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PHAS0030

Projorbits - Mini Project

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An Investigation into the Stability of N gravitational interacting bodies

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Abstract—**The evolution of a series of n-body systems were simulated through python code with the aims of investigating their stability over long periods. Experiments were run on a set of systems that exhibited stability for long periods with no external interaction, by testing how close an external body had to approach before the stability of each devolved into chaotic motion. It was found that each of the tested systems were less stable than the solar system, with a dependence on symmetry being associated with lower measured stability.**

I. Introduction

The three body problem has been known since the days of Kepler to have no analytical solution. Modern computing however allows us to run detailed simulations of systems of these sorts with ease, allowing us to investigate the evolution of these systems easily. In this paper we analyse different forms of N-body systems, focusing on their stability.

Obviously, stable systems can exist with any number of bodies, such as planets in orbits around stars, or even stars in orbit about galactic cores. Systems of this sort are stable due to order of magnitudes differences in mass of the bodies.

More complex examples include systems of multiple stars. For systems with 3 or more stars, the system must be structured in a hierarchy, with bodies separated by large distances. This results in stability, as the multi-body core from the perspective of a further out body can be treated as a singular, more massive, object, such that the system appears as multiple sets of two body systems. [1] Both of these types of system have a high level of stability, with hundreds of multiple star systems having been recorded throughout our galaxy [2].

The final types of system with any form of stability are those with extreme levels of symmetry[3]. A 2013 study was able to find a set of systems that are stable. With the code written in this project we were also able to find orbits that were deemed meta stable - that is they exhibited stability for a long period of time.

Each of these types of system have varying level of stability, and our aim is to investigate how stable these systems are. This can be done in two ways. First through altering starting conditions, to investigate what amount of perturbation is required to break the system stability. The next is the addition of an extra body into the system, investigating how close and how massive an external body must be to break the systems in question.

II. Method

Python code was written that used Matplotlib[4], NumPy[5], and SciPy[6] to solve the ODEs that arise from newtons laws of gravitation. The main body of the code is based on [7]. The second order differential was solved by creating two coupled first order differential equations:

TABLE I
Table containing initial starting conditions for the stable solar system simulation

Body Name	Mass (Kg)	x_0 ($m \times 10^{10}$)	v_{y_0} (km/s)
Sun	1.989×10^{30}	0	**0**
Mercury	3.285×10^{23}	5.2868	48
Venus	4.867×10^{24}	10.763	35
Earth	5.972×10^{24}	14.807	29.72

$$\frac{dr_i}{dt} = v_i, \tag{1}$$
$$\frac{dv_i}{dt} = G\sum_j \frac{m_j}{r_{ij}^2}. \tag{2}$$

To keep the orders of magnitude of all values close to unity, scale values were used for the mass (M), distances (D), velocity (V), and time (T), such that all related values used in the simulations are relative to these. This results in a set of constants used when solving the coupled differential equations. These were labeled k_1 and k_2. The gravitational constant was also absorbed into the value of k_1:

$$k_1 = G\frac{TM}{D^2V} \Rightarrow m_i\frac{dv_i}{dt} = k_1\sum_j \frac{m_i m_j}{r_{ij}^3}\hat{r}_{ij},$$
$$k_2 = \frac{VT}{D} \Rightarrow \frac{dr_i}{dt} = k_2 v_i.$$

In this way, the code was able to perform simulations for any number of bodies supplied.

