1 Language

1.1 Syntax

$$
\begin{array}{rrcl}
\textbf{\textit{Value}} & v & ::= & c \mid \lambda x.e \mid \mathit{TODO}\ D\ v^{\perp} \dots v^{\perp} \\
\textbf{\textit{Lifted Value}} & v^{\perp} & ::= & v \mid \perp \\
\textbf{\textit{Expressions}} & e & ::= & c \mid \lambda x.e \mid \perp \mid x \mid e\ e \\
 & & & \text{let } x = e \text{ in } e \\
\textbf{\textit{Basic Types}} & b & ::= & \text{int} \mid \text{bool} \\
\textbf{\textit{Types}} & \tau & ::= & \{v{:}b \mid e\} \mid x{:}\tau \to \tau \\
\textbf{\textit{Environment}} & \Gamma & ::= & \emptyset \mid x{:}\tau, \Gamma
\end{array}
$$

1.2 Operational Semantics

$$
\begin{array}{lll}
e_1\ e_2 \hookrightarrow e_1'\ e_2 & \text{if } e_1 \hookrightarrow e_1' & \lambda x.e\ e_x \hookrightarrow e\,[e_x/x] \\
c\ e \hookrightarrow c\ e' & \text{if } e \hookrightarrow e' & c\ v \hookrightarrow [|c|](v) \\
\text{let } x = e_x \text{ in } e \hookrightarrow e\,[e_x/x] & & e\ \hookrightarrow \perp
\end{array}
$$

NV:♣ $c([| \perp |]) = \perp$?♣

2 Undecidable System

2.1 Erasing

$$
\begin{aligned}
\lfloor \{v{:}b \mid e\} \rfloor &= b \\
\lfloor x{:}\tau_x \to \tau \rfloor &= \lfloor \tau_x \rfloor \to \lfloor \tau \rfloor
\end{aligned}
$$

$$
\begin{aligned}
\lfloor \emptyset \rfloor &= \emptyset \\
\lfloor x{:}\tau, \Gamma \rfloor &= x{:}\lfloor \tau \rfloor, \lfloor \Gamma \rfloor
\end{aligned}
$$

2.2 Substitutions

$$
\begin{aligned}
(\{v{:}b \mid e\})\,[e_y/y] &= \{v{:}b \mid e\,[e_y/y]\} \\
(x{:}\tau_x \to \tau)\,[e_y/y] &= x{:}(\tau_x\,[e_y/y]) \to (\tau\,[e_y/y])
\end{aligned}
$$

2.3 Interpretations

Definition 1 ($v^{\perp} \sqsubset v^{\perp}$)**.**

$$
\perp \sqsubset v \qquad\qquad\qquad v^{\perp} \sqsubseteq v^{\perp}
$$

$$
\begin{array}{lll}
[|\,\{v{:}b \mid e_v\}\,|] & = \{e \mid & \vdash e : b \wedge e \hookrightarrow^{\star} v \Rightarrow \mathit{lub}\{v_p^{\perp} \mid e_v\,[e/v] \hookrightarrow^{\star} v_p^{\perp}\} = \text{true}\} \\
[|x{:}\tau_x \to \tau|] & = \{e \mid & \vdash e : \lfloor \tau_x \rfloor \to \lfloor \tau \rfloor \wedge \forall e_x \in [|\tau_x|].\ e\ e_x \in [|\tau\,[e_x/x]\,|]\}
\end{array}
$$

2.4 Typing

NV:♣ The undecideble system is needed for preservation $e\ e \hookrightarrow e$♣ $\quad \Gamma \vdash e : \tau$

$$\frac{\Gamma \vdash e : \{v{:}b \mid e'\}}{\Gamma \vdash e : \{v{:}b \mid v =_b e\}}\ \text{T-Ex}$$

$$\frac{(x, \{v{:}b \mid e\}) \in \Gamma}{\Gamma \vdash x : \{v{:}b \mid v =_b x\}}\ \text{T-Var-Base} \qquad \frac{(x, \tau) \in \Gamma \quad \tau \neq (x, \{v{:}b \mid e\})}{\Gamma \vdash x : \tau}\ \text{T-Var}$$

$$\frac{}{\Gamma \vdash c : \mathrm{ty}(c)}\ \text{T-Const} \qquad \frac{\Gamma \vdash e : \tau' \quad \Gamma \vdash \tau' \preceq \tau \quad \Gamma \vdash \tau}{\Gamma \vdash e : \tau}\ \text{T-Sub}$$

$$\frac{\Gamma, x{:}\tau_x \vdash e : \tau \quad \Gamma \vdash \tau_x}{\Gamma \vdash \lambda x.e : (x{:}\tau_x \to \tau)}\ \text{T-Fun} \qquad \frac{\Gamma \vdash e_1 : (x{:}\tau_x \to \tau) \quad \Gamma \vdash e_2 : \tau_x}{\Gamma \vdash e_1\ e_2 : \tau\,[e_2/x]}\ \text{T-App}$$

$$\frac{\Gamma \vdash e_x : \tau_x \quad \Gamma, x{:}\tau_x \vdash e_2 : \tau \quad \Gamma \vdash \tau}{\Gamma \vdash \mathrm{let}\ x = e_x\ \mathrm{in}\ e : \tau}\ \text{T-Let} \qquad \frac{\Gamma \vdash \tau}{\Gamma \vdash \perp : \tau}\ \text{T-Bot}$$

