Liquidate your Assets

Reasoning about resource usage in Liquid Haskell

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```
insert :: Ord a => x:a -> xs:[a] -> [a]


insert x []   = [x]
insert x (y:ys)
  | x <= y     = x:y:ys
  | otherwise = y:insert x ys
```

Refinement types for length preservation

```
insert :: Ord a => x:a -> xs:[a]
       -> {os:[a]|len os == 1 + len xs}

insert x []   = [x]
insert x (y:ys)
  | x <= y     = x:y:ys
  | otherwise = y:insert x ys
```

SMT automated checking.
Even for list sortedness.

Refinement types for length preservation

```haskell
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       -> {os:[a]|len os == 1 + len xs}

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  | x <= y      = x:y:ys
  | otherwise = y:insert x ys
```

What about resources?
(here number of comparisons)

Tracking resources

```
insert :: Ord a => x:a -> xs:[a]
       -> {os:[a]|len os == 1 + len xs}
        cost o <= len xs
insert x []   = [x]
insert x (y:ys)
  | x <= y    = x:y:ys
  | otherwise = y:insert x ys
```

Tracking resources using The Tick data type

```
 insert :: Ord a => x:a -> xs:[a]
-> {o:Tick {os:[a]|len os == 1 + len xs}
         | tcost o <= len xs }
 insert x []   = [x]
 insert x (y:ys)
  | x <= y      = x:y:ys
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```

The Tick data type

```
data Tick a = Tick { tcost :: Int,
                     tval  :: a  }
```

The Tick data type

```
data Tick a = Tick { tcost :: Int,
                     tval  :: a   }
```

The Applicative Instance

```
pure :: x:a -> {t:Tick a | tcost t == 0}
pure x = Tick 0 x

(<*>) :: f:(Tick (a -> b)) -> x:Tick a
      -> {t:Tick b | tcost t == tcost x + tcost f}
Tick i f <*> Tick j x = Tick (i+j) (f x)
```

Zero resources using the Tick data type

```
 insert :: Ord a => x:a -> xs:[a]
-> {o:Tick {os:[a]|len os == 1 + len xs}
        | tcost o <= len xs }
 insert x []   = pure [x]
 insert x (y:ys)
  | x <= y     = pure (x:y:ys)
  | otherwise = (pure (y:)) <*> insert x ys
```

The Tick data type

```
data Tick a = Tick { tcost :: Int,
                     tval  :: a  }
```

Resource tracking

```
step :: x:a -> {t:Tick a | tcost t == 1}
step x = Tick 1 x

(</>) :: f:(Tick (a -> b)) -> x:Tick a
      -> {t:Tick b | tcost t == 1 + tcost x + tcost f}
Tick i f </> Tick j x = Tick (1+i+j) (f x)
```

Actual resources using the Tick data type

```
 insert :: Ord a => x:a -> xs:[a]
-> {o:Tick {os:[a]|len os == 1 + len xs}
           | tcost o <= len xs }
 insert x []   = pure [x]
 insert x (y:ys)
  | x <= y     = step (x:y:ys)
  | otherwise = (pure (y:)) </> insert x ys
```

Let’s define insertion sort!

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```
isort :: Ord a => xs:[a]
      -> {o:Tick {os:[a]|len os == len xs}
           | tcost o <= (len xs)² }
isort []     = pure []
isort (x:xs) = isort xs >>= insert x
```

Tick is a Monad!

The Tick data type

```
data Tick a = Tick { tcost :: Int,
                     tval  :: a   }
```

The Monad Instance

```
return :: x:a -> {t:Tick a | tcost t == 0}
return x = Tick 0 x

(>>=) :: x:Tick a -> f:(a -> Tick b)
   -> {t:Tick b | tcost t == tcost x
                           + tcost (f (tval x)) }
Tick i x >>= f = case f x of
                    Tick j y -> Tick (i+j) y
```

Resources of insertion sort

