How to guess an n-digit number

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2. The metric dimension of a graph is the minimum cardinality of a subset S of vertices such that all other vertices are uniquely determined by their distances to the vertices in S

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The metric dimension (2) of the Hamming graph (1), denoted by $\boldsymbol{m(K_q, n)}$

1. The Hamming graph is the Cartesian product of n complete graphs K_q
2. The metric dimension of a graph is the minimum cardinality of a subset S of vertices such that all other vertices are uniquely determined by their distances to the vertices in S

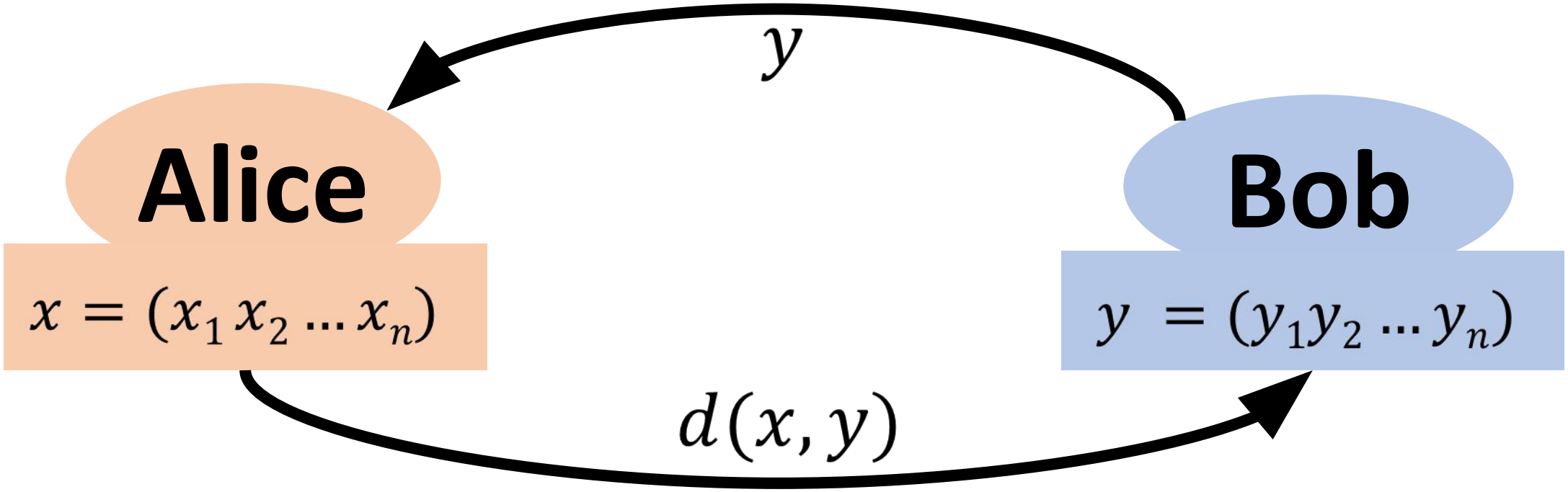
Our contribution: the asymptotic (q is fixed, n is large) of $m(K_q, n)$ *is*