After much experimentation, a set of systems to test were chosen to investigate further. We wished to make a quantitative measurement of the stability of the systems being tested. To do this, we defined the rogue star approach value, R_{rs}

This value is defined by taking a body of mass equal to the total mass of the system being tested, and placing it a distance R from the centre of mass of the system, with an initial velocity equal to 1.5 times the escape velocity at this radius. These starting conditions are detailed in Equation 3, Equation 4, and Equation 5:

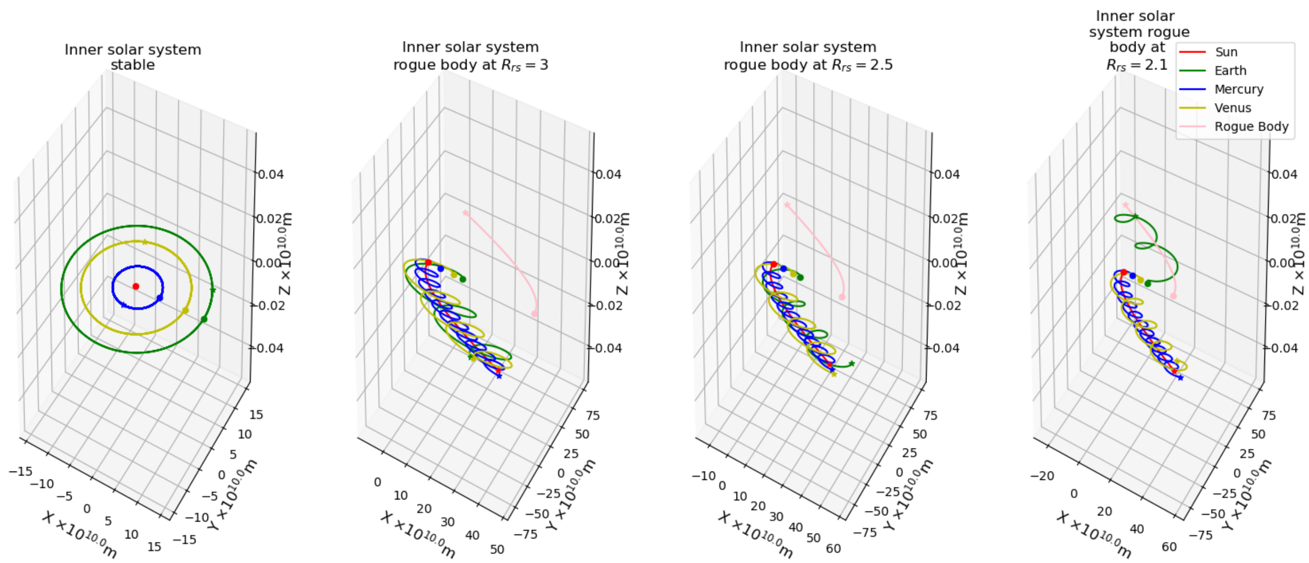

Fig. 1. Figure showing effects of a rogue body approaching at varying values of R_{rs} on the inner solar system

$$M_{rs} = \sum_i M_i, \tag{3}$$

$$R = R_{rs} \times \max |\mathbf{r}_i|\hat{i}, \tag{4}$$

$$v_{rs} = \frac{3}{2}\sqrt{2G\frac{M_{rs}}{R}}\hat{j}. \tag{5}$$

This value of R is then divided by the distance of the furthest body from the systems centre at the start of the simulation. This provides a slightly crude but effective measurement of the stability of the systems being checked. We shall use the inner solar system as a control value, defining it as a stable system.

A systems resulting stability can then be defined by its stability factor, S compared to the stable inner solar system, calculated as seen in Equation 6:

$$S = 1 - \frac{R_{rs_{M_\odot}} - R_{rs}}{R_{rs}} \tag{6}$$

Where the value $R_{rs_{M_\odot}}$ represents R_{rs} for the inner solar system.

III. Results

A. Inner Solar System

A model of the solar system was simulated as a known stable orbital system, and as a control group to define a high level of stability. We can see the effect of adding a rogue body in Figure 1. The first plot shows the stable system, with the other 3 showing varying values of R_{rs}.

We see instability beginning to occur from values of $R_{rs} = 3.0$. In this case, the system remains stable (with long term stability tested for the case of >100 years) but with a slight increase in orbital eccentricity for all bodies, most strongly the outer most body (Earth).

