Mechanizing Refinement Types

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Practical checkers based on refinement types use the combination of implicit semantic subtyping and parametric polymorphism to simplify the specification and automate the verification of sophisticated properties of programs. However, a formal meta-theoretic accounting of the *soundness* of refinement type systems using this combination has proved elusive. We present λ_{RF}, a core refinement calculus that combines semantic subtyping and parametric polymorphism. We develop a metatheory for this calculus and prove soundness of the type system. Finally, we give two full mechanizations of our metatheory. First, we introduce *data propositions*, a novel feature that enables encoding derivation trees for inductively defined judgments as refined data types, and use them to show that LiquidHaskell's refinement types can be used *for* mechanization. Second, we mechanize our results in Coq, which comes with stronger soundness guarantees than LiquidHaskell, thereby laying the foundations for mechanizing the metatheory *of* LiquidHaskell.

1 INTRODUCTION

Refinements constrain types with logical predicates to specify new concepts. For example, the refinement type `Pos` $\doteq$ `Int`$\{v : 0 < v\}$ describes *positive* integers and `Nat` $\doteq$ `Int`$\{v : 0 \leq v\}$ specifies natural numbers. Refinements on types have been successfully used to define sophisticated concepts (*e.g.* secrecy [Fournet et al. 2011], resource constraints [Knoth et al. 2020], security policies [Lehmann et al. 2021]) that can then be verified in programs developed in various programming languages like Haskell [Vazou et al. 2014b], Scala [Hamza et al. 2019], and Racket [Kent et al. 2016].

The success of refinement types relies on the combination of two essential features. First, *implicit* semantic subtyping uses semantic (SMT-based) reasoning to automatically convert the types of expressions without troubling the programmer for explicit type casts. For example, consider a positive expression e : `Pos` and a function expecting natural numbers f : `Nat` $\rightarrow$ `Int`. To type check the application $f\ e$, the refinement type system will implicitly convert the type of e from `Pos` to `Nat`, because $0 < v \implies 0 \leq v$ semantically holds. Importantly, refinement types propagate semantic subtyping inside type constructors to, for example, treat function arguments in a contravariant manner. Second, *parametric polymorphism* allows the propagation of the refined types through polymorphic function interfaces, without the need for extra reasoning. As a trivial example, once we have established that e is positive, parametric polymorphism should let us conclude that $g\ e$: `Pos` if, for example, g is the identity function $g : a \rightarrow a$.

As is often the case with useful ideas, the engineering of practical tools has galloped far ahead of the development of the meta-theoretical foundations for refinements with subtyping and polymorphism. In fact, semantic subtyping is very tricky as it is mutually defined with typing, leading to metatheoretic proofs with circular dependencies. Unsurprisingly, the addition of polymorphism poses further challenges. As Sekiyama et al. [2017] observe, a naïve definition of type instantiation can lose potentially contradicting refinements leading to unsoundness. Existing formalizations of refinement types drop semantic subtyping [Hamza et al. 2019; Sekiyama et al. 2017] or polymorphism [Flanagan 2006; Swamy et al. 2016], or have problematic metatheory [Belo et al. 2011].

In this paper we formalize λ_{RF}, a core calculus with a refinement type system that combines semantic subtyping with polymorphism. Our development has four concrete contributions.

1. Reconciliation Our first contribution is a language that combines refinements and polymorphism in a way that ensures the metatheory remains sound without sacrificing the expressiveness needed for practical verification. To this end, λ_{RF} introduces a kind system that distinguishes the type variables that can be soundly refined (without the risk of losing refinements at instantiation) from the rest, which are then left unrefined. In addition our design includes a form of existential

typing [Knowles and Flanagan 2009b] which is essential to *synthesize* the types – in the sense of bidirectional typing – for applications and let-binders in a compositional manner (§ 3, 4).

2. Foundation Our second contribution is to establish the foundations of λ_{RF} by proving soundness, which says that if e has a type, then either e is a value or it can step to another term of the same type. The combination of semantic subtyping, polymorphism, and existentials makes the soundness proof challenging with circular dependencies that do not arise in standard (unrefined) calculi. To ease the presentation and tease out the essential ingredients of the proof we stage the metatheory. First, we review an unrefined *base* language λ_F, a classic System F [Pierce 2002] with primitive `Int` and `Bool` types (§ 5). Next, we show how refinements (kinds, subtyping, and existentials) must be accounted for to establish the soundness of λ_{RF} (§ 6).

3. Reification Our third contribution is to introduce *data propositions*, a novel feature that enables the encoding of derivation trees for inductively defined judgments as refined data types, by first reifying the propositions and evidence as plain Haskell data, and then using refinements to connect the two. Hence, data propositions let us write plain Haskell functions over refined data to provide explicit, constructive proofs (§ 7). Without data propositions reasoning about potentially non-terminating computations was not possible in LiquidHaskell, thereby precluding even simple meta-theoretic developments such as the soundness of λ_F let alone λ_{RF}.

4. Mechanization Our final contribution is to fully mechanize the metatheory of λ_{RF} *twice*: using LiquidHaskell and Coq. We formalized λ_{RF} in LiquidHaskell (§ 8) to evaluate the feasibility of such substantial meta-theoretical formalizations. Our proof is non-trivial, requiring 9,400 lines of code, 30 minutes to verify, and various modifications in the internals of LiquidHaskell. We translated the same proof to Coq (§ 9) to compare the two alternatives. The Coq development is slightly shorter (about 7,800 lines), much faster (about 30 seconds to verify), but more difficult to manipulate various partial and mutual recursive definitions of the formalization. Finally, Coq comes with stronger foundational soundness guarantees than LiquidHaskell. While the metatheory of Coq is well studied, λ_{RF} lays the foundation for the mechanized metatheory of LiquidHaskell.

2 REFINEMENT TYPES

We start by an informal overview of the refined core calculus λ_{RF}, which we later formally presented (§ 3.1) and prove sound (§ 6). Concretely, we present the goals of refinement types (§ 2.1) and how they are achieved via the three essential features of semantic subtyping, existential types, and polymorphism (§ 2.2). We explain how the typing judgements are designed to accommodate these features (§ 2.3) and how we addressed the challenged these features impose in the mechanization of the soundness proof (§ 2.4).

2.1 The goal of Refinement Types

Refinement types refine the types of an existing programming language with logical predicates to define abstractions not expressible by the underlying type system. These abstractions are used to ensure both 1) static error detection and 2) functional correctness.

Static Error Detection. Figure 1 presents the interface of a bound size array that is encoded in the core calculus λ_{RF} as a function. The function `new n x` returns an array that contains `x` when indexed with an integer between `0` and `n` and otherwise throws an "out of bounds" error. To statically ensure that this error will never occur, `new` returns the refined array `ArrayN a n`, *i.e.* a function whose domain is restricted to integers less than `n`. The `set` and `get` operators manipulate the refined arrays on the index `i:{Nat | i < n}`, *i.e.* refined to be in-bounds of the array.

With this refined interface, out-of-bounds indexing is statically ruled out:

```
type Array a = Int → a
{-@ type ArrayN a N = {i:Nat | i < N} → a @-}

{-@ new :: n:Nat → a → ArrayN a n @-}
new :: Int → a → Array a
new n x = \i → if 0 ≤ i && i < n then x else error "Out of Bounds"

{-@ set :: n:Nat → i:{Nat | i < n} → a → ArrayN a n → ArrayN a n @-}
set :: Int → Int → a → Array a → Array a
set n i x a = \j → if i == j then x else a j

{-@ get :: n:Nat → i:{Nat | i < n} → ArrayN a n → a @-}
get :: Int → Int → Array a → a
get n i a = a i
```

Fig. 1. Array Refined for save indexing

```
{-@ array10 :: ArrayN Int 10 @-}
array10 = new 10 0

good = get 10 4 array10 -- OK
bad = get 10 42 array10 -- Refinement Type Error
```

Functional Correctness. Refinement types are also used to ensure that the program has the intended behavior. To achieve this, we use uninterpreted functions to *specify* behaviors and rely on the type system to propagate them. For example, below using the uninterpreted function `isPrime` we *specify* that some integers behave as primes:

```
uninterpreted isPrime :: Int → Bool
{-@ type Prime = {v:Int | isPrime v} @-}
```

Refinement types are not the best tool to verify complex properties such as the generation of a primality certificate. Yet, assuming that a function checks primality, they can easily propagate this behavior:

```
assume checkPrime :: x:Int → {v:Bool | v ⇔isPrime x}

nextPrime :: Nat → Prime
nextPrime x = if checkPrime x then x else nextPrime (x+1)
```

The case sensitivity of refinement types (Rule T-If of Fig. 7) ensures that the function `nextPrime` returns only values that pass check primality check.

todo add rule

Primes Array Example. As a bigger example, we define a function that ensures both static error detection and functional correctness. Concretely, we define the function `primes n` that generates an array with the first `n` prime numbers:

```
{-@ primes :: n:Nat → ArrayN Prime n @-}
primes n = go 1 0 (new n (nextPrime 1))
  where
```

```
    go i p acc = if i < n then let p' = nextPrime (p+1)
                               in go (i+1) p' (set n i p' acc)
                          else acc
```

Since `primes` typechecks under the safe array interface of Fig. 1, no out-of-bounds errors will occur. At the same time, all elements of the array are `set` by a result `nextPrime` and thus `primes` returns an array of prime numbers.

Typechecking of `primes` requires no user annotations, even the type of the `go` local function is inferred! Next, we see the three essential features of refinement types that make this possible.

2.2 The essence of Refinement Types

The practicality of refinement types is due to the combination of three essential features:

(1) ***Semantic Subtyping:*** The user does not need to provide any explicit type casts, because subtyping is implicit and semantic. For example, to type check `get 10 4 array10` (from § 2.1), the type of `4 :: {v:Int | v == 4}` is implicitly converted to `{v:Int | 0 ≤ v < 10}`.

(2) ***Decidability:*** The semantic casts are reduced to logical implications that are automatically checked by an SMT solver. Refinement types are designed to generate decidable logical implications, thus ensure predictable and fast SMT-automated verification and also permit type inference [Rondon et al. 2008] that (as illustrated in § 2.1) makes verification practical.

(3) ***Polymorphism:*** Polymorphism on refinement types permits instantiation of type variables with any refined type. For example, the same array interface can be used to describe prime numbers, functions with integer domains, and any other concept that can be encoded as a refinement type.

Next, we present how refinement types are designed to ensure these features.

2.3 The design of Refinement Types

The core calculus λ_{RF} has four static judgements that relate expressions (e), types (t), kinds (k), predicates (p), and environments (Γ): typing ($\Gamma \vdash e : t$), subtyping ($\Gamma \vdash t_1 \preceq t_2$), well-formedness ($\Gamma \vdash_w t : k$), and implication checking ($\Gamma \vdash p_1 \Rightarrow p_2$). In § 4 we define the judgements in detail. Here, we present the design decisions that ensure the three essential features of refinement types.

