Bounds on the rate of superimposed codes

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***Abstract*—A binary code is called a superimposed cover-free (s,ℓ)-code if the code is identified by the incidence matrix of a family of finite sets in which no intersection of ℓ sets is covered by the union of s others. A binary code is called a superimposed list-decoding s_L-code if the code is identified by the incidence matrix of a family of finite sets in which the union of any s sets can cover not more than $L-1$ other sets of the family. For $L=\ell=1$, both of the definitions coincide and the corresponding binary code is called a superimposed s-code. Our aim is to obtain new lower and upper bounds on the rate of the given codes. The most interesting result is a lower bound on the rate of superimposed cover-free (s,ℓ)-codes based on the ensemble of constant weight binary codes. If the parameter $\ell\ge 1$ is fixed and $s\to\infty$, then the ratio of this lower bound to the best known upper bound converges to the limit $2\,e^{-2}=0.271$. For the classical case $\ell=1$ and $s\ge 2$, the given statement means that the upper bound on the rate of superimposed s-codes obtained by A.G. Dyachkov and V.V. Rykov (1982) is asymptotically attained to within a constant factor a, $2\,e^{-2}\le a\le 1$.**

I. Notations and Definitions

Let N, t, s, L and ℓ be integers, $1\le s<t$, $1\le L\le t-s$, $1\le \ell\le t-s$. Let $\triangleq$ denote the equality by definition, $|A|$ – the size of A and $[N]\triangleq\{1,2,\dots,N\}$ - the set of integers from 1 to N. A binary $(N\times t)$-matrix

$$X=\|x_i(j)\|,\quad x_i(j)=0,1,\ i\in[N],\ j\in[t] \tag{1}$$

with N rows and t columns (codewords) is called a *code of length N and size t*. The standard symbol $\bigvee$ denotes the *disjunct* (Boolean) sum of two binary numbers:

$$0\bigvee 0=0,\quad 0\bigvee 1=1\bigvee 0=1\bigvee 1=1,$$

as well as the component-wise disjunct sum of two binary columns. We say that a column $\mathbf{u}$ covers column $\mathbf{v}$ ($\mathbf{u}\succeq\mathbf{v}$) if $\mathbf{u}\bigvee\mathbf{v}=\mathbf{u}$. The standard symbol $\lfloor a\rfloor$ ($\lceil a\rceil$) will be used to denote the largest (least) integer $\le a$ ($\ge a$).

Definition 1. [1]. A code X is called a *superimposed cover-free (s,ℓ)-code* (briefly, *CF (s,ℓ)-code*) if for any two non-intersecting sets $\mathcal{S},\,\mathcal{L}\subset[t]$, $|\mathcal{S}|=s$, $|\mathcal{L}|=\ell$, $\mathcal{S}\cap\mathcal{L}=\varnothing$, there exists a row $\boldsymbol{x}_i$, $i\in[N]$, for which

$$x_i(j)=0 \text{ for any } j\in\mathcal{S} \text{ and } x_i(k)=1 \text{ for any } k\in\mathcal{L}.$$

Taking into account the evident symmetry over s and ℓ, we introduce $t_{cf}(N,s,\ell)=t_{cf}(N,\ell,s)$ - the maximal size of CF (s,ℓ)-codes of length N and define the *rate* of CF (s,ℓ)-codes:

$$R(s,\ell)=R(\ell,s)\triangleq\varlimsup_{N\to\infty}\frac{\log_2 t_{cf}(N,s,\ell)}{N} \tag{2}$$

Definition 2. [2]. A code X is called a *list-decoding superimposed code* of *strength s and list size L* (briefly, *LD s_L-code*), if the disjunct sum of any s-subset of codewords X can cover not more than $L-1$ codewords that are not components of the given s-subset. We introduce $t_{ld}(N,s,L)$ – the maximal size of LD s_L-codes of length N and define the *rate* of LD s_L-codes:

$$R_L(s)\triangleq\varlimsup_{N\to\infty}\frac{\log_2 t_{ld}(N,s,L)}{N} \tag{3}$$

If $L=\ell=1$, then Definitions 1 and 2 coincide, i.e., $R_1(s)=R(s,1)$, $s=1,2,\dots$, and the corresponding code is called a *superimposed s-code*. Superimposed s-codes were introduced in the initial paper [3], where the first nontrivial properties, applications and constructions[1] were developed. In addition, the problem of obtaining bounds on the rate $R(s,1)$ was suggested.