$\Gamma \vdash \tau$

$$\frac{\lfloor\Gamma\rfloor, v{:}b \vdash_B e : \mathrm{bool}}{\Gamma \vdash \{v{:}b \mid e\}}\ \text{WF-Base} \qquad \frac{\Gamma \vdash \tau_x \quad \Gamma, x{:}\tau_x \vdash \tau}{\Gamma \vdash x{:}\tau_x \to \tau}\ \text{WF-Fun}$$

$\Gamma \vdash \tau \preceq \tau$

$$\frac{\Gamma, v : b \vdash e_1 \Rightarrow e_2}{\Gamma \vdash \{v{:}b \mid e_1\} \preceq \{v{:}b \mid e_2\}}\ \preceq\text{-Base} \qquad \frac{\Gamma \vdash \tau_x' \preceq \tau_x \quad \Gamma, x{:}\tau_x' \vdash \tau \preceq \tau'}{\Gamma \vdash x{:}\tau_x \to \tau \preceq x{:}\tau_x' \to \tau'}\ \preceq\text{-Fun}$$

$\Gamma \vdash e \Rightarrow e$

$$\frac{\forall \theta.\Gamma \vdash \theta \wedge \mathit{lub}\{v^\perp \mid \theta\ e_1 \hookrightarrow^\star v^\perp\} = \mathrm{true} \Rightarrow \mathit{lub}\{v^\perp \mid \theta\ e_2 \hookrightarrow^\star v^\perp\} = \mathrm{true}}{\Gamma \vdash e_1 \Rightarrow e_2}\ \Rightarrow\text{-Base}$$

$\vdash \Gamma$

$$\frac{\vdash \Gamma \quad \Gamma \vdash \tau}{\vdash x{:}\tau, \Gamma} \qquad \frac{}{\vdash \emptyset}$$

$\Gamma \vdash \theta$

$$\frac{\forall x \in \mathrm{Dom}(\Gamma).\theta(x) \in [|\theta\ \Gamma(x)|]}{\Gamma \vdash \theta}$$

2.5 Constants and Data Constructors

Definition 2. `crash` *is an untyped constant.*
For each constant $c \neq$ `crash`

1. $\emptyset \vdash c : ty(c)$ *and* $\vdash ty(c)$
2. *If* $ty(c) = x{:}\tau_x \to \tau$, *then for each* v $[|c|](v)$ *is defined and if* $\emptyset \vdash v : \tau_x$ *then* $\vdash [|c|](v) : \tau\,[v/x]$, *otherwise* $[|c|](v) =$ `crash`
3. *If* $ty(c) = \{v{:}b \mid e\}$, *then* $lub\{v \mid e\,[c/v] \hookrightarrow^\star v\} = true$ *and* $\forall c'\ c' \neq c.\neg(e\,[c'/v] \hookrightarrow^\star true)$

Moreover, for any base type b, $=_b$ is a constant and

- *For any expression e we have*

$$lub\{v \mid e =_b e \hookrightarrow^\star v\} = true$$

- *For any base type b*

$$ty(=_b) \equiv x{:}b \rightarrow y{:}b \rightarrow bool$$

2.6 Semantic Typing

$$\Gamma \vdash e \in \tau \doteq \forall \theta . \Gamma \vdash \theta \Rightarrow \theta\ e \in [|\theta\ \tau|]$$
$$\Gamma \vdash \tau_1 \subseteq \tau_2 \doteq \forall \theta . \Gamma \vdash \theta \Rightarrow [|\theta\ \tau_1|] \subseteq [|\theta\ \tau_2|]$$

Lemma 1. *.*

1. *If $\Gamma \vdash \tau_1 \preceq \tau_2$ then $\Gamma \vdash \tau_1 \subseteq \tau_2$*
2. *If $\Gamma \vdash e : \tau$ then $\Gamma \vdash e \in \tau$*

`TODO`

Lemma 2 (Substitution)**.** *If $\Gamma \vdash e_x \in \tau_x$ and $\vdash \Gamma, x{:}\tau_x, \Gamma'$, then*

1. *If $\Gamma, x{:}\tau_x, \Gamma' \vdash \tau_1 \preceq \tau_2$ then $\Gamma, [e_x/x]\,\Gamma' \vdash [e_x/x]\,\tau_1 \preceq [e_x/x]\,\tau_2$*
2. *If $\Gamma, x{:}\tau_x, \Gamma' \vdash e : \tau$ then $\Gamma, [e_x/x]\,\Gamma' \vdash [e_x/x]\,e : [e_x/x]\,\tau$*
3. *If $\Gamma, x{:}\tau_x, \Gamma' \vdash \tau$ then $\Gamma, [e_x/x]\,\Gamma' \vdash [e_x/x]\,\tau$*