2.3.1 Semantic Subtyping. Refinement types rely on implicit semantic subtyping, that is, type conversion (from subtypes) happens without any explicit casts and is checked semantically via logical validity. For example, in the application `get 10 4 array10` (of Fig. 1), the type of `4` was implicitly converted. To see how, consider an environment Γ that contains the array interface. Let $\Gamma \subseteq \{\texttt{get} : n : \texttt{Int} \rightarrow i : \texttt{Int}\{v : v < n\} \rightarrow \texttt{ArrayN}\ a\ n \rightarrow a\}$ (for space here, we ignore the requirement that i and n are natural numbers and, as in Fig. 1, we use `ArrayN` a n as shorthand for $\texttt{Int}\{v : v < n\} \rightarrow a$). The application (`get` 10) 4 will type check as below, using the T-Sub rule to implicitly convert the type of the argument and the S-Base rule to check that 4 is a valid index by checking the validity of the formula $\forall v.\ v = 4 \Rightarrow v < 10$.

$$\dfrac{\dfrac{\ldots}{\Gamma \vdash \texttt{get}\ 10 : \texttt{Int}\{v : v < 10\} \rightarrow \ldots} \quad \dfrac{\dfrac{}{\Gamma \vdash 4 : \texttt{Int}\{v : v = 4\}} \quad \dfrac{\forall v. v = 4 \Rightarrow v < 10}{\Gamma \vdash \texttt{Int}\{v : v = 4\} \preceq \texttt{Int}\{v : v < 10\}}\ \text{S-Base}}{\Gamma \vdash 4 : \texttt{Int}\{v : v < 10\}}\ \text{T-Sub}}{\Gamma \vdash \texttt{get}\ 10\ 4 : \texttt{ArrayN}\ a\ 10 \rightarrow a}$$

Importantly, most refinement type systems use type-constructor directed rules to destruct subtyping obligations into basic (semantic) implications. For example, in Fig. 8 the rule S-Fun states that functions are covariant on the result and contravariant on the arguments. Thus, a refinement

type system can, without any annotations or casts, decide that a_{20} : `ArrayN` a 20 is a suitable argument for the higher order function `get` 10 4 : `ArrayN` a 10 $\longrightarrow$ a.

2.3.2 Decidability. As illustrated in the previous type derivation, refinement type checking essentially generates a set of verification conditions (VCs) whose validity implies type safety. Importantly, the refinement type checking rules are designed to generate VCs in the logical language used by the user-provided specifications. In general, let $\mathcal{L}$ be a logical language that contains equality and conjunction. If all the user-specified predicates belong to $\mathcal{L}$, then the VCs will be in $\mathcal{L}$ as well. In practice (*e.g.* in Liquid Haskell [Vazou et al. 2014a] and Flux [Nico Lehmann 2023]), $\mathcal{L}$ is the decidable logic of booleans, linear arithmetic, and uninterpreted functions.

To achieve this logical-language preservation, special care is taken in type checking function declarations and application.

Function Declarations. Function declarations are checked using the refinement type rule for let bindings (Rule T-Let) (We note that recursive definitions are encoded using a fixpoint operator and do not require a special rule.):

$$\frac{\Gamma \vdash e_f : t_f \qquad \Gamma \vdash_w t : k \qquad f{:}t_f, \Gamma \vdash e : t}{\Gamma \vdash \texttt{let}\ f = e_f\ \texttt{in}\ e : t}\ \textsc{T-Let}$$

The rule infers the type t_f of the function, that could be user-annotated (*e.g.* e_f could be $e'_f : t_f$).

Importantly, the body e is checked without knowledge of the definition of f. The exact encoding of the body of the function definitions (for example, as done in Dafny [Leino 2010] or Prusti [Astrauskas et al. 2022]) requires the use of $\forall$-quantifiers in the SMT solver, thus potentially leading to undecidability. Instead, refinement types only use the refinement type specifications of functions, providing a fast but incomplete verification technique. For example, given only specifications of `get` and `set`, and not their exact definitions, it is not possible to show that `get` after `set` returns the value that was set.

```
{-@ getSet :: n:Int → i:{Nat|i<n} → x:a → ArrayN a n → {v:a|x == v} @-}
getSet n i x a = get n i (set n i x a) -- Refinement Type Error
```

Function Application. For decidable type checking, refinement types use an existential type [Knowles and Flanagan 2009a] to check dependent function application, *i.e.* the TApp-Exists rule below, instead of the standard type-theoretic TApp-Exact rule.

this is mostly as before, can we instead use an array related example?

$$\frac{\Gamma \vdash f : x{:}t_x \longrightarrow t \qquad \Gamma \vdash e : t_x}{\Gamma \vdash f\ e : t[e/x]}\ \textsc{TApp-Exact} \qquad \frac{\Gamma \vdash f : x{:}t_x \longrightarrow t \qquad \Gamma \vdash e : t_x}{\Gamma \vdash f\ e : \exists\, x{:}t_x.\, t}\ \textsc{TApp-Exists}$$

To understand the difference, consider some expression e of type `Pos` and the identity function f

$$e : \texttt{Pos} \qquad\qquad f : x{:}\texttt{Int} \longrightarrow \texttt{Int}\{v : v = x\}$$

The application $f\ e$ is typed as `Int`$\{v : v = e\}$ with the TApp-Exact rule, which has two problems. First, the information that e is positive is lost. To regain this information the system needs to re-analyze the expression e breaking compositional reasoning. Second, the arbitrary expression e enters the refinement logic potentially breaking decidability. Using the TApp-Exists rule, both of these problems are addressed. Typing first uses subtyping on f to track the actual type of the argument, thus weakening the type of f to f : x:`Pos` $\longrightarrow$ `Int`$\{v : v = x\}$. With this, the type of $f\ e$ becomes $\exists\, x{:}\texttt{Pos}.\, \texttt{Int}\{v : v = x\}$ preserving the information that the application argument is positive, while the variable x cannot break any carefully crafted decidability guarantees.

Knowles and Flanagan [2009a] introduce the existential application rule and show that it preserves the decidability and completeness of the refinement type system. An alternative approach for

decidable and compositional type checking is to ensure that all the application arguments are variables by ANF transforming the original program [Flanagan et al. 1993]. ANF is more amicable to *implementation* as it does not require the definition of one more type form. However, ANF is more problematic for the *metatheory*, as ANF is not preserved by evaluation. Additionally, existentials let us *synthesize* types for let-binders in a bidirectional style: when typing `let` $x = e_1$ `in` e_2, the existential lets us eliminate x from the type synthesized for e_2, yielding a precise, algorithmic system [Cosman and Jhala 2017]. Thus, we choose to use existential types in λ_{RF}.

Same: this is mostly as before, can we instead use an array related example?

2.3.3 *Polymorphism.*

Polymorphism is a precious type abstraction [Wadler 1989], but combined with refinements, it can lead to imprecise or, worse, unsound systems. As an example, below we present the function `max` with four potential type signatures.

	Definition	`max`	=	$\lambda x\, y.$`if` $x < y$ `then` y `else` x
Attempt 1:	*Monomorphism*	`max`	::	x:`Int` $\rightarrow$ y:`Int` $\rightarrow$ `Int`$\{v : x \leq v \wedge y \leq v\}$
Attempt 2:	*Unrefined Polymorphism*	`max`	::	$x{:}\alpha \rightarrow y{:}\alpha \rightarrow \alpha$
Attempt 3:	*Refined Polymorphism*	`max`	::	$x{:}\alpha \rightarrow y{:}\alpha \rightarrow \alpha\{v : x \leq v \wedge y \leq v\}$
λ_{RF}:	*Kinded Polymorphism*	`max`	::	$\forall\, \alpha{:}B.\, x{:}\alpha \rightarrow y{:}\alpha \rightarrow \alpha\{v : x \leq v \wedge y \leq v\}$

As a first attempt, we give `max` a monomorphic type, stating that the result of `max` is an integer greater than or equal to each of its arguments. This type is insufficient because it forgets any information known for `max`'s arguments. For example, if both arguments are positive, the system cannot decide that `max x y` is also positive. To preserve the argument information we give `max` a polymorphic type, as a second attempt. Now the system can deduce that `max x y` is positive, but forgets that it is also greater than or equal to both `x` and `y`. In a third attempt, we naively combine the benefits of polymorphism with refinements to give `max` a very precise type that is sufficient to propagate the arguments' properties (positivity) and `max` behavior (inequality).

Unfortunately, refinements on arbitrary type variables are dangerous for two reasons. First, the type of `max` implies that the system allows comparison between any values (including functions). Second, if refinements on type variables are allowed, then, for soundness [Belo et al. 2011], all the types that substitute variables should be refined. For example, as detailed in §6 of [Jhala and Vazou 2021], if a type variable is refined with `false` (that is, $\alpha\{v : \texttt{false}\}$) and gets instantiated with an unrefined function type ($x{:}t_x \rightarrow t$), then the `false` refinement is lost and the system becomes unsound.

Base Kind when Refined To preserve the benefits of refinements on type variables, without the complications of refining function types, we introduce a kind system that separates the type variables that can be refined from the ones that cannot. To do so, we extend the standard well-formedness rule of refinement types to also perform kind checking ($\Gamma \vdash_w t : k$). Variables with the base kind B can be refined, compared, and only substituted by base, refined types. The other type variables have kind $\star$ and can only be trivially refined with `true`. With this kind system, we have a simple and convenient way to encode comparable values and we can give `max` a polymorphic and precise type that naturally rejects non-comparable (*e.g.* function) arguments. This simple kind system could be further stratified, *i.e.* if some base types did not support comparison, and it could be implemented via typeclass constraints, if our system contained data types.

2.4 The soundness of Refinement Types

The soundness of a type system gives meaning to the static typing judgement $\Gamma \vdash e : t$. In this work we mechanized two soundness theorems for refinement types that give meaning to typing using type safety and denotational semantics.

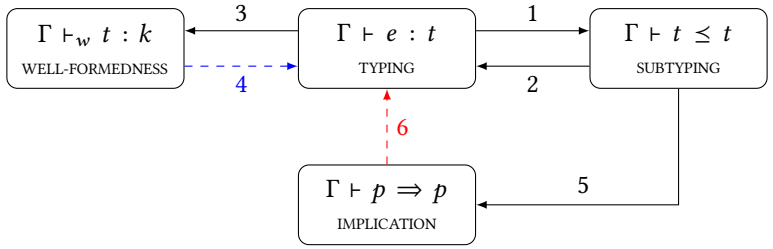

Fig. 2. Dependencies of Typing Judgements in Refinement Types. (Dashed lines do not exist in our formalism.)

Type Safety ensures that well-typed programs do not get stuck at runtime. It says that if an expression has a type ($\emptyset \vdash e : t$) and evaluates to another expression ($e \hookrightarrow^* e'$), then either evaluation reached a value or it can take another step ($e' \hookrightarrow e''$). In λ_{RF}, we use the primitive `error` to denote program errors (such as out-of-bounds indexing of Fig. 1). The `error` primitive neither is a value nor takes a step. Thus, if an expression type checks, via type safety, we know that `error` will not be reached at runtime. Theorem **??** formally defines type safety and it is proved via the preservation and progress lemmas.

Type safety ensures that programs will not get stuck, but does not ensure that they satisfy their functional specifications. This is ensured by the second soundness theorem.

Denotational Soundness states that if an expression has a type ($\emptyset \vdash e : t$), then it belongs in the denotations of this type ($e \in [\![t]\!]$). For example, the denotation of the type `{i:Nat | i ≤ 42}` is the set of integers between between 0 and 42. In § **??** we inductively define the denotations of each type and Theorem **??** formally encodes denotational soundness.

But how to we know that evaluation preserves semantics? Revisit after we add the denotations

The challenges. This work, for the first time, mechanizes the soundness of refinement types with semantic subtyping, existential types, and polymorphism. This mechanization was challenging for three main reasons:

1. Circularities. Figure 2 presents the dependencies of the four typing judgements in refinement types. As we saw in the example of § 2.3.1 (and can be confirmed in the rules defined in § Section 4), typing depends on subtyping (arrow 1) which in turn depends on implication checking (arrow 5). Subtyping depends on typing (arrow 2; because of rule S-Wit of Fig. 8), so typing and subtyping have a circular dependency we cannot break.

Typing also depends on well-formedness (arrow 3) that checks that types, especially the ones inferred by the system are well-formed: all the variables appearing in the refinements are bound in the type environment and refinements are of boolean type. To check the type of the refinements the system could use typing thus introducing one more dependency (arrow 4) and yet another circle. We break this dependency by using an unrefined core calculus (system λ_F) to check that refinements are typed as booleans. The system λ_F is obtained by removing the refinements and required extensions from the core calculus λ_{RF}.

The final potential circle is introduced when implication depends on typing (arrow 6). In § **??** we define implication via type denotations, but as observed by Greenberg [2013], in this case, special care should be taken so that the system is monotonic and thus well-defined. To avoid this dangerous circularity we again use λ_F (and not its refinement λ_{RF}) to define denotations and thus implication.