In the given article, we present a brief survey of known results and formulate new upper and lower bounds on $R(s,\ell)$ and $R_L(s)$. A preprint containing their detailed proofs is available at: arXiv: 1401.0050 [cs.IT].

II. Survey of Results

A. Lower and Upper Bounds on $R(s,1)$

The best known lower bound on the rate $R(s,1)$ was obtained in paper [6], where using a random coding method based on the ensemble of binary constant weight codes, we proved that

$$R(s,1)\ge\underline{R}(s,1)\triangleq s^{-1}\cdot\max_{0<Q<1}A(s,Q),\ s\ge 1, \tag{4}$$

$$A(s,Q)\triangleq\log_2\frac{Q}{1-y}-sK(Q,1-y)-\ K\left(Q,\frac{1-y}{1-y^s}\right),$$

$$K(a,b)\triangleq a\cdot\log_2\frac{a}{b}+(1-a)\cdot\log_2\frac{1-a}{1-b} \tag{5}$$

and $y=y(s,Q)$ is the unique root of the equation:

$$y=1\,-\,Q\,+\,Q\,y^s\cdot\frac{1-y}{1-y^s},\qquad 1-Q\le y<1. \tag{6}$$

If $s\to\infty$, then the asymptotic behavior of (4)-(6) has the form:

$$R(s,1)\ \ge\ \underline{R}(s,1)=\frac{1}{s^2\log_2 e}(1+o(1)). \tag{7}$$

Here and below, $e=2,718$ is the base of the natural logarithm.

[1]Later on, the constructions were essentially extended in [4]-[5]

Obviously [3], $R(s,1) \le 1/s$, $s = 1,2,\dots$, and the best known upper bound on $R(s,1)$ was proved in paper [7]. This upper bound is called a *recurrent bound* and it will be denoted by the symbol $\overline{R}(s,1)$, $s = 1,2,\dots$. For its description, we introduce the standard notation of binary entropy

$$h(v) \triangleq -v\log_2 v - (1-v)\log_2(1-v), \ 0 < v < 1, \quad (8)$$

and for each integer s, $s \ge 1$, define the following function:

$$f_s(v) \triangleq h(v/s) - v\,h(1/s), \quad 0 < v < 1. \quad (9)$$

Evidently [7], for any value of argument v, $0 < v < 1$, the function $f_s(v)$ is positive and $\cap$-convex. In addition, its maximal value

$$\max_{0<v<1} f_s(v) = f_s(v_s), \text{ where } v_s \triangleq \frac{s}{1 + 2^{s\cdot h\left(\frac{1}{s}\right)}}. \quad (10)$$

Put $\overline{R}(1,1) \triangleq 1$ and

$$\overline{R}(2,1) \triangleq \max_{0<v<1} f_2(v) = f_2(v_2) = 0.322. \quad (11)$$

Then for $s = 3,4,\dots$, the sequence $\overline{R}(s,1)$ is defined [7] as the unique root of the following recurrent equation:

$$\overline{R}(s,1) = f_s\left(1 - \frac{\overline{R}(s,1)}{\overline{R}(s-1,1)}\right). \quad (12)$$

For $s = 2,3,\dots$, we proved [7] the inequalities

$$R(s,1) \le \overline{R}(s,1) \le \frac{2\log_2[e(s+1)/2]}{s^2}, \quad (13)$$

which yield the asymptotic upper bound:

$$R(s,1) \le \frac{2\log_2 s}{s^2}(1+o(1)), \quad s\to\infty. \quad (14)$$

Several numerical values of the lower bound $\underline{R}(s,1)$, defined by (4)-(6) and the upper bound $\overline{R}(s,1)$ defined by (11)-(12) are given in Table 1.

Our first new result is given by

Theorem 1. *If $s \ge 8$, then the recurrent sequence $\overline{R}(s,1)$ satisfies the inequality*

$$\overline{R}(s,1) \ge \frac{2\log_2[(s+1)/8]}{(s+1)^2}, \quad s \ge 8. \quad (15)$$

From (13) and (15), it follows that the asymptotic equality is

$$\overline{R}(s,1) = \frac{2\log_2 s}{s^2}(1+o(1)), \quad s\to\infty. \quad (16)$$

For classical superimposed s-codes, the main result of our work is presented by

Theorem 2. *For the rate $R(s,1)$, the asymptotic inequality holds*:

$$R(s,1) \ge \frac{4e^{-2}\log_2 s}{s^2}(1+o(1)), \quad s\to\infty. \quad (17)$$

The bound (17) essentially improves inequality (7). To within a constant factor a, $4e^{-2} \le a \le 2$, bounds (14) and (17) establish the asymptotic behavior for the rate $R(s,1)$ of superimposed s-codes. It is important to note that Theorem 2 is obtained as a consequence of lower bounds on the rate $R(s,\ell)$ for CF (s,ℓ)-codes at $\ell \ge 2$. These lower bounds formulated in Sect. B are constructed using a random coding method based on the ensemble of binary constant weight codes.

