SUPPLEMENTARY MATERIAL FOR THE PAPER"A FIRST RUNTIME ANALYSIS OF THE PAES-25: AN ENHANCED VARIANT OF THE PARETO ARCHIVED EVOLUTION STRATEGY"

This document contains the proofs that we omitted in the main paper, due to space restrictions.

Lemma 3.2. *Let S be a maximum cardinality set of mutually incomparable solutions for $f := m$-LOTZ. Then $|S| = n+1$ if $m = 2$ and*

$$\frac{(2n/m+1)^{m-1}}{4(m-2)^{m/2-1}} \le |S| \le (2n/m+1)^{m-1}$$

if $m \ge 4$.

Proof. Let $V := f(S)$. We only show $|V| \ge k^{m-1}/(4(m-2)^{m/2-1})$ where $k := 2n/m + 1$ as the upper bound is Lemma 4.2 in [42]. For $m = 2$ a set S with $f(S) = \{(n,0),(1,n-1),\dots,(0,n)\}$ and cardinality $n+1$ is a set of mutually incomparable solutions. Suppose that $m \ge 4$. We construct a set $V' \subset \mathbb{N}_0^m$ with $|V'| \ge k^{m-1}/(4(m-2)^{m/2-1})$ such that there is a set S' of mutually incomparable solutions with $f(S') = V'$ (which implies $|S'| = |V'|$ since $f(x) \ne f(y)$ for two distinct $x, y \in S'$, and $|S| \ge |S'|$ as S has maximum possible cardinality). At first define for $w := (w_1, \dots, w_{m/2}) \in \{0,\dots,k\}^{m/2}$

$$M_w := \{v \in \{0,\dots,k\}^m \mid v_{2i-1} + v_{2i} = w_i \text{ for } i \in \{1,\dots,m/2\}\}.$$

Then two search points x, y with $f(x) = u$, $f(y) = v$ for $u, v \in M_w$ with $u \ne v$ are incomparable: Fix $i \in \{1,\dots,m\}$ with $u_i \ne v_i$. If $u_i < v_i$, then $u_{i-1} > v_{i-1}$ if i is even and $u_{i+1} > v_{i+1}$ if i is odd. If $u_i > v_i$, then $u_{i-1} < v_{i-1}$ if i is even and $u_{i+1} < v_{i+1}$ if i is odd.

For $w \in \{0,\dots,k\}^{m/2}$ we have that $|M_w| = \prod_{i=1}^{m/2}(w_i+1)$ (since $v_{2i-1} + v_{2i} = w_i$ is possible for $i \in \{1,\dots,m\}$ if and only if $(v_{2i-1}, v_{2i}) \in \{(w_i,0),(w_i-1,1),\dots,(1,w_i-1),(0,w_i)\}$). Further $M_w \cap M_{w'} = \emptyset$ for $w \ne w'$ as the sum $v_{2i-1} + v_{2i}$ is uniquely determined for a vector $v \in \mathbb{R}^m$. Now consider for $r := \lceil (m-3)k/(m-2) \rceil$

$$W := \left\{ w \in \{0,\dots,k\}^{m/2} \mid w_i \in [r,k] \text{ for } i \in \{1,\dots,m/2-1\} \text{ and } w_{m/2} = k - \sum_{i=1}^{m/2-1} (w_i - r) \right\}.$$

Then we have that

$$w_1 + \ldots + w_{m/2} = k + \sum_{i=1}^{m/2-1} r = k + (m/2-1)r$$

implying $v_1 + \ldots + v_m = k + (m/2-1)r$ for every $v \in M_w$ where $w \in W$. Consequently, two search points x, y with $f(x) = u$ and $f(y) = v$ for $u \in M_{w_1}$ and $v \in M_{w_2}$ with distinct $w_1, w_2 \in W$ are incomparable: If there is a dominance relation between x and y we have that $u_1 + \ldots + u_m > v_1 + \ldots + v_m$ or $u_1 + \ldots + u_m < v_1 + \ldots + v_m$, but these both sums are $k + (m/2-1)r$. There is also no weak dominace relation between x and y since $u \ne v$ (because w_1 and w_2 are distinct). Thus for

