List-decodable codes for the Z–channel

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… codes with large asymmetric list-decoding radius

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1. Code = set of binary words of length $\boldsymbol{n}$
2. “Adversarial” Z-channel injects up to $\boldsymbol{n\tau}$ asymmetric errors to codeword
3. Decoding radius for list size $\boldsymbol{L-1}$ is the maximum $\boldsymbol{R} > \boldsymbol{0}$ such that any Z-ball with radius $\boldsymbol{R}$ contains $< \boldsymbol{L}$ codewords

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Main result (Informal):

- Largest list-decodable code $\boldsymbol{\varepsilon}$-above the *Plotkin point* has size of order $\boldsymbol{\varepsilon^{-3/2}}$ *irrespective* of list size

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fraction of errors $\boldsymbol{\tau=\varepsilon+L^{-\frac{1}{L-1}}-L^{-\frac{L}{L-1}}}$

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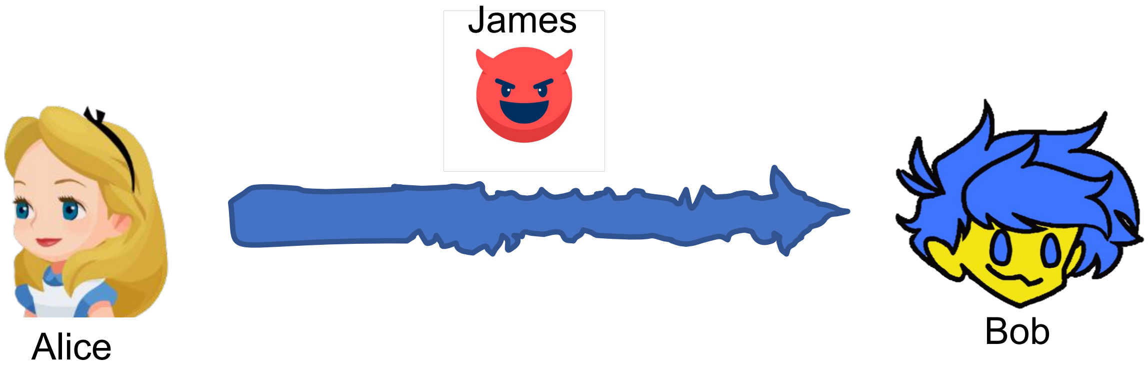
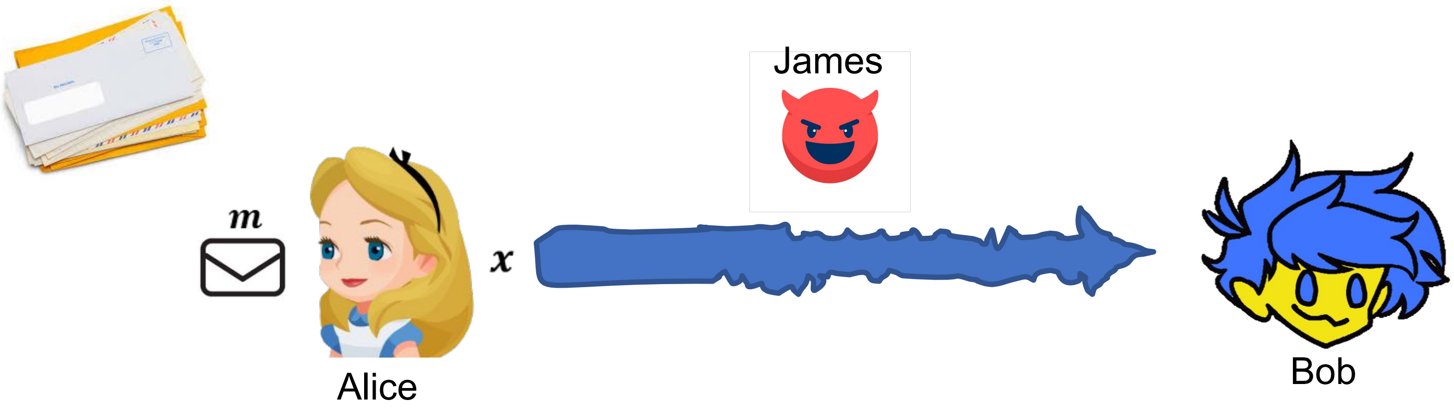
Outline