B. Upper and Lower Bounds on $R(s,\ell)$ for $2 \le \ell \le s$

Superimposed cover-free (s,ℓ)-codes (CF (s,ℓ)-codes) were introduced in [1]. The first upper bounds on $R(s,\ell)$ for CF (s,ℓ)-codes, $2 \le \ell \le s$, were obtained in [8]-[9]. In papers [10]-[11], the following recurrent inequality was proved:

$$R(s,\ell) \le \frac{R(s-i,\ell-j)}{R(s-i,\ell-j) + \frac{(i+j)^{i+j}}{i^i\cdot j^j}}, \quad (18)$$

where $i \in [s-1]$, $j \in [\ell-1]$, that can be considered as an improvement of the recurrent inequality

$$R(s,\ell) \le R(s-i,\ell-j)\cdot\frac{i^i\cdot j^j}{(i+j)^{i+j}}, \quad i\in[s-1],\ j\in[\ell-1],$$

established in [12]. The recurrent inequality (18) and the recurrent upper bound $\overline{R}(s,1)$, $s \ge 1$, defined by (9)-(12), yield the best known upper bound on $R(s,\ell)$, $2 \le \ell \le s$, having the following recurrent form:

$$R(s,\ell) \le \overline{R}(s,\ell) \triangleq$$

$$\triangleq \min_{i\in[s-1]}\min_{j\in[\ell-1]} \frac{\overline{R}(s-i,\ell-j)}{\overline{R}(s-i,\ell-j) + \frac{(i+j)^{i+j}}{i^i\cdot j^j}}. \quad (19)$$

The asymptotic consequence of (19) is given [13] by

Theorem 3. *If $s\to\infty$ and $\ell \ge 2$ is fixed, then*

$$R(s,\ell) \le \overline{R}(s,\ell) \le \frac{(\ell+1)^{\ell+1}}{2\,e^{\ell-1}}\cdot\frac{\log_2 s}{s^{\ell+1}}\cdot(1+o(1)). \quad (20)$$

The best known lower bound for $R(s,\ell)$, $2 \le \ell \le s$, was obtained in [8] with the help of a random coding method based on the standard ensemble with independent components of binary codewords and a special ensemble with independent constant-weight codewords suggested in [14]. For fixed $\ell \ge 2$ and $s\to\infty$, the asymptotic behavior of this lower bound can be written [8] as follows

$$R(s,\ell) \ge \frac{e^{-\ell}\ell^{\ell+1}\log_2 e}{s^{\ell+1}}(1+o(1)). \quad (21)$$

The central result of our paper is a new random coding bound for $R(s,\ell)$, $2 \le \ell \le s$, formulated below as Theorem 4. The given lower bound is based on the ensemble with binary independent constant weight codewords.

Theorem 4. (Random coding bound $\underline{R}(s,\ell)$.) *The following two statements hold.* **1.** *Let $2 \le \ell \le s$. Then the rate of CF (s,ℓ)-codes*

$$R(s,\ell) \ge \underline{R}(s,\ell) \triangleq \frac{1}{s+\ell-1}\max_{0<z<1} T(z,s,\ell), \quad (22)$$

$$T(z,s,\ell) \triangleq \frac{\ell z^s(1-z)^\ell}{1-z^s(1-z)^\ell}\log_2\left[\frac{z}{1-z}\right] +$$

$$+(s+\ell-1)\cdot\log_2\left[1-z^s(1-z)^\ell\right] -$$

$$-(s+\ell)\frac{z-z^s(1-z)^\ell}{1-z^s(1-z)^\ell}\log_2\left[1-z^{s-1}(1-z)^\ell\right]. \quad (23)$$

2. *If $s \to \infty$ and $\ell \geq 2$ is fixed, then the lower bound $\underline{R}(s,\ell)$ satisfies the asymptotic equality:*

$$\underline{R}(s,\ell) = \frac{e^{-\ell}\ell^{\ell+1}\log_2 s}{s^{\ell+1}}(1+o(1)). \qquad (24)$$