2. Encoding of Implication The second mechanization challenge was the encoding of implication. In the bibliography of refinement types, implication has been defined in three ways:

(1) Using denotations and thus operational semantics [Flanagan 2006; Vazou et al. 2018]. This encoding is more convenient when proving the soundness of the system, since implication

$$
\begin{array}{rrcl}
\textbf{\textit{Primitives}} & c & ::= & \texttt{true} \mid \texttt{false} \mid 0, 1, 2, \ldots \mid \wedge, \neg \mid \leq, c\leq, =, c= \\
\textbf{\textit{Values}} & v & ::= & c \mid x, y, \ldots \mid \lambda x.e \mid \Lambda\alpha{:}k.e \\
\textbf{\textit{Terms}} & e & ::= & v \mid e_1\ e_2 \mid e[t] \mid \texttt{let}\ x = e_1\ \texttt{in}\ e_2 \mid e : t
\end{array}
$$

Fig. 3. Syntax of Primitives, Values, and Expressions.

and thus subtyping and typing, directly connect with operational semantics, and thus the proof of soundness is more direct. However, the implementation of this encoding of implication is not realistic, since it is not decidable.

(2) Using logical implication [Gordon and Fournet 2010; Rondon et al. 2008]. The encoding of the implication as a logical implication is the closest to the implementation of a refinement system, where an SMT is used to check logical implications. Yet, to prove soundness, a claim should be made that logical implication correctly approximates the runtime semantics of the system (*i.e.* that $[\![\Gamma]\!] \Rightarrow p \Rightarrow q$ approximates $\Gamma \vdash p \Rightarrow q$) which is not known how to mechanize.

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(3) By axiomatization [Lehmann and Tanter 2016]. A final approach is to leave the implication uninterpreted and axiomatize it with all the properties required to prove soundness. This approach is the easiest to mechanize, but it is dangerous, since in the past the axioms assumed for implication were "flawed", *i.e.* inconsistent, thus soundness was proved "proved with flawed premises" (Table 1 of [Sekiyama et al. 2017]).

Our mechanization follows a combination of the first and the third approach. We axiomatize the interface of implication (via Requirement 2 encoded as an inductive data type in the Coq mechanization) to distill the exact properties required by the soundness proof. Then, we provide an instance of the implication interface using the denotational semantics of the system. This encoding has two major benefits. First, the denotational implementation ensures that our axioms are consistent. Second, the development of the interface leaves room for the implementation of alternative implication "oracles", *e.g.* closer to SMT solvers. Even though we did not explore this alternative implementation, in § **??** we present how logical implications are derived from the implication judgement.

I think this confused many reviews and plan to add it

3. Proof Complexity. All the three essential features of refinement types add complexity to the mechanization of the soundness proof. Polymorphism requires the extension of well-formedness to kind checking. Semantic subtyping makes type checking non-deterministic (thus inversion is not trivial **??**) and dependent to subtyping. In turn, the existential types required for decidability make subtyping dependent to type checking. Due to this mutual dependency, all the standard metatheoretical lemmas (Substitution, Weakening, Narrowing, *etc.*) require versions for both typing and subtyping, which are proved by mutual induction. Thus, with the combination of the three essential for refinement types features and once all the potential sources of unsoundness we discussed are carefully eliminated, the mechanization of the proof is unsurprising, yet strenuous.

3 LANGUAGE

To cut the circularities in the metatheory, we formalize refinements using two calculi. The first is the *base* language λ_F: a classic System F [Pierce 2002] with call-by-value semantics extended with primitive `Int` and `Bool` types and operations. The second is the *refined* language λ_{RF} which extends λ_F with refinements. By using the first calculus to express the typing judgments for our refinements, we avoid making the well-formedness (in rule WF-Refn in § 4.1) and typing judgments be mutually dependent. We use the grey highlights for the extensions to λ_F required for λ_{RF}.

Kinds	k	$::=$	$B \mid \star$	*base and star kind*
Predicates	p	$::=$	$\{e \mid \exists \Gamma.\, \Gamma \vdash_F e : \texttt{Bool}\}$	*boolean-typed terms*
Base Types	b	$::=$	$\texttt{Bool} \mid \texttt{Int} \mid \alpha$	*booleans, integers, and type variables*
Types	t	$::=$	$b\,\{v : p\}$	*refined base type*
		$\mid$	$x{:}t_x \to t$	*function type*
		$\mid$	$\exists\, x{:}t_x.\, t$	*existential type*
		$\mid$	$\forall\, \alpha{:}k.\, t$	*polymorphic type*
Environments	Γ	$::=$	$\emptyset \mid \Gamma, x{:}t \mid \Gamma, \alpha{:}k$	*variable and type bindings*

Fig. 4. Syntax of Types. The grey boxes are the extensions to λ_F needed by λ_{RF}. We use τ for λ_F-only types.

3.1 Syntax

We start by describing the syntax of terms and types in the two calculi.

Constants, Values and Terms Fig. 3 summarizes the syntax of terms in both calculi. The *primitives* c include `Int` and `Bool` constants, boolean operations, the polymorphic comparison and equality, and their curried versions. *Values* v are constants, binders and λ- and type- abstractions. Finally, the *terms* e comprise values, value- and type- applications, let-binders and annotated expressions. The types in annotations are, potentially wrong, specifications writen by the user and checked by the type checker.

Kinds & Types Fig. 4 shows the syntax of the types, with the grey boxes indicating the extensions to λ_F required by λ_{RF}. In λ_{RF}, only base types `Bool` and `Int` can be refined: we do not permit refinements for functions and polymorphic types. λ_{RF} enforces this restriction using two kinds which denote types that may (B) or may not ($\star$) be refined. The (unrefined) *base* types b comprise `Int`, `Bool`, and type variables α. The simplest type is of the form $b\{v : p\}$ comprising a base type b and a *refinement* that restricts b to the subset of values v that satisfy p *i.e.* for which p evaluates to `true`. We use refined base types to build up dependent function types (where the input parameter x can appear in the output type's refinement), existential and polymorphic types. In the sequel, we write b to abbreviate $b\{v : \texttt{true}\}$ and call types refined with only `true` "trivially refined" types.

Refinement Erasure The reduction semantics of our polymorphic primitives are defined using an *erasure* function that returns the unrefined, λ_F version of a refined λ_{RF} type:

$$\lfloor b\{v : p\} \rfloor \doteq b, \quad \lfloor x{:}t_x \to t \rfloor \doteq \lfloor t_x \rfloor \to \lfloor t \rfloor, \quad \lfloor \exists\, x{:}t_x.\, t \rfloor \doteq \lfloor t \rfloor, \quad \text{and} \quad \lfloor \forall\, \alpha{:}k.\, t \rfloor \doteq \forall\, \alpha{:}k.\, \lfloor t \rfloor$$

Environments Fig. 4 describes the syntax of typing environments Γ which contain both term variables bound to types and type variables bound to kinds. These variables may appear in types bound later in the environment. In our formalism, environments grow from right to left.

Note on Variable Representation Our metatheory requires that all variables bound in the environment are distinct. Our mechanization enforces this invariant via the locally nameless representation [Aydemir et al. 2005]: free and bound variables are distinct objects in the syntax, as are type and term variables. All free variables have unique names which never conflict with bound variables represented as de Bruijn indices. This eliminates the possibility of capture in substitution and the need to perform alpha-renaming during substitution. The locally nameless representation avoids technical manipulations such as index shifting by using names instead of indices for free variables (we discuss alternatives in § 10). To simplify the presentation of the syntax and rules, we use names for bound variables to make the dependent nature of the function arrow clear.

Operational Semantics $\boxed{e \hookrightarrow e'}$

$$\frac{}{c\ v \hookrightarrow \delta(c,v)}\text{E-Prim} \quad \frac{}{c[t] \hookrightarrow \delta_T(c,\lfloor t \rfloor)}\text{E-TPrim} \quad \frac{e \hookrightarrow e'}{e : t \hookrightarrow e' : t}\text{E-PAnn} \quad \frac{}{v : t \hookrightarrow v}\text{E-Ann}$$

$$\frac{e \hookrightarrow e'}{e\ e_1 \hookrightarrow e'\ e_1}\text{E-PLApp} \quad \frac{e \hookrightarrow e'}{v\ e \hookrightarrow v\ e'}\text{E-PRApp} \quad \frac{}{(\lambda x.e)\ v \hookrightarrow e[v/x]}\text{E-App} \quad \frac{}{(\Lambda\alpha{:}k.e)[t] \hookrightarrow e[t/\alpha]}\text{E-TApp}$$

$$\frac{e \hookrightarrow e'}{e[t] \hookrightarrow e'[t]}\text{E-PTApp} \quad \frac{e_x \hookrightarrow e_x'}{\texttt{let}\ x = e_x\ \texttt{in}\ e \hookrightarrow \texttt{let}\ x = e_x'\ \texttt{in}\ e}\text{E-PLet} \quad \frac{}{\texttt{let}\ x = v\ \texttt{in}\ e \hookrightarrow e[v/x]}\text{E-Let}$$

$$\begin{array}{rcl@{\qquad}rcl}
\beta\{x:p\}[t_\alpha/\alpha] & \doteq & \beta\{x:p[t_\alpha/\alpha]\}, \alpha \neq \beta & \mathsf{refine}(\alpha\{z:q\}, p, x) & \doteq & \alpha\{z:p[z/x] \wedge q\} \\
(x{:}t_x \to t)[t_\alpha/\alpha] & \doteq & x{:}(t_x[t_\alpha/\alpha]) \to t[t_\alpha/\alpha] & \mathsf{refine}(\exists\, z{:}t_z.\, t, p, x) & \doteq & \exists\, z{:}t_z.\, \mathsf{refine}(t, p, x) \\
(\exists\, x{:}t_x.\, t)[t_\alpha/\alpha] & \doteq & \exists\, x{:}(t_x[t_\alpha/\alpha]).\, t[t_\alpha/\alpha] & \mathsf{refine}(x{:}t_x \to t, _, _) & \doteq & x{:}t_x \to t \\
(\forall\, \beta{:}k.\, t)[t_\alpha/\alpha] & \doteq & \forall\, \beta{:}k.\, t[t_\alpha/\alpha] & \mathsf{refine}(\forall\, \alpha{:}k.\, t, _, _) & \doteq & \forall\, \alpha{:}k.\, t \\
\alpha\{x:p\}[t_\alpha/\alpha] & \doteq & \mathsf{refine}(t_\alpha, p[t_\alpha/\alpha], x) & & &
\end{array}$$

Fig. 5. The small-step semantics and type substitution.

3.2 Dynamic Semantics

Fig. 5 summarizes the substitution-based, call-by-value, contextual, small-step semantics for both calculi. We specify the reduction semantics of the primitives using the functions δ and δ_T.

Substitution The key difference with standard formulations is the notion of substitution for type variables at (polymorphic) type-application sites as shown in rule E-TApp. Type substitution is defined at the bottom left of Fig. 5 and it is standard except for the last line which defines the substitution of a type variable α in a refined type variable $\alpha\{x:p\}$ with a (potentially refined) type t_α. To do this substitution, we combine p with the type t_α by using $\mathsf{refine}(t_\alpha, p, x)$ which essentially conjoins the refinement p to the top-level refinement of a base-kinded t_α. For existential types, refine *pushes* the refinement through the existential quantifier. Function and quantified types are left unchanged as they cannot instantiate a *refined* type variable (which must be of base kind).