```
isort :: Ord a => xs:[a]
      -> {o:Tick {os:[a]|len os == len xs}
           | tcost o <= (len xs)² }
isort []     = pure []
isort (x:xs) = isort xs >>= insert x
```

```
(>>=) :: x:Tick a -> f:(a -> Tick b)
   -> {t:Tick b | tcost t == tcost x
                          + tcost (f (tval x)) }
```

Problem: type level computations!

Solution I: ghost parameter

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            | tcost o <= (len xs)² }
isort []     = pure []
isort (x:xs) = isort xs >>={len xs} insert x
```

```
(>>={n}) :: x:Tick a
   -> f:(a -> {t:Tick b | tcost t <= n})
   -> {t:Tick b | tcost t <= tcost x + n}
```

Solution I: ghost parameter

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isort :: Ord a => xs:[a]
      -> {o:Tick {os:[a]|len os == len xs}
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(>>={n}) :: x:Tick a
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```

Cost of monadic function is bound by n

Solution I: ghost parameter

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```
(>>={n}) :: x:Tick a
   -> f:(a -> {t:Tick b | tcost t <= n})
   -> {t:Tick b | tcost t <= tcost x + n}
```

No type level computations…

Solution I: ghost parameter

```
isort :: Ord a => xs:[a]
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```
(>>={n}) :: x:Tick a
    -> f:(a -> {t:Tick b | tcost t <= n})
    -> {t:Tick b | tcost t <= tcost x + n}
```

✓

… but explicit parameter should be provided.

Resource Tracking using Refinement Types

Tick monad lets you track resources in refinement types

Problem:

The bind operation breaks automatic verification

Solution I: ghost parameter

Solution II: extrinsic proofs

Extrinsic resource analysis proofs

```
isortCostSorted :: Ord a => xs:OList a
      -> { tcost (isort xs) <= len xs }
isortCostSorted []     = () – automated
isortCostSorted [_]    = () – automated
isortCostSorted (x:xs@(y:ys))
 =   tcost (isort (x:xs))
 ==. tcost (isort xs >>={len xs} insert x)
 ==. tcost (isort xs) + tcost (insert x (tval (isort xs)))
     ? isortSortedVal xs – tval (isort xs) == xs
 ==. tcost (isort xs) + tcost (insert x xs)
     ? isortCostSorted xs
 <=. len xs + tcost (insert x xs)
 <=. len xs + tcost (insert x (y:ys))
 <=. len xs + 1
 <=. len (x:xs)
 *** QED
```

Extrinsic resource analysis proofs can be used for arbitrary properties

Relational Properties,
e.g., sorted lists are sorted faster.

```
  xs:OList a
→ ys:{[a] | len xs == len ys}
→ { tcost (isort xs) <= tcost (isort ys) }
```

Extrinsic resource analysis proofs can be used for arbitrary properties

Relational Properties,
e.g., sorted lists are sorted faster.

We encoded all the examples from the relational refinement types[1, 2, 3] work.

): But, required manual proofs :(

[1]Aguirre et al: A Relational Logic for Higher-Order Programs. ICFP'17.
[2]Çiçek et al: Relational Cost Analysis. POPL'17.
[3]Radiček et al:Monadic Refinements for Relational Cost Analysis. POPL'18.

Extrinsic resource analysis proofs can be used for arbitrary properties

Relational Properties,
e.g., sorted lists are sorted faster.

Function optimization,
e.g., [] ++ xs is faster than xs ++ []

We developed operators to simultaneously reason about
1/ resource modification and 2/program equivalence.

Extrinsic resource analysis proofs can be used for arbitrary properties

Function optimization,
e.g., [] ++ xs is faster than xs ++ []

```
xs:[a] → { [] ++ xs <~< xs ++ [] }
```

value preservation &
cost improvement!

Extrinsic resource analysis proofs can be used for arbitrary properties

Relational Properties,
e.g., sorted lists are sorted faster.

Function optimization,
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Higher Order Properties,
e.g., map fusion is an optimization

Extrinsic resource analysis proofs can be used for arbitrary properties

Higher Order Properties,
e.g., map fusion is an optimization

