Syntax

$$
\begin{array}{rrcl}
\textbf{\textit{Value}} & v & ::= & c \;|\; \lambda x.e \\
\textbf{\textit{Expressions}} & e & ::= & v \;|\; x \;|\; e\ e \\
\textbf{\textit{Basic Types}} & b & ::= & \text{int} \;|\; \text{bool} \\
\textbf{\textit{Types}} & \tau & ::= & \{v{:}b \mid e\} \;|\; x{:}\tau \to \tau \\
\textbf{\textit{Environment}} & \Gamma & ::= & \emptyset \;|\; x{:}\tau, \Gamma
\end{array}
$$

Operational Semantic

$$
\begin{array}{ll}
e_1\ e_2 \hookrightarrow e_1'\ e_2 & \text{if } e_1 \hookrightarrow e_1' \\
\lambda x.e\ e_x \hookrightarrow e\,[e_x/x] & \\
c\ e \hookrightarrow c\ e' & \text{if } e \hookrightarrow e' \\
c\ v \hookrightarrow [|c|](v) &
\end{array}
$$

Erasing

$$
\begin{aligned}
\lfloor \{v{:}b \mid e\} \rfloor &= b \\
\lfloor x{:}\tau_x \to \tau \rfloor &= \lfloor \tau_x \rfloor \to \lfloor \tau \rfloor
\end{aligned}
$$

Interpretations

$$
\begin{aligned}
[|\ \{v{:}b \mid e_v\}\ |] &= \{e \mid \vdash e{:}b \wedge e \hookrightarrow^\star v \Rightarrow e_v\,[e/v] \hookrightarrow^\star \text{true}\} \\
[|x{:}\tau_x \to \tau|] &= \{e \mid \vdash e{:}\lfloor \tau \rfloor \to \lfloor \tau_x \rfloor \wedge \forall e_x \in [|\tau_x|].\ e\ e_x \in [|\tau\,[e_x/x]\,|]\}
\end{aligned}
$$

Typing

$$\boxed{\Gamma \vdash e{:}\tau}$$

$$
\frac{(x,\tau) \in \Gamma}{\Gamma \vdash x{:}\tau}
\qquad
\frac{}{\Gamma \vdash c{:}ty(c)}
\qquad
\frac{\Gamma \vdash e{:}\tau' \quad \Gamma \vdash \tau' \preceq \tau \quad \Gamma \vdash \tau}{\Gamma \vdash e{:}\tau}
$$

$$
\frac{\Gamma, x{:}\tau_x \vdash e{:}\tau \quad \Gamma \vdash \tau_x}{\Gamma \vdash \lambda x.e{:}(x{:}\tau_x \to \tau)}
\qquad
\frac{\Gamma \vdash e_1{:}(x{:}\tau_x \to \tau) \quad \Gamma \vdash e_2{:}\tau_x}{\Gamma \vdash e_1\ e_2{:}\tau\,[e_2/x]}
$$

$$\boxed{\Gamma \vdash \tau}$$

$$
\frac{\Gamma, v{:}b \vdash e{:}\text{bool}}{\Gamma \vdash \{v{:}b \mid e\}}
\qquad
\frac{\Gamma \vdash \tau_x \quad \Gamma, x{:}\tau_x \vdash \tau}{\Gamma \vdash x{:}\tau_x \to \tau}
$$

$$\boxed{\Gamma \vdash \tau \preceq \tau}$$

$$
\frac{\Gamma, v : b \vdash e_1 \Rightarrow e_2}{\Gamma \vdash \{v{:}b \mid e_1\} \preceq \{v{:}b \mid e_2\}}
\qquad
\frac{\Gamma \vdash \tau_x' \preceq \tau_x \quad \Gamma, x{:}\tau_x' \vdash \tau \preceq \tau'}{\Gamma \vdash x : \tau_x \to \tau \preceq x : \tau_x' \to \tau'}
$$

$$\Gamma \vdash e \Rightarrow e$$

$$\frac{\forall \theta . \Gamma \vdash \theta \wedge \theta\ e_1 \hookrightarrow^{\star} \text{true} \Rightarrow \theta\ e_2 \hookrightarrow^{\star} \text{true}}{\Gamma \vdash e_1 \Rightarrow e_2}$$

$$\Gamma \vdash \theta$$

$$\frac{\forall x \in \mathrm{Dom}(\Gamma) . \theta(x) \in [|\theta\ \Gamma(x)|]}{\Gamma \vdash \theta}$$

Constants

For each constant c,

1. $\emptyset \vdash c{:}ty(c)$
2. If $ty(c) = x{:}\tau_x \to \tau$, then for each v such that $\emptyset \vdash v{:}\tau_x$ $[|c|](v)$ is defined and $\vdash [|c|](v){:}\tau\,[v/x]$
3. If $ty(c) = \{v{:}b \mid e\}$, then $e\,[c/v] \hookrightarrow^{\star}$ true and $\forall c'\ c' \neq c . \neg(e\,[c'/v] \hookrightarrow^{\star} \text{true})$

Semantic Typing

$$\Gamma \vdash e \in \tau = \forall \theta . \Gamma \vdash \theta \Rightarrow \theta\ e \in [|\theta\ \tau|]$$
$$\Gamma \vdash \tau_1 \subseteq \tau_2 = \forall \theta . \Gamma \vdash \theta \Rightarrow [|\tau_1|] \subseteq [|\tau_2|]$$

Lemma 1 *1. If $\Gamma \vdash \tau_1 \preceq \tau_2$ then $\Gamma \vdash \tau_1 \subseteq \tau_2$*

2. If $\Gamma \vdash e{:}\tau$ then $\Gamma \vdash e \in \tau$

Lemma 2 *If $e \hookrightarrow e'$ then $\Gamma \vdash \tau\,[e'/x] \preceq \tau\,[e/x]$*

Lemma 3 (Substitution) *If $\Gamma \vdash e_x{:}\tau_x$, then*

1. If $\Gamma, x{:}\tau_x, \Gamma' \vdash \tau_1 \preceq \tau_2$ then $\Gamma, [e_x/x]\,\Gamma' \vdash [e_x/x]\,\tau_1 \preceq [e_x/x]\,\tau_2$

2. If $\Gamma, x{:}\tau_x, \Gamma' \vdash e{:}\tau$ then $\Gamma, [e_x/x]\,\Gamma' \vdash [e_x/x]\,e{:}\,[e_x/x]\,\tau$

3. If $\Gamma, x{:}\tau_x, \Gamma' \vdash \tau$ then $\Gamma, [e_x/x]\,\Gamma' \vdash [e_x/x]\,\tau$

Lemma 4 (Preservation) *If $\emptyset \vdash e{:}\tau$ and $e \hookrightarrow e'$ then $\emptyset \vdash e'{:}\tau$.*

Lemma 5 (Progress) *If $\emptyset \vdash e{:}\tau$ and $e \neq v$ then there exists an e' such that $e \hookrightarrow e'$.*