TABLE II
Table containing initial starting conditions for the stable figure of 8 system simulation

Body Name	Mass (Kg)	x_0, y_0 $(m \times 10^{10})$	v_{x0}, v_{y0} (km/s)
Body 1	1.5×10^{29}	-0.97000346, 0.24308753	4.66203685, 4.3236573
Body 2	1.5×10^{29}	0.97000346, -0.24308753	4.66203685, 4.3236573
Body 3	1.5×10^{29}	0,0	-9.3240737, -8.6473164

A similar effect is seen in the plot of $R_{rs} = 2.5$. The long term stability was checked for the same period, but in this plot we see a very significant change in orbital shapes for all the bodies.

In this system a value of $R_{rs} = 2.1$ is required for a total breakdown of the system, which in this specific case results in a singular body being 'stolen' by the external body.

All further systems shall have their stability compared to that of the inner solar system, with a value of $R_{rs} = 2.1$ being defined as a stable point.

B. Figure of 8

This orbital system is the first of the 3 non-hierarchy systems approached, sourced from a paper released in the year 2000 [8], with initial starting conditions as described in Table II The system displayed only a slightly larger value of R_{rs} for a point of complete instability when compared to the first experiment. We can see this being demonstrated in Figure 2. The point of chaotic evolution occurs at $R_{rs} = 2.6$, meaning this system is only has a stability factor of 0.81 relative to our solar system This is an interesting result, with this orbital system being significantly more stable than one would expect for such a symmetric solution to the three body problem.

One possible reason for this relatively high level of stability

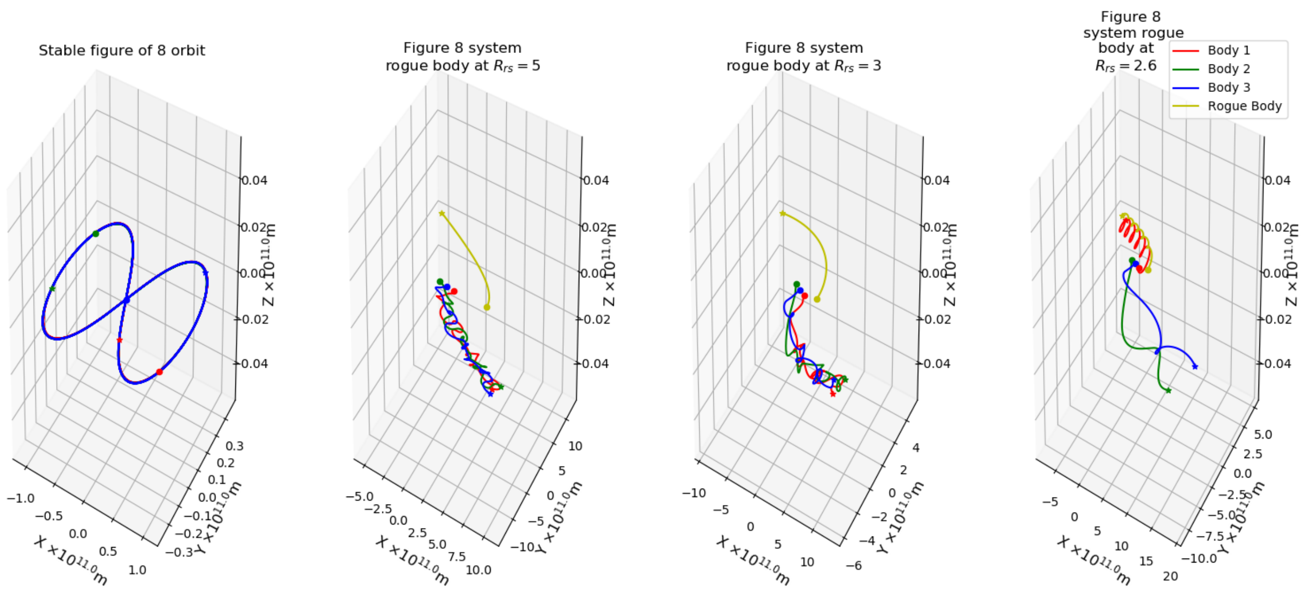

Fig. 2. Figure showing effects of a rogue body approaching at varying values of R_{rs} on the figure of 8 system