$$m(K_q, n) \sim 2n/\log_q n$$

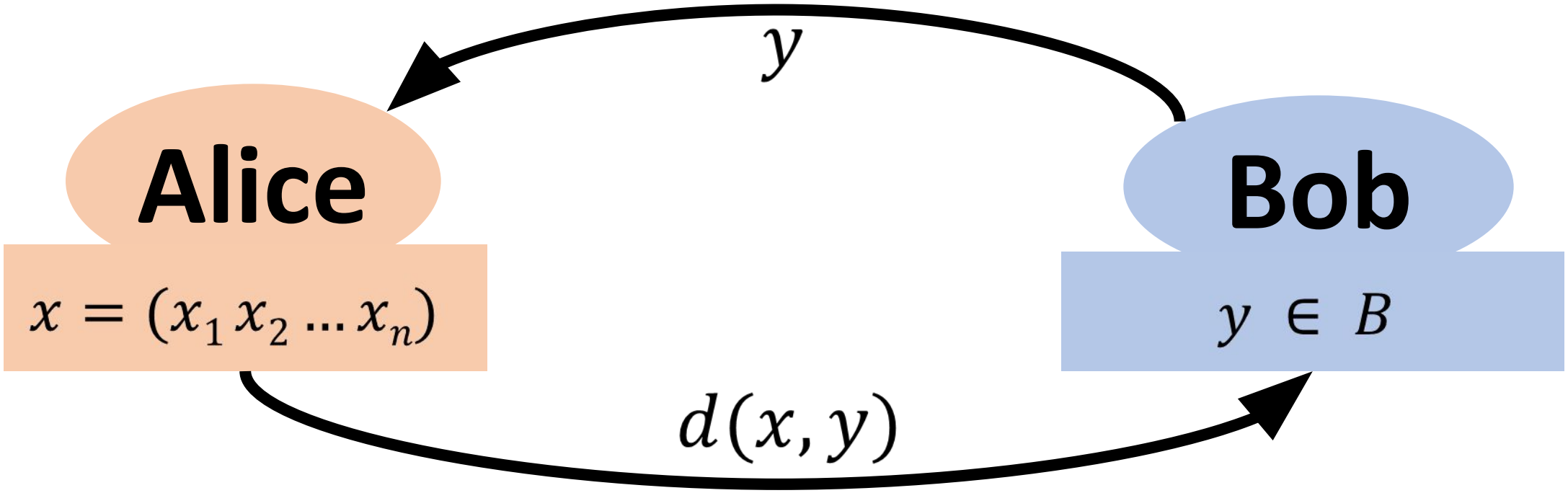
Outline

1. Introduction and related problems
   - Coin-weighing problem, Mastermind game
2. Construction (upper bound)
   - is based on the theory of Mobius functions
3. Lower bound
   - is based on the information-theoretic method
4. A more general problem…
   - The metric dimension of the Cartesian powers of a graph
5. The End

1. Introduction and related problems


y
Alice
$x = (x_1 x_2 \dots x_n)$
Bob
$y = (y_1 y_2 \dots y_n)$
$d(x, y)$

Goal: Bob must identify any hidden n-digit x based on $(d(x,y))_{y\in B}$



Definition: If so, B is called a ***base*** (of the q-ary n-dim. Hamming space).

Definition: The ***metric dimension*** $m(K_q, n)$ of the q-ary n-dim. Hamming space (or equivalently the Cartesian n-th power of K_q) is

$$m(K_q, n) := \min_{\text{base } B} |B|.$$

Problem: Given $q \geq 2$, find the asymptotic behavior $m(K_q, n)$ as $n \to \infty$.

Binary case (coin-weighing problem)

The problem was posed by Söderberg-Shapiro’63

Binary case (coin-weighing problem)

$m(K_2, n) \gtrsim 2n/\log_2 n$
by Erdös-Rényi'63
Moser'70, Pippenger'77

$m(K_2, n) \lesssim 2n/\log_2 n$
by Lindström'64
Cantor-Mills'66
Martirosyan-Khachatryan'89
Bshouty'09

q-ary case (Mastermind game)

- The commercial version was invented by Meirowitz'70 and was solved y Knuth'76
- The study of **non-adaptive** Mastermind game was initiated by Chvatal'83

q-ary case (Mastermind game)

$$m(K_q, n) \gtrsim 2n/\log_q n$$

by Kabatianskii et al'99

for $q = 3,4$, $m(K_q, n) \lesssim 2n/\log_q n$

by Kabatianskii-Lebedev'18

2. Construction (upper bound)

Let $(\mathbb{N}, \prec)$ be a *poset*, where $i \preccurlyeq j$ iff $i = i\&j$.