With the help of Theorem 4 and recurrent inequality (18) we essentially improve the asymptotic behavior of lower bound (21) and prove

Theorem 5. *For any fixed $\ell = 1,2,\dots$ and $s \to \infty$, the rate $R(s,\ell)$ satisfies the asymptotic inequality*

$$R(s,\ell) \;\geq\; \left(\frac{\ell+1}{e}\right)^{\ell+1} \frac{\log_2 s}{s^{\ell+1}}(1+o(1)). \qquad (25)$$

It is evident that Theorem 2 is a direct corollary of Theorem 5. From the evident comparison of upper bound (20) with lower bound (25) the result formulated in the paper abstract follows.

For fixed $s \geq 2$, any $i = 1,2,\dots$ and any integer parameter j, $2 \leq j \leq s$, inequality (18) can be written in the form

$$R(s,1) \;\geq\; \frac{R(s+i,j)}{1-R(s+i,j)}\,\frac{(i+j-1)^{i+j-1}}{i^i\,(j-1)^{j-1}},$$

where $2 \leq j \leq s,\ i = 1,2,\dots$. Therefore, applying the lower bound of Theorem 4, we get the following lower bound on the rate of classical superimposed s-codes:

$$R(s,1) \;\geq\; \underline{R'}(s,1) \;\triangleq$$

$$\triangleq \max_{i\geq 1,\,2\leq j\leq s}\left\{\frac{\underline{R}(s+i,j)}{1-\underline{R}(s+i,j)}\,\frac{(i+j-1)^{i+j-1}}{i^i(j-1)^{j-1}}\right\}. \qquad (26)$$

$(s,1)$	$(2,1)$	$(3,1)$	$(4,1)$	(5,1)	(6,1)
$\overline{R}(s,1)$	.322	.199	.140	.106	.083
$\underline{R}(s,1)$	.182	.079	.044	.028	.019
$\underline{R'}(s,1)$	.128	.082	.0566	.0420	.0325
(i,j)	$(1,2)$	$(2,2)$	$(3,2)$	$(3,2)$	$(4,2)$
(s,ℓ)	$(2,2)$	$(3,2)$	$(4,2)$	$(5,2)$	$(6,2)$
$\overline{R}(s,\ell)$	.161	.0744	.0455	.0286	.0203
$\underline{R}(s,\ell)$	.0584	.0310	.0185	.0120	.00825
$Q(s,\ell)$	.32	.27	.24	.21	.19
(s,ℓ)	$(3,3)$	$(4,3)$	$(5,3)$	$(6,3)$	$(4,4)$
$\overline{R}(s,\ell)$	.0387	.0183	.0109	.00669	.00958
$\underline{R}(s,\ell)$	.0098	.0055	.0034	.00215	.00192
$Q(s,\ell)$	.34	.31	.28	.26	.35
(s,ℓ)	$(5,4)$	$(6,4)$	$(5,5)$	$(6,5)$	$(6,6)$
$\overline{R}(s,\ell)$	.0045	.00256	.00239	.00114	.00060
$\underline{R}(s,\ell)$	.0011	.00067	.00040	.00023	.00008
$Q(s,\ell)$	.32	.30	.37	.35	.38

Table 1.

In Table 1, for $\ell = 1$ and $2 \leq s \leq 6$, we give numerical values of the lower bound $\underline{R'}(s,1)$ along with optimal parameters (i,j) from definition (26). For $3 \leq s \leq 6$, the given values improve the lower bound $\underline{R}(s,1)$ defined by (4)-(6). For several parameters $2 \leq \ell \leq s \leq 6$, Table 1 also gives the values of the upper bound $\overline{R}(s,\ell)$, defined by (19), and lower bound $\underline{R}(s,\ell)$ along with the values of optimal relative weight $Q(s,\ell)$ for the ensemble used in Theorem 4. In the proof of Theorem 4, the following asymptotic equality is established

$$Q(s,\ell) = \frac{\ell}{s}\,(1+o(1)), \quad s \to \infty, \quad \ell = 2,3,\dots. \qquad (27)$$