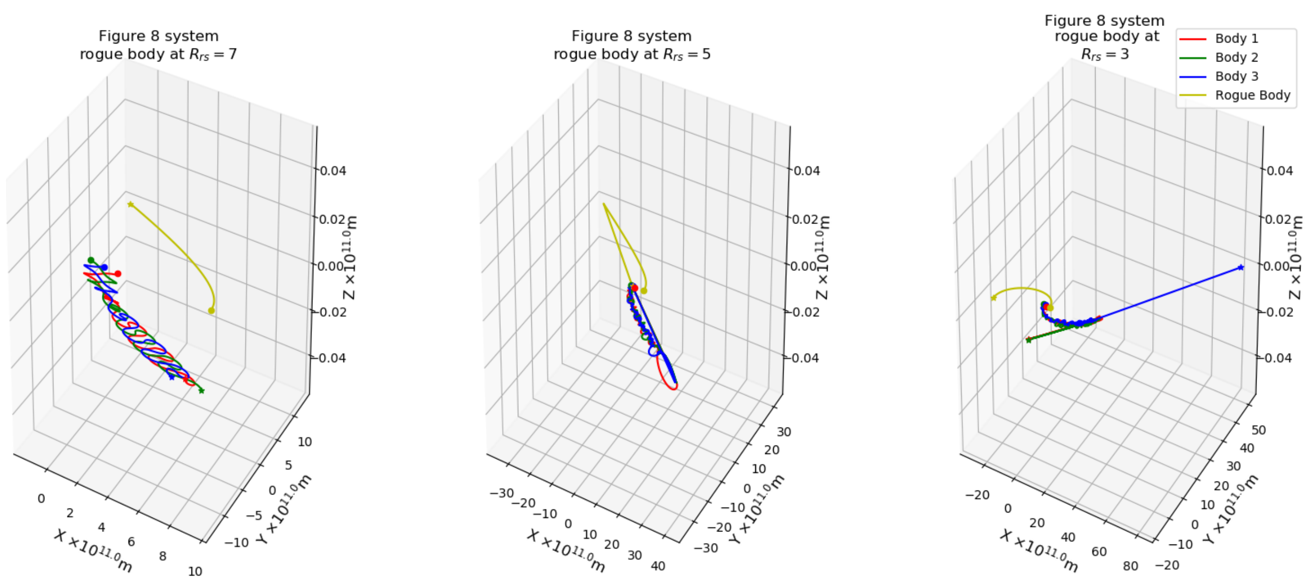

Fig. 3. Figure showing effects of a rogue body approaching at varying values of R_{rs} on the figure of 8 system

could be a result of how we define the value of R_{rs}. Due to the symmetry of this system, all bodies are relatively close to the centre of mass of the system, and so interact with each other strongly.

However, similarly to the solar system simulation, we see a great breakdown in the structure of the orbits with larger values of R_{rs}. At a value of 5, we can see the orbits start to exhibit some slight chaotic behavior, but at a value of 3 we see a significant breakdown in the symmetry of the system, with the final orbits being quite randomly aligned.

Due to this, we ran further simulations for significantly longer time frames. We can see the results of this in Figure 3. It was found that for all values of $R_{rs} < 7$ there was a total breakdown in stability when longer time frames were simulated. This is far closer to what we would expect for systems of this type.

The limit value of $R_{rs} > 7$ as a defining point for stability was only tested for a maximum period of 250 years. At values of $R_{rs} = 6.5$ the system breaks down within 50 years, and so it can be concluded that stability after the 250 year point has a high probability of remaining stable indefinitely.

This final result defines the figure of 8 system to have a stability of 0.81.