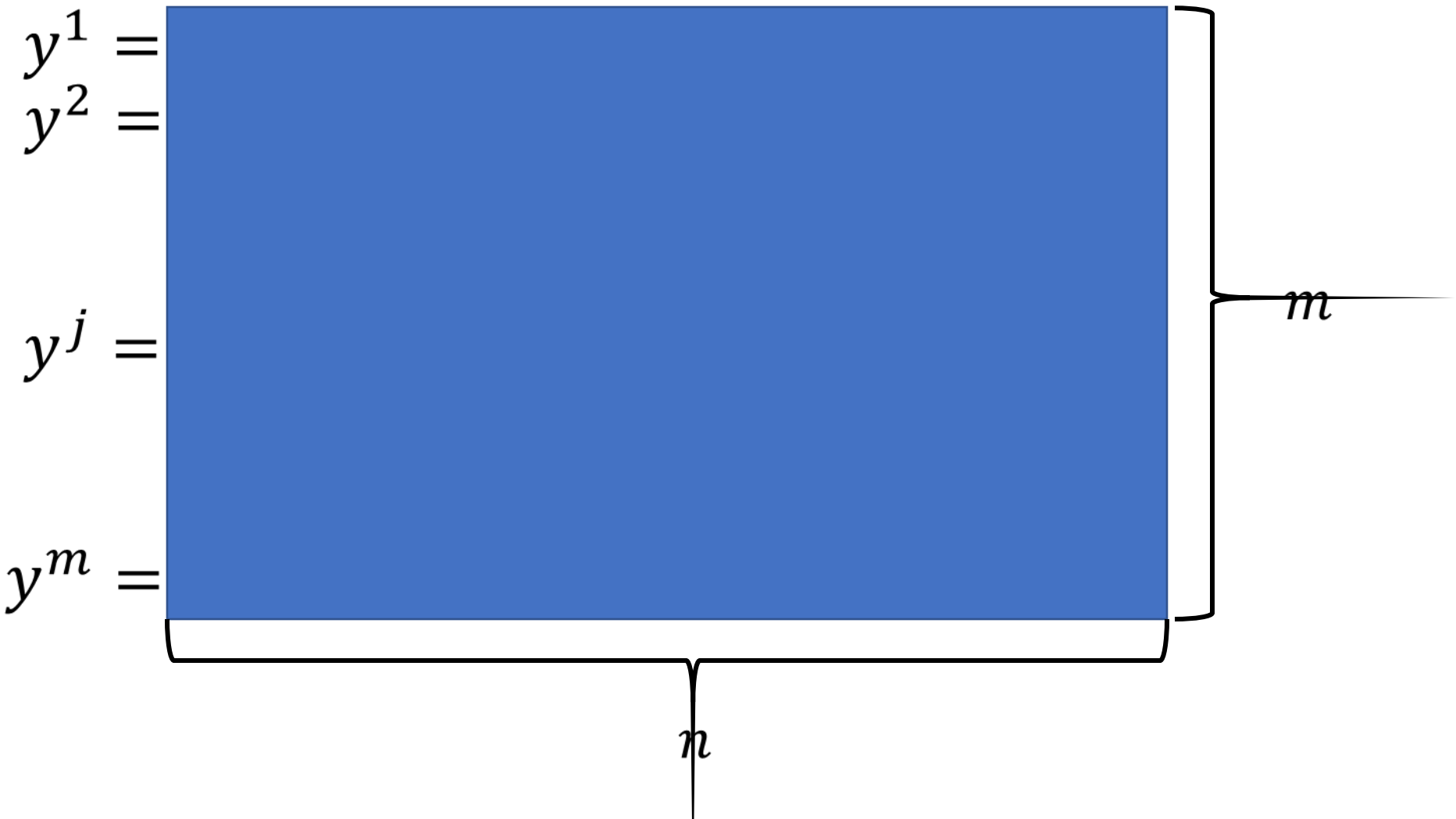
BITWISE AND

Define the *Mobius function*

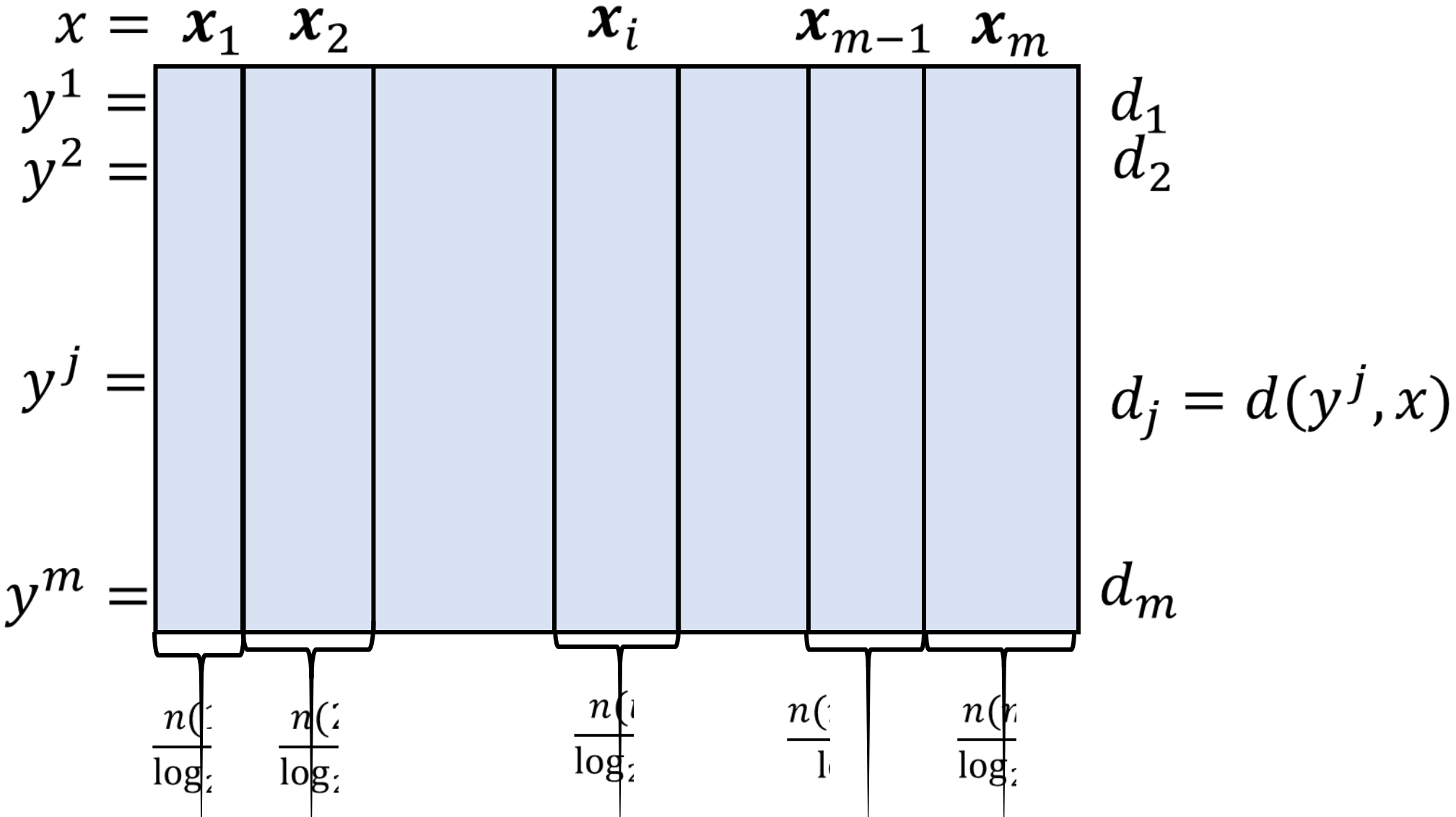
$$\mu(i,j) := \begin{cases} 1 & \text{if } i = j \\ -\sum_{i \preccurlyeq k \prec j} \mu(i,k) & \text{if } i \prec j \end{cases}$$