Introduction	Construction
Converse bound	Conclusion

List-decoding for adversarial channel

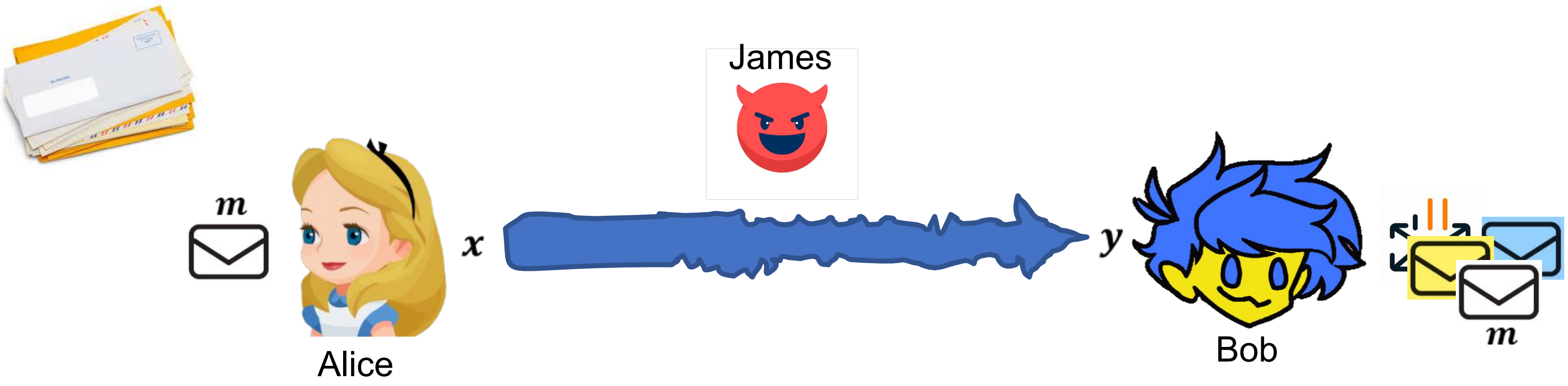


List-decoding for adversarial channel



- Alice picks a message m, encode it to x and transmit x over the noisy channel

List-decoding for adversarial channel



- Alice picks a message $\boldsymbol{m}$, encode it to $\boldsymbol{x}$ and transmit $\boldsymbol{x}$ over the noisy channel
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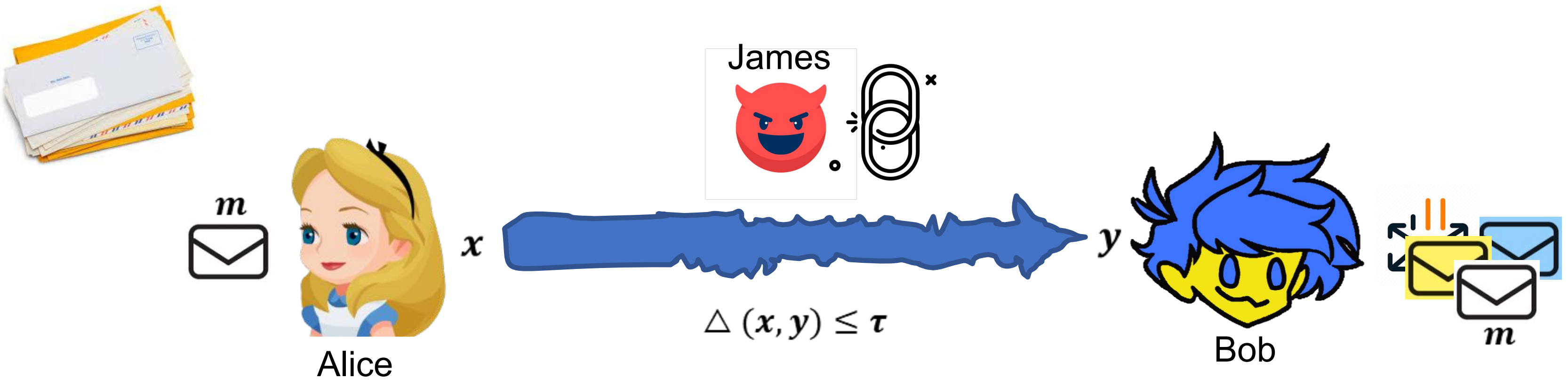