Primitives The function $\delta(c, v)$ evaluates the application $c\ v$ of built-in monomorphic primitives. The reductions are defined in a curried manner, *i.e.* $\leq\ m\ n$ evaluates to $\delta(\delta(\leq, m), n)$. Currying gives us unary relations like $m\leq$ which is a partially evaluated version of the $\leq$ relation. The function $\delta_T(c, \lfloor t \rfloor)$ specifies the reduction rules for type application on the polymorphic built-in primitives.

$$\begin{array}{rcl@{\qquad}rcl@{\qquad}rcl}
\delta(\wedge, \texttt{true}) & \doteq & \lambda x.\, x & \delta(\leq, m) & \doteq & m\leq & \delta_T(=, \texttt{Bool}) & \doteq & = \\
\delta(\wedge, \texttt{false}) & \doteq & \lambda x.\, \texttt{false} & \delta(m\leq, n) & \doteq & (m \leq n) & \delta_T(=, \texttt{Int}) & \doteq & = \\
\delta(\neg, \texttt{true}) & \doteq & \texttt{false} & \delta(=, m) & \doteq & m= & \delta_T(\leq, \texttt{Bool}) & \doteq & \leq \\
\delta(\neg, \texttt{false}) & \doteq & \texttt{true} & \delta(m=, n) & \doteq & (m = n) & \delta_T(\leq, \texttt{Int}) & \doteq & \leq
\end{array}$$

Determinism Our soundness proof uses the determinism property of the operational semantics.

Lemma 3.1 (Determinism). *For every expression e, 1) there exists at most one term e' s.t. $e \hookrightarrow e'$, 2) there exists at most one value v s.t. $e \hookrightarrow^* v$, and 3) if e is a value there is no term e' s.t. $e \hookrightarrow e'$.*

4 STATIC SEMANTICS

The static semantics of our calculi comprise four main judgment forms: *well-formedness* judgments that determine when a type or environment is syntactically well-formed (in λ_F and λ_{RF}); *typing* judgments that stipulate that a term has a particular type in a given context (in λ_F and λ_{RF});

subtyping judgments that establish when one type can be viewed as a subtype of another (in λ_{RF}); and *implication* judgments that establish when one predicate implies another (in λ_{RF}). Next, we present the static semantics of λ_{RF} by describing the rules that establish each of these judgments. We use grey to highlight the antecedents and rules specific to λ_{RF}.

4.1 Well-formedness

Judgments The judgment $\Gamma \vdash_w t : k$ says that the type t is well-formed in the environment Γ and has kind k. The judgment $\vdash_w \Gamma$ says that the environment Γ is well formed, meaning that it only binds to well-formed types. Well-formedness is also used in the (unrefined) system λ_F, where $\Gamma \vdash_w \tau : k$ means that the (unrefined) λ_F type τ is well-formed in environment Γ and has kind k and $\vdash_w \Gamma$ means that the free type variables of the environment Γ are bound earlier in the environment.

Rules Fig. 6 summarizes the rules that establish the well-formedness of types and environments. Rule WF-Base states that the two closed base types (`Int` and `Bool`, refined with `true` in λ_{RF}) are well-formed and have base kind. Similarly, rule WF-Var says that a type variable α is well-formed with kind k so long as $\alpha{:}k$ is bound in the environment. The rule WF-Refn stipulates that a refined base type $b\{x : p\}$ is well-formed with base kind in some environment if the unrefined base type b has base kind in the same environment and if the refinement predicate p has type `Bool` in the environment augmented by binding a fresh variable to type b. Note that if $b \equiv \alpha$ then we can only form the antecedent $\Gamma \vdash_w \alpha\{x : \texttt{true}\} : B$ when $\alpha{:}B \in \Gamma$ (rule WF-Var), which prevents us from refining star-kinded type variables. To break a circularity in which well-formedness judgments appear in the antecedents of typing judgments and a typing judgment appears in the antecedents of WF-Refn, we use the λ_F judgment to check that p has type `Bool`. Finally, rule WF-Kind simply states that if a type t is well-formed with base kind in some environment, then it is also well-formed with star kind. This rule is required by our metatheory to convert base to star kinds in type variables.

As for environments, the empty environment is well-formed. A well-formed environment remains well-formed after binding a fresh term or type variable to *resp.* any well-formed type or kind.

4.2 Typing

The judgment $\Gamma \vdash e : t$ states that the term e has type t in the context of environment Γ. We write $\Gamma \vdash_F e : \tau$ to indicate that term e has the (unrefined) λ_F type τ in the (unrefined) context Γ. Fig. 7 summarizes the rules that establish typing for both λ_F and λ_{RF}, with grey for the λ_{RF} extensions.

Typing Primitives The type of a built-in primitive c is given by the function $\mathrm{ty}(c)$, which is defined for every constant of our system. Below we present essential examples of the $\mathrm{ty}(c)$ definition.

$$
\begin{array}{rcl@{\qquad}rcl}
\mathrm{ty}(\texttt{true}) & \doteq & \texttt{Bool}\{x : x = \texttt{true}\} & \mathrm{ty}(\wedge) & \doteq & x{:}\texttt{Bool} \to y{:}\texttt{Bool} \to \texttt{Bool}\{v : v = x \wedge y\} \\
\mathrm{ty}(3) & \doteq & \texttt{Int}\{x : x = 3\} & \mathrm{ty}(\leq) & \doteq & \forall\, \alpha{:}B.\, x{:}\alpha \to y{:}\alpha \to \texttt{Bool}\{v : v = (x \leq y)\} \\
\mathrm{ty}(m\leq) & \doteq & y{:}\texttt{Int} \to \texttt{Bool}\{v : v = (m \leq y)\} & \mathrm{ty}(=) & \doteq & \forall\, \alpha{:}B.\, x{:}\alpha \to y{:}\alpha \to \texttt{Bool}\{v : v = (x = y)\}
\end{array}
$$

We note that the = used in the refinements is the polymorphic equals with type applications elided. Further, we use $m\leq$ to represent an arbitrary member of the infinite family of primitives $0\leq$, $1\leq$, $2\leq$, ... For λ_F we erase the refinements using $\lfloor \mathrm{ty}(c) \rfloor$. The rest of the definition is similar.

Our choice to make the typing and reduction of constants external to our language, *i.e.* given by the functions $\mathrm{ty}(c)$ and $\delta(c)$, makes our system easily extensible with further constants, including a `fix` constant to encode induction. The requirement, for soundness, is that these two functions together satisfy the following four conditions.

Requirement 1. *(Primitives) For every primitive c,*

(1) If $\mathrm{ty}(c) = b\{x : p\}$, *then* $\varnothing \vdash_w \mathrm{ty}(c) : B$ *and* $\varnothing \vdash \texttt{true} \Rightarrow p[c/x]$.
(2) If $\mathrm{ty}(c) = x{:}t_x \to t$ *or* $\mathrm{ty}(c) = \forall\, \alpha{:}k.\, t$, *then* $\varnothing \vdash_w \mathrm{ty}(c) : \star$.

Well-formed Type $\boxed{\Gamma \vdash_w t : k}$

$$\frac{b \in \{\texttt{Bool}, \texttt{Int}\}}{\Gamma \vdash_w b\,\{x : \texttt{true}\} : B}\text{WF-Base} \qquad \frac{\alpha : k \in \Gamma}{\Gamma \vdash_w \alpha\,\{x : \texttt{true}\} : k}\text{WF-Var} \qquad \frac{\Gamma \vdash_w t : B}{\Gamma \vdash_w t : \star}\text{WF-Kind}$$

$$\frac{\Gamma \vdash_w b\{x : \texttt{true}\} : B \quad \forall y \notin \Gamma. y{:}b, \lfloor\Gamma\rfloor \vdash_F p[y/x] : \texttt{Bool}}{\Gamma \vdash_w b\{x : p\} : B}\text{WF-Refn}$$

$$\frac{\Gamma \vdash_w t_x : k_x \quad \forall y \notin \Gamma.\ y{:}t_x, \Gamma \vdash_w t\,[y/x] : k}{\Gamma \vdash_w x{:}t_x \to t : \star}\text{WF-Func}$$

$$\frac{\Gamma \vdash_w t_x : k_x \quad \forall y \notin \Gamma.\ y{:}t_x, \Gamma \vdash_w t[y/x] : k}{\Gamma \vdash_w \exists\, x{:}t_x.\, t : k}\text{WF-Exis} \qquad \frac{\forall \alpha' \notin \Gamma.\ \alpha'{:}k, \Gamma \vdash_w t[\alpha'/\alpha] : k_t}{\Gamma \vdash_w \forall\, \alpha{:}k.\, t : \star}\text{WF-Poly}$$

Well-formed Environment $\boxed{\vdash_w \Gamma}$

$$\frac{}{\vdash_w \emptyset}\text{WFE-Emp} \qquad \frac{\Gamma \vdash_w t_x : k_x \quad \vdash_w \Gamma \quad x \notin \Gamma}{\vdash_w x{:}t_x, \Gamma}\text{WFE-Bind} \qquad \frac{\vdash_w \Gamma \quad \alpha \notin \Gamma}{\vdash_w \alpha{:}k, \Gamma}\text{WFE-TBind}$$

Fig. 6. Well-formedness of types and environments. The rules for λ_F exclude the grey boxes.

Typing $\boxed{\Gamma \vdash e : t}$

$$\frac{\mathrm{ty}(c) = t}{\Gamma \vdash c : t}\text{T-Prim} \qquad \frac{x{:}t \in \Gamma \quad \Gamma \vdash_w t : k}{\Gamma \vdash x : \mathrm{self}(t, x, k)}\text{T-Var} \qquad \frac{\Gamma \vdash e : t \quad \Gamma \vdash_w t : k}{\Gamma \vdash e : t : t}\text{T-Ann} \qquad \frac{\Gamma \vdash_w t : k \quad \Gamma \vdash e : s \quad \Gamma \vdash s \preceq t}{\Gamma \vdash e : t}\text{T-Sub}$$

$$\frac{\Gamma \vdash e_x : t_x \quad \Gamma \vdash e : x{:}t_x \to t}{\Gamma \vdash e\ e_x : \exists\, x{:}t_x.\ t}\text{T-App} \qquad \frac{\Gamma \vdash_w t_x : k_x \quad \forall y \notin \Gamma. y{:}t_x, \Gamma \vdash e[y/x] : t\,[y/x]}{\Gamma \vdash \lambda x.e : x{:}t_x \to t}\text{T-Abs} \qquad \frac{\Gamma \vdash_w t : k \quad \Gamma \vdash e : \forall\, \alpha{:}k.\, s}{\Gamma \vdash e[t] : s[t/\alpha]}\text{T-TApp}$$

$$\frac{\forall \alpha' \notin \Gamma. \quad \alpha'{:}k, \Gamma \vdash e[\alpha'/\alpha] : t[\alpha'/\alpha]}{\Gamma \vdash \Lambda\alpha{:}k.e : \forall\, \alpha{:}k.\, t}\text{T-TAbs} \qquad \frac{\Gamma \vdash e_x : t_x \quad \Gamma \vdash_w t : k \quad \forall y \notin \Gamma. y{:}t_x, \Gamma \vdash e[y/x] : t\,[y/x]}{\Gamma \vdash \texttt{let}\ x = e_x\ \texttt{in}\ e : t}\text{T-Let}$$

Fig. 7. Typing rules. The judgment $\Gamma \vdash_F e : \tau$ is defined by excluding the grey boxes.

(3) *If* $\mathrm{ty}(c) = x{:}t_x \to t$, *then for all* v_x *such that* $\emptyset \vdash v_x : t_x$, $\emptyset \vdash \delta(c, v_x) : t[v_x/x]$.
(4) *If* $\mathrm{ty}(c) = \forall\, \alpha{:}k.\, t$, *then for all* t_α *such that* $\emptyset \vdash_w t_\alpha : k$, $\emptyset \vdash \delta_T(c, t_\alpha) : t[t_\alpha/\alpha]$.

To type constants, rule T-Prim gives the type $ty(c)$ to any built-in primitive c, in any context.

Typing Variables with Selfification Rule T-Var establishes that any variable x that appears as $x{:}t$ in environment Γ can be given the *selfified* type [Ou et al. 2004] $\mathrm{self}(t, x, k)$ provided that $\Gamma \vdash_w t : k$. This rule is crucial in practice, to enable path-sensitive "occurrence" typing [Tobin-Hochstadt and Felleisen 2008], where the types of variables are refined by control-flow guards.