Here, $\mu(i,j) := (-1)^{n(i)-n(j)}$, where $n(u) :=$ #of $1's$ in the binary representation of u.

Let us put vectors from a base B of the q-ary n-dim.
Hamming space in rows of matrix $Y = (y^1, y^2, \dots, y^m)^T$



Divide hidden vector $x = (x_1 x_2 \ldots x_n)$ into m subvectors



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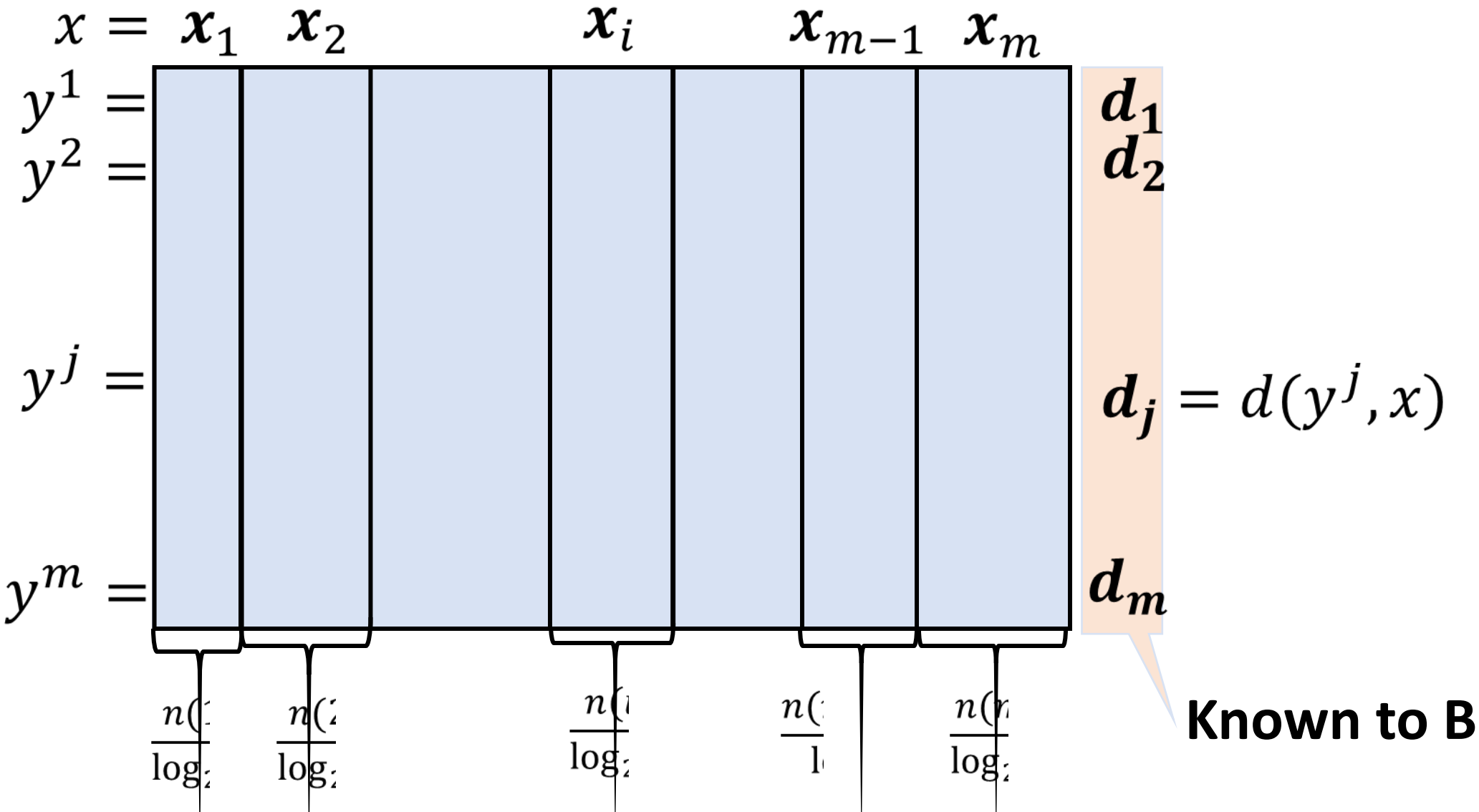

Known to Bob!