- Alice picks a message m, encode it to x and transmit x over the noisy channel
- Bob's goal is, based on y, to reconstruct a list of messages of size $< L$, which includes m
- James can inflict to the codeword only a fraction τ of errors

List-decodable codes for Z-channel

$$\boldsymbol{x} = (x_1, \dots, x_n)$$

$$\downarrow$$

$$\boldsymbol{y} = (y_1, \dots, y_n)$$

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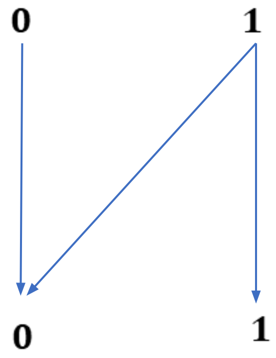

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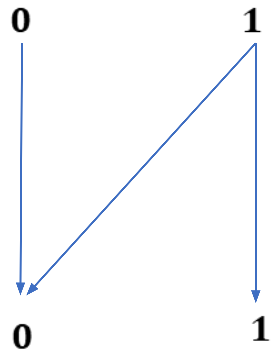

$\triangle(\boldsymbol{x}, \boldsymbol{y}) \le \boldsymbol{\tau}$, the number of positions where 1 changed to 0 is at most $\boldsymbol{\tau n}$

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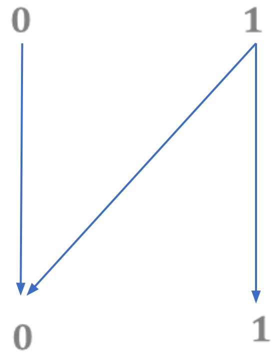

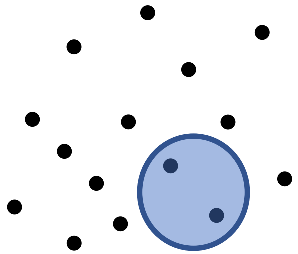

any Z-ball contains $< L$ codewords

$\triangle(x, y) \le \tau$, the number of positions where 1 changed to 0 is at most τn

Definition:

- $C \subseteq \{0,1\}^n$ is a $(\tau, L-1)$-list decodable code if for any word $y \in \{0,1\}^n$, the Z-ball centered at y with radius τn contains $< L$ codewords.

Related works

- Bit-flip errors

Theorem (Blinovsky`86, Polyanskiy`16, ABP`18, ZBJ`20)

1. For $L = 2k, 2k+1$, exponential-sized $(\tau, L-1)$-list-decodable codes exist when
$$\tau < \frac{1}{2} - 2^{-(2k+1)}\binom{2k}{k}$$
2. For $\tau = \varepsilon + \frac{1}{2} - 2^{-(2k+1)}\binom{2k}{k}$, the largest $(\tau, L-1)$-list-decodable code has size
$$\begin{cases} \Theta(\varepsilon^{-1}) \text{ for even } L \\ \Theta(\varepsilon^{-3/2}) \text{ for } L = 3 \end{cases}$$

Related works

- Asymmetric errors (Z-channel)

Theorem (ZBJ`20, LLP`21, DG`21)

Exponential-sized $(\tau, L-1)$-list-decodable codes exist when

$$\tau < L^{-\frac{1}{L-1}} - L^{-\frac{L}{L-1}}$$

Related works

- Asymmetric errors (Z-channel)

Theorem (ZBJ`20, LLP`21, DG`21)

Exponential-sized $(\tau, L-1)$-list-decodable codes exist when

$$\tau < L^{-\frac{1}{L-1}} - L^{-\frac{L}{L-1}}$$

What can we say when $\tau = \varepsilon + L^{-\frac{1}{L-1}} - L^{-\frac{L}{L-1}}$

Constant-weight codes

Consider $\boldsymbol{w}$-constant-weight $(\boldsymbol{\tau}, \boldsymbol{L}-\mathbf{1})$-list-decodable code of size $\boldsymbol{M}$.

