_1 \leq 0$). Labor supply elasticities are important for determining how workers will change the number of hours they desire to work when tax rates on wage income change. If z_1 is the manufacturing wage, we expect $\beta_1 \leq 0$: other factors equal, if the manufacturing wage increases, more workers will go into manufacturing than into agriculture.

When we graph labor supply, we sketch hours as a function of wage, with z_1 and u_1 held fixed. A change in z_1 shifts the labor supply function, as does a change in u_1. The difference is that z_1 is observed while u_1 is not. Sometimes, z_1 is called an *observed supply shifter*, and u_1 is called an *unobserved supply shifter*.

How does equation (16.1) differ from those we have studied previously? The difference is subtle. Although equation (16.1) is supposed to hold for all possible values of wage, we cannot generally view wage as varying exogenously for a cross section of counties. If we could run an experiment where we vary the level of agricultural and manufacturing wages across a sample of counties and survey workers to obtain the labor supply h_s for each county, then we could estimate (16.1) by OLS. Unfortunately, this is not a manageable experiment. Instead, we must collect data on average wages in these two sectors along with how many person hours were spent in agricultural production. In deciding how to analyze these data, we must understand that they are best described by the interaction of labor supply *and* demand. Under the assumption that labor markets clear, we actually observe *equilibrium* values of wages and hours worked.

To describe how equilibrium wages and hours are determined, we need to bring in the demand for labor, which we suppose is given by

$$h_d = \alpha_2 w + \beta_2 z_2 + u_2, \qquad [16.2]$$

where h_d is hours demanded. As with the supply function, we graph hours demanded as a function of wage, w, keeping z_2 and u_2 fixed. The variable z_2—say, agricultural land area—is an *observable demand shifter*, while u_2 is an *unobservable demand shifter*.

Just as with the labor supply equation, the labor demand equation is a structural equation: it can be obtained from the profit maximization considerations of farmers. If h_d and w are in logarithmic form, α_2 is the labor demand elasticity. Economic theory tells us that $\alpha_2 < 0$. Because labor and land are complements in production, we expect $\beta_2 > 0$.

Notice how equations (16.1) and (16.2) describe entirely different relationships. Labor supply is a behavioral equation for workers, and labor demand is a behavioral relationship for farmers. Each equation has a ceteris paribus interpretation and stands on its own. They become linked in an econometric analysis only because *observed* wage and hours are determined by the intersection of supply and demand. In other words, for each county i, observed hours h_i and observed wage w_i are determined by the equilibrium condition

$$h_{is} = h_{id}. \qquad [16.3]$$

Because we observe only equilibrium hours for each county i, we denote observed hours by h_i.

When we combine the equilibrium condition in (16.3) with the labor supply and demand equations, we get

$$h_i = \alpha_1 w_i + \beta_1 z_{i1} + u_{i1} \qquad [16.4]$$

and

$$h_i = \alpha_2 w_i + \beta_2 z_{i2} + u_{i2}, \qquad [16.5]$$

where we explicitly include the i subscript to emphasize that h_i and w_i are the equilibrium observed values for county i. These two equations constitute a **simultaneous equations model** (**SEM**), which has several important features. First, given z_{i1}, z_{i2}, u_{i1}, and u_{i2}, these two equations determine h_i and w_i. (Actually, we must assume that $\alpha_1 \neq \alpha_2$, which means that the slopes of the supply and demand functions differ; see Problem 1.) For this reason, h_i and w_i are the **endogenous variables** in this SEM. What about z_{i1} and z_{i2}? Because they are determined outside of the model, we view them as **exogenous variables**. From a statistical standpoint, the key assumption concerning z_{i1} and z_{i2} is that they are both uncorrelated with the supply and demand errors, u_{i1} and u_{i2}, respectively. These are examples of **structural errors** because they appear in the structural equations.

A second important point is that, without including z_1 and z_2 in the model, there is no way to tell which equation is the supply function and which is the demand function. When z_1 represents manufacturing wage, economic reasoning tells us that it is a factor in agricultural labor supply because it is a measure of the opportunity cost of working in agriculture; when z_2 stands for agricultural land area, production theory implies that it appears in the labor demand function. Therefore, we know that (16.4) represents labor supply and (16.5) represents labor demand. If z_1 and z_2 are the same—for example, average education level of adults in the county, which can affect both supply and demand—then the equations look identical, and there is no hope of estimating either one. In a nutshell, this illustrates the identification problem in simultaneous equations models, which we will discuss more generally in Section 16.3.

The most convincing examples of SEMs have the same flavor as supply and demand examples. Each equation should have a behavioral, ceteris paribus interpretation on its own. Because we only observe equilibrium outcomes, specifying an SEM requires us to ask such counterfactual questions as: How much labor *would* workers provide if the wage were different from its equilibrium value? Example 16.1 provides another illustration of an SEM where each equation has a ceteris paribus interpretation.

EXAMPLE 16.1 MURDER RATES AND SIZE OF THE POLICE FORCE

Cities often want to determine how much additional law enforcement will decrease their murder rates. A simple cross-sectional model to address this question is

$$murdpc = \alpha_1 polpc + \beta_{10} + \beta_{11} incpc + u_1, \quad [16.6]$$

where *murdpc* is murders per capita, *polpc* is number of police officers per capita, and *incpc* is income per capita. (Henceforth, we do not include an *i* subscript.) We take income per capita as exogenous in this equation. In practice, we would include other factors, such as age and gender distributions, education levels, perhaps geographic variables, and variables that measure severity of punishment. To fix ideas, we consider equation (16.6).

The question we hope to answer is: If a city exogenously increases its police force, will that increase, on average, lower the murder rate? If we could exogenously choose police force sizes for a random sample of cities, we could estimate (16.6) by OLS. Certainly, we cannot run such an experiment. But can we think of police force size as being exogenously determined, anyway? Probably not. A city's spending on law enforcement is at least partly determined by its expected murder rate. To reflect this, we postulate a second relationship:

$$polpc = \alpha_2 murdpc + \beta_{20} + other\ factors. \quad [16.7]$$

We expect that $\alpha_2 > 0$: other factors being equal, cities with higher (expected) murder rates will have more police officers per capita. Once we specify the other factors in (16.7), we have a two-equation simultaneous equations model. We are really only interested in equation (16.6), but, as we will see in Section 16.3, we need to know precisely how the second equation is specified in order to estimate the first.

An important point is that (16.7) describes behavior by city officials, while (16.6) describes the actions of potential murderers. This gives each equation a clear ceteris paribus interpretation, which makes equations (16.6) and (16.7) an appropriate simultaneous equations model.

We next give an example of an inappropriate use of SEMs.

EXAMPLE 16.2 HOUSING EXPENDITURES AND SAVING

Suppose that, for a random household in the population, we assume that annual housing expenditures and saving are jointly determined by

$$housing = \alpha_1 saving + \beta_{10} + \beta_{11} inc + \beta_{12} educ + \beta_{13} age + u_1 \quad [16.8]$$

and

$$saving = \alpha_2 housing + \beta_{20} + \beta_{21} inc + \beta_{22} educ + \beta_{23} age + u_2, \quad [16.9]$$

where *inc* is annual income and *educ* and *age* are measured in years. Initially, it may seem that these equations are a sensible way to view how housing and saving expenditures are determined. But we have to ask: What value would one of these equations be without the other? Neither has a ceteris paribus interpretation because *housing* and *saving* are chosen by the same household. For example, it makes no sense to ask this question: If annual income increases by \$10,000, how would housing expenditures change, *holding saving fixed*? If family income increases, a household will generally change the optimal mix of housing expenditures and saving. But equation (16.8) makes it seem as if we want to know the effect of changing *inc*, *educ*, or *age* while keeping *saving* fixed. Such a thought experiment is not interesting. Any model based on economic principles, particularly utility maximization, would have households optimally choosing *housing* and *saving* as functions of *inc* and the relative prices of housing and saving. The variables *educ* and *age* would affect preferences for consumption, saving, and risk. Therefore, *housing* and *saving* would each be functions of income, education, age, and other variables that affect the utility maximization problem (such as different rates of return on housing and other saving).