For example, suppose we want to establish $\alpha{:}B \vdash (\lambda x.x) : x{:}\alpha \to \alpha\{y : x = y\}$, and not just $\alpha{:}B \vdash (\lambda x.x) : \alpha \to \alpha$. The latter would result if T-Var merely stated that $\Gamma \vdash x : t$ whenever $x{:}t \in \Gamma$. Instead, we strengthen the T-Var rule to be *selfified*. Informally, to get information about x into the refinement level, we need to say that x is constrained to elements of type α that are equal to x itself. In order to express the exact type of variables, below we define the "selfification" function that strengthens a refinement with the condition that a value is equal to itself. Since abstractions do not admit equality, we only selfify the base types and the existential quantifications of them.

$$\begin{aligned} \mathsf{self}(\exists\, z{:}t_z.\, t, x, k) &\doteq \exists\, z{:}t_z.\, \mathsf{self}(t, x, k) & \mathsf{self}(b\{z : p\}, x, B) &\doteq b\{z : p \wedge z = x\} \\ \mathsf{self}(x{:}t_x \to t, \_, \_) &\doteq x{:}t_x \to t & \mathsf{self}(b\{z : p\}, x, \star) &\doteq b\{z : p\} \\ \mathsf{self}(\forall\, \alpha{:}k.\, t, \_, \_) &\doteq \forall\, \alpha{:}k.\, t \end{aligned}$$

Typing Applications with Existentials Our rule T-App states the conditions for typing a term application $e\ e_x$. Under the same environment, we must be able to type e at some function type $x{:}t_x \to t$ and e_x at t_x. Then we can give $e\ e_x$ the existential type $\exists\, x{:}t_x.\, t$. The use of existential types in rule T-App is one of the distinctive features of our language and was introduced by Knowles and Flanagan [2009b]. As overviewed in § **??**, we chose this form of T-App over the conventional form of $\Gamma \vdash e\ e_x : t[e_x/x]$ because our version prevents the substitution of arbitrary expressions (*e.g.* functions and type abstractions) into refinements. As an alternative, we could have used ANF (A-Normal Form [Flanagan et al. 1993]), but our metatheory would be more complex since ANF is not preserved under the small step operational semantics.

Other Typing Rules Our rule T-TApp states that whenever a term e has polymorphic type $\forall\, \alpha{:}k.\, s$, then for any well-formed type t with kind k, we can give the type $s[t/\alpha]$ to the type application $e[t]$. For the λ_F variant of T-TApp, we erase the refinements (via $\lfloor t \rfloor$) before checking well-formedness and performing the substitution. Rule T-Ann establishes that an explicit annotation $e : t$ indeed has type t when the underlying e has type t and t is well-formed. The λ_F version of the rule erases the refinements and uses $\lfloor t \rfloor$. Finally, rule T-Sub tells us that we can exchange a subtype s for a supertype t in a judgment $\Gamma \vdash s : t$ provided t is well-formed and $\Gamma \vdash s \preceq t$, which we present next.

4.3 Subtyping

The *subtyping* judgment $\Gamma \vdash s \preceq t$, defined in Fig. 8, stipulates that the type s is a subtype of the type t in the environment Γ and is used in the subsumption typing rule T-Sub (of Fig. 7).

Subtyping Rules Rules S-Bind and S-Wit establish subtyping for existential types [Knowles and Flanagan 2009b], *resp.* when the existential appears on the left or right. Rule S-Bind allows us to exchange a universal quantifier (a variable bound to some type t_x in the environment) for an existential quantifier. If we have a judgment of the form $y{:}t_x, \Gamma \vdash t[y/x] \preceq t'$ where y does *not* appear free in either t' or in the context Γ, then we can conclude that $\exists\, x{:}t_x.\, t$ is a subtype of t'. Rule S-Wit states that if type t is a subtype of $t'[v_x/x]$ for some value v_x of type t_x, then we can discard the specific *witness* for x and quantify existentially to obtain that t is a subtype of $\exists\, x{:}t_x.\, t'$.

Refinements enter the scene in the rule S-Base which specifies that a refined base type $b\{x_1 : p_1\}$ is a subtype of another $b\{x_2 : p_2\}$ in context Γ when p_1 *implies* p_2 in the environment Γ augmented by binding a fresh variable to the unrefined type b. Next, we describe how we formalized implication.

4.4 Implication

The *implication* judgment $\Gamma \vdash p_1 \Rightarrow p_2$ states that the implication $p_1 \Rightarrow p_2$ is (logically) valid under the assumptions captured by the context Γ. In refinement type implementations [Swamy et al. 2016; Vazou et al. 2014a], this relation is implemented as an external automated (usually SMT) solver. In non-mechanized refinement type formalizations, there have been two approaches to formalize predicate implication: either directly reduce it into a logical implication (*e.g.* in Gordon

Subtyping $\boxed{\Gamma \vdash s \preceq t}$

$$\frac{\Gamma \vdash t_{x2} \preceq t_{x1} \quad \forall y \notin \Gamma.y{:}t_{x2}, \Gamma \vdash t_1[y/x] \preceq t_2[y/x]}{\Gamma \vdash x{:}t_{x1} \to t_1 \preceq x{:}t_{x2} \to t_2}\text{S-Fun} \qquad \frac{\Gamma \vdash v_x : t_x \quad \Gamma \vdash t \preceq t'[v_x/x]}{\Gamma \vdash t \preceq \exists\, x{:}t_x.\, t'}\text{S-Wit} \qquad \frac{\forall y \notin \Gamma. \quad y{:}b, \Gamma \vdash p_1[y/x] \Rightarrow p_2[y/x]}{\Gamma \vdash b\{x : p_1\} \preceq b\{x : p_2\}}\text{S-Base}$$

$$\frac{\forall y \notin \text{free}(t) \cup \Gamma.y{:}t_x, \Gamma \vdash t[y/x] \preceq t'}{\Gamma \vdash \exists\, x{:}t_x.\, t \preceq t'}\text{S-Bind} \qquad \frac{\forall \alpha' \notin \Gamma.\alpha'{:}k, \Gamma \vdash t_1[\alpha'/\alpha] \preceq t_2[\alpha'/\alpha]}{\Gamma \vdash \forall\, \alpha{:}k.\, t_1 \preceq \forall\, \alpha{:}k.\, t_2}\text{S-Poly}$$

Fig. 8. Subtyping Rules.

and Fournet [2010]) or define it using operational semantics (*e.g.* in Vazou et al. [2018]). It turns out that none of these approaches can be directly encoded in a mechanized proof. The former approach is insufficient because it requires a formal connection between the (deeply embedded) terms of λ_{RF} and the terms of the logic, which has not yet been clearly established. The second approach is more direct, since it gives meaning to implication directly using the terms of λ_{RF}, via denotational semantics. Sadly, the definition of denotational semantics for our polymorphic calculus is not currently possible: encoding type denotations as an inductive data type (or proposition in our LiquidHaskell encoding § 8) requires a negative occurrence which is not currently admitted.

Axiomatization of Implication In our mechanization, following Lehmann and Tanter [2016], we encode implication as an axiomatized judgment that satisfies the requirements below.

Requirement 2. *The implication relation satisfies the following statements:*

(1) (Reflexivity) $\Gamma \vdash p \Rightarrow p$.
(2) (Transitivity) If $\Gamma \vdash p_1 \Rightarrow p_2$ *and* $\Gamma \vdash p_2 \Rightarrow p_3$, *then* $\Gamma \vdash p_1 \Rightarrow p_3$.
(3) (Faithfulness) $\Gamma \vdash p \Rightarrow \texttt{true}$.
(4) (Introduction) If $\Gamma \vdash p_1 \Rightarrow p_2$ *and* $\Gamma \vdash p_1 \Rightarrow p_3$, *then* $\Gamma \vdash p_1 \Rightarrow p_2 \wedge p_3$.
(5) (Conjunction) $\Gamma \vdash p_1 \wedge p_2 \Rightarrow p_1$ *and* $\Gamma \vdash p_1 \wedge p_2 \Rightarrow p_2$.
(6) (Repetition) $\Gamma \vdash p_1 \wedge p_2 \Rightarrow p_1 \wedge p_1 \wedge p_2$.
(7) (Evaluation) If $p_1 \hookrightarrow^* p_2$, *then* $\Gamma \vdash p_1 \Rightarrow p_2$ *and* $\Gamma \vdash p_2 \Rightarrow p_1$.
(8) (Narrowing) If $\Gamma_1, x{:}t_x, \Gamma_2 \vdash p_1 \Rightarrow p_2$ *and* $\Gamma_2 \vdash s_x \preceq t_x$, *then* $\Gamma_1, x{:}s_x, \Gamma_2 \vdash p_1 \Rightarrow p_2$.
(9) (Weaken) If $\Gamma_1, \Gamma_2 \vdash p_1 \Rightarrow p_2$, $a, x \notin \Gamma$, *then* $\Gamma_1, x{:}t_x, \Gamma_2 \vdash p_1 \Rightarrow p_2$ *and* $\Gamma_1, a{:}k, \Gamma_2 \vdash p_1 \Rightarrow p_2$.
(10) (Subst I) If $\Gamma_1, x{:}t_x, \Gamma_2 \vdash p_1 \Rightarrow p_2$ *and* $\Gamma_2 \vdash v_x : t_x$, *then* $\Gamma_1[v_x/x], \Gamma_2 \vdash p_1[v_x/x] \Rightarrow p_2[v_x/x]$.
(11) (Subst II) If $\Gamma_1, a{:}k, \Gamma_2 \vdash p_1 \Rightarrow p_2$ *and* $\Gamma_2 \vdash_w t : k$, *then* $\Gamma_1[t/a], \Gamma_2 \vdash p_1[t/a] \Rightarrow p_2[t/a]$.
(12) (Strengthening) If $y{:}b\{x : q\}, \Gamma \vdash p_1 \Rightarrow p_2$, *then* $y{:}b, \Gamma \vdash q[y/x] \wedge p_1 \Rightarrow q[y/x] \wedge p_2$.

This axiomatic approach precisely explicates the requirements of the implication checker to establish the soundness of the entire refinement type system. In the future, we could verify that these properties hold for SMT solvers or even build other implication oracles that satisfy this contract.

5 λ_F SOUNDNESS

Next, we present the metatheory of the underlying (unrefined) λ_F. Even though it follows the textbook techniques of Pierce [2002], it is a convenient stepping stone *towards* the metatheory for (refined) λ_{RF}. In addition, the soundness results for λ_F are used *for* our full metatheory, as our well-formedness judgments require the refinement predicate to have the λ_F type `Bool` thereby avoiding the circularity of using a regular typing judgment in the antecedents of the well-formedness rules. The light grey boxes in **??** show the high level outline of the metatheory for λ_F which provides a miniaturized model for λ_{RF} but without the challenges of subtyping and existentials. Next, we

describe the top-level type safety result, how it is decomposed into progress (Lemma 5.2) and preservation (Lemma 5.3) lemmas, and the various technical results that support the lemmas.

The main type safety theorem for λ_F states that a well-typed term does not get stuck: *i.e.* either evaluates to a value or can step to another term (progress) of the same type (preservation). The judgment $\Gamma \vdash_F e : \tau$ is defined in Fig. 7 without the grey boxes, and for clarity we use τ for λ_F types.

Theorem 5.1. *(Type Safety) If* $\emptyset \vdash_F e : \tau$ *and* $e \hookrightarrow^* e'$*, then* e' *is a value or* $e' \hookrightarrow e''$ *for some* e''*.*

We prove type safety by induction on the length of the sequence of steps comprising $e \hookrightarrow^* e'$, using the preservation and progress lemmas.

Progress The progress lemma says a well-typed term is a value or steps to some other term.