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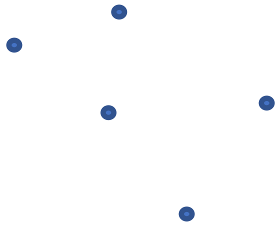
By double counting arguments (distances to center of Z-ball covering each $\boldsymbol{L}$-tuple)

$$\frac{\boldsymbol{M}^{\boldsymbol{L}}}{\boldsymbol{M}(\boldsymbol{M}-\mathbf{1})\ldots(\boldsymbol{M}-\boldsymbol{L}+\mathbf{1})} \geq \frac{\boldsymbol{\tau}}{\boldsymbol{\tau}_{\boldsymbol{L}}(\boldsymbol{w})}$$



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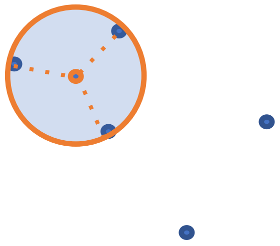
$$\frac{M^L}{M(M-1)\dots(M-L+1)} \geq \frac{\tau}{\tau_L(w)}$$



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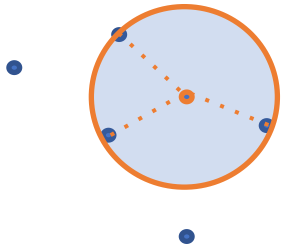

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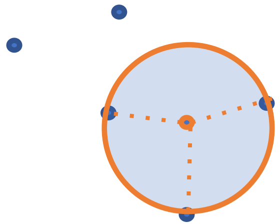

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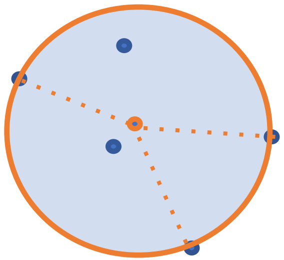

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Lemma:

- Code with $\tau = \varepsilon + \tau_L(w)$ has size $O_L(1/\varepsilon)$

Almost constant-weight codes

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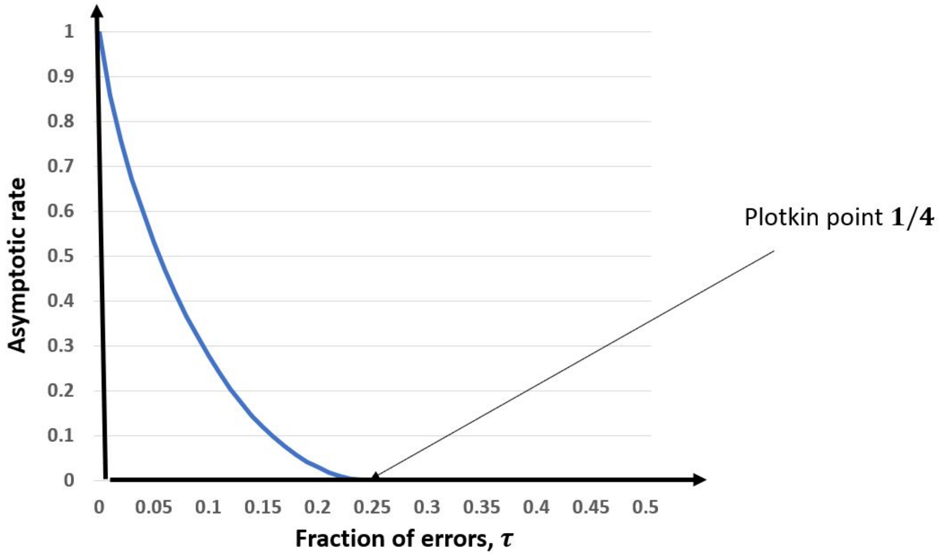
Plotkin point



