Analog and Digital Control
2^{nd} Group of Theoretical Exercises

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Using the Partial Fraction Expansion method, evaluate the inverse Laplace transforms of the following quantities

$k=2$

1. $F(s)=\dfrac{2\cdot s+k}{(s+1)(s+2)(s+1+k)}$

2. $F(s)=\dfrac{s+k+1}{(s+1)(s+2)(s+12+k)^3}$

3. $F(s)=\dfrac{s+k+3}{s^2+s+k+15}$

4. $F(s)=\dfrac{s+k+4}{(s^2+s+k+12)(s+1)}$

5. $F(s)=\dfrac{s+k+5}{(s^2+s+k+12)\cdot s^3}$

6. $F(s)=\dfrac{s+k+6}{(s+1)(s^2+s+k+2)(s+2)^2}$

Λυσεις:

1. $F(s)=\dfrac{Q(s)}{P(s)}=\dfrac{2s+k}{(s+1)(s+2)(s+1+k)}=\dfrac{2s+2}{(s+1)(s+2)(s+3)}=\dfrac{k_1}{s+1}+\dfrac{k_2}{s+2}+\dfrac{k_3}{s+3}$

$$k_1=(s+1)F(s)\Big|_{s=-1}=(s+1)\frac{2s+2}{(s+1)(s+2)(s+3)}\Big|_{s=-1}=\frac{2s+2}{(s+2)(s+3)}=0$$

$$k_2=(s+2)F(s)\Big|_{s=-2}=\frac{2s+2}{(s+1)(s+3)}=2$$

$$k_3=(s+3)F(s)\Big|_{s=-3}=\frac{2s+2}{(s+1)(s+2)}=-2$$

Αρα:

$$F(s) = \frac{0}{s+1} + \frac{2}{s+2} + \frac{-2}{s+3} = \frac{2}{s+2} + \frac{-2}{s+3}$$

οπότε:

$$f(t) = 2e^{-2t} - 2e^{-3t}$$

2. $$F(s) = \frac{s+k+1}{(s+1)(s+2)(s+12+k)^3} = \frac{s+3}{(s+1)(s+2)(s+14)^3} = \frac{A}{(s+1)} + \frac{B}{s+2} + \frac{C_1}{s+14} + \frac{C_2}{(s+14)^2}$$

$$+ \frac{C_3}{(s+14)^3}$$

$$A = (s+1)F(s)\Big|_{s=-1} = \frac{s+3}{(s+2)(s+14)^3} = \frac{2}{2197}$$

$$B = (s+2)F(s)\Big|_{s=-2} = \frac{s+3}{(s+1)(s+14)^3} = \frac{1}{-1728}$$

$$C_1 = \frac{1}{(3-1)!}\frac{d^{(3-1)}}{d_s^{(3-1)}}\left[(s-14)^3 F(s)\right]\Big|_{s=14} = \frac{1}{2}\frac{d^2}{d_s^2}\left[(s-14)^3 \cdot \frac{s+3}{(s+1)(s+2)(s+14)^3}\right]\Big|_{s=14}$$

$$\frac{1}{2}\frac{d^2}{d^2_s}\left[\frac{s+3}{(s+1)(s+2)}\right]\Big|_{s=14} = \frac{1}{2} \cdot \frac{2s^3+18s^2+42s+30}{(s^2+3s+2)^3}\Big|_{s=14} =$$

$$\frac{1}{2} \cdot \frac{5488+3528+56+30}{(196+42+2)^3} = \frac{9102}{13144256 \cdot 2} = \frac{4551}{13144256} = 0.0034$$

$$C_2 = \frac{1}{(2-1)!}\frac{d^{(2-1)}}{d_s^{(2-1)}}\left[(s-14)^3 F(s)\right]\Big|_{s=+14} = \frac{d}{ds}\left((s-14)^3 \frac{s+3}{(s+1)(s+2)(s+14)^3}\right)\Big|_{s=14} =$$

$$\frac{d}{ds}\left(\frac{s+3}{(s+1)(s+2)}\right)\Big|_{s=14} = \frac{-x^2-6x-7}{(x^2+3x+2)^2}\Big|_{s=14} = \frac{-196-84-7}{57600} = -0.00498$$

$$C_3 = \frac{1}{0!}\left[(s-14)^3 F(s)\right]\Big|_{s=14} = \frac{s+3}{(s+1)(s+2)}\Big|_{s=14} = \frac{17}{240} = 0.0703$$