C. Bounds on the Rate $R_L(s)$ for LD s_L-Codes

Superimposed list-decoding codes (LD s_L-codes) were introduced in [2] where nontrivial bounds on the rate $R_L(s)$ were obtained. Some constructions were considered in [5] (see, also [15]-[17]) in connection with two-stage pooling designs arising from the potentialities of molecular biology to identify any p-subset, $p \leq s$, of positive clones in the clone-library of size t. From Definition 2, follows the possibility of applying an LD s_L-code X of size t and length N at the first screening stage. Then $\leq s+L-1$ candidates are confirmed individually in a confirmatory (second) screening stage. In other words, if the number of positive clones $\leq s$, then the two stage list decoding algorithm needs to carry out $\leq N+s+L-1$ tests (pools). Note, that at fixed $s \geq 2$, the rate of two-stage pooling designs $R_L(s)$ is an increasing function of parameter $L \geq 1$ and, hence, the number

$$R_\infty(s) \;\triangleq\; \lim_{L\to\infty} R_L(s) \qquad (28)$$

can be interpreted as the *maximal rate* for two-stage group testing in the disjunct search model of p, $p \leq s$, positives.

The following important properties of LD s_L-codes arise immediately from Definition 2.

Proposition 1. [2]. For any $s \geq 1$ and $L \geq 1$, the rate $R_L(s)$ of LD s_L-codes satisfies the inequality

$$R_L(s) \leq \frac{1}{s}, \quad s \geq 1, \quad L \geq 1. \qquad (29)$$

Proposition 2. If $s > L \geq 2$, then the maximal size $t_{ld}(N,s,L)$ and the rate $R_L(s)$ of LD s_L-codes satisfy the inequalities

$$t_{ld}(N,s,L) \;\leq\; t_{ld}\left(N,\lfloor s/L\rfloor,1\right)+L-1,$$

$$R_L(s) \;\leq\; R\left(\lfloor s/L\rfloor,1\right), \quad L \leq s. \qquad (30)$$

Proposition 3. [2]. If all $\binom{t}{s}$ disjunct sums corresponding to different s-collections of columns of a code X are distinct, then the code X is an LD $(s-1)_2$-code. The given sufficient condition for LD $(s-1)_2$-code is evidently proved by contradiction.

The first results about the upper and lower bounds on the rate $R_L(s)$ for $L \geq 2$ were published in [2]. The upper bound on $R_L(s)$ was obtained as an obvious consequence of the second inequality in (30) and upper bound (13). The lower bound on $R_L(s)$ was proved by a random coding method based on the standard ensemble of binary codewords with independent components.

In consequent works [18]-[19] the given bounds were improved. Other our results concerning new lower and upper bounds on the rate $R_L(s)$ are presented below in the form of Theorem 6 and 7.

Theorem 6. (Recurrent upper bound $\overline{R}_L(s)$). *The following three statements hold.* **1.** *For any fixed $L \ge 1$, the rate of LD s_L-codes $R_L(s) \le \overline{R}_L(s)$, $s = 1, 2, \ldots$, and the right-hand side sequence $\overline{R}_L(s)$, $s = 1, 2, \ldots$, is defined recurrently*:

- *if $1 \le s \le L$, then*

$$\overline{R}_L(s) \triangleq 1/s, \qquad s = 1, 2, \ldots, L; \tag{31}$$

- *if $s = L+1, L+2, \ldots$, then*

$$\overline{R}_L(s) \triangleq \min\{1/s;\, r_L(s)\} \tag{32}$$

and $r_L(s)$ is the unique root of the equation

$$r_L(s) \triangleq \max_{(34)} f_{\lfloor s/L \rfloor}(v), \tag{33}$$

where the function $f_n(v)$, $n = 1, 2, \ldots$, of parameter v, $0 < v < 1$, is defined by $(8) - (9)$ and the maximum is taken over all v satisfying the condition

$$0 < v < 1 - \frac{r_L(s)}{\overline{R}_L(s-1)}; \tag{34}$$

- *if $s > 2L$ and $L \ge 1$, then equation (33) can be written in the form of the equality*

$$r_L(s) = f_{\lfloor s/L \rfloor}\left(1 - \frac{r_L(s)}{\overline{R}_L(s-1)}\right). \tag{35}$$

2. *For any $L \ge 1$, there exists an integer $s(L) \ge 2$, such that*

$$\overline{R}_L(s) = \begin{cases} 1/s & \text{if} \quad s = s(L) - 1, \\ < 1/s & \text{if} \quad s \ge s(L), \end{cases}$$

and $s(L) = L \log_2 L$ as $L \to \infty$. **3.** *If $L \ge 1$ is fixed and $s \to \infty$, then*

$$\overline{R}_L(s) = \frac{2L \log_2 s}{s^2}(1 + o(1)). \tag{36}$$

The recurrent bound (31)-(35) and asymptotic behavior (36) are generalizations of the recurrent bound (11)-(12) and asymptotic behavior (16).