```
mapM f xs >>= mapM g
  >== len xs ==>
 mapM (f >=> g) xs
```

value preservation &
exact cost improvement!

Extrinsic resource analysis proofs can be used for arbitrary properties

Relational Properties,
e.g., sorted lists are sorted faster.

Function optimization,
e.g., [] ++ xs is faster than xs ++ []

Higher Order Properties,
e.g., map fusion is an optimization

In the paper:

More benchmarks

Soundness of the Tick library

What about laziness?
Lazy ADTs should be explicitly defined using Ticks[*]

[*]Danielsson: Lightweight Semiformal Time Complexity Analysis for Purely Functional Data Structures. POPL'08.

Resource Tracking using Refinement Types

Tick monad lets you track resources in refinement types

Problem:

The bind operation breaks automatic verification

Solution I: ghost parameter

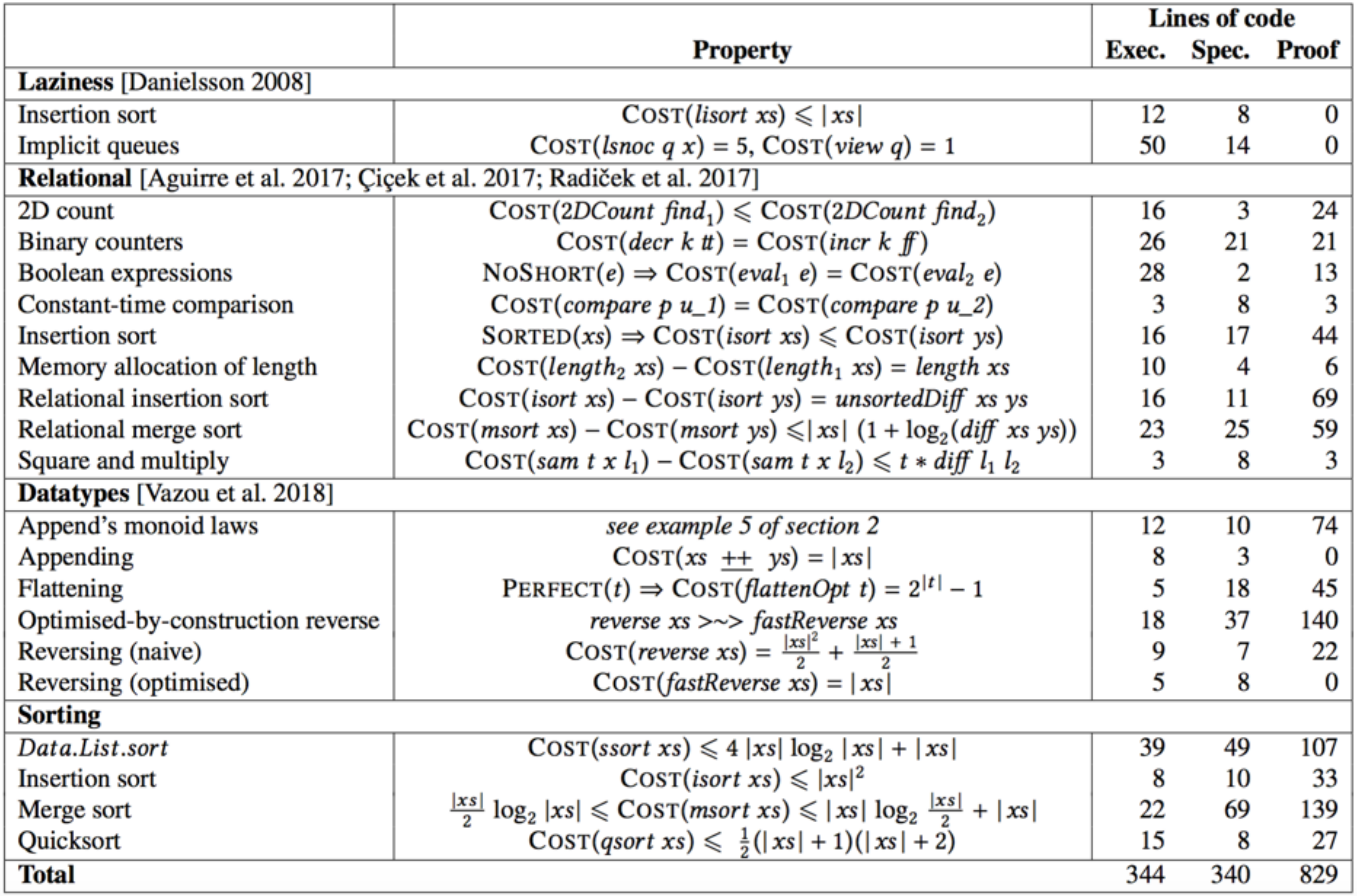
Solution II: extrinsic proofs

Extrinsic proofs can prove arbitrary resource properties

Thanks!

The End

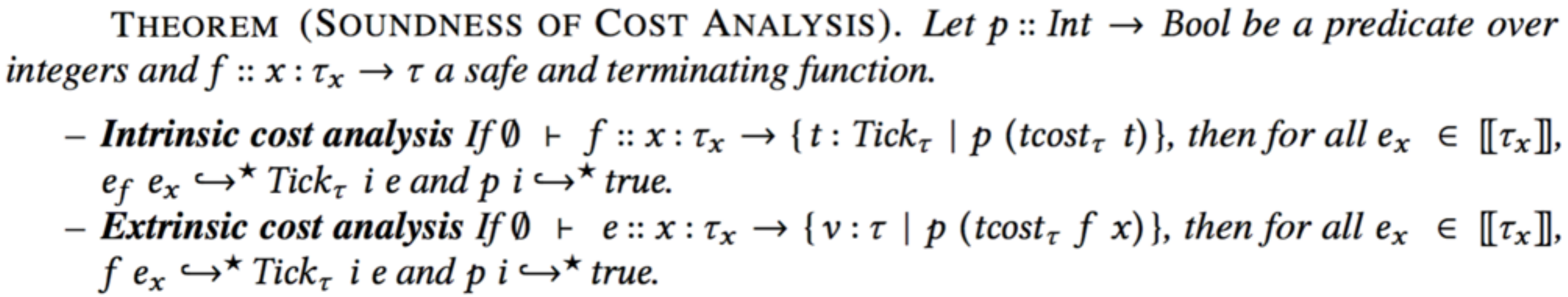
Benchmarks



	Property	Lines of code		
		Exec.	Spec.	Proof
Laziness [Danielsson 2008]				
Insertion sort	$\textsc{Cost}(\mathit{lisort}\ xs) \leqslant \lvert xs \rvert$	12	8	0
Implicit queues	$\textsc{Cost}(\mathit{lsnoc}\ q\ x) = 5$, $\textsc{Cost}(\mathit{view}\ q) = 1$	50	14	0
Relational [Aguirre et al. 2017; Çiçek et al. 2017; Radiček et al. 2017]				
2D count	$\textsc{Cost}(\mathit{2DCount}\ \mathit{find}_1) \leqslant \textsc{Cost}(\mathit{2DCount}\ \mathit{find}_2)$	16	3	24