`TODO`

2.7 Soundness

Lemma 3 (Preservation)**.** *If $\emptyset \vdash e : \tau$ and $e \hookrightarrow e'$ then $\emptyset \vdash e' : \tau$.*

`TODO`

Lemma 4 (Progress)**.** *If $\emptyset \vdash e : \tau$ and $e \neq v$ then there exists an e' such that $e \hookrightarrow e'$.*

`TODO`

3 Decidable System

Interpretations

Definition 3 (Interpretation)**.** $[|e|]$

$$[|x|] = x \qquad [|c|] = c \qquad [|e_1\ e_2|] = [|e_1|]([|e_2|])$$

$[|\Gamma|]$

$$[|\Gamma|] = \bigwedge \{ whnf\ x \Rightarrow [|e\,[x/v]\,|] \mid (x, \{v{:}b \mid e\}) \in \Gamma \}$$

Typing

$$\Gamma \vdash e : \tau$$

$$\frac{(x, \{v{:}b \mid e\}) \in \Gamma}{\Gamma \vdash x : \{v{:}b \mid v =_b x\}}\ \textsc{T-Var-Base} \qquad \frac{(x, \tau) \in \Gamma \qquad \tau \neq (x, \{v{:}b \mid e\})}{\Gamma \vdash x : \tau}\ \textsc{T-Var}$$

$$\frac{}{\Gamma \vdash c : \mathrm{ty}(c)}\ \textsc{T-Const} \qquad \frac{\Gamma \vdash e : \tau' \qquad \Gamma \vdash \tau' \preceq \tau \qquad \Gamma \vdash \tau}{\Gamma \vdash e : \tau}\ \textsc{T-Sub}$$

$$\frac{\Gamma, x{:}\tau_x \vdash e : \tau \qquad \Gamma \vdash \tau_x}{\Gamma \vdash \lambda x.e : (x{:}\tau_x \to \tau)}\ \textsc{T-Fun} \qquad \frac{\Gamma \vdash e_1 : (x{:}\tau_x \to \tau) \qquad \Gamma \vdash y : \tau_x}{\Gamma \vdash e_1\ y : \tau\,[y/x]}\ \textsc{T-App}$$

$$\frac{\Gamma \vdash e_x : \tau_x \qquad \Gamma, x{:}\tau_x \vdash e_2 : \tau \qquad \Gamma \vdash \tau}{\Gamma \vdash \text{let } x = e_x \text{ in } e : \tau}\ \textsc{T-Let} \qquad \frac{\Gamma \vdash \tau}{\Gamma \vdash \bot : \tau}\ \textsc{T-Bot}$$

$$\Gamma \vdash \tau$$

$$\frac{\lfloor \Gamma \rfloor, v{:}b \vdash_B e : \mathrm{bool} \qquad \lfloor \Gamma \rfloor, v{:}b \vdash_{\mathrm{pure}} e}{\Gamma \vdash \{v{:}b \mid e\}}\ \textsc{WF-Base} \qquad \frac{\Gamma \vdash \tau_x \qquad \Gamma, x{:}\tau_x \vdash \tau}{\Gamma \vdash x{:}\tau_x \to \tau}\ \textsc{WF-Fun}$$

$$\Gamma \vdash \tau \preceq \tau$$

$$\frac{\Gamma, v : b \vdash e_1 \Rightarrow e_2}{\Gamma \vdash \{v{:}b \mid e_1\} \preceq \{v{:}b \mid e_2\}}\ \preceq\textsc{-Base} \qquad \frac{\Gamma \vdash \tau'_x \preceq \tau_x \qquad \Gamma, x{:}\tau'_x \vdash \tau \preceq \tau'}{\Gamma \vdash x{:}\tau_x \to \tau \preceq x{:}\tau'_x \to \tau'}\ \preceq\textsc{-Fun}$$

$$\Gamma \vdash e \Rightarrow e$$

$$\frac{[|\Gamma|] \Rightarrow [|e_1|] \Rightarrow [|e_2|]}{\Gamma \vdash e_1 \Rightarrow e_2}\ \Rightarrow\textsc{-Base}$$

$$\vdash \Gamma$$

$$\frac{\vdash \Gamma \qquad \Gamma \vdash \tau}{\vdash x{:}\tau, \Gamma} \qquad \frac{}{\vdash \emptyset}$$

$$\Gamma \vdash \theta$$

$$\frac{\forall x \in \mathrm{Dom}(\Gamma). \theta(x) \in [|\theta\ \Gamma(x)|]}{\Gamma \vdash \theta}$$

3.1 TODO

Now, we should prove that

- each constant we define respects the definition of constants.
- the decideble judjement implies the undecidable