Lemma 5.2. *(Progress) If* $\emptyset \vdash_F e : \tau$*, then* e *is a value or* $e \hookrightarrow e'$ *for some* e'*.*

Preservation The preservation lemma states that λ_F typing is preserved by evaluation.

Lemma 5.3. *(Preservation) If* $\emptyset \vdash_F e : \tau$ *and* $e \hookrightarrow e'$*, then* $\emptyset \vdash_F e' : \tau$*.*

The proof is by structural induction on the derivation of the typing judgment. We use the determinism of the operational semantics (Lemma 3.1) and the canonical forms lemma to case split on e to determine e'. The interesting cases are for T-App and T-TApp that require a Substitution Lemma 5.4.

Substitution Lemma To prove type preservation when a lambda or type abstraction is applied, we proved that the substituted result has the same type, as established by the Substitution Lemma:

Lemma 5.4. *(Substitution) If* $\Gamma \vdash_F v_x : \tau_x$ *and* $\Gamma \vdash_w \lfloor t_\alpha \rfloor : k_\alpha$*, then*

(1) if $\Gamma', x{:}\tau_x, \Gamma \vdash_F e : \tau$ *and* $\vdash_w \Gamma$*, then* $\Gamma', \Gamma \vdash_F e[v_x/x] : \tau$ *and*
(2) if $\Gamma', \alpha{:}k_\alpha, \Gamma \vdash_F e : \tau$ *and* $\vdash_w \Gamma$*, then* $\Gamma'[\lfloor t_\alpha \rfloor/\alpha], \Gamma \vdash_F e[t_\alpha/\alpha] : \tau[\lfloor t_\alpha \rfloor/\alpha]$*.*

The proof goes by induction on the derivation tree. Because we encoded our typing rules using cofinite quantification the proof does not require a renaming lemma, but the rules that lookup environments (rules T-Var and WF-Var) do need a *Weakening Lemma*:

Lemma 5.5. *(Weakening) If* $\Gamma_1, \Gamma_2 \vdash_F e : \tau$ *and* $x, \alpha \notin \Gamma_1, \Gamma_2$*, then* $\Gamma_1, x{:}\tau_x, \Gamma_2 \vdash_F e : \tau$ *and* $\Gamma_1, \alpha{:}k, \Gamma_2 \vdash_F e : \tau$*.*

6 λ_{RF} SOUNDNESS

We proceed to the metatheory of λ_{RF} by fleshing out the skeleton of light grey lemmas in **??** (which are similar to the λ_F versions) and describing the three regions (§ **??**) that establish the inversion, substitution, and narrowing properties. Type safety combines progress and preservation.

Theorem 6.1. *(Type Safety of* λ_{RF}*)*

(1) (Type Safety) If $\emptyset \vdash e : t$ *and* $e \hookrightarrow^* e'$*, then* e' *is a value or* $e' \hookrightarrow e''$ *for some* e''*.*
(2) (Progress) If $\emptyset \vdash e : t$*, then* e *is a value or* $e \hookrightarrow e'$ *for some* e'*.*
(3) (Preservation) If $\emptyset \vdash e : t$ *and* $e \hookrightarrow e'$*, then* $\emptyset \vdash e' : t$*.*

Next, let's see the three main ways in which the proof of Progress differs from λ_F.

6.1 Inversion of Typing Judgments

The vertical lined region of **??** accounts for the fact that, due to subtyping chains, the typing judgment in λ_{RF} is not syntax-directed. First, we establish that subtyping is transitive

Lemma 6.2. *(Transitivity) If* $\Gamma \vdash_w t_1 : k_1, \Gamma \vdash_w t_3 : k_3, \vdash_w \Gamma, \Gamma \vdash t_1 \preceq t_2, \Gamma \vdash t_2 \preceq t_3$*, then* $\Gamma \vdash t_1 \preceq t_3$*.*

The proof consists of a case-split on the possible rules for $\Gamma \vdash t_1 \preceq t_2$ and $\Gamma \vdash t_2 \preceq t_3$. When the last rule used in the former is S-Wit and the latter is S-Bind, we require the Substitution Lemma 6.4. As Aydemir et al. [2005], we use the Narrowing Lemma 6.6 for the transitivity for function types.

Inverting Typing Judgments We use the transitivity of subtyping to prove some non-trivial lemmas that let us "invert" the typing judgments to recover information about the underlying terms and types. We describe the non-trivial case which pertains to type and value abstractions:

Lemma 6.3. *(Inversion of T-Abs, T-TAbs)*

(1) If $\Gamma \vdash (\lambda w.e) : x{:}t_x \to t$ *and* $\vdash_w \Gamma$, *then for all* $y \notin \Gamma$, $y{:}t_x, \Gamma \vdash e[y/w] : t[y/x]$.
(2) If $\Gamma \vdash (\Lambda\alpha_1{:}k_1.e) : \forall\, \alpha{:}k.\, t$ *and* $\vdash_w \Gamma$, *then for all* $\alpha' \notin \Gamma$, $\alpha'{:}k, \Gamma \vdash e[\alpha'/\alpha_1] : t[\alpha'/\alpha]$.

If $\Gamma \vdash (\lambda w.e) : x{:}t_x \to t$, then we cannot directly invert the typing judgment to get a judgment for the body e of $\lambda w.e$. Perhaps the last rule used was T-Sub, and inversion only tells us that there exists a type t_1 such that $\Gamma \vdash (\lambda w.e) : t_1$ and $\Gamma \vdash t_1 \preceq x{:}t_x \to t$. Inverting again, we may in fact find a chain of types $t_{i+1} \preceq t_i \preceq \cdots \preceq t_2 \preceq t_1$ which can be arbitrarily long. But the proof tree must be finite so eventually we find a type $w{:}s_w \to s$ such that $\Gamma \vdash (\lambda w.e) : w{:}s_w \to s$ and $\Gamma \vdash w{:}s_w \to s \preceq x{:}t_x \to t$ (by transitivity) and the last rule was T-Abs. Then inversion gives us that for any $y \notin \Gamma$ we have $y{:}s_w, \Gamma \vdash e : s[y/w]$. To get the desired typing judgment, we must use the Narrowing Lemma 6.6 to obtain $y{:}t_x, \Gamma \vdash e : s[y/w]$. Finally, we use T-Sub to derive $y{:}t_x, \Gamma \vdash e : t[y/w]$.

6.2 Substitution Lemma

The main result in the diagonal lined region of **??** is the Substitution Lemma. The biggest difference between the λ_F and λ_{RF} metatheories is the introduction of a mutual dependency between the lemmas for typing and subtyping. Due to this dependency, both the substitution the weakening lemmas must now be proven in a mutually recursive form for both typing and subtyping:

Lemma 6.4. *(Substitution)*

(1) If $\Gamma_1, x{:}t_x, \Gamma_2 \vdash s \preceq t$, $\vdash_w \Gamma_2$, *and* $\Gamma_2 \vdash v_x : t_x$, *then* $\Gamma_1[v_x/x], \Gamma_2 \vdash s[v_x/x] \preceq t[v_x/x]$.
(2) If $\Gamma_1, x{:}t_x, \Gamma_2 \vdash e : t$, $\vdash_w \Gamma_2$, *and* $\Gamma_2 \vdash v_x : t_x$, *then* $\Gamma_1[v_x/x], \Gamma_2 \vdash e[v_x/x] : t[v_x/x]$.
(3) If $\Gamma_1, \alpha{:}k, \Gamma_2 \vdash s \preceq t$, $\vdash_w \Gamma_2$, *and* $\Gamma_2 \vdash_w t_\alpha : k$, *then* $\Gamma_1[t_\alpha/\alpha], \Gamma_2 \vdash s[t_\alpha/\alpha] \preceq t[t_\alpha/\alpha]$.
(4) If $\Gamma_1, \alpha{:}k, \Gamma_2 \vdash e : t$, $\vdash_w \Gamma_2$, *and* $\Gamma_2 \vdash_w t_\alpha : k$, *then* $\Gamma_1[t_\alpha/\alpha], \Gamma_2 \vdash e[t_\alpha/\alpha] : t[t_\alpha/\alpha]$.

The main difficulty arises in substituting some type t_α for variable α in $\Gamma_1, \alpha{:}k, \Gamma_2 \vdash \alpha\{x_1 : p\} \preceq \alpha\{x_2 : q\}$ because t_α must be strengthened by the refinements p and q respectively. As with the λ_F version, the proof requires the *Weakening* Lemma 6.5 but now both for typing and subtyping.

Lemma 6.5. *(Weakening) If* $x, \alpha \notin \Gamma_1, \Gamma_2$, *then*

(1) if $\Gamma_1, \Gamma_2 \vdash e : t$ *then* $\Gamma_1, x{:}t_x, \Gamma_2 \vdash e : t$ *and* $\Gamma_1, \alpha{:}k, \Gamma_2 \vdash e : t$.
(2) if $\Gamma_1, \Gamma_2 \vdash s \preceq t$ *then* $\Gamma_1, x{:}t_x, \Gamma_2 \vdash s \preceq t$ *and* $\Gamma_1, \alpha{:}k, \Gamma_2 \vdash s \preceq t$.

The proof is by mutual induction on the derivation of the typing and subtyping judgments.

6.3 Narrowing

The narrowing lemma says that whenever we have a judgment where a binding $x{:}t_x$ appears in the binding environment, we can replace t_x by any subtype s_x. The intuition here is that the judgment holds under the replacement because we are making the context more specific.

Lemma 6.6. *(Narrowing) If* $\Gamma_2 \vdash s_x <: t_x$, $\Gamma_2 \vdash_w s_x : k_x$, *and* $\vdash_w \Gamma_2$ *then*

(1) if $\Gamma_1, x{:}t_x, \Gamma_2 \vdash_w t : k$, *then* $\Gamma_1, x{:}s_x, \Gamma_2 \vdash_w t : k$.
(2) if $\Gamma_1, x{:}t_x, \Gamma_2 \vdash t_1 <: t_2$, *then* $\Gamma_1, x{:}s_x, \Gamma_2 \vdash t_1 <: t_2$.
(3) if $\Gamma_1, x{:}t_x, \Gamma_2 \vdash e : t$, *then* $\Gamma_1, x{:}s_x, \Gamma_2 \vdash e : t$.

The narrowing proof requires an Exact Typing Lemma 6.7 which says that both subtyping and typing is preserved after selfification.

Lemma 6.7. *(Exact Typing)*

(1) If $\Gamma \vdash e : t, \vdash_w \Gamma, \Gamma \vdash_w t : k$, *and* $\Gamma \vdash s \preceq t$, *then* $\Gamma \vdash \text{self}(s, v, k) \preceq \text{self}(t, v, k)$.

(2) If $\Gamma \vdash v : t, \vdash_w \Gamma$, *and* $\Gamma \vdash_w t : k$, *then* $\Gamma \vdash v : \text{self}(t, v, k)$.

7 REFINED DATA PROPOSITIONS

In § 8 we will present how soundness λ_{RF} is proved in LiquidHaskell. Here we present *refined data propositions*, a novel feature of LiquidHaskell that made such a meta-theoretic proof possible. Intuitively, refined data propositions encode Coq-style inductive predicates to permit constructive reasoning about potentially non-terminating properties, as required for meta-theoretic proofs.

Refined data propositions encode inductive predicates in LiquidHaskell by refining Haskell's data types, allowing the programmer to write plain Haskell functions to provide constructive proofs for user-defined propositions. Here, for exposition, we present the four steps we followed in the mechanization of λ_{RF} to define the "has-type" proposition and then use it to type the primitive one.

Step 1: Reifying Propositions as Data Our first step is to represent the propositions of interest as plain Haskell data. For example, we can define the following types (suffixed `Pr` for "proposition"):

```
data HasTyPr   = HasTyPr   Env  Expr Type
data IsSubTyPr = IsSubTyPr Env  Type Type
```

Thus, `HasTyPr γ e t` and `IsSubTyPr γ s t` *resp.* represent the *propositions* $\gamma \vdash e : t$ and $\gamma \vdash s \preceq t$.