Decoding:

1. Consider the value

$$D_m := \sum_{i=1}^{m} \mu(i,m) d_i \ = f_m(\boldsymbol{x}_m),$$

and $\boldsymbol{x}_m$ can be extracted ($\boldsymbol{x}_m$ is a somewhat q-ary expansion of D_m).

2. Consider the value

$$D_{m-1} := \sum_{i=1}^{m} \mu(i,m-1) d_i = f_{m-1}(\boldsymbol{x}_m, \boldsymbol{x}_{m-1}),$$

and $\boldsymbol{x}_{m-1}$ can be extracted.

And so on…

3. Lower bound

Claim (informal)

For any base B, its size m is at least $2n/\log_q n$

Proof idea is a modification of the Erdös-Rényi'63 arguments…

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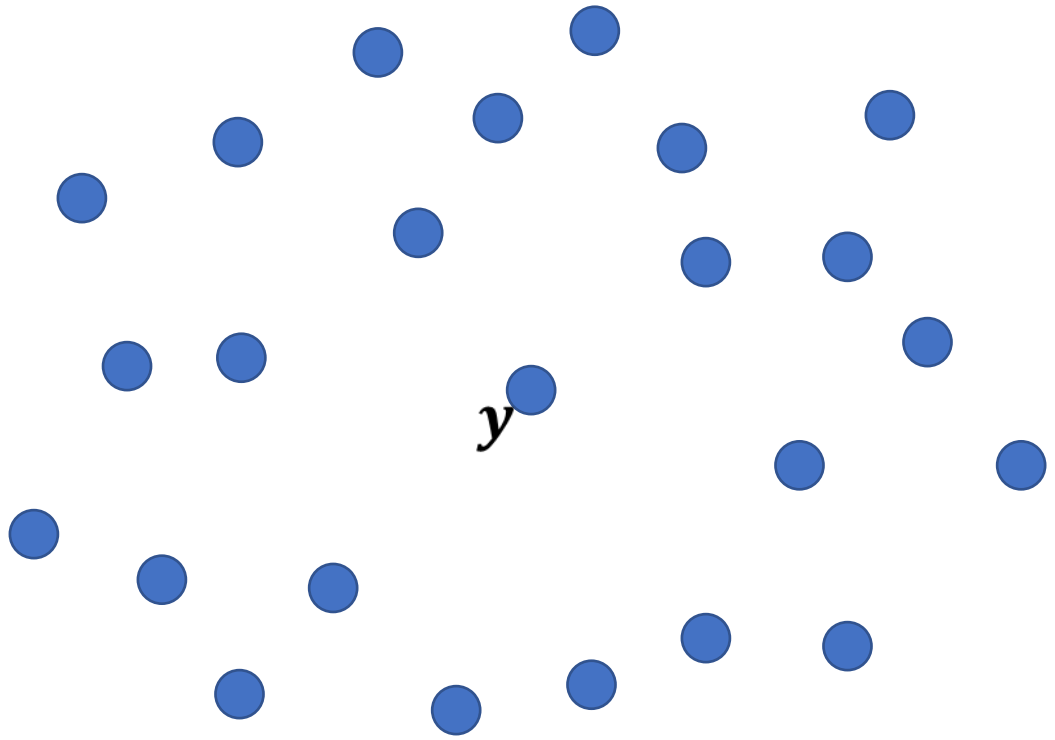