Theorem 7. (Random coding bound $\underline{R}_L(s)$). *The following three statements hold.* **1.** *For any $s \ge 1$ and $L \ge 1$, the rate of LD s_L-codes*

$$R_L(s) \ge \underline{R}_L(s) \triangleq \frac{1}{s + L - 1} \max_{0<Q<1} A_L(s, Q), \tag{37}$$

$$A_L(s, Q) \triangleq \log_2 \frac{Q}{1-y} - sK(Q, 1-y) - L\, K\left(Q, \frac{1-y}{1-y^s}\right),$$

where we use the notation (5) and parameter y, $1 - Q \le y < 1$, is defined as the unique root of the equation

$$y = 1 - Q + Qy^s\left[1 - \left(\frac{y - y^s}{1 - y^s}\right)^L\right]. \tag{38}$$

2. *For any fixed $L = 1, 2, \ldots$ and $s \to \infty$, the asymptotic behavior of the random coding bound $\underline{R}_L(s)$ has the form*

$$\underline{R}_L(s) = \frac{L}{s^2 \log_2 e}(1 + o(1)). \tag{39}$$

3. *At fixed $s = 2, 3, \ldots$ and $L \to \infty$, there exists*

$$\underline{R}_\infty(s) \triangleq \lim_{L \to \infty} \underline{R}_L(s) = \log_2\left[\frac{(s-1)^{s-1}}{s^s} + 1\right]. \tag{40}$$

If $s \to \infty$, then

$$\underline{R}_\infty(s) = \frac{\log_2 e}{e \cdot s}(1 + o(1)) = \frac{0,5307}{s}(1 + o(1)).$$

Remark 1. For the particular case $L = 1$, the lower bound (37)-(38) and asymptotic behavior (39) coincide with the lower bound (4)-(6) and (7). In the proofs of Theorems 4 and 7, we analyze our random coding method for a constant weight code ensemble and observe why the random coding bound (24) for CF (s, ℓ)-codes essentially differs from the random coding bound (7) for classical superimposed s-codes.

The right-hand side of (40) gives the best known lower bound on the maximal rate $R_\infty(s)$ defined by (28) for two-stage group testing in the disjunct search model. An open problem is to obtain an upper bound on $R_\infty(s)$ improving the evident upper bound $R_\infty(s) \le 1/s$ which follows from (29).

Remark 2. We would like to mention paper [20] yielding a lower bound on $R_\infty(s)$ that is better than (40) but, unfortunately, its proof contains a principal mistake.

(s, L)	$(2,2)$	$(2,3)$	$(2,4)$	$(2,5)$	$(2,6)$
$\underline{R}_L(s)$	.235	.259	.272	.281	.287
$Q_L(s)$	.24	.23	.23	.22	.22
(s, L)	$(3,2)$	$(3,3)$	$(3,4)$	$(3,5)$	$(3,6)$
$\underline{R}_L(s)$	.114	.134	.146	.155	.161
$Q_L(s)$	.18	.17	.16	.16	.15
(s, L)	$(4,2)$	$(4,3)$	$(4,4)$	$(4,5)$	$(4,6)$
$\underline{R}_L(s)$	.0684	.0837	.0940	.101	.106
$Q_L(s)$	.14	.13	.13	.12	.12
(s, L)	$(5,2)$	$(5,3)$	$(5,4)$	$(5,5)$	$(5,6)$
$\underline{R}_L(s)$	.0455	.0574	.0659	.0722	.0771
$Q_L(s)$	0.12	0.11	0.11	0.10	0.10
(s, L)	$(6,2)$	$(6,3)$	$(6,4)$	$(6,5)$	$(6,6)$
$\underline{R}_L(s)$	.0325	.0420	.0490	.0544	.0586
$Q_L(s)$	.10	.09	.09	.09	.09
s	2	3	4	5	6
$\underline{R}_\infty(s)$	.322	.199	.145	.114	.094

Table 2.