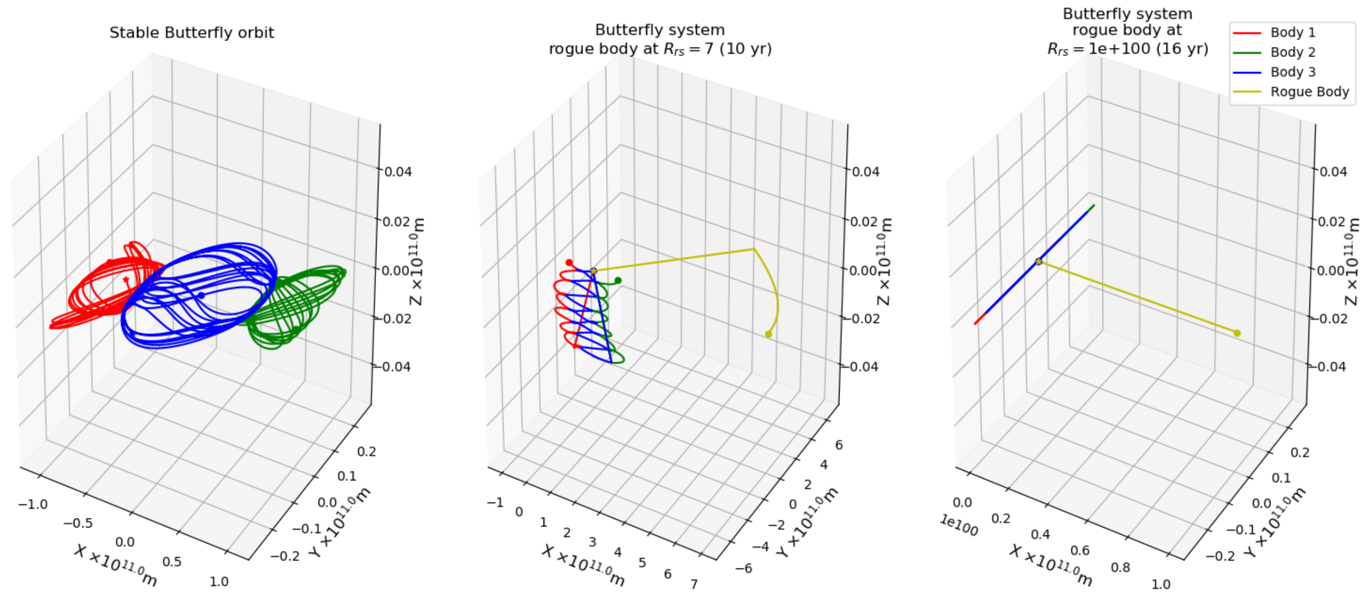

Fig. 4. Figure showing effects of a rogue body approaching at varying values of R_{rs} on the butterfly system

TABLE III
TABLE CONTAINING INITIAL STARTING CONDITIONS FOR THE STABLE BUTTERFLY SYSTEM SIMULATION

Body Name	Mass (Kg)	x_0, y_0 $(m \times 10^{11})$	v_{x_0}, v_{y_0} (km/s)
Body 1	1.0×10^{29}	-1, 0	3.0689, 1.2551
Body 2	1.0×10^{29}	1, 0	3.0689, 1.2551
Body 3	1.0×10^{29}	0,0	-6.1378, -2.5102

C. Butterfly Orbit

In 2013 another set of solutions to this problem was found [3], with initial starting conditions seen in Table III. All systems in this paper were ones based on geometric symmetry, similarly to the case of the figure of 8 orbit. As a result, we expected to see a roughly approximate level of stability.

However, upon running simulations we found that this system exhibited such extreme instability that no value of R_{rs} could be determined. In Figure 4 we can see the cause of this instability.

If we look at the left most plot - the stable orbit, we can see that this system contains points where the bodies come within extreme proximity of each other. As a result, even the slightest change in the orbital path of any body results in two of the bodies colliding.

Our code made no attempt to simulate the collisions of two bodies, instead resulting in a divide by 0 error, signified by the sudden jump in each bodies path as seen most clearly in the middle plot.