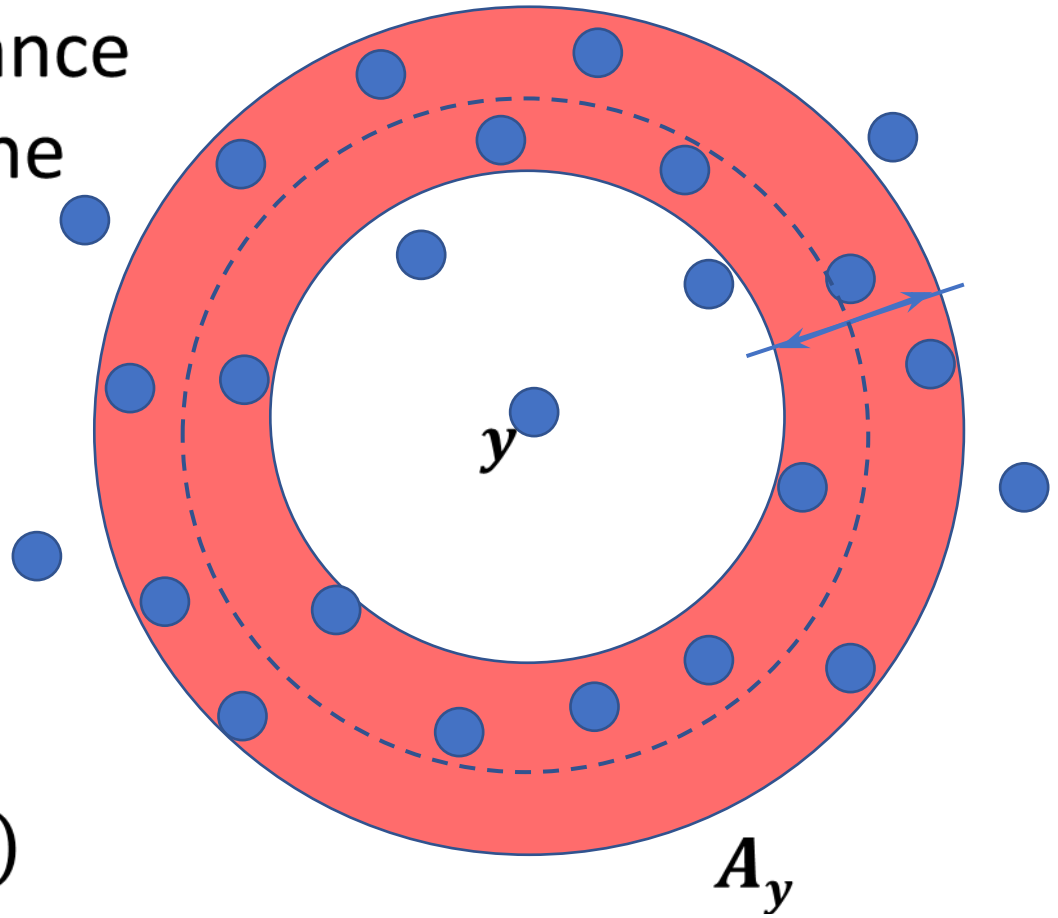
point in Base B

$y = (y_1, \ldots, y_n)$

Claim (informal)

For any base B, its size m is at least $2n/\log_q n$

A_y - set of points at distance "close" to expected one



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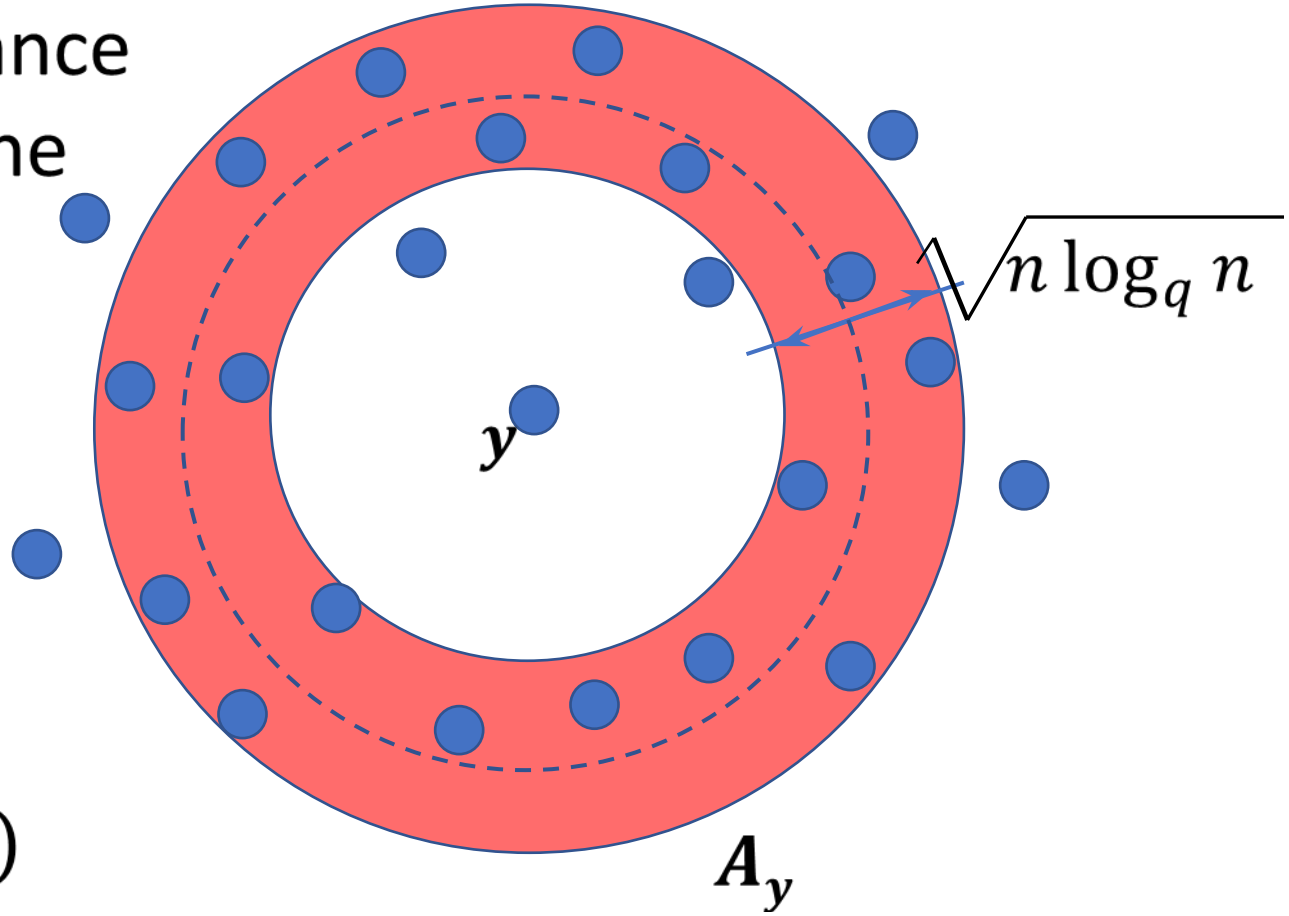
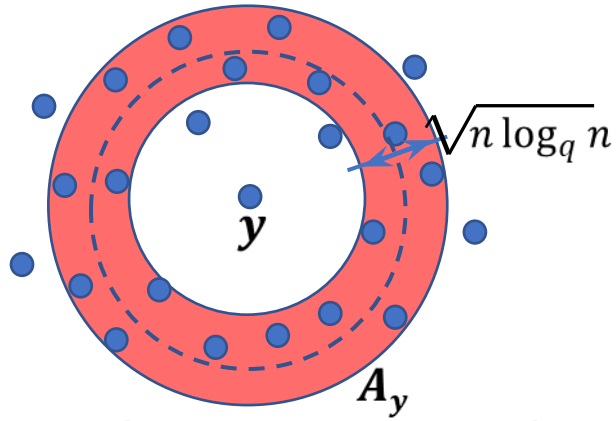