Step 2: Reifying Evidence as Data Next, we reify evidence, *i.e. derivation trees* as data by defining Haskell data types with a *single constructor per derivation rule*. For example, we define the data type `HasTyEv` to encode the typing rules of Fig. 7, with constructors that match the names of each rule.

```
data HasTyEv where
  TPrim :: Env → Prim → HasTyEv
  TSub  :: Env → Expr → Type → Type → HasTyEv → IsSubTyEv →
  HasTyEv
  ...
```

Using these data one can construct derivation trees. For instance, `TPrim Empty (PInt 1)::HasTyEv` is the tree that types the primitive one under the empty environment.

Step 3: Relating Evidence to its Propositions Next, we specify the relationship between the evidence and the proposition that it establishes, via a refinement-level *uninterpreted function*:

```
measure hasTyEvPr   :: HasTyEv → HasTyPr
measure isSubTyEvPr :: IsSubTyEv → IsSubTyPr
```

The above signatures declare that `hasTyEvPr` (resp. `isSubTyEvPr`) is a refinement-level function that maps has-type (resp. is-subtype) evidence to its corresponding proposition. We can now use these uninterpreted functions to define *type aliases* that denote well-formed evidence that establishes a proposition. For example, consider the (refined) type aliases

```
type HasTy   γ e t = {ev:HasTyEv   | hasTyEvPr ev == HasTyPr γ e t }
type IsSubTy γ s t = {ev:IsSubTyEv | isSubTyEvPr ev == IsSubTyPr γ s t }
```

The definition stipulates that the type `HasTy γ e t` is inhabited by evidence (of type `HasTyEv`) that establishes the typing proposition `HasTyPr γ e t`. Similarly `IsSubTy γ s t` is inhabited by evidence (of type `IsSubTyEv`) that establishes the subtyping proposition `IsSubTyPr γ s t`. Note that the first three steps have only defined separate data types for propositions and evidence, and *specified* the relationship between them via uninterpreted functions in the refinement logic.

Step 4: Refining Evidence to Establish Propositions Finally, we *implement* the relationship between evidence and propositions *refining* the types of the evidence data constructors (rules)

with pre-conditions that require the rules' premises and post-conditions that ensure the rules' conclusions. For example, we connect the evidence and proposition for the typing relation by refining the data constructors for `HasTyEv` using their respecting typing rule from Fig. 7.

```
data HasTyEv where
  TPrim :: γ:Env →  c:Prim →  HasTy γ (Prim c) (ty c)
  TSub  :: γ:Env →  e:Expr →  s:Type →  t:Type
        →  HasTy γ e s →  IsSubTy γ s t →  HasTy γ e t
  ...
```

The constructors `TPrim` and `TSub` respectively encode the rules T-Prim and T-Sub (with well-formedness elided for simplicity). The refinements on the input types, which encode the premises of the rules, are checked whenever these constructors are used. The refinement on the output type (being evidence of a specific proposition) is axiomatized to encode the conclusion of the rules. For example, the type for `TSub` says that "for all γ, e, s, t, given evidence that $\gamma \vdash e : s$ and $\gamma \vdash s \preceq t$", the constructor returns "evidence that $\gamma \vdash e : t$".

Implementation of Data Propositions Data propositions are a novel feature required to encode inductive propositions in the mechanization of λ_{RF}. (Parker et al. [2019] developed a LiquidHaskell meta-theoretic proof but before data propositions and thus had to axiomatize a terminating evaluation relation; see § 10.) To implement this feature, we had to extend the refinement logic of LiquidHaskell to use existing SMT support to make data constructors *injective*, *i.e.* if C is a constructor then $\forall x, y.\, C(x) = C(y) \implies x = y$. Thus, refined data types and injectivity are the two required components to implement data propositions.

8 LIQUIDHASKELL MECHANIZATION

We mechanized soundness of λ_{RF} in both Coq 8.15.1 and LiquidHaskell 8.10.7.1, and have submitted these as anonymous supplementary material. In LiquidHaskell we use refined data propositions (§ 7) to specify the static (*e.g.* typing, subtyping, well-formedness) and dynamic (*i.e.* small-step transitions and their closure) semantics of λ_{RF}. Other that the development of data propositions, we extended LiquidHaskell with two more features during the development of this proof. First, we implemented an interpreter that critically dropped the verification time from 10 hours to only 29 minutes (§8.3). Second, we implemented a (Coq-style) strictly-positive-occurrence checker to ensure data propositions are well defined, since early versions of our proof used negative occurrences.

The LiquidHaskell mechanization is simplified by SMT-automation (§ 8.1) and consists of proofs implemented as recursive functions that construct evidence to establish propositions by induction (§ 8.2). Note that while Haskell types are inhabited by diverging $\perp$ values, LiquidHaskell's totality, termination, and type checks ensure that all cases are handled, the induction (recursion) is well-founded, and that the proofs (programs) indeed inhabit the propositions (types).

8.1 SMT Solvers, Arithmetic, and Set Theory

The most tedious part in mechanization of metatheories is the establishment of invariants about variables, for example uniqueness and freshness. LiquidHaskell offers a built-in, SMT automated support for the theory of sets, which simplifies establishing such invariants.

Intrinsic Verification LiquidHaskell embeds the functions of the standard `Data.Set` Haskell library as SMT set operators. Given a Haskell function, *e.g.* the set of free variables in an expression, this embedding, combined with SMT's support for set theory, lets LiquidHaskell prove properties about free variables "for free". For example, consider the function `subFV x vx e` which substitutes the variable `x` with `vx` in `e`. The refinement type of `subFV` describes the free variables of the result.

```
subFV :: x:VName → vx:{Expr | isVal vx } → e:Expr
      → {e':Expr | fv e' ⊆ (fv vx ∪ (fv e \ x)) && (isVal e ⇒ isVal e')}

subFV x vx (EVar y)    = if x == y then vx else EVar y
subFV x vx (ELam   e)  = ELam (subFV x vx e)
subFV x vx (EApp e e') = EApp (subFV x vx e) (subFV x vx e')
... -- other cases
```

The refinement type specifies that the free variables after substitution is a subset of the free variables in the two argument expressions, excluding `x`, *i.e.* $\texttt{fv}(e[v_x/x]) \subseteq \texttt{fv}(v_x) \cup (\texttt{fv}(e) \setminus \{x\})$. This specification is proved *intrinsically*, *i.e.* the definition of `subFV` is the proof (no user aid is required) and, importantly, the specification is automatically established each time the function `subFV` is called without any need for explicit hints. The specification of `subFV` above shows another example of SMT-based proof simplification. It intrinsically proves that the value property is preserved by substitution, using the Haskell boolean function `isVal` that defines when an expression is a *value*.

8.2 Inductive Proofs as Recursive Functions

The majority of our proofs are by induction on derivations. These proofs are recursive Haskell functions that operate over refined data propositions. LiquidHaskell ensures the proofs are valid by checking that they are inductive (*i.e.* the recursion is well-founded), handle all cases (*i.e.* the function is total) and establish the desired properties (*i.e.* witnesses the appropriate proposition).

Preservation (Theorem 6.1) relates the `HasTy` data proposition of § 7 with a `Step` data proposition that encodes Fig. 5 and is proved by induction on the type derivation tree. Below we present a snippet of the proof, where the subtyping case is by induction while the primitive case is impossible:

```
preservation :: e:Expr → t:Type → e':Expr → HasTy Empty e t → Step
  e e'
             → HasTy Empty e' t
preservation _e _t e' (TSub Empty e t' t e_has_t' t'_sub_t) e_step_e'
  = TSub Empty e' t' t (preservation e t' e' e_has_t' e_step_e') t'_sub_t
preservation  e _t e' (TPrim _ _) step
  = impossible "value" ? lemValStep e e' step -- e ↪ e' ⇒ ¬(isVal e)
...
impossible :: {v:String | false} → a
lemValStep :: e:Expr → e':Expr → Step e e' → {¬(isVal e)}
```

In the `TSub` case we note that LiquidHaskell knows that the argument `_e` is equal to the subtyping parameter `e`. The termination checker ensures the inductive call happens on a smaller derivation subtree. The `TPrim` case is by contradiction since primitives cannot step: we proved values cannot step in the `lemValStep` lemma, which is combined with the fact that `e` is a value to allow the call of the false-precondition `impossible`. LiquidHaskell's totality checker ensures all cases of `HasTyEv` are covered and the termination checker ensures the proof is well-founded.

8.3 Quantitative Results

We provide a full, mechanically checked proof of the metatheory in § 5 and § 6. The only facts that are assumed are Req. 2, *i.e.* the implication relation which we encoded as a data proposition, and Req. 1, *i.e.* assumptions about built-in primitives. Concretely, we assumed Req. 1 for some constants of λ_{RF} because it was obvious but too strenuous to mechanically prove without interactive aid.

	LiquidHaskell Mechanization				Coq Mechanization	
Subject	**Files**	**Time (m)**	**Spec**	**Proof**	**Spec**	**Proof**
Definitions	6	1	1805	374	890	170
Basic Properties	8	4	646	2117	1094	2056
λ_F Soundness	4	3	138	685	173	744
Weakening	4	1	379	467	88	366
Substitution	4	7	458	846	122	574
Exact Typing	2	4	70	230	33	180
Narrowing	1	1	88	166	44	147
Inversion	1	1	124	206	57	255
Primitives	3	4	120	277	88	500
λ_{RF} Soundness	1	1	14	181	12	198
Total	35	29	3842	5549	2601	5190

Table 1. Empirical mechanization details. We split each development into sets of modules pertaining to regions of **??**, and for each we count lines of specification (definitions, lemma statements) and of proof.

Table 1 summarizes the development of our metatheory, which was checked using LiquidHaskell 8.10.7.1 and a Lenovo ThinkPad T15p laptop with an Intel Core i7-11800H processor. Our mechanized proofs are substantial, each over 8000 lines across about 35 files. Currently, the whole LiquidHaskell proof can be checked in about 30 minutes, which makes interactive development difficult, especially compared to the Coq proof (§ 9) that is checked in about 30 seconds. While incremental modular checking provides a modicum of interactivity, improving the ergonomics of LiquidHaskell, *i.e.* verification time and actionable error messages, remains an important direction for future work.

9 COQ MECHANIZATION

Our Coq mechanization is a translation from LiquidHaskell and was built to compare the two developments. All theorems from § 6 are proven in Coq. Req. 1 is proved (using Coq's interactive development) and Req. 2 (*i.e.* the implication judgement) is axiomatized. To fairly compare the two developments in terms of effort and ergonomics, we did not use external Coq libraries because no such libraries exist yet for LiquidHaskell. Vazou et al. [2017] previously compared LiquidHaskell and Coq as theorem provers, but their mechanizations were an order of magnitude smaller than ours and did not use data propositions (§ 7), which permit constructive LiquidHaskell proofs.

Coq vs. LiquidHaskell Coq has a tiny TCB and strong foundational mechanized soundness guarantees [Sozeau et al. 2020]. In contrast, LiquidHaskell trusts the Haskell compiler (GHC), the SMT solver (Z3), and its constraint generation rules which have not been formalized. This work, λ_{RF}, serves precisely that purpose: by formalizing and mechanizing a significant subset of LiquidHaskell, leaving out literals, casts, and data types. As far as the user experience is concerned, Coq metatheoretical developments are much faster to check, which was expected since LiquidHaskell comes with expensive inference, and can be aided by relevant libraries. The two tools come with different kinds of automation: tactics *vs.* SMT, which we found to be useful in *complementary* parts of the proofs, pointing the way to possible improvements for both verification styles. Finally, LiquidHaskell greatly facilitates reasoning over mutually defined and partial functions. Next, we expand upon the last two points with snippets from our mechanizations.