Even if we decided that the SEM in (16.8) and (16.9) made sense, there is no way to estimate the parameters. (We discuss this problem more generally in Section 16.3.) The two equations are indistinguishable, unless we assume that income, education, or age appears in one equation but not the other, which would make no sense.

Though this makes a poor SEM example, we might be interested in testing whether, other factors being fixed, there is a tradeoff between housing expenditures and saving. But then we would just estimate, say, (16.8) by OLS, unless there is an omitted variable or measurement error problem.

Example 16.2 has the characteristics of all too many SEM applications. The problem is that the two endogenous variables are chosen by the same economic agent. Therefore, neither equation can stand on its own. Another example of an inappropriate use of an SEM would be to model weekly hours spent studying and weekly hours working. Each student will choose these variables simultaneously—presumably as a function of the wage that can be earned working, ability as a student, enthusiasm for college, and so on. Just as in Example 16.2, it makes no sense to specify two equations where each is a function of the other. The important lesson is this: just because two variables are determined simultaneously does *not* mean that a simultaneous equations model is suitable. For an SEM to make sense, each equation in the SEM should have a ceteris paribus interpretation in isolation from the other equation. As we discussed earlier, supply and demand examples, and Example 16.1, have this feature. Usually, basic economic reasoning, supported in some cases by simple economic models, can help us use SEMs intelligently (including knowing when not to use an SEM).

EXPLORING FURTHER 16.1

Pindyck and Rubinfeld (1992, Section 11.6) describe a model of advertising where monopolistic firms choose profit maximizing levels of price and advertising expenditures. Does this mean we should use an SEM to model these variables at the firm level?

16.2 Simultaneity Bias in OLS

It is useful to see, in a simple model, that an explanatory variable that is determined simultaneously with the dependent variable is generally correlated with the error term, which leads to bias and inconsistency in OLS. We consider the two-equation structural model

$$y_1 = \alpha_1 y_2 + \beta_1 z_1 + u_1 \quad [16.10]$$

$$y_2 = \alpha_2 y_1 + \beta_2 z_2 + u_2 \quad [16.11]$$

and focus on estimating the first equation. The variables z_1 and z_2 are exogenous, so that each is uncorrelated with u_1 and u_2. For simplicity, we suppress the intercept in each equation.

To show that y_2 is generally correlated with u_1, we solve the two equations for y_2 in terms of the exogenous variables and the error term. If we plug the right-hand side of (16.10) in for y_1 in (16.11), we get

$$y_2 = \alpha_2(\alpha_1 y_2 + \beta_1 z_1 + u_1) + \beta_2 z_2 + u_2$$

or

$$(1 - \alpha_2\alpha_1)y_2 = \alpha_2\beta_1 z_1 + \beta_2 z_2 + \alpha_2 u_1 + u_2. \quad [16.12]$$

Now, we must make an assumption about the parameters in order to solve for y_2:

$$\alpha_2\alpha_1 \neq 1. \quad [16.13]$$

Whether this assumption is restrictive depends on the application. In Example 16.1, we think that $\alpha_1 \leq 0$ and $\alpha_2 \geq 0$, which implies $\alpha_1\alpha_2 \leq 0$; therefore, (16.13) is very reasonable for Example 16.1.

Provided condition (16.13) holds, we can divide (16.12) by $(1 - \alpha_2\alpha_1)$ and write y_2 as

$$y_2 = \pi_{21} z_1 + \pi_{22} z_2 + v_2, \quad [16.14]$$

where $\pi_{21} = \alpha_2\beta_1/(1 - \alpha_2\alpha_1)$, $\pi_{22} = \beta_2/(1 - \alpha_2\alpha_1)$, and $v_2 = (\alpha_2 u_1 + u_2)/(1 - \alpha_2\alpha_1)$. Equation (16.14), which expresses y_2 in terms of the exogenous variables and the error terms, is the **reduced form equation** for y_2, a concept we introduced in Chapter 15 in the context of instrumental variables estimation. The parameters π_{21} and π_{22} are called **reduced form parameters**; notice how they are nonlinear functions of the **structural parameters**, which appear in the structural equations, (16.10) and (16.11).

The **reduced form error**, v_2, is a linear function of the structural error terms, u_1 and u_2. Because u_1 and u_2 are each uncorrelated with z_1 and z_2, v_2 is also uncorrelated with z_1 and z_2. Therefore, we can consistently estimate π_{21} and π_{22} by OLS, something that is used for two stage least squares estimation (which we return to in the next section). In addition, the reduced form parameters are sometimes of direct interest, although we are focusing here on estimating equation (16.10).

A reduced form also exists for y_1 under assumption (16.13); the algebra is similar to that used to obtain (16.14). It has the same properties as the reduced form equation for y_2.

We can use equation (16.14) to show that, except under special assumptions, OLS estimation of equation (16.10) will produce biased and inconsistent estimators of α_1 and β_1 in equation (16.10). Because z_1 and u_1 are uncorrelated by assumption, the issue is whether y_2 and u_1 are uncorrelated. From the reduced form in (16.14), we see that y_2 and u_1 are correlated if and only if v_2 and u_1 are correlated (because z_1 and z_2 are assumed exogenous). But v_2 is a linear function of u_1 and u_2, so it is generally correlated with u_1. In fact, if we assume that u_1 and u_2 are uncorrelated, then v_2 and u_1 *must* be correlated whenever $\alpha_2 \neq 0$. Even if α_2 equals zero—which means that y_1 does not appear in equation (16.11)—v_2 and u_1 will be correlated if u_1 and u_2 are correlated.

When $\alpha_2 = 0$ *and* u_1 and u_2 are uncorrelated, y_2 and u_1 are also uncorrelated. These are fairly strong requirements: if $\alpha_2 = 0$, y_2 is not simultaneously determined with y_1. If we add zero correlation between u_1 and u_2, this rules out omitted variables or measurement errors in u_1 that are correlated with y_2. We should not be surprised that OLS estimation of equation (16.10) works in this case.

When y_2 is correlated with u_1 because of simultaneity, we say that OLS suffers from **simultaneity bias**. Obtaining the direction of the bias in the coefficients is generally complicated, as we saw with omitted variables bias in Chapters 3 and 5. But in simple models, we can determine the direction of the bias. For example, suppose that we simplify equation (16.10) by dropping z_1 from the equation, and we assume that u_1 and u_2 are uncorrelated. Then, the covariance between y_2 and u_1 is

$$\begin{aligned}\mathrm{Cov}(y_2,u_1) = \mathrm{Cov}(v_2,u_1) &= [\alpha_2/(1-\alpha_2\alpha_1)]\mathrm{E}(u_1^2)\\ &= [\alpha_2/(1-\alpha_2\alpha_1)]\sigma_1^2,\end{aligned}$$

where $\sigma_1^2 = \mathrm{Var}(u_1) > 0$. Therefore, the asymptotic bias (or inconsistency) in the OLS estimator of α_1 has the same sign as $\alpha_2/(1 - \alpha_2\alpha_1)$. If $\alpha_2 > 0$ and $\alpha_2\alpha_1 < 1$, the asymptotic bias is positive. (Unfortunately, just as in our calculation of omitted variables bias from Section 3.3, the conclusions do not carry over to more general models. But they do serve as a useful guide.) For example, in Example 16.1, we think $\alpha_2 > 0$ and $\alpha_2\alpha_1 \leq 0$, which means that the OLS estimator of α_1 would have a positive bias. If $\alpha_1 = 0$, OLS would, on average, estimate a *positive* impact of more police on the murder rate; generally, the estimator of α_1 is biased upward. Because we expect an increase in the size of the police force to reduce murder rates (ceteris paribus), the upward bias means that OLS will underestimate the effectiveness of a larger police force.