point in Base B

$y = (y_1, \ldots, y_n)$

Need two things:



1. From Hoeffding's inequality it follows A_y is large enough. Moreover, the cardinality $|\bigcap_{y \in B} A_y| > (1 - \varepsilon)q^n$.

2. Define the map d_B: $\bigcap_{y \in B} A_y \to \mathbb{N}^m$ as $d_B := \big(d(x, y)\big)_{y \in S}$

This map is **injective**. So the cardinality of the domain is less than or equal to the cardinality of the range

This logic (with some computation) proves the claim!

4. A more general problem…

Definition: A subset S of vertices in a graph $G = (V, E)$ is called a ***base*** of G, if the map

$$d_S\colon V \to \mathbb{N}^{|S|} \text{ defined as } d_S(x) \coloneqq (d(x, y))_{y \in S}$$

is injective.

Definition: Given a connected graph G , the ***metric dimension*** $m(G, n)$ of the Cartesian n-th power of G is

$$m(G, n) \coloneqq \min_{\text{base } B} |B|.$$

Problem: Given G, find the asymptotic behavior $m(G, n)$ as $n \to \infty$.

Theorem (Jiang-P.'18)

Given a connected graph G on q vertices, let M be the distance matrix of G. If the matrix

$$\begin{pmatrix} M & \mathbf{1} \\ \mathbf{1}^T & 0 \end{pmatrix}$$

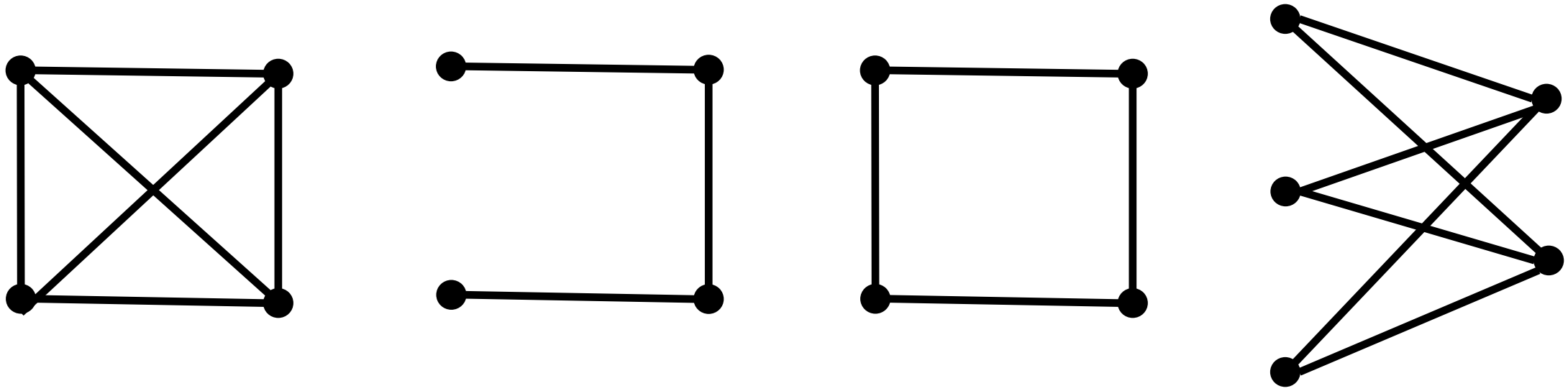
is invertible, then the metric dimension $m(G, n)$ is