Plotkin point

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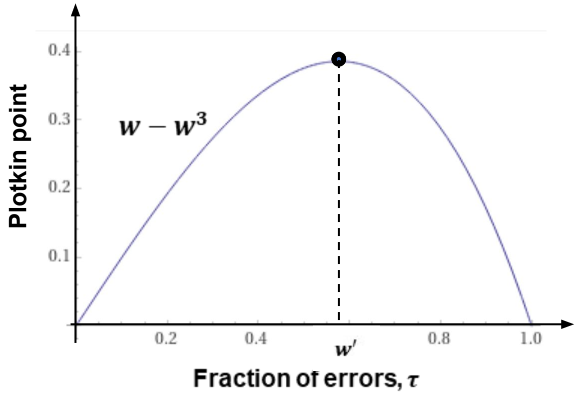
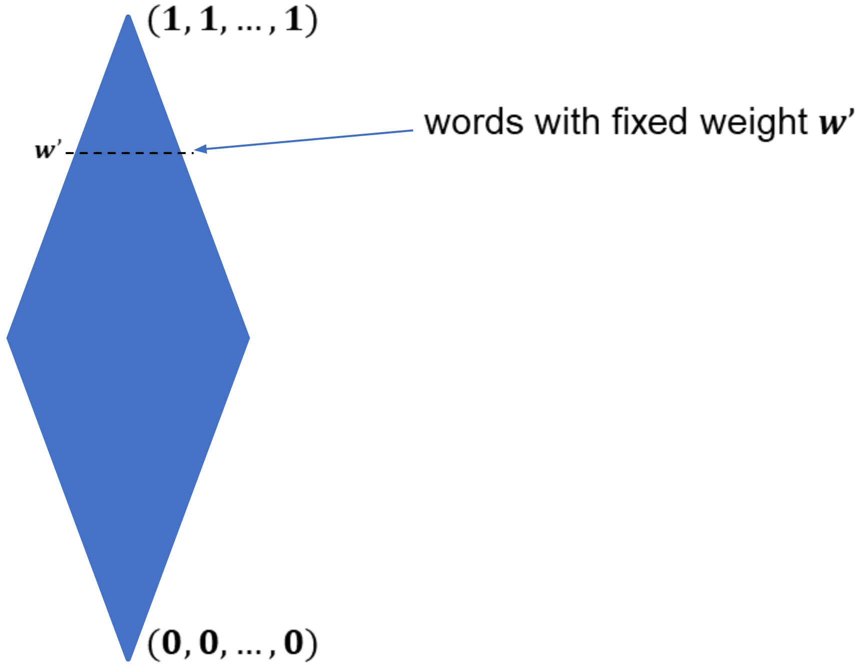


Plotkin point

The Plotkin point for $\boldsymbol{w}$-constant-weight codes for Z-channel is described by $\boldsymbol{\tau_L}(\boldsymbol{w}) = \boldsymbol{w} - \boldsymbol{w}^{\boldsymbol{L}}$, which is is maximized at $\boldsymbol{w}' = \boldsymbol{L}^{-\frac{1}{L-1}}$.

Thus, focus on $(\boldsymbol{\tau}, \boldsymbol{L} - \mathbf{1})$-list-decodable codes $\boldsymbol{\varepsilon}$-far from the Plotkin point for Z-channel, i.e. $\boldsymbol{\tau} = \boldsymbol{\varepsilon} + \boldsymbol{\tau_L}(\boldsymbol{w}')$

Partition the space



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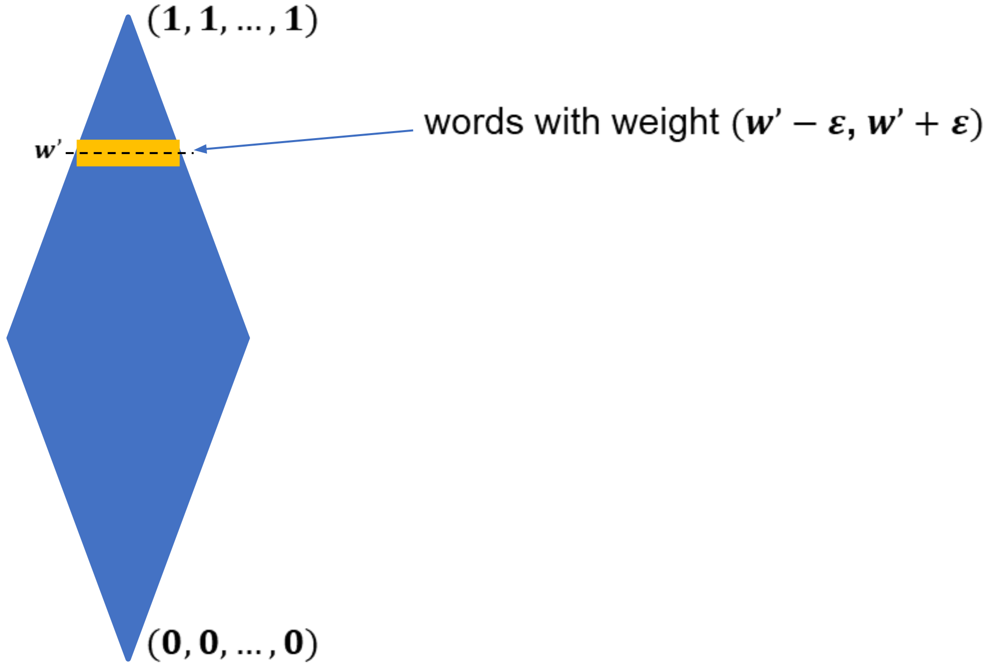