16.3 Identifying and Estimating a Structural Equation

As we saw in the previous section, OLS is biased and inconsistent when applied to a structural equation in a simultaneous equations system. In Chapter 15, we learned that the method of two stage least squares can be used to solve the problem of endogenous explanatory variables. We now show how 2SLS can be applied to SEMs.

The mechanics of 2SLS are similar to those in Chapter 15. The difference is that, because we specify a structural equation for each endogenous variable, we can immediately see whether sufficient IVs are available to estimate either equation. We begin by discussing the identification problem.

Identification in a Two-Equation System

We mentioned the notion of identification in Chapter 15. When we estimate a model by OLS, the key identification condition is that each explanatory variable is uncorrelated with the error term. As we demonstrated in Section 16.2, this fundamental condition no longer holds, in general, for SEMs. However, if we have some instrumental variables, we can still identify (or consistently estimate) the parameters in an SEM equation, just as with omitted variables or measurement error.

Before we consider a general two-equation SEM, it is useful to gain intuition by considering a simple supply and demand example. Write the system in equilibrium form (that is, with $q_s = q_d = q$ imposed) as

$$q = \alpha_1 p + \beta_1 z_1 + u_1 \qquad [16.15]$$

and

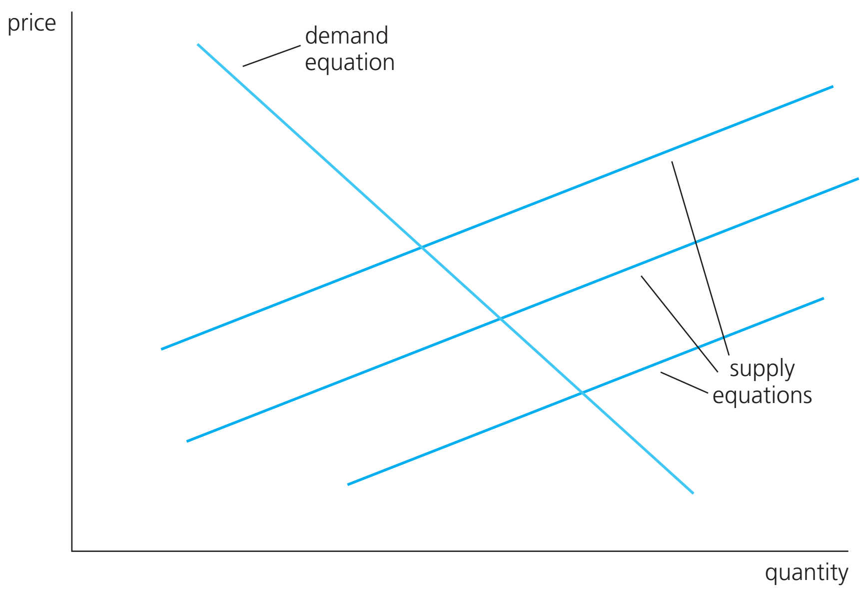
$$q = \alpha_2 p + u_2. \qquad [16.16]$$

For concreteness, let q be per capita milk consumption at the county level, let p be the average price per gallon of milk in the county, and let z_1 be the price of cattle feed, which we assume is exogenous to the supply and demand equations for milk. This means that (16.15) must be the supply function, as the price of cattle feed would shift supply ($\beta_1 < 0$) but not demand. The demand function contains no observed demand shifters.

Given a random sample on (q, p, z_1), which of these equations can be estimated? That is, which is an **identified equation**? It turns out that the *demand* equation, (16.16), is identified, but the supply equation is not. This is easy to see by using our rules for IV estimation from Chapter 15: we can use z_1 as an IV for price in equation (16.16). However, because z_1 appears in equation (16.15), we have no IV for price in the supply equation.

Intuitively, the fact that the demand equation is identified follows because we have an observed variable, z_1, that shifts the supply equation while not affecting the demand equation. Given variation in z_1 and no errors, we can trace out the demand curve, as shown in Figure 16.1. The presence of the unobserved demand shifter u_2 causes us to estimate the demand equation with error, but the estimators will be consistent, provided z_1 is uncorrelated with u_2.

FIGURE 16.1 Shifting supply equations trace out the demand equation. Each supply equation is drawn for a different value of the exogenous variable, z_1.



The supply equation cannot be traced out because there are no exogenous observed factors shifting the demand curve. It does not help that there are unobserved factors shifting the demand function; we need something observed. If, as in the labor demand function (16.2), we have an observed exogenous demand shifter—such as income in the milk demand function—then the supply function would also be identified.

To summarize: *In the system of* (*16.15*) *and* (*16.16*), *it is the presence of an exogenous variable in the supply equation that allows us to estimate the demand equation.*

Extending the identification discussion to a general two-equation model is not difficult. Write the two equations as

$$y_1 = \beta_{10} + \alpha_1 y_2 + \mathbf{z}_1\boldsymbol{\beta}_1 + u_1 \qquad [16.17]$$

and

$$y_2 = \beta_{20} + \alpha_2 y_1 + \mathbf{z}_2\boldsymbol{\beta}_2 + u_2, \qquad [16.18]$$

where y_1 and y_2 are the endogenous variables, and u_1 and u_2 are the structural error terms. The intercept in the first equation is β_{10}, and the intercept in the second equation is β_{20}. The variable $\mathbf{z}_1$ denotes a set of k_1 exogenous variables appearing in the first equation: $\mathbf{z}_1 = (z_{11}, z_{12}, \ldots, z_{1k_1})$. Similarly, $\mathbf{z}_2$ is the set of k_2 exogenous variables in the second equation: $\mathbf{z}_2 = (z_{21}, z_{22}, \ldots, z_{2k_2})$. In many cases, $\mathbf{z}_1$ and $\mathbf{z}_2$ will overlap. As a shorthand form, we use the notation

$$\mathbf{z}_1\boldsymbol{\beta}_1 = \beta_{11}z_{11} + \beta_{12}z_{12} + \ldots + \beta_{1k_1}z_{1k_1}$$

and

$$\mathbf{z}_2\boldsymbol{\beta}_2 = \beta_{21}z_{21} + \beta_{22}z_{22} + \ldots + \beta_{2k_2}z_{2k_2};$$

that is, $\mathbf{z}_1\boldsymbol{\beta}_1$ stands for all exogenous variables in the first equation, with each multiplied by a coefficient, and similarly for $\mathbf{z}_2\boldsymbol{\beta}_2$. (Some authors use the notation $\mathbf{z}_1'\boldsymbol{\beta}_1$ and $\mathbf{z}_2'\boldsymbol{\beta}_2$ instead. If you have an interest in the matrix algebra approach to econometrics, see Appendix E.)

The fact that $\mathbf{z}_1$ and $\mathbf{z}_2$ generally contain different exogenous variables means that we have imposed **exclusion restrictions** on the model. In other words, we *assume* that certain exogenous variables do not appear in the first equation and others are absent from the second equation. As we saw with the previous supply and demand examples, this allows us to distinguish between the two structural equations.