$$m(G, n) \sim 2n/\log_q n$$

Corollary (*some graph classes*)

If G is a complete graph, a path, a cycle or a complete bipartite graph on q vertices, then the asymptotic of $m(G, n)$ is as follows

$$m(G\,, n) \sim 2n/\log_q n$$



Corollary (*small graphs*)

For any $q \leq 9$ and any* G on q vertices, the asymptotic of $m(G, n)$ is as follows

$$m(G, n) \sim 2n / \log_q n$$

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For any $q \leq 9$ and any* G on q vertices, the asymptotic of $m(G,n)$ is as follows

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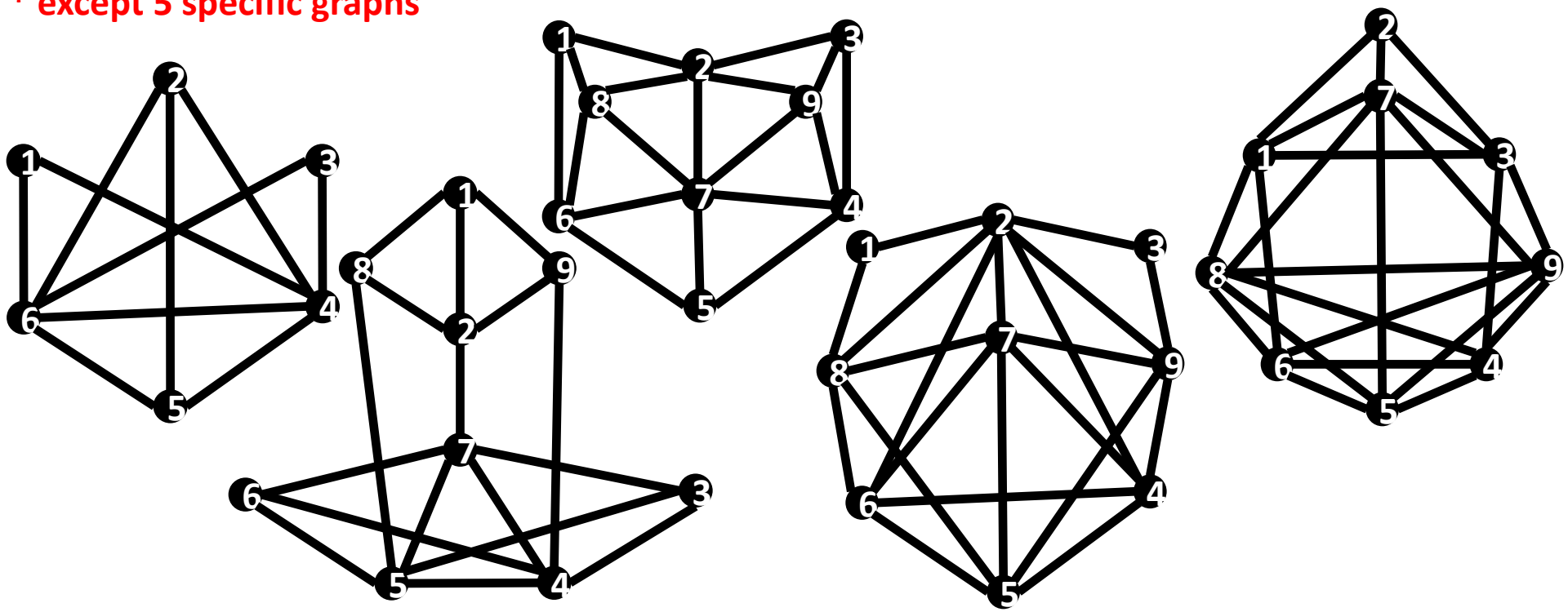
* except 5 specific graphs



Open questions

- Can we get the same asymptotic of $m(G, n)$ for any graph G?
  - A subproblem: given n distinct vectors $u_1, u_2, \dots, u_n \in \mathbb{Z}^n$, how to minimize $v \in \mathbb{Z}^n$ so that $< v, u_i >$ are different. Here, $\min \| v \|$ should be $O(n)$.
- What can we say about the metric dimension of other metric spaces?
  - E.g. Johnson and Kneser graphs.

Thanks!

Questions?