Tactics and Automation Coq's tactics and automation often permit shorter proofs as lemmas and constructors can be used with the `apply` tactic without writing out all arguments. For example, in LiquidHaskell soundness (Thm. 6.1) is encoded using Haskell's `Either` for disjunction and dependent pairs for existentials. (`Steps` is defined, using data propositions, as the closure of `Step`.)

```
soundness :: e0:Expr → t:Type → e:Expr → HasTy Empty e0 t → Steps e
  0 e
          → Either {isVal e} (ei::Expr, Step e ei)
soundness _e0 t _e e0_has_t e0_evals_e = case e0_evals_e of
   Refl e0 → progress e0 t e0_has_t        -- e0 = e
   AddStep e0 e1 e0_step_e1 e e1_eval_e →   -- e0 ↪ e1 ↪* e
     soundness e1 t e (preservation e0 t e0_has_t e1 e0_step_e1) e1_eval_e
```

In Coq soundness is proved without any of the three fully applied calls above:

```
Theorem soundness : forall (e0 e:expr) (t:type),
   Steps e0 e → HasTy Empty e0 t → isVal e \/ exists ei, Steps e ei.
Proof. intros; induction H.
  - (* Refl *) apply progress with t; assumption.
  - (* Add  *) apply IHSteps; apply preservation with e; assumption. Qed.
```

Automation tactics could make this proof even shorter, but we retain the essential proof structure.

Mutual Recursion LiquidHaskell makes it easy to define and work with mutually recursive data types, such as our typing and subtyping judgments, and to prove mutually inductive lemmas. Mutually recursive types are not a natural fit for Coq: the automatically generated induction principles do not work, so we need to use the `Scheme` keyword to generate suitable principles. Theorems involving these types cannot be broken up into separate lemmas for each type involved. Rather, one combined statement must be given, which is difficult to use in the `rewrite` tactic.

Another weakness of Coq is that all information about the hypothesis is lost during the induction tactic, which means that structural induction using the normal `induction` tactic only works when a judgment contains no information, *i.e.* the data constructor is instantiated solely with universally quantified variables. For instance, in the proof of the Weakening Lemma 6.5, to do structural induction on `HasTy (concat g g')e t` we must introduce a universally quantified variable `g0` and strengthen our theorem statement with the additional hypothesis `g0 = concat g g'`. While the standard library contains an "experimental" tactic `dependent induction`, we also need to work with the special mutual induction principles that we generate for our types, so we have to directly instantiate the principle with a complex hypothesis and state the lemma as:

```
Lemma lem_weaken_typ' : ( forall (g0 : env) (e : expr) (t : type),
HasTy g0 e t →  ( forall (g g' : env) (x : vname) (t_x : type),
      g0 = concatE g g' →  unique g →  unique g' →
      (binds g) ∩ (binds g') = empty →  ~ (in_env x g) →  ~ (in_env x g')

      →  HasTy (concatE (Cons x t_x g) g') e t ) ) /\ (
forall (g0 : env) (t : type) (t' : type),
  Subtype g0 t t' →  ( forall (g g' : env) (x : vname) (t_x : type),
      g0 = concatE g g' →  unique g →  unique g' →
      (binds g) ∩ (binds g') = empty →  ~ (in_env x g) →  ~ (in_env x g')

      →  Subtype (concatE (Cons x t_x g) g') t t') ).
```

By contrast, LiquidHaskell allows us to state two separate mutually recursive lemma functions for (term variable) weakening: one for typing and one for subtyping judgments. Then we may call either lemma in their own proofs on any smaller instance of the typing (resp. subtyping) judgment.

Partial Functions LiquidHaskell makes it easy to define partial Haskell functions and to prove totality with respect to the refined input types, usually automatically, without having to reason about impossible cases in mechanized proofs. For instance, our syntax does not contain an explicit `error` value, so we only want the function $\delta(c, v)$ to be defined where $c\ v$ can step in our semantics. This is straightforward in LiquidHaskell: we define a predicate `isCompat :: Prim → Value → Bool` and refine the input types of δ to satisfy `isCompat`. In Coq a more roundabout approach is needed: we have to define `isCompat` as an inductive type and include this object as an explicit argument to our `delta` function. However, this makes it harder to prove the determinism of our semantics due to the dependence on the proof object. The difficulty can be sidestepped by defining a partial version of δ with type `Prim → Expr → option Expr` and proving the two functions always agree regardless of proof object, *e.g.* using *subset types*; but since each value comes wrapped with a term-level proof object, agreement proofs would require a *Proof Irrelevance* axiom.

10 RELATED WORK

We discuss the most closely related work on the meta-theory of unrefined and refined type systems.

Representing Binders Our development for λ_F (§ 5) follows the standard presentation of System F's metatheory by Pierce [2002]. The main difference is that ours includes well-formedness of types and environments, which help with mechanization [Rémy 2021] and are crucial when formalizing refinements. One main challenge in the mechanized metatheory is the syntactic representation of variables and binders [Aydemir et al. 2005]. The *named* representation has severe difficulties because of variable capturing substitutions and the *nameless* (*a.k.a.* de Bruijn) requires heavy index shifting. The variable representation of λ_{RF} is *locally nameless representation* [Aydemir et al. 2008; Pollack 1993], where free variables are named, but bound variables are represented by deBruijn indices. Our metatheory still resembles the paper and pencil proofs (performed before mechanization), yet it clearly addresses the following two problems with named bound variables. First, when different refinements are strengthened (as in Fig. 5) the variable capturing problem reappears because we are substituting underneath a binder. Second, subtyping usually permits alpha-renaming of binders, which breaks a required invariant that each λ_{RF} derivation tree is a valid λ_F tree after erasure.

Hybrid & Contract Systems Flanagan [2006] formalizes on paper a monomorphic lambda calculus with refinement types that differs from our λ_{RF} in three ways. First, the denotational soundness methodology of Flanagan [2006] connects subtyping with expression evaluation. We could not follow this approach because encoding type denotations as a data proposition requires a negative occurrence (§ 4.4). Second, in [Flanagan 2006] type checking is hybrid: the developed system is undecidable and inserts runtime casts when subtyping cannot be statically decided. Third, the original system lacks polymorphism. Sekiyama et al. [2017] extended hybrid types with polymorphism, but unlike λ_{RF}, their system does not support semantic subtyping. For example, consider a divide by zero-error. The refined types for `div` and 0 could be given by div :: `Int` → `Int`$\{n : n \neq 0\}$ → `Int` and 0 :: `Int`$\{n : n = 0\}$. This system will compile `div 1 0` by inserting a cast on 0: $\langle$`Int`$\{n : n = 0\} \Rightarrow$ `Int`$\{n : n \neq 0\}\rangle$, causing a definite runtime failure that could have easily been prevented statically. Having removed semantic subtyping, the metatheory of [Sekiyama et al. 2017] is highly simplified. Static refinement type systems (as summarized by Jhala and Vazou [2021]) usually restrict the definition of predicates to quantifier-free first-order formulae that can be *decided* by SMT solvers. This restriction is not preserved by evaluation that can substitute variables with any value, thus allowing expressions that cannot be encoded in decidable logics, like lambdas, to seep into the predicates of types. In contrast, we allow predicates to be any language term (including lambdas) to prove soundness via preservation and progress: our meta-theoretical results trivially apply to systems that, for efficiency of implementation, restrict

their source languages. Finally, none of the above systems (hybrid, contracts or static refinement types) come with a machine checked soundness proof.

Refinement Types in Coq Our soundness formalization follows the axiomatized implication relation of Lehmann and Tanter [2016] that decides subtyping (our rule S-Base) without formally connecting implication and expression evaluation. Lehmann and Tanter [2016]'s Coq formalization of a monomorphic lambda calculus with refinement types differs from λ_{RF} in two ways. First, their axiomatized implication allows them to restrict the language of refinements. We allow refinements to be arbitrary program terms and intend, in the future, to connect our axioms to SMT solvers or other oracles. Second, λ_{RF} includes polymorphism, existentials, and selfification which are critical for context-sensitive refinement typing, but make the metatheory more challenging. Hamza et al. [2019] present System FR, a polymorphic, refined language with a mechanized metatheory of about 30K lines of Coq. Compared to our system, their notion of subtyping is not semantic, but relies on a reducibility relation. For example, even though System FR will deduce that `Pos` is a subtype of `Int`, it will fail to derive that `Int` $\rightarrow$ `Pos` is subtype of `Pos` $\rightarrow$ `Int` as reduction-based subtyping cannot reason about contra-variance. Because of this more restrictive notion of subtyping, their mechanization requires neither the indirection of denotational soundness nor an implication proving oracle. Further, System FR's support for polymorphism is limited in that it disallows refinements on type variables, thereby precluding many practically useful specifications.

Metatheory in LiquidHaskell `LWeb` [Parker et al. 2019] also used LiquidHaskell to prove metatheory, the non-interference of λ_{LWeb}, a core calculus that extends the LIO formalism with database access. The `LWeb` proof did not use refined data propositions, which were not present at development time, and thus it has two major weaknesses compared to our present development. First, `LWeb` *assumes* termination of λ_{LWeb}'s evaluation function; without refined data propositions metatheory can be developed only over terminating functions. This was not a critical limitation since non-interference was only proved for terminating programs. However, in our proof the requirement that evaluation of λ_{RF} terminates would be too strict. In our encoding with refined data propositions such an assumption was not required. Second, the `LWeb` development is not constructive: the structure of an assumed evaluation tree is logically inspected instead of the more natural case splitting permitted only with refined data propositions. This constructive way to develop metatheories is more compact (*e.g.* there is no need to logically inspect derivation trees) and akin to the standard meta-theoretic developments of constructive tools like Coq and Isabelle.

11 CONCLUSIONS & FUTURE WORK

We presented and formalized soundness of λ_{RF}, a core refinement calculus that combines semantic subtyping, existential types, and parametric polymorphism, which are critical for practical refinement typing but whose combination has never been formalized. Our meta-theory is mechanized in both Coq and LiquidHaskell, the latter using the novel feature of refined data propositions to reify derivations as (refined) Haskell datatypes and exploiting SMT to automate various tedious invariants about variables.

While our proof can be mechanized in other specialized proof systems like Agda [Norell 2007] or Isabelle [Nipkow et al. 2002] or those equipped with SMT-based automation like Dafny [Leino 2010] or F* [Martínez et al. 2019], our goal here is not to compare LiquidHaskell against every system. Instead, our primary contribution is to, for the first time, *establish the soundness* of the combination of features critical for practical refinement typing and show that such a proof can be *mechanized as a plain program* with refinement types. To achieve this mechanization we had to develop data propositions that allow constructive verification in LiquidHaskell, which naturally led to a comparison with a Coq mechanization.

Looking ahead, we envision two lines of work on mechanizing metatheory *of* and *with* refinement types.

1. Mechanization of Refinements λ_{RF} covers a crucial but small fragment of the features of modern refinement type checkers. The immediate next step is to extend the language to include literals, casts, and data types, thus covering *all* GHC's core calculus. Next, λ_{RF} can be extended to more sophisticated features of refinement types, such as abstract and bounded refinements and refinement reflection. Similarly, our current work axiomatizes the requirements of the semantic implication checker (*i.e.* SMT solver). It would be interesting to implement a solver and verify that it satisfies that contract, or alternatively, show how proof certificates [Necula 1997] could be used in place of such axioms.

2. Mechanization with Refinements While this work shows that non-trivial meta-theoretic proofs are *possible* with SMT-based refinement types, our experience is that much remains to make such developments *pleasant*. For example, programming would be far more convenient with support for automatically *splitting cases* or filling in *holes* as done in Agda [Norell 2007] and envisioned by Redmond et al. [2021]. Similarly, when a proof fails, the user has little choice but to think really hard about the internal proof state and what extra lemmas are needed to prove their goal. Finally, the stately pace of verification — 9400 lines across 35 files take about 30 minutes — hinders interactive development. Thus, rapid incremental checking, lightweight synthesis, and actionable error messages would go a long way towards improving the ergonomics of verification, and hence remain important directions for future work.

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