Table 2 presents several numerical values of $\underline{R}_L(s)$ for small parameters s and L along with values $Q_L(s)$ for the corresponding optimal relative weight in the right-hand side of (37). In Table 2, some numerical values of the lower bound (40) are given as well. In proofs of Statements 2 and 3, we establish the following asymptotic equalities:

$$Q_L(s) = \frac{\ln 2}{s} + \frac{L \ln^2 2}{s^2} + o\left(\frac{1}{s^2}\right), \; s \to \infty, \; L = 1, 2, \ldots,$$

$$Q_L(s) = \left[\frac{s^s}{(s-1)^{s-1}} + 1\right]^{-1} + o(1), \; L \to \infty, \; s = 2, 3, \ldots.$$

Evidently, for any $s \ge 1$ and $L \ge 1$, the rate of LD s_L-codes $R_L(s)$ satisfies the inequality

$$\underline{R}'(s, 1) \le R(s, 1) = R_1(s) \le R_L(s),$$

where the lower bound $\underline{R}'(s,1)$ is defined by (26). Hence, one can compare the bound $\underline{R}_L(s)$ defined by (37)-(38) with the bound $\underline{R}'(s,1)$. Tables 1-2 show that for $L=2$ and $2 \le s \le 6$, the values of $\underline{R}_2(s)$ improve (exceed) the values of $\underline{R}'(s,1)$, and it is easy to check that for $s \ge 7$, the values of $\underline{R}'(s,1)$ become greater than values of $\underline{R}_2(s)$. This corresponds to the asymptotic behavior of the given bounds. The same is also true when $L \ge 2$.

D. Disjunct Search Designs

Definition 3. [2]-[3]. A code X is called a *disjunct s-design* (*($\le s$)-design*), if the disjunct (Boolean) sum of any collection containing s ($\le s$) columns of code X differs from the disjunct sum of any other collection containing s ($\le s$) columns of code X. Let $N(t,=s)$ ($N(t,\le s)$) be the minimal number of rows for disjunct s-designs (($\le s$)-designs) of size t. Introduce the *rate* of disjunct s-designs (($\le s$)-designs) as:

$$R(=s) \triangleq \varlimsup_{t\to\infty} \frac{\log_2 t}{N(t,=s)}, \; \left(R(\le s) \triangleq \varlimsup_{t\to\infty} \frac{\log_2 t}{N(t,\le s)} \right). \tag{41}$$

Obviously [3], the rate

$$R(\le s) \;\le\; R(=s) \le 1/s, \qquad s = 1,2,\ldots. \tag{42}$$

In the non-adaptive disjunct search model of s ($\le s$) defects among a set of t elements, Definition 3 gives the necessary and sufficient condition for identification. Any disjunct s-design can be considered as the incidence matrix for a *union-free family* [21] containing t subset of the set $[N]$. For any $s = 2,3,\ldots$, the rates (2)-(3) and (41) satisfy the following inequalities

$$R(s,1) \;\le R(\le s) \;\le\; R(s-1,1), \; R(=s) \;\le\; \overline{R}_2(s-1). \tag{43}$$

The first and second inequalities were observed in [3] and the third inequality is an evident consequence of Proposition 3.

Applying formulas (31)-(35), we calculated: $s(1) = 2$, $s(2) = 6$, $s(3) = 12$, $s(4) = 20$, $s(5) = 25$, $s(6) = 36, \ldots$. In addition, for $L = 2$ and $s \ge 7$, we have the following values of $\overline{R}_2(s-1)$:

s	7	8	9	10	11	12
$1/s$	.143	.125	.111	.100	.091	.083
$\overline{R}_2(s-1)$	.163	.141	.117	.102	.086	.076

Table 3.

Table 3 shows that the upper bound $\overline{R}_2(s-1) < 1/s$ if $s \ge 11$. Therefore, the third inequality in (43) means that the rate of disjunct s-designs $R(=s) < 1/s$ if $s \ge 11$. For $s = 2$, the nontrivial inequality $R(=2) \le 0,4998 < 1/2$ was proved in [21]. For $3 \le s \le 10$, the inequality $R(=s) < 1/s$ is our conjecture.

References

[1] *Mitchell C.J*, *Piper F.C.* Key storage in Secure Networks // Discrete Applied Mathematics. 1988. V. 21. pp. 215-228.

[2] *D'yachkov A.G.*, *Rykov V.V.* A Survey of Superimposed Code Theory // Problems of Control and Inform. Theory. 1983. V. 12. n 4. pp. 229-242.

[3] *Kautz W.H.*, *Singleton R.C.* Nonrandom Binary Superimposed Codes// IEEE Trans. Inform. Theory. 1964. V. 10. n 4. pp. 363-377.