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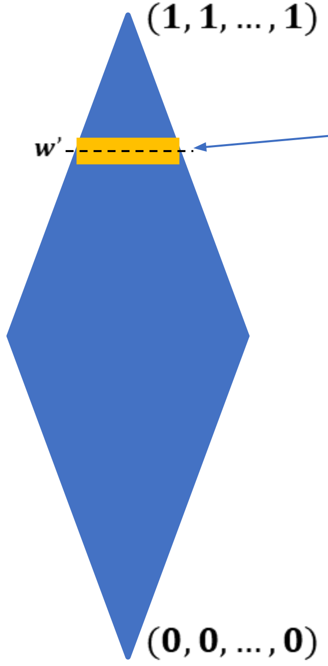
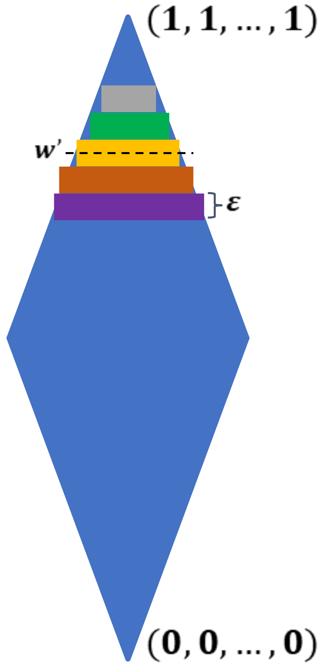


words with weight $(w' - \varepsilon, w' + \varepsilon)$

By Lemma, $O_L(\varepsilon)$ codewords could lie in this slice

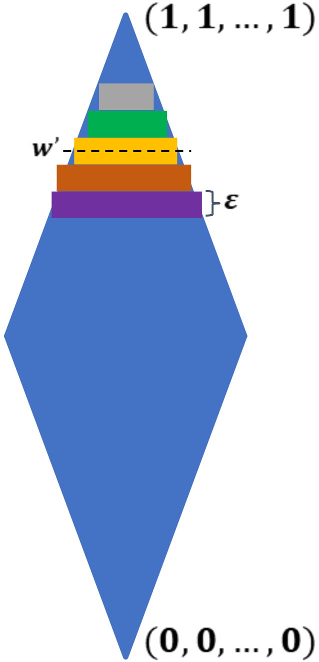
Partition the space



Partition the space into $O(1/\varepsilon)$ slices

Partition the space

Introduction | Construction | **Converse bound** | Conclusion



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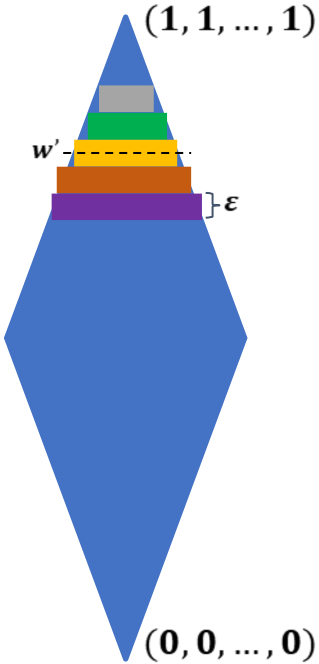
Each slice contains $O_L(1/\varepsilon)$ codewords