When can we solve equations (16.17) and (16.18) for y_1 and y_2 (as linear functions of all exogenous variables and the structural errors, u_1 and u_2)? The condition is the same as that in (16.13), namely, $\alpha_2\alpha_1 \neq 1$. The proof is virtually identical to the simple model in Section 16.2. Under this assumption, reduced forms exist for y_1 and y_2.

The key question is: Under what assumptions can we estimate the parameters in, say, (16.17)? This is the identification issue. The **rank condition** for identification of equation (16.17) is easy to state.

Rank Condition for Identification of a Structural Equation. The first equation in a two-equation simultaneous equations model is identified if, and only if, the *second* equation contains at least one exogenous variable (with a nonzero coefficient) that is excluded from the first equation.

This is the necessary and sufficient condition for equation (16.17) to be identified. The **order condition**, which we discussed in Chapter 15, is necessary for the rank condition. The order condition for identifying the first equation states that at least one exogenous variable is excluded from this equation. The order condition is trivial to check once both equations have been specified. The rank condition requires more: at least one of the exogenous variables excluded from the first equation must have a nonzero population coefficient in the second equation. This ensures that at least one of the exogenous variables omitted from the first equation actually appears in the reduced form of y_2, so that we can use these variables as instruments for y_2. We can test this using a t or an F test, as in Chapter 15; some examples follow.

Identification of the second equation is, naturally, just the mirror image of the statement for the first equation. Also, if we write the equations as in the labor supply and demand example in Section 16.1—so that y_1 appears on the left-hand side in *both* equations, with y_2 on the right-hand side—the identification condition is identical.

EXAMPLE 16.3 LABOR SUPPLY OF MARRIED, WORKING WOMEN

To illustrate the identification issue, consider labor supply for married women already in the workforce. In place of the demand function, we write the wage offer as a function of hours and the usual productivity variables. With the equilibrium condition imposed, the two structural equations are

$$\begin{aligned} hours = {} & \alpha_1 \log(wage) + \beta_{10} + \beta_{11} educ + \beta_{12} age + \beta_{13} kidslt6 \\ & + \beta_{14} nwifeinc + u_1 \end{aligned} \quad [16.19]$$

and

$$\begin{aligned} \log(wage) = {} & \alpha_2 hours + \beta_{20} + \beta_{21} educ + \beta_{22} exper \\ & + \beta_{23} exper^2 + u_2. \end{aligned} \quad [16.20]$$

The variable *age* is the woman's age, in years, *kidslt6* is the number of children less than six years old, *nwifeinc* is the woman's nonwage income (which includes husband's earnings), and *educ* and *exper* are years of education and prior experience, respectively. All variables except *hours* and log(*wage*) are assumed to be exogenous. (This is a tenuous assumption, as *educ* might be correlated with omitted ability in either equation. But for illustration purposes, we ignore the omitted ability problem.) The functional form in this system—where *hours* appears in level form but *wage* is in logarithmic form—is popular in labor economics. We can write this system as in equations (16.17) and (16.18) by defining $y_1 = hours$ and $y_2 = \log(wage)$.

The first equation is the supply function. It satisfies the order condition because two exogenous variables, *exper* and $exper^2$, are omitted from the labor supply equation. These exclusion restrictions are crucial assumptions: we are assuming that, once wage, education, age, number of small children, and other income are controlled for, past experience has no effect on current labor supply. One could certainly question this assumption, but we use it for illustration.

Given equations (16.19) and (16.20), the rank condition for identifying the first equation is that at least one of *exper* and $exper^2$ has a nonzero coefficient in equation (16.20).

If $\beta_{22} = 0$ and $\beta_{23} = 0$, there are no exogenous variables appearing in the second equation that do not also appear in the first (*educ* appears in both). We can state the rank condition for identification of (16.19) equivalently in terms of the reduced form for log(*wage*), which is

$$\log(wage) = \pi_{20} + \pi_{21}educ + \pi_{22}age + \pi_{23}kidslt6 + \pi_{24}nwifeinc + \pi_{25}exper + \pi_{26}exper^2 + v_2. \quad [16.21]$$

For identification, we need $\pi_{25} \neq 0$ or $\pi_{26} \neq 0$, something we can test using a standard F statistic, as we discussed in Chapter 15.

The wage offer equation, (16.20), is identified if at least one of *age*, *kidslt6*, or *nwifeinc* has a nonzero coefficient in (16.19). This is identical to assuming that the reduced form for *hours*—which has the same form as the right-hand side of (16.21)—depends on at least one of *age*, *kidslt6*, or *nwifeinc*. In specifying the wage offer equation, we are *assuming* that *age*, *kidslt6*, and *nwifeinc* have no effect on the offered wage, once hours, education, and experience are accounted for. These would be poor assumptions if these variables somehow have direct effects on productivity, or if women are discriminated against based on their age or number of small children.

In Example 16.3, we take the population of interest to be married women who are in the workforce (so that equilibrium hours are positive). This excludes the group of married women who choose not to work outside the home. Including such women in the model raises some difficult problems. For instance, if a woman does not work, we cannot observe her wage offer. We touch on these issues in Chapter 17; but for now, we must think of equations (16.19) and (16.20) as holding only for women who have $hours > 0$.

EXAMPLE 16.4 INFLATION AND OPENNESS

Romer (1993) proposes theoretical models of inflation that imply that more "open" countries should have lower inflation rates. His empirical analysis explains average annual inflation rates (since 1973) in terms of the average share of imports in gross domestic (or national) product since 1973—which is his measure of openness. In addition to estimating the key equation by OLS, he uses instrumental variables. While Romer does not specify both equations in a simultaneous system, he has in mind a two-equation system:

$$inf = \beta_{10} + \alpha_1 open + \beta_{11}\log(pcinc) + u_1 \quad [16.22]$$

$$open = \beta_{20} + \alpha_2 inf + \beta_{21}\log(pcinc) + \beta_{22}\log(land) + u_2, \quad [16.23]$$

where *pcinc* is 1980 per capita income, in U.S. dollars (assumed to be exogenous), and *land* is the land area of the country, in square miles (also assumed to be exogenous). Equation (16.22) is the one of interest, with the hypothesis that $\alpha_1 < 0$. (More open economies have lower inflation rates.) The second equation reflects the fact that the degree of openness might depend on the average inflation rate, as well as other factors. The variable log(*pcinc*)

EXPLORING FURTHER 16.2

If we have money supply growth since 1973 for each country, which we assume is exogenous, does this help identify equation (16.23)?

appears in both equations, but log(*land*) is *assumed* to appear only in the second equation. The idea is that, ceteris paribus, a smaller country is likely to be more open (so $\beta_{22} < 0$).

Using the identification rule that was stated earlier, equation (16.22) is identified, provided $\beta_{22} \neq 0$. Equation (16.23) is *not* identified because it contains both exogenous variables. But we are interested in (16.22).

Estimation by 2SLS

Once we have determined that an equation is identified, we can estimate it by two stage least squares. The instrumental variables consist of the exogenous variables appearing in either equation.