Binary counters	$\textsc{Cost}(\mathit{decr}\ k\ \mathit{tt}) = \textsc{Cost}(\mathit{incr}\ k\ \mathit{ff})$	26	21	21
Boolean expressions	$\textsc{NoShort}(e) \Rightarrow \textsc{Cost}(\mathit{eval}_1\ e) = \textsc{Cost}(\mathit{eval}_2\ e)$	28	2	13
Constant-time comparison	$\textsc{Cost}(\mathit{compare}\ p\ u\_1) = \textsc{Cost}(\mathit{compare}\ p\ u\_2)$	3	8	3
Insertion sort	$\textsc{Sorted}(xs) \Rightarrow \textsc{Cost}(\mathit{isort}\ xs) \leqslant \textsc{Cost}(\mathit{isort}\ ys)$	16	17	44
Memory allocation of length	$\textsc{Cost}(\mathit{length}_2\ xs) - \textsc{Cost}(\mathit{length}_1\ xs) = \mathit{length}\ xs$	10	4	6
Relational insertion sort	$\textsc{Cost}(\mathit{isort}\ xs) - \textsc{Cost}(\mathit{isort}\ ys) = \mathit{unsortedDiff}\ xs\ ys$	16	11	69
Relational merge sort	$\textsc{Cost}(\mathit{msort}\ xs) - \textsc{Cost}(\mathit{msort}\ ys) \leqslant \lvert xs \rvert\ (1 + \log_2(\mathit{diff}\ xs\ ys))$	23	25	59
Square and multiply	$\textsc{Cost}(\mathit{sam}\ t\ x\ l_1) - \textsc{Cost}(\mathit{sam}\ t\ x\ l_2) \leqslant t * \mathit{diff}\ l_1\ l_2$	3	8	3
Datatypes [Vazou et al. 2018]				
Append's monoid laws	*see example 5 of section 2*	12	10	74
Appending	$\textsc{Cost}(xs\ \underline{++}\ ys) = \lvert xs \rvert$	8	3	0
Flattening	$\textsc{Perfect}(t) \Rightarrow \textsc{Cost}(\mathit{flattenOpt}\ t) = 2^{\lvert t \rvert} - 1$	5	18	45
Optimised-by-construction reverse	*reverse xs >~> fastReverse xs*	18	37	140
Reversing (naive)	$\textsc{Cost}(\mathit{reverse}\ xs) = \frac{\lvert xs \rvert^2}{2} + \frac{\lvert xs \rvert + 1}{2}$	9	7	22