$$V' := \{v \in \{0,\dots,k\}^m \mid v \in M_w \text{ for a } w \in W\}$$

there is a set S' of mutually incomparable solutions with $f(S') = V'$. We show $|V'| \ge (2n/m+1)^{m-1}/(4(m-2)^{m/2-1})$ and obtain the result. Since w_i is bounded from below by r for $i \in \{1,\dots,m/2-1\}$ if $w \in W$, we obtain

$$\begin{aligned}
|V'| = \sum_{w\in W} |M_w| &= \sum_{w\in W} \prod_{i=1}^{m/2} (w_i+1) \\
&\ge \sum_{w\in W} (r+1)^{m/2-1}(w_{m/2}+1) \\
&= (r+1)^{m/2-1} \sum_{w\in W} (w_{m/2}+1).
\end{aligned}$$

Since we have $w_{m/2} = k - \sum_{i=1}^{m/2-1}(w_i - r)$ and w_i has range in $r,\dots,k$ if $w \in W$, we obtain

$$\begin{aligned}
\sum_{w\in W} (w_{m/2}+1) &= \sum_{w_1=r}^{k} \cdots \sum_{w_{m/2-1}=r}^{k} \left(k + 1 - \sum_{i=1}^{m/2-1} (w_i - r) \right) \\
&= \sum_{w_1=0}^{k-r} \cdots \sum_{w_{m/2-1}=0}^{k-r} \left(k + 1 - \sum_{i=1}^{m/2-1} w_i \right)
\end{aligned}$$

and for every $j \in \{1, \ldots, m/2-1\}$

$$\sum_{w_1=0}^{k-r} \cdots \sum_{w_{m/2-1}=0}^{k-r} w_j = (k-r+1)^{m/2-2} \cdot \sum_{w_j=0}^{k-r} w_j$$
$$= \frac{k-r}{2}(k-r+1)^{m/2-1} =: q$$

where the latter equality is due to the Gaussian sum. Since $r = \lceil (m-3)k/(m-2) \rceil$, we see $k - r \leq k - (m-3)k/(m-2) = k/(m-2)$ and we obtain

$$\begin{aligned}
\sum_{w \in W} (w_{m/2} + 1) &= (k+1) \cdot (k-r+1)^{m/2-1} - (m/2-1)q \\
&= (k-r+1)^{m/2-1} \left(k + 1 - \frac{(m/2-1)(k-r)}{2} \right) \\
&\geq (k-r+1)^{m/2-1} \left(k + 1 - \frac{(m/2-1)k}{4(m/2-1)} \right) \\
&\geq (k-r+1)^{m/2-1} (k + 1 - k/4) \\
&\geq k/2 \cdot (k-r+1)^{m/2-1}
\end{aligned}$$

and consequently due to $\lceil x \rceil \geq x$ and $-x \leq -\lceil x \rceil + 1$

$$\begin{aligned}
|V'| &\geq \frac{k}{2} \cdot (r+1)^{m/2-1} \cdot (k-r+1)^{m/2-1} \\
&\geq \frac{k}{2} \cdot \left(\frac{m-3}{m-2} \cdot k + 1 \right)^{m/2-1} \cdot \left(k - \frac{m-3}{m-2} \cdot k \right)^{m/2-1} \\
&\geq \frac{k}{2} \cdot \left(\frac{m-3}{m-2} \cdot k \right)^{m/2-1} \cdot \left(\frac{k}{m-2} \right)^{m/2-1} \\
&= \frac{k^{m-1}}{2} \left(1 - \frac{1}{m-2} \right)^{(m-2)/2} \left(\frac{1}{m-2} \right)^{m/2-1} \\
&\geq \frac{k^{m-1}}{4(m-2)^{m/2-1}} = \frac{(2n/m+1)^{m-1}}{4(m-2)^{m/2-1}}
\end{aligned}$$

where the last inequality holds, as $(1 - 1/\ell)^{\ell/2} \geq 1/2$ for every $\ell \geq 2$ and $m \geq 4$. □