EXAMPLE 16.5 LABOR SUPPLY OF MARRIED, WORKING WOMEN

We use the data on working, married women in MROZ.RAW to estimate the labor supply equation (16.19) by 2SLS. The full set of instruments includes *educ*, *age*, *kidslt6*, *nwifeinc*, *exper*, and $exper^2$. The estimated labor supply curve is

$$\begin{aligned}
\widehat{hours} &= \underset{(574.56)}{2{,}225.66} + \underset{(470.58)}{1{,}639.56}\,\log(wage) - \underset{(59.10)}{183.75}\,educ \\
&\quad - \underset{(9.38)}{7.81}\,age - \underset{(182.93)}{198.15}\,kidslt6 - \underset{(6.61)}{10.17}\,nwifeinc \\
n &= 428,
\end{aligned} \tag{16.24}$$

which shows that the labor supply curve slopes upward. The estimated coefficient on log(*wage*) has the following interpretation: holding other factors fixed, $\Delta\widehat{hours} \approx 16.4(\%\Delta wage)$. We can calculate labor supply elasticities by multiplying both sides of this last equation by 100/*hours*:

$$100\cdot(\Delta\widehat{hours}/hours) \approx (1{,}640/hours)(\%\Delta wage)$$

or

$$\%\Delta\widehat{hours} \approx (1{,}640/hours)(\%\Delta wage),$$

which implies that the labor supply elasticity (with respect to wage) is simply 1,640/*hours*. [The elasticity is not constant in this model because *hours*, not log(*hours*), is the dependent variable in (16.24).] At the average hours worked, 1,303, the estimated elasticity is 1,640/1,303 ≈ 1.26, which implies a greater than 1% increase in hours worked given a 1% increase in wage. This is a large estimated elasticity. At higher hours, the elasticity will be smaller; at lower hours, such as $hours = 800$, the elasticity is over two.

For comparison, when (16.19) is estimated by OLS, the coefficient on log(*wage*) is −2.05 (se = 54.88), which implies no wage effect on hours worked. To confirm that log(*wage*) is in fact endogenous in (16.19), we can carry out the test from Section 15.5. When we add the reduced form residuals $\hat{v}_2$ to the equation and estimate by OLS, the t statistic on $\hat{v}_2$ is −6.61, which is very significant, and so log(*wage*) appears to be endogenous.

The wage offer equation (16.20) can also be estimated by 2SLS. The result is

$$\begin{aligned}\widehat{\log(wage)} = &-.656 + .00013\ hours + .110\ educ \\ &(.338)\quad (.00025) \qquad\quad (.016) \\ &+ .035\ exper - .00071\ exper^2 \\ &\quad (.019) \qquad\quad (.00045) \\ n = 428.\end{aligned} \tag{16.25}$$

This differs from previous wage equations in that *hours* is included as an explanatory variable and 2SLS is used to account for endogeneity of *hours* (and we assume that *educ* and *exper* are exogenous). The coefficient on *hours* is statistically insignificant, which means that there is no evidence that the wage offer increases with hours worked. The other coefficients are similar to what we get by dropping *hours* and estimating the equation by OLS.

Estimating the effect of openness on inflation by instrumental variables is also straightforward.

EXAMPLE 16.6 INFLATION AND OPENNESS

Before we estimate (16.22) using the data in OPENNESS.RAW, we check to see whether *open* has sufficient partial correlation with the proposed IV, log(*land*). The reduced form regression is

$$\begin{aligned}\widehat{open} &= 117.08 + .546\log(pcinc) - 7.57\log(land) \\ &\quad\ (15.85)\quad (1.493) \qquad\qquad\qquad (.81) \\ n &= 114, R^2 = .449.\end{aligned}$$

The t statistic on log(*land*) is over nine in absolute value, which verifies Romer's assertion that smaller countries are more open. The fact that log(*pcinc*) is so insignificant in this regression is irrelevant.

Estimating (16.22) using log(*land*) as an IV for *open* gives

$$\begin{aligned}\widehat{inf} &= 26.90 - .337\ open + .376\log(pcinc) \\ &\quad\ (15.40)\quad (.144) \qquad\quad (2.015) \\ n &= 114.\end{aligned} \tag{16.26}$$

EXPLORING FURTHER 16.3

How would you test whether the difference between the OLS and IV estimates on *open* are statistically different?

The coefficient on *open* is statistically significant at about the 1% level against a one-sided alternative ($\alpha_1 < 0$). The effect is economically important as well: for every percentage point increase in the import share of GDP, annual inflation is about one-third of a percentage point lower. For comparison, the OLS estimate is $-.215$ (se $= .095$).

16.4 Systems with More Than Two Equations

Simultaneous equations models can consist of more than two equations. Studying general identification of these models is difficult and requires matrix algebra. Once an equation in a general system has been shown to be identified, it can be estimated by 2SLS.

Identification in Systems with Three or More Equations

We will use a three-equation system to illustrate the issues that arise in the identification of complicated SEMs. With intercepts suppressed, write the model as

$$y_1 = \alpha_{12}y_2 + \alpha_{13}y_3 + \beta_{11}z_1 + u_1 \quad [16.27]$$

$$y_2 = \alpha_{21}y_1 + \beta_{21}z_1 + \beta_{22}z_2 + \beta_{23}z_3 + u_2 \quad [16.28]$$

$$y_3 = \alpha_{32}y_2 + \beta_{31}z_1 + \beta_{32}z_2 + \beta_{33}z_3 + \beta_{34}z_4 + u_3, \quad [16.29]$$

where the y_g are the endogenous variables and the z_j are exogenous. The first subscript on the parameters indicates the equation number, and the second indicates the variable number; we use α for parameters on endogenous variables and β for parameters on exogenous variables.

Which of these equations can be estimated? It is generally difficult to show that an equation in an SEM with more than two equations is identified, but it is easy to see when certain equations are *not* identified. In system (16.27) through (16.29), we can easily see that (16.29) falls into this category. Because every exogenous variable appears in this equation, we have no IVs for y_2. Therefore, we cannot consistently estimate the parameters of this equation. For the reasons we discussed in Section 16.2, OLS estimation will not usually be consistent.

What about equation (16.27)? Things look promising because z_2, z_3, and z_4 are all excluded from the equation—this is another example of *exclusion restrictions*. Although there are two endogenous variables in this equation, we have three potential IVs for y_2 and y_3. Therefore, equation (16.27) passes the order condition. For completeness, we state the order condition for general SEMs.

Order Condition for Identification. An equation in any SEM satisfies the order condition for identification if the number of *excluded* exogenous variables from the equation is at least as large as the number of right-hand side endogenous variables.

The second equation, (16.28), also passes the order condition because there is one excluded exogenous variable, z_4, and one right-hand side endogenous variable, y_1.

As we discussed in Chapter 15 and in the previous section, the order condition is only necessary, not sufficient, for identification. For example, if $\beta_{34} = 0$, z_4 appears nowhere in the system, which means it is not correlated with y_1, y_2, or y_3. If $\beta_{34} = 0$, then the second equation is not identified, because z_4 is useless as an IV for y_1. This again illustrates that identification of an equation depends on the values of the parameters (which we can never know for sure) in the other equations.

There are many subtle ways that identification can fail in complicated SEMs. To obtain sufficient conditions, we need to extend the rank condition for identification in two-equation systems. This is possible, but it requires matrix algebra [see, for example, Wooldridge (2010, Chapter 9)]. In many applications, one assumes that, unless there is obviously failure of identification, an equation that satisfies the order condition is identified.

The nomenclature on overidentified and just identified equations from Chapter 15 originated with SEMs. In terms of the order condition, (16.27) is an **overidentified equation** because we need only two IVs (for y_2 and y_3) but we have three available (z_2, z_3, and z_4); there is one overidentifying restriction in this equation. In general, the number of overidentifying restrictions equals the total number of exogenous variables in the system, minus the total number of explanatory variables in the equation. These can be tested using the overidentification test from Section 15.5. Equation (16.28) is a **just identified equation**, and the third equation is an **unidentified equation**.

Estimation

Regardless of the number of equations in an SEM, each identified equation can be estimated by 2SLS. The instruments for a particular equation consist of the exogenous variables appearing anywhere in the system. Tests for endogeneity, heteroskedasticity, serial correlation, and overidentifying restrictions can be obtained, just as in Chapter 15.