It was found that after a time period of 16 years no stable system could be found, tested up to a value of $R_{rs} = 10^{100}$. As a result this system has a stability factor $S = 0$

D. The Halo System

The final system simulated was one we were able to create after attempting various initial conditions, and fine tuning them until a stable system was found. The result can be seen in Figure 5, based on the conditions in Table IV.

As with our other systems, extreme geometric symmetry is required for the system to function. The central body only requires a very small in balance of force until to initiate a strong divergence from its central path. We were able to determine an approximate value for R_{rs} of 10^{45}, simulated over a time period of 250 years to check stability.

This system is interesting in its relative level of stability when compared to the butterfly system tested. This may be due in part to to the slight difference in masses of the bodies, but is likely caused mostly by the fact that all bodies remain a relatively consistent distance from one another when in the stable mode. This means that slight perturbations due not result in collisions, with issues only arising when the central body moves too far off its central path, resulting in a sudden and drastic runaway effect as the asymmetry of forces pulls it further from the central path.

As a result of this, we would perhaps expect the system to exhibit more stability if a large radius. We tested this and found our assumption to be true, as seen in Figure 6. The new starting conditions can be seen in Table V, and are based on increasing the radial distance of the two bodies travelling on the xy plane by a factor of 5, and scaling down their initial velocities to keep the overall orbital shape constant.

We found that by changing just this small factor, our value of R_{rs} dropped by a factor of a thousand, down to 10^{42} from the initial point of 10^{45}. This difference is quite striking for such a small change in initial starting conditions, and highlights the

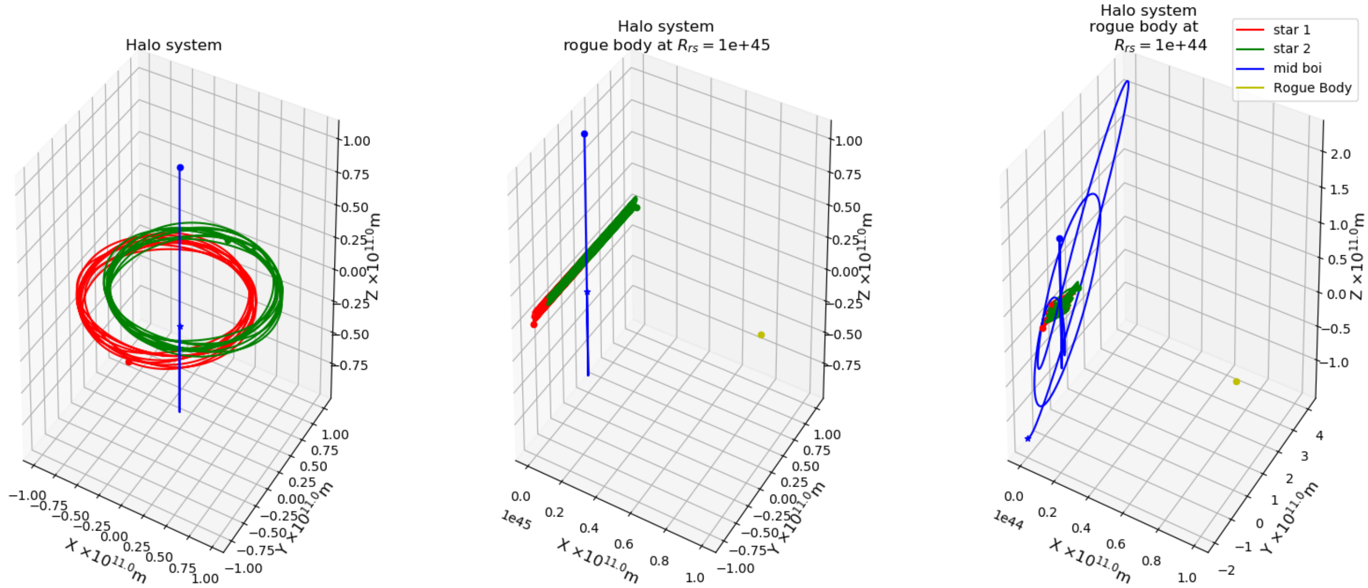

Fig. 5. Figure showing effects of a rogue body approaching at varying values of R_{rs} on the halo system