Αρα $$F(s) = \frac{\frac{2}{2197}}{s+1} + \frac{\frac{1}{-1728}}{s+2} + \frac{0.0034}{(s+14)^1} + \frac{0.00498}{(s+14)^2} + \frac{0.0703}{(s+14)^3}$$

οπότε $$f(t) = \frac{2}{2197}e^{-t} + \frac{1}{-1728}e^{-2t} + 0.0034 \cdot e^{-14t} + 0.00498 \cdot t \cdot e^{-14t} +$$
$$0.0703 \cdot t^2 \cdot e^{-14t}$$

3. $F(s) = \frac{s+k+3}{s^2+s+k+15} = \frac{s+5}{s^2+s+17} = \frac{r_1}{(s-p_1)} + \frac{r_2}{(s-p_2)}$

$r_1 = (s-p_1)F(s)\Big|_{s=p_1} = (s-p_1)\frac{s+5}{s^2+s+17}\Big|_{s=r_1} = \cancel{(s-p_1)}\frac{s+5}{\cancel{(s-p_1)}(s-p_2)} = \frac{s+5}{s-p_2}\Big|_{s=\frac{67^{0.5}i}{2}-\frac{1}{2}} =$

$\frac{\frac{67^{0.5}i}{2}-\frac{1}{2}+5}{\frac{67^{0.5}i}{2}-\frac{1}{2}-p_2} = \frac{4.0927i+\frac{9}{2}}{4.0927i-\frac{1}{2}-p_2} = \frac{(0.8529-1.2037i)}{-p_2} =$

$\frac{0.8529-1.2037i}{-\left(\frac{67^{0.5}i}{2}-\frac{1}{2}\right)} = 0.2647+0.2407i$

Αρα $x=0.2647$, $y=0.2407$ $\alpha=0.5$ $B=4.0927$

οπότε $f(t) = 2(0.2647)e^{-0.5t}\cos(4.0927\cdot t) - 2(0.2407)e^{-0.5t}\sin(4.0927t)$
$= 0.5294e^{-\frac{1}{2}t}\cos(4.0927t) - 0.4814e^{-\frac{1}{2}t}\sin(4.0927t)$

4. $F(s) = \frac{s+k+4}{(s^2+s+k+12)(s+1)} = \frac{s+6}{(s^2+s+14)(s+1)}$, $p_1=-1$, $p_{2,3} = 0.5000 \pm 3.7081i$

$F(s) = \frac{A_1}{s+1} + \frac{r_1}{s-p_2} + \frac{r_2}{s-p_3}$

$A_1 = (s+1)F(s)\Big|_{s=-1} = \frac{s+6}{(s^2+s+14)}\Big|_{s=-2} = 0.25$

$r_1 = (s-p_2)F(s)\Big|_{s=p_2} = \frac{s+6}{(s+1)(s-p_3)}\Big|_{s=p_2} = \frac{(0.5+3.7081i)+6}{(0.5+3.7081i+1)(0.5+3.7081i-(0.5-3.7081i))}$

$= \frac{6.5+3.7081i}{(1.5+3.7081i)(0+7.4162i)} = \frac{6.5+3.7081i}{-27.5+11.1243i} = -0.1562-0.1980i$

Αρα: $\alpha=0.5$, $B=3.7081$ $x=0.1562$ $y=0.1980$
οπότε $f(t) = 0.25\cdot e^{-t} + 0.3124e^{-0.5t}\cos(3.7081\cdot t) - 0.396e^{-0.5t}\sin(3.7081t)$