Reversing (optimised)	$\textsc{Cost}(\mathit{fastReverse}\ xs) = \lvert xs \rvert$	5	8	0
Sorting				
Data.List.sort	$\textsc{Cost}(\mathit{ssort}\ xs) \leqslant 4\ \lvert xs \rvert \log_2 \lvert xs \rvert + \lvert xs \rvert$	39	49	107
Insertion sort	$\textsc{Cost}(\mathit{isort}\ xs) \leqslant \lvert xs \rvert^2$	8	10	33
Merge sort	$\frac{\lvert xs \rvert}{2} \log_2 \lvert xs \rvert \leqslant \textsc{Cost}(\mathit{msort}\ xs) \leqslant \lvert xs \rvert \log_2 \frac{\lvert xs \rvert}{2} + \lvert xs \rvert$	22	69	139
Quicksort	$\textsc{Cost}(\mathit{qsort}\ xs) \leqslant \frac{1}{2}(\lvert xs \rvert + 1)(\lvert xs \rvert + 2)$	15	8	27
Total		344	340	829

Metatheory

A corollary of monadic encapsulation + metatheory of refinement types:



THEOREM (SOUNDNESS OF COST ANALYSIS). *Let $p :: Int \to Bool$ be a predicate over integers and $f :: x : \tau_x \to \tau$ a safe and terminating function.*

- ***Intrinsic cost analysis*** *If $\emptyset \vdash f :: x : \tau_x \to \{t : Tick_\tau \mid p\ (tcost_\tau\ t)\}$, then for all $e_x \in [\![\tau_x]\!]$, $e_f\ e_x \hookrightarrow^\star Tick_\tau\ i\ e$ and $p\ i \hookrightarrow^\star true$.*
- ***Extrinsic cost analysis*** *If $\emptyset \vdash e :: x : \tau_x \to \{v : \tau \mid p\ (tcost_\tau\ f\ x)\}$, then for all $e_x \in [\![\tau_x]\!]$, $f\ e_x \hookrightarrow^\star Tick_\tau\ i\ e$ and $p\ i \hookrightarrow^\star true$.*

Future Directions

Resource Bound Inference

Automate lifting or, at least, erasure

Turn Tick into a monad transformer
(e.g., to combine with Parallel Monad)