Partition the space

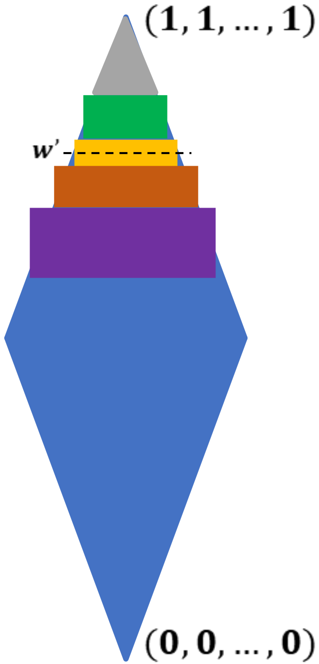


Partition the space into $O(1/\varepsilon)$ slices

Each slice contains $O_L(1/\varepsilon)$ codewords

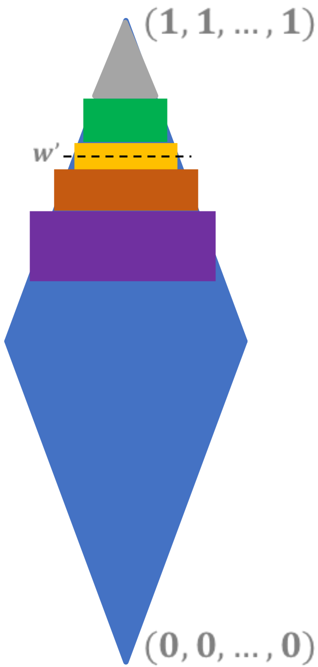
Thus, largest code has size $O_L(1/\varepsilon^2)$

Partition the space



But, this bound can be improved by non-uniform partition!!

Partition the space



But, this bound can be improved by non-uniform partition!!

Theorem:

- Codes with $\boldsymbol{\tau = \varepsilon + \tau_L(w')}$ have size $\boldsymbol{O_L\left(\varepsilon^{-3/2}\right)}$

Constant-weight codes

$$
\left.\begin{array}{ccccccc}
1 & 1 & 1 & 0 & 0 & \dots & 0 \\
1 & 0 & 0 & 1 & 1 & \dots & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 1 & 0 & 1 & 0 & \dots & 1 \\
0 & 0 & 1 & 0 & 1 & \dots & 1
\end{array}\right\}
$$

$\boldsymbol{m}$ rows
$\boldsymbol{wm}$ ones in each column

Constant-weight codes

1	1	1	0	0	…	0
1	0	0	1	1	…	0
…	…	…	…	…	…	…
0	1	0	1	0	…	1
0	0	1	0	1	…	1

$\boldsymbol{m}$ rows
$\boldsymbol{wm}$ ones in each column

$\binom{m}{wm}$ columns

Constant-weight codes

codewords

$$\underbrace{\begin{matrix} 1 & 1 & 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 1 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & 0 & 1 & 0 & \dots & 1 \\ 0 & 0 & 1 & 0 & 1 & \dots & 1 \end{matrix}}_{\binom{m}{wm} \text{ columns}}$$

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- $\boldsymbol{w}$-constant-weight code of size $\boldsymbol{m}$ and length $\binom{\boldsymbol{m}}{\boldsymbol{wm}}$

Constant-weight codes

codewords

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1	0	0	1	1	…	0
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- $\boldsymbol{w}$-constant-weight code of size $\boldsymbol{m}$ and length $\binom{\boldsymbol{m}}{\boldsymbol{wm}}$
- $(\boldsymbol{\tau}, \boldsymbol{L}-\mathbf{1})$-list-decodable with $\boldsymbol{\tau} = \boldsymbol{\tau}_{\boldsymbol{L}}(\boldsymbol{w}) + \boldsymbol{\Omega}(\boldsymbol{m}^{-1})$

Constant-weight codes

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Lemma:

- Exist w-constant-weight codes with $\tau = \varepsilon + \tau_L(w)$ of size $O_L(\varepsilon^{-1})$

❖ w-constant-weight code of size m and length $\binom{m}{wm}$

❖ $(\tau, L-1)$-list-decodable with $\tau = \tau_L(w) + \Omega(m^{-1})$

Almost constant-weight codes

1	1	1	0	0	...	0
1	0	0	1	1	...	0
...	..	...	..	...	...	...
0	1	0	1	0	...	1
0	0	1	0	1	...	1

$\boldsymbol{w}'$-constant weight with $\boldsymbol{\tau} = \boldsymbol{\varepsilon} + \boldsymbol{\tau}_{L}(\boldsymbol{w}')$

Almost constant-weight codes

1	1	1	0	0	...	0
1	0	0	1	1	...	0
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$(\boldsymbol{w}' + \boldsymbol{\varepsilon})$-constant weight with $\boldsymbol{\tau} = \boldsymbol{\Omega}(\boldsymbol{\varepsilon}) + \boldsymbol{\tau}_{\boldsymbol{L}}(\boldsymbol{w}')$

1	1	1	0	0	...	0
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Almost constant-weight codes

$\Omega(\varepsilon^{-1/2})$ codes

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Repetition + independent permutation

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Theorem:

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  Q: Non-optimal except the Plotkin point $\tau = \tau_L(w')$ and noiseless case $\tau = 0$.

  How to improve?

Thanks!

Questions?