It turns out that, when any system with two or more equations is correctly specified and certain additional assumptions hold, *system estimation methods* are generally more efficient than estimating each equation by 2SLS. The most common system estimation method in the context of SEMs is *three stage least squares*. These methods, with or without endogenous explanatory variables, are beyond the scope of this text. [See, for example, Wooldridge (2010, Chapters 7 and 8).]

16.5 Simultaneous Equations Models with Time Series

Among the earliest applications of SEMs was estimation of large systems of simultaneous equations that were used to describe a country's economy. A simple Keynesian model of aggregate demand (that ignores exports and imports) is

$$C_t = \beta_0 + \beta_1(Y_t - T_t) + \beta_2 r_t + u_{t1} \qquad [16.30]$$

$$I_t = \gamma_0 + \gamma_1 r_t + u_{t2} \qquad [16.31]$$

$$Y_t \equiv C_t + I_t + G_t, \qquad [16.32]$$

where

C_t = consumption
Y_t = income
T_t = tax receipts
r_t = the interest rate
I_t = investment
G_t = government spending

[See, for example, Mankiw (1994, Chapter 9).] For concreteness, assume t represents year.

The first equation is an aggregate consumption function, where consumption depends on disposable income, the interest rate, and the unobserved structural error u_{t1}. The second equation is a very simple investment function. Equation (16.32) is an *identity* that is a result of national income accounting: it holds by definition, without error. Thus, there is no sense in which we estimate (16.32), but we need this equation to round out the model.

Because there are three equations in the system, there must also be three endogenous variables. Given the first two equations, it is clear that we intend for C_t and I_t to be endogenous. In addition, because of the accounting identity, Y_t is endogenous. We would assume, at least in this model, that T_t, r_t, and G_t are exogenous, so that they are uncorrelated with u_{t1} and u_{t2}. (We will discuss problems with this kind of assumption later.)

If r_t is exogenous, then OLS estimation of equation (16.31) is natural. The consumption function, however, depends on disposable income, which is endogenous because Y_t is. We have two instruments available under the maintained exogeneity assumptions: T_t and G_t. Therefore, if we follow our prescription for estimating cross-sectional equations, we would estimate (16.30) by 2SLS using instruments (T_t,G_t,r_t).

Models such as (16.30) through (16.32) are seldom estimated now, for several good reasons. First, it is very difficult to justify, at an aggregate level, the assumption that taxes, interest rates, and government spending are exogenous. Taxes clearly depend directly on income; for example, with a single marginal income tax rate τ_t in year t, $T_t = \tau_t Y_t$. We can easily allow this by replacing $(Y_t - T_t)$ with $(1 - \tau_t)Y_t$ in (16.30), and we can still estimate the equation by 2SLS if we assume that government spending is exogenous. We could also add the tax rate to the instrument list, if it is exogenous. But are government spending and tax rates really exogenous? They certainly could be in principle, if the government sets spending and tax rates independently of what is happening in the economy. But it is a difficult case to make in reality: government spending generally depends on the level of income, and at high levels of income, the same tax receipts are collected for lower marginal tax rates. In addition, assuming that interest rates are exogenous is extremely questionable. We could specify a more realistic model that includes money demand and supply, and then interest rates could be jointly determined with C_t, I_t, and Y_t. But then finding enough exogenous variables to identify the equations becomes quite difficult (and the following problems with these models still pertain).

Some have argued that certain components of government spending, such as defense spending—see, for example, Hall (1988) and Ramey (1991)—are exogenous in a variety of simultaneous equations applications. But this is not universally agreed upon, and, in any case, defense spending is not always appropriately correlated with the endogenous explanatory variables [see Shea (1993) for discussion and Computer Exercise C6 for an example].

A second problem with a model such as (16.30) through (16.32) is that it is completely static. Especially with monthly or quarterly data, but even with annual data, we often expect adjustment lags. (One argument in favor of static Keynesian-type models is that they are intended to describe the long run without worrying about short-run dynamics.) Allowing dynamics is not very difficult. For example, we could add lagged income to equation (16.31):

$$I_t = \gamma_0 + \gamma_1 r_t + \gamma_2 Y_{t-1} + u_{t2}. \quad [16.33]$$

In other words, we add a **lagged endogenous variable** (but not I_{t-1}) to the investment equation. Can we treat Y_{t-1} as exogenous in this equation? Under certain assumptions on u_{t2}, the answer is yes. But we typically call a lagged endogenous variable in an SEM a **predetermined variable**. Lags of exogenous variables are also predetermined. If we assume that u_{t2} is uncorrelated with current exogenous variables (which is standard) and all *past* endogenous and exogenous variables, then Y_{t-1} is uncorrelated with u_{t2}. Given exogeneity of r_t, we can estimate (16.33) by OLS.

If we add lagged consumption to (16.30), we can treat C_{t-1} as exogenous in this equation under the same assumptions on u_{t1} that we made for u_{t2} in the previous paragraph. Current disposable income is still endogenous in

$$C_t = \beta_0 + \beta_1(Y_t - T_t) + \beta_2 r_t + \beta_3 C_{t-1} + u_{t1}, \quad [16.34]$$

so we could estimate this equation by 2SLS using instruments (T_t,G_t,r_t,C_{t-1}); if investment is determined by (16.33), Y_{t-1} should be added to the instrument list. [To see why, use (16.32), (16.33), and (16.34) to find the reduced form for Y_t in terms of the exogenous and predetermined variables: T_t, r_t, G_t, C_{t-1}, and Y_{t-1}. Because Y_{t-1} shows up in this reduced form, it should be used as an IV.]

The presence of dynamics in aggregate SEMs is, at least for the purposes of forecasting, a clear improvement over static SEMs. But there are still some important problems with estimating SEMs using aggregate time series data, some of which we discussed in Chapters 11 and 15. Recall that the validity of the usual OLS or 2SLS inference procedures in time series applications hinges on the notion of *weak dependence*. Unfortunately, series such as aggregrate consumption, income, investment, and even interest rates seem to violate the weak dependence requirements. (In the terminology of Chapter 11, they have *unit roots*.) These series also tend to have exponential trends, although this can be partly overcome by using the logarithmic transformation and assuming different functional forms. Generally, even the large sample, let alone the small sample, properties of OLS and 2SLS are complicated and dependent on various assumptions when they are applied to equations with I(1) variables. We will briefly touch on these issues in Chapter 18. An advanced, general treatment is given by Hamilton (1994).

Does the previous discussion mean that SEMs are not usefully applied to time series data? Not at all. The problems with trends and high persistence can be avoided by specifying systems in first differences or growth rates. But one should recognize that this is a different SEM than one specified in levels. [For example, if we specify consumption growth as a function of disposable income growth and interest rate changes, this is different from (16.30).] Also, as we discussed earlier, incorporating dynamics is not especially difficult. Finally, the problem of finding truly exogenous variables to include in SEMs is often easier with disaggregated data. For example, for manufacturing industries, Shea (1993) describes how output (or, more precisely, growth in output) in other industries can be used as an instrument in estimating supply functions. Ramey (1991) also has a convincing analysis of estimating industry cost functions by instrumental variables using time series data.

The next example shows how aggregate data can be used to test an important economic theory, the permanent income theory of consumption, usually called the *permanent income hypothesis* (PIH). The approach used in this example is not, strictly speaking, based on a simultaneous equations model, but we can think of consumption and income growth (as well as interest rates) as being jointly determined.

EXAMPLE 16.7 TESTING THE PERMANENT INCOME HYPOTHESIS

Campbell and Mankiw (1990) used instrumental variables methods to test various versions of the permanent income hypothesis. We will use the annual data from 1959 through 1995 in CONSUMP.RAW to mimic one of their analyses. Campbell and Mankiw used quarterly data running through 1985.