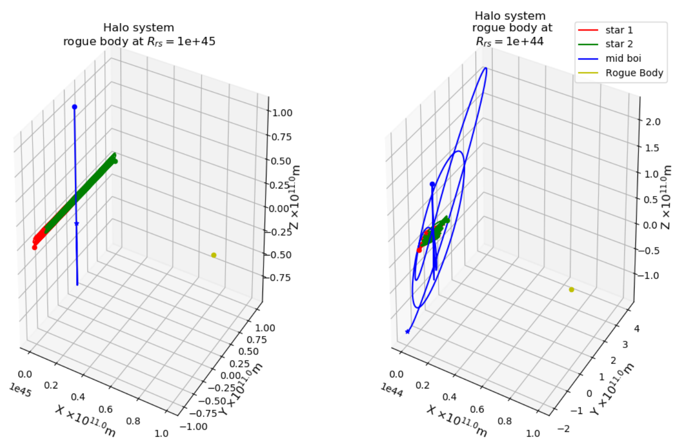

Fig. 6. Figure showing effects of a rogue body approaching at varying values of R_{rs} on the modified radius halo system

TABLE IV
TABLE CONTAINING INITIAL STARTING CONDITIONS FOR THE STABLE HALO SYSTEM SIMULATION

Body Name	Mass (Kg)	x_0, y_0, z_0 $(m \times 10^{11})$	v_{x0}, v_{y0}, v_{z0} (km/s)
Star 1	1.0×10^{30}	$0, -1, 0$	12.92, 0, 0
Star 2	1.0×10^{30}	0, 1, 0	-12.92, 0, 0
Mid	0.1×10^{30}	0,0, 1	0, 0, 0

effect that causes the system to breakdown.
This system is still significantly too unstable for a stability factor to have any meaningful value, and so is excluded for this data set, with us only being able to confirm $S > 0$

TABLE V
TABLE CONTAINING INITIAL STARTING CONDITIONS FOR THE SECOND HALO ORBITAL SYSTEM, WITH LARGER ORBITAL SYSTEM SIZE

Body Name	Mass (Kg)	x_0, y_0, z_0 $(m \times 10^{11})$	v_{x0}, v_{y0}, v_{z0} (km/s)
Star 1	1.0×10^{30}	$0, -5, 0$	5.778, 0, 0
Star 2	1.0×10^{30}	0, 5, 0	-5.778, 0, 0
Mid	0.1×10^{30}	0,0, 1	0, 0, 0

IV. CONCLUSIONS

Throughout all the un-stable systems analysed, we saw that chaotic motion was initiated by only a slight perturbation from the stable mode. As shown in the case of the halo system, this fact is likely caused by the strong tidal forces felt between the bodies, with any movement away from the point of 0 gravitational potential causing a drastic breakdown in the orbits.
This idea is further highlighted by looking at the stability of the figure of 8 system compared to that of the butterfly or halo systems. We believe that the high level of stability is due to the on average constant distances between each of the bodies. However, it is still unknown exactly what mechanism causes the stability to be so high when compared to the halo system, which also keeps bodies at a relatively constant distance from one another.
Further research could be done honing in on the stability of systems similar to the figure of 8 system with the aim of finding exactly why its stability is so high.
It could also be interesting to see if a system of this sort could ever naturally occur. Normal multiple star systems are thought to form either through the generation of multiple protostars

from the same initial gas cloud [9], or through a star being captured at a later point in the stellar evolution.
Due to the shape of the figure of 8 system, we believe it to be unlikely that this system could form directly in such a way, but it is not outside the realms of possibility that the system could form as a result of a third star penetrating a stable binary system.

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