5. $F(s)=\dfrac{s+k+5}{(s^2+s+k+12)\cdot s^3}=\dfrac{s+7}{(s^2+s+14)\,s^3}$; $p_{1,2}=0.5\pm 3.7081j$, $p_{3,4,5}=0$

$$F(s)=\frac{A_1}{s}+\frac{A_2}{s^2}+\frac{A_3}{s^3}+\frac{r_1}{s-p_1}+\frac{r_2}{s-p_2}$$

$$A_1=\frac{1}{(3-1)!}\cdot\frac{d^{3-1}}{d s^{3-1}}\left[s^3F(s)\right]\Big|_{s=0}=\frac{1}{2}\frac{d^2}{ds^2}\left(\frac{s+7}{s^2+s+14}\right)\Big|_{s=0}=$$
$$\frac{1}{2}\left(\frac{2s^3+42s^2-42s-210}{(x^2+x+14)^3}\right)\Big|_{s=0}=-0.0382$$

$$A_2=\frac{1}{(2-1)!}\,\frac{d^{2-1}}{d s^{2-1}}\left[s^3F(x)\right]\Big|_{s=0}=\frac{d}{ds}\left(\frac{s+7}{s^2+s+14}\right)\Big|_{s=0}=$$
$$\frac{-s^2-14s+7}{(s^2+s+14)^2}\Big|_{s=0}=0.0357$$

$$A_3=\frac{1}{(1-1)!}\cdot\frac{d^{1-1}}{d s^{1-1}}\left[s^3F(s)\right]\Big|_{s=0}=\frac{s+7}{s^2+s+14}\Big|_{s=0}=0.5$$

$$r_1=(s-p_1)F(s)\Big|_{s=p_1}=\cancel{(s-p_1)}\,\frac{s+7}{\cancel{(s-p_1)}(s-p_2)\,s^3}\Big|_{s=p_1}=0.0140+0.0163i$$

Άρα $x=0.0140$, $y=0.0163$, $\alpha=0.5$, $\beta=3.7081$

Οπότε: $f(t)=0.0382\cdot e^{-0t}+0.0357\cdot e^{-0t}+0.5\cdot t^{3-1}e^{-0t}+$
$2(0.0140)e^{-0.5t}\cos(3.7081t)-2\cdot(0.0163)e^{-0.5t}\sin(3.7081t)=$
$0.0382+0.0357t+0.5t^2+0.028e^{-0.5t}\cos(3.7081t)-$
$0.0326e^{-0.5t}\sin(3.7081t)$

6. $F(s) = \dfrac{s+k+6}{(s+1)(s^2+s+k+2)(s+2)^2} = \dfrac{s+8}{(s+1)(s^2+s+4)(s+2)^2}$

poles:

$p_1 = -1 \quad p_{2,3} = -0.5 \pm 1.9365i \quad p_4 = -2 \quad p_5 = -2$

$$F(s) = \frac{A}{s+1} + \frac{B1}{s-p_2} + \frac{B2}{s-p_3} + \frac{C1}{s+2} + \frac{C2}{(s+2)^2}$$

$$A_1 = (s+1)F(s) = \left.\frac{s+8}{(s^2+s+4)(s+2)^2}\right|_{s=-1} = \frac{7}{4} = 1.75$$

$$r_1 = (s-p_2)F(s)\Big|_{s=p_2} = \left.\frac{s+8}{(s+1)(s-p_3)(s+2)^2}\right|_{s=p_2} = \frac{-0.5+1.9365i+8}{(-0.5+1.9365i+1)(-0.5+1.9365i-p_3)(s+2)^2} =$$

$-0.0417 + 0.1614i$

$$C_1 = \frac{1}{(2-1)!} \cdot \frac{d^{2-1}}{d^{2-1}x}[(s+2)^2 F(s)] = \frac{d}{ds}\left.\frac{s+8}{(s+1)(s^2+s+4)}\right|_{s=-2} =$$

$$\left.\frac{-2s^3-26s^2-32s-36}{(s^3+2s^2+5s+4)^2}\right|_{s=-2} = \frac{-60}{36} = -1.66$$

$$C_2 = \frac{1}{0!}[(s+2)^2 F(s)] = \left.\frac{s+8}{(s+1)(s^2+s+4)}\right|_{s=-2} = -1$$

$x = 0.0417 \quad y = 0.1614 \quad \alpha = 0.5 \quad B = 1.9365$

Ap $f(t) = 1.75 \cdot e^{-1t} + 0.0834 \cdot e^{-0.5t} \cos(1.9365t) - 0.3228 e^{-0.5t} \sin($
$1.9365t) + 1.66 \cdot e^{-2t} + (-1) \cdot t \cdot e^{-2t}$