One equation estimated by Campbell and Mankiw (using our notation) is

$$gc_t = \beta_0 + \beta_1 gy_t + \beta_2 r3_t + u_t, \tag{16.35}$$

where

$gc_t = \Delta\log(c_t) =$ annual growth in real per capita consumption (excluding durables).

$gy_t =$ growth in real disposable income.

$r3_t =$ the (ex post) real interest rate as measured by the return on three-month T-bill rates: $r3_t = i3_t - inf_t$, where the inflation rate is based on the Consumer Price Index.

The growth rates of consumption and disposable income are not trending, and they are weakly dependent; we will assume this is the case for $r3_t$ as well, so that we can apply standard asymptotic theory.

The key feature of equation (16.35) is that the PIH implies that the error term u_t has a zero mean conditional on all information observed at time $t - 1$ or earlier: $\mathrm{E}(u_t|I_{t-1}) = 0$. However, u_t is *not* necessarily uncorrelated with gy_t or $r3_t$; a traditional way to think about this is that these variables are jointly determined, but we are not writing down a full three-equation system.

Because u_t is uncorrelated with all variables dated $t - 1$ or earlier, valid instruments for estimating (16.35) are lagged values of gc, gy, and $r3$ (and lags of other observable variables, but we will not use those here). What are the hypotheses of interest? The pure form of the PIH has $\beta_1 = \beta_2 = 0$. Campbell and Mankiw argue that β_1 is positive if some fraction of the population consumes current income, rather than permanent income. The PIH with a nonconstant real interest rate implies that $\beta_2 > 0$.

When we estimate (16.35) by 2SLS, using instruments gc_{-1}, gy_{-1}, and $r3_{-1}$ for the endogenous variables gy_t and $r3_t$, we obtain

$$\begin{aligned} \widehat{gc}_t &= \underset{(.0032)}{.0081} + \underset{(.135)}{.586}\, gy_t - \underset{(.00076)}{.00027}\, r3_t \\ n &= 35,\ R^2 = .678. \end{aligned} \tag{16.36}$$

Therefore, the pure form of the PIH is strongly rejected because the coefficient on gy is economically large (a 1% increase in disposable income increases consumption by over .5%) and statistically significant ($t = 4.34$). By contrast, the real interest rate coefficient is very small and statistically insignificant. These findings are qualitatively the same as Campbell and Mankiw's.

The PIH also implies that the errors $\{u_t\}$ are serially uncorrelated. After 2SLS estimation, we obtain the residuals, $\hat{u}_t$, and include $\hat{u}_{t-1}$ as an additional explanatory variable in (16.36); we still use instruments gc_{t-1}, gy_{t-1}, $r3_{t-1}$, *and* $\hat{u}_{t-1}$ acts as its own instrument (see Section 15.7). The coefficient on $\hat{u}_{t-1}$ is $\hat{\rho} = .187$ (se $= .133$), so there is some evidence of positive serial correlation, although not at the 5% significance level. Campbell and Mankiw discuss why, with the available quarterly data, positive serial correlation might be found in the errors even if the PIH holds; some of those concerns carry over to annual data.

EXPLORING FURTHER 16.4

Suppose that for a particular city you have monthly data on per capita consumption of fish, per capita income, the price of fish, and the prices of chicken and beef; income and chicken and beef prices are exogenous. Assume that there is no seasonality in the demand function for fish, but there is in the supply of fish. How can you use this information to estimate a constant elasticity demand-for-fish equation? Specify an equation and discuss identification. (*Hint:* You should have 11 instrumental variables for the price of fish.)

Using growth rates of trending or I(1) variables in SEMs is fairly common in time series applications. For example, Shea (1993) estimates industry supply curves specified in terms of growth rates.

If a structural model contains a time trend—which may capture exogenous, trending factors that are not directly modeled—then the trend acts as its own IV.

16.6 Simultaneous Equations Models with Panel Data

Simultaneous equations models also arise in panel data contexts. For example, we can imagine estimating labor supply and wage offer equations, as in Example 16.3, for a group of people working over a given period of time. In addition to allowing for simultaneous determination of variables within each time period, we can allow for unobserved effects in each equation. In a labor supply function, it would be useful to allow an unobserved taste for leisure that does not change over time.

The basic approach to estimating SEMs with panel data involves two steps: (1) eliminate the unobserved effects from the equations of interest using the fixed effects transformation or first differencing and (2) find instrumental variables for the endogenous variables in the transformed equation. This can be very challenging because, for a convincing analysis, we need to find instruments that change over time. To see why, write an SEM for panel data as

$$y_{it1} = \alpha_1 y_{it2} + \mathbf{z}_{it1}\boldsymbol{\beta}_1 + a_{i1} + u_{it1} \quad [16.37]$$

$$y_{it2} = \alpha_2 y_{it1} + \mathbf{z}_{it2}\boldsymbol{\beta}_2 + a_{i2} + u_{it2}, \quad [16.38]$$

where i denotes cross section, t denotes time period, and $\mathbf{z}_{it1}\boldsymbol{\beta}_1$ or $\mathbf{z}_{it2}\boldsymbol{\beta}_2$ denotes linear functions of a set of exogenous explanatory variables in each equation. The most general analysis allows the unobserved effects, a_{i1} and a_{i2}, to be correlated with *all* explanatory variables, even the elements in z. However, we assume that the idiosyncratic structural errors, u_{it1} and u_{it2}, are uncorrelated with the $\mathbf{z}$ in both equations and across all time periods; this is the sense in which the $\mathbf{z}$ are exogenous. Except under special circumstances, y_{it2} is correlated with u_{it1}, and y_{it1} is correlated with u_{it2}.

Suppose we are interested in equation (16.37). We cannot estimate it by OLS, as the composite error $a_{i1} + u_{it1}$ is potentially correlated with all explanatory variables. Suppose we difference over time to remove the unobserved effect, a_{i1}:

$$\Delta y_{it1} = \alpha_1 \Delta y_{it2} + \Delta\mathbf{z}_{it1}\boldsymbol{\beta}_1 + \Delta u_{it1}. \quad [16.39]$$

(As usual with differencing or time-demeaning, we can only estimate the effects of variables that change over time for at least some cross-sectional units.) Now, the error term in this equation is uncorrelated with $\Delta \mathbf{z}_{it1}$ by assumption. But Δy_{it2} and Δu_{it1} are possibly correlated. Therefore, we need an IV for Δy_{it2}.

As with the case of pure cross-sectional or pure time series data, possible IVs come from the *other* equation: elements in $\mathbf{z}_{it2}$ that are not also in $\mathbf{z}_{it1}$. In practice, we need *time-varying* elements in $\mathbf{z}_{it2}$ that are not also in $\mathbf{z}_{it1}$. This is because we need an instrument for Δy_{it2}, and a change in a variable from one period to the next is unlikely to be highly correlated with the *level* of exogenous variables. In fact, if we difference (16.38), we see that the natural IVs for Δy_{it2} are those elements in $\Delta \mathbf{z}_{it2}$ that are not also in $\Delta \mathbf{z}_{it1}$.

As an example of the problems that can arise, consider a panel data version of the labor supply function in Example 16.3. After differencing, suppose we have the equation

$$\Delta hours_{it} = \beta_0 + \alpha_1 \Delta \log(wage_{it}) + \Delta(other\ factors_{it}),$$

and we wish to use $\Delta exper_{it}$ as an instrument for $\Delta \log(wage_{it})$. The problem is that, because we are looking at people who work in every time period, $\Delta exper_{it} = 1$ for all i and t. (Each person gets another year of experience after a year passes.) We cannot use an IV that is the same value for all i and t, and so we must look elsewhere.