[4] *D'yachkov A.G.*, *Macula A.J.*, *Rykov V.V.* New Constructions of Superimposed Codes // IEEE Trans. Inform. Theory. 2000. V. 46. n 1. pp. 284-290.

[5] *D'yachkov A.G.*, *Macula A.J.*, *Rykov V.V.* New Applications and Results of Superimposed Code Theory Arising from the Potentialities of Molecular Biology // In the book "Numbers, Information and Complexity". pp. 265-282, Kluwer Academic Publishers, 2000.

[6] *D'yachkov A.G.*, *Rykov V.V.*, *Rashad A.M.* Superimposed Distance Codes // Problems of Control and Inform. Theory. 1989. V. 18. n 4. pp. 237-250.

[7] *D'yachkov A.G.*, *Rykov V.V.* Bounds on the Length of Disjunctive Codes // Problems of Information Transmission. 1982. V. 18. n 3. pp. 166-171.

[8] *D'yachkov A.*, *Vilenkin P.*, *Macula A.*, *Torney D.* Families of Finite Sets in Which No Intersection of ℓ Sets Is Covered by the Union of s Others // J. Combin. Theory. Ser. A. 2002. V. 99. pp. 195-218.

[9] *Stinson D.R.*, *Wei R.*, *Zhu L.* Some New Bounds for Cover-Free Families // J. Combin. Theory. Ser. A. 2000. V. 90. pp. 224-234.

[10] *Lebedev V.S.* Some Tables for (w,r)-Superimposed Codes // Proc. Eighth Int. Workshop on Algebraic and Combinatorial Coding Theory. Tsarskoe Selo, Russia. September 8-14, 2002. pp. 185-189.

[11] *Lebedev V.S.* Asymptotic Upper Bound for the Rate of (w,r) Cover-Free Codes // Problems of Information Transmission. 2003. V. 39. n 4. pp. 317-323.

[12] *Engel K.* Interval Packing and Covering in the Boolean Lattice // Combinatorics Prob. and Computing. 1996. V. 5. pp. 373-384.

[13] *D'yachkov A.G.*, *Vilenkin P.A.*, *Yekhanin S.M.* Upper Bounds on the Rate of Superimposed (s,ℓ)-Codes Based on Engel's Inequality // Proc. Eighth Int. Workshop on Algebraic and Combinatorial Coding Theory. Tsarskoe Selo, Russia. September 8-14, 2002. pp. 95-99.

[14] *Nguyen Quang A.*, *Zeisel T.* Bounds on Constant Weight Binary Superimposed Codes // Problems of Control and Inform. Theory. 1988. V. 17. n 4. pp. 223-230.

[15] *Vilenkin P.A.* On Constructions of List-Decoding Superimposed Codes // Proc. Sixth Int. Workshop on Algebraic and Combinatorial Coding Theory. Pskov, Russia. September 6-12, 1998. pp. 228-231.

[16] *D'yachkov A.G.*, *Vilenkin P.A.*, *Macula A.J.*, *Torney D.C.*, *Yekhanin S.M.* New Results in the Theory of Superimposed Codes // Proc. Seventh Int. Workshop on Algebraic and Combinatorial Coding Theory. Bansko, Bulgaria. 2000. pp. 126-136.

[17] *D'yachkov A.G.*, *Vilenkin P.A.*, *Macula A.J.*, *Torney D.C.* Two Models of Non-Adaptive Group Testing for Designing Screening Experiments // Advances in Model Oriented Design and Analysis: Proceedings of the 6th International Workshop on Model Oriented Design and Analysis. Puchberg/Schneeberg, Austria. June 25-29, 2001. pp. 63-75. Physica-Verlag Heidelberg, 2001.

[18] *Rashad A.M.* Random Coding Bounds on the Rate for List-Decoding Superimposed Codes // Problems of Control and Inform. Theory. 1990. V.19. n 2. pp. 141-149.

[19] *D'yachkov A.G.* Lectures on Designing Screening Experiments // Lecture Note Series 10, Feb. 2003, Combinatorial and Computational Mathematics Center, Pohang University of Science and Technology (POSTECH), Korea Republic, (monograph, 112 pages).

[20] *Cheng Y.X.*, *Du D.Z.* New Constructions of One- and Two-Stage Pooling Designs // Journal of Computational Biology. 2008. V. 15. n 2. pp. 195-205.

[21] *Copperersmith D.*, *Shearer J.* New Bounds for Union-free Families of Sets // The Electronic Journal of Combinatorics. 1998. V.5. n 1. R39.