Often, participation in an experimental program can be used to obtain IVs in panel data contexts. In Example 15.10, we used receipt of job training grants as an IV for the change in hours of training in determining the effects of job training on worker productivity. In fact, we could view that in an SEM context: job training and worker productivity are jointly determined, but receiving a job training grant is exogenous in equation (15.57).

We can sometimes come up with clever, convincing instrumental variables in panel data applications, as the following example illustrates.

EXAMPLE 16.8 EFFECT OF PRISON POPULATION ON VIOLENT CRIME RATES

In order to estimate the causal effect of prison population increases on crime rates at the state level, Levitt (1996) used instances of prison overcrowding litigation as instruments for the growth in prison population. The equation Levitt estimated is in first differences; we can write an underlying fixed effects model as

$$\log(crime_{it}) = \theta_t + \alpha_1 \log(prison_{it}) + \mathbf{z}_{it1}\boldsymbol{\beta}_1 + a_{i1} + u_{it1}, \quad [16.40]$$

where θ_t denotes different time intercepts, and *crime* and *prison* are measured per 100,000 people. (The prison population variable is measured on the last day of the previous year.) The vector $\mathbf{z}_{it1}$ contains log of police per capita, log of income per capita, the unemployment rate, proportions of black and those living in metropolitan areas, and age distribution proportions.

Differencing (16.40) gives the equation estimated by Levitt:

$$\Delta \log(crime_{it}) = \xi_t + \alpha_1 \Delta \log(prison_{it}) + \Delta \mathbf{z}_{it1}\boldsymbol{\beta}_1 + \Delta u_{it1}. \quad [16.41]$$

Simultaneity between crime rates and prison population, or more precisely in the growth rates, makes OLS estimation of (16.41) generally inconsistent. Using the violent crime rate and a subset of the data from Levitt (in PRISON.RAW, for the years 1980 through 1993, for $51 \cdot 14 = 714$ total observations), we obtain the pooled OLS estimate of

α_1, which is $-.181$ (se $= .048$). We also estimate (16.41) by pooled 2SLS, where the instruments for $\Delta\log(prison)$ are two binary variables, one each for whether a final decision was reached on overcrowding litigation in the current year or in the previous two years. The pooled 2SLS estimate of α_1 is -1.032 (se $= .370$). Therefore, the 2SLS estimated effect is much larger; not surprisingly, it is much less precise, too. Levitt found similar results when using a longer time period (but with early observations missing for some states) and more instruments.

Testing for AR(1) serial correlation in $r_{it1} = \Delta u_{it1}$ is easy. After the pooled 2SLS estimation, obtain the residuals, $\hat{r}_{it1}$. Then, include one lag of these residuals in the original equation, and estimate the equation by 2SLS, where $\hat{r}_{it1}$ acts as its own instrument. The first year is lost because of the lagging. Then, the usual 2SLS t statistic on the lagged residual is a valid test for serial correlation. In Example 16.8, the coefficient on $\hat{r}_{it1}$ is only about .076 with $t = 1.67$. With such a small coefficient and modest t statistic, we can safely assume serial independence.

An alternative approach to estimating SEMs with panel data is to use the fixed effects transformation and then to apply an IV technique such as pooled 2SLS. A simple procedure is to estimate the time-demeaned equation by pooled 2SLS, which would look like

$$\ddot{y}_{it1} = \alpha_1 \ddot{y}_{t2} + \ddot{\mathbf{z}}_{it1}\boldsymbol{\beta}_1 + \ddot{u}_{it1}, \quad t = 1, 2, \ldots, T, \qquad [16.42]$$

where $\ddot{\mathbf{z}}_{it1}$ and $\ddot{\mathbf{z}}_{it2}$ are IVs. This is equivalent to using 2SLS in the dummy variable formulation, where the unit-specific dummy variables act as their own instruments. Ayres and Levitt (1998) applied 2SLS to a time-demeaned equation to estimate the effect of LoJack electronic theft prevention devices on car theft rates in cities. If (16.42) is estimated directly, then the df needs to be corrected to $N(T-1) - k_1$, where k_1 is the total number of elements in α_1 and $\boldsymbol{\beta}_1$. Including unit-specific dummy variables and applying pooled 2SLS to the original data produces the correct df. A detailed treatment of 2SLS with panel data is given in Wooldridge (2010, Chapter 11).

Summary

Simultaneous equations models are appropriate when each equation in the system has a ceteris paribus interpretation. Good examples are when separate equations describe different sides of a market or the behavioral relationships of different economic agents. Supply and demand examples are leading cases, but there are many other applications of SEMs in economics and the social sciences.

An important feature of SEMs is that, by fully specifying the system, it is clear which variables are assumed to be exogenous and which ones appear in each equation. Given a full system, we are able to determine which equations can be identified (that is, can be estimated). In the important case of a two-equation system, identification of (say) the first equation is easy to state: at least one exogenous variable must be excluded from the first equation that appears with a nonzero coefficient in the second equation.

As we know from previous chapters, OLS estimation of an equation that contains an endogenous explanatory variable generally produces biased and inconsistent estimators. Instead, 2SLS can be used to estimate any identified equation in a system. More advanced system methods are available, but they are beyond the scope of our treatment.

The distinction between omitted variables and simultaneity in applications is not always sharp. Both problems, not to mention measurement error, can appear in the same equation. A good example is the labor supply of married women. Years of education (*educ*) appears in both the labor supply and the wage offer functions [see equations (16.19) and (16.20)]. If omitted ability is in the error term of the labor supply function, then wage and education are both endogenous. The important thing is that an equation estimated by 2SLS can stand on its own.

SEMs can be applied to time series data as well. As with OLS estimation, we must be aware of trending, integrated processes in applying 2SLS. Problems such as serial correlation can be handled as in Section 15.7. We also gave an example of how to estimate an SEM using panel data, where the equation is first differenced to remove the unobserved effect. Then, we can estimate the differenced equation by pooled 2SLS, just as in Chapter 15. Alternatively, in some cases, we can use time-demeaning of all variables, including the IVs, and then apply pooled 2SLS; this is identical to putting in dummies for each cross-sectional observation and using 2SLS, where the dummies act as their own instruments. SEM applications with panel data are very powerful, as they allow us to control for unobserved heterogeneity while dealing with simultaneity. They are becoming more and more common and are not especially difficult to estimate.

Key Terms

Endogenous Variables
Exclusion Restrictions
Exogenous Variables
Identified Equation
Just Identified Equation
Lagged Endogenous Variable
Order Condition
Overidentified Equation
Predetermined Variable
Rank Condition
Reduced Form Equation
Reduced Form Error
Reduced Form Parameters
Simultaneity
Simultaneity Bias
Simultaneous Equations Model (SEM)
Structural Equation
Structural Errors
Structural Parameters
Unidentified Equation

Problems

1 Write a two-equation system in "supply and demand form," that is, with the same variable y_1 (typically, "quantity") appearing on the left-hand side:

$$y_1 = \alpha_1 y_2 + \beta_1 z_1 + u_1$$
$$y_1 = \alpha_2 y_2 + \beta_2 z_2 + u_2.$$

(i) If $\alpha_1 = 0$ or $\alpha_2 = 0$, explain why a reduced form exists for y_1. (Remember, a reduced form expresses y_1 as a linear function of the exogenous variables and the structural errors.) If $\alpha_1 \neq 0$ and $\alpha_2 = 0$, find the reduced form for y_2.

(ii) If $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, and $\alpha_1 \neq \alpha_2$, find the reduced form for y_1. Does y_2 have a reduced form in this case?

(iii) Is the condition $\alpha_1 \neq \alpha_2$ likely to be met in supply and demand examples? Explain.