Functional Extensionality for Refinement Types

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Functional Extensionality for Refinement Types

"Two functions are equal,
if their values are equal at every argument."

$$\text{funExt} :: \forall a\ b.\ f{:}(a \to b) \to g{:}(a \to b) \to (x{:}a \to \underline{\{f\ x = g\ x\}}) \to \{f = g\}$$

short for
$\{v{:}()\ |\ f\ x = g\ x\}$

```
funExt :: ∀a b. f:(a → b) → g:(a → b) → (x:a → {f x = g x}) → {f = g}

incrNat :: Int → Int                              incrInt :: Int → Int
incrNat x = if 0 ≤ x then 0 else x + 1            incrInt x = x + 1

type Nat = {v:Int | 0 ≤ v}
incrEq :: x:Nat → {incrNat x = incrInt x}
incrEq _ = ()

incrFEq :: () → { incrNat = incrInt }
incrFEq _ = funExt incrNat incrInt incrEq

incrEqMap :: xs:[Nat] → {map incrNat xs = map incrInt xs}
incrEqMap _ = incrFEq ()
```

```
funExt :: ∀a b. f:(a → b) → g:(a → b) → (x:a → {f x = g x}) → {f = g}
```

Problem 1: Result type forgets the domain

```
incrEq :: x:Nat → {incrNat x = incrInt x}
incrEq _ = ()

incrFEq :: () → { incrNat = incrInt }
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$$\text{funExt} :: \forall a\ b.\ f{:}(a \to b) \to g{:}(a \to b) \to (x{:}a \to \{f\ x = g\ x\}) \to \{f = g\}$$

Problem 1: Result type forgets the domain

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incrEq :: x:Nat → {incrNat x = incrInt x}
incrEq _ = ()
incrFEq :: () → { incrNat = incrInt }
incrFEq _ = funExt incrNat incrInt incrEq
incrEqMap :: xs:[Nat] → {map incrNat xs = map incrInt xs}
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```
funExt :: ∀a b. f:(a → b) → g:(a → b) → (x:a → {f x = g x}) → {f = g}
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Problem 1: Result type forgets the domain

```
incrEq :: x:Nat → {incrNat x = incrInt x}
incrEq _ = ()

incrFEq :: () → { incrNat = incrInt }
incrFEq _ = funExt incrNat incrInt incrEq

incrEqMap :: xs:[Int] → {map incrNat xs = map incrInt xs}
incrEqMap _ = incrFEq ()

incrBad :: { incrNat (-2) = incrInt (-2) } —  i.e., 0 = -1
incrBad _ = incrFEq ()
```

```
funExt :: ∀a b. f:(a → b) → g:(a → b) → (x:a → {f x = g x}) → {f = g}
```

Problem 2: Domain only appears positively

At instantiation we can pick **any** domain, e.g.,

```
a := {v:Int | false}
```

Under the empty domain, we can trivially prove anything

```
incrFEq :: () → { incrInt = plus2 }
incrFEq _ = funExt incrInt plus2
                 (\_ → ()) — x:{v:Int | false) → {incrInt x = plus2 x}
eqBad :: () → { incrInt 0 = plus2 0 }    — i.e., 1 = 2
eqBad _ = incrFEq ()
```

Problem 1: Result type forgets the domain

Problem 2: Domain only appears positively

Solution: Type-indexed equality

$$\text{PEq}\ a\ \{e_l\}\ \{e_r\}$$

"e_l is equal to e_r on type a"

Problem 1: Result type forgets the domain

Problem 2: Domain only appears positively

Solution: Type-indexed equality

$$\mathrm{PEq}\ a\ \{e_l\}\ \{e_r\}$$

$$\begin{aligned} & f : (a \to b) \to g : (a \to b) \\ & \to (x : a \to \mathrm{PEq}\ b\ \{f\ x\}\ \{g\ x\}) \\ & \to \mathrm{PEq}\ (a \to b)\ \{f\}\ \{g\} \end{aligned}$$

Domain appears

negatively in the result!

PEq: Type-indexed equality

```
data PEq :: * → * where
  XEq ::  f:(a → b) → g:(a → b)
        → (x:a → PEq b {f x} {g x})
        → PEq (a → b) {f} {g}
```

e.g.,

```
XEq incrNat incrInt ...
:: PEq (Nat → Int) {incrNat} {incrInt}
```

PEq: Type-indexed equality

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  BEq :: x:a → y:a → { x = y }
       → PEq a {x} {y}
```

e.g.,

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XEq incrNat incrInt ...
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What if a is function???

PEq: Type-indexed equality

```
data PEq :: * → * where
  XEq ::  f:(a → b) → g:(a → b)
        → (x:a → PEq b {f x} {g x})
        → PEq (a → b) {f} {g}
  BEq :: AEq a ⇒ x:a → y:a → { x ≡ y }
       → PEq a {x} {y}
```

What if a is function???

AEq: axiomatised equality

AEq: axiomatised equality

$$
\begin{array}{l}
\textbf{class}\ \mathrm{AEq}\ a\ \textbf{where} \\
\quad (\equiv) :: a \rightarrow a \rightarrow \mathrm{Bool} \\
\quad \mathrm{reflP} :: x{:}a \rightarrow \{x \equiv x\} \\
\quad \mathrm{symmP} :: x{:}a \rightarrow y{:}a \rightarrow \{x \equiv y \Rightarrow y \equiv x\} \\
\quad \mathrm{transP} :: x{:}a \rightarrow y{:}a \rightarrow z{:}a \rightarrow \{x \equiv y \wedge y \equiv z \Rightarrow x \equiv z\}
\end{array}
$$

e.g., $x :: \mathrm{Nat} \vdash \mathrm{reflP}\ (\mathrm{incrInt}\ x) :: \{\mathrm{incrInt}\ x \equiv \mathrm{incrInt}\ x\}$

$$:: \{\mathrm{incrNat}\ x \equiv \mathrm{incrInt}\ x\}$$

PEq: Type-indexed equality

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data PEq :: * → * where
  XEq ::   f:(a → b) → g:(a → b)
         → (x:a → PEq b {f x} {g x})
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  BEq :: AEq a ⇒ x:a → y:a → { x ≡ y }
        → PEq a {x} {y}
```

e.g.,

```
XEq incrNat incrInt $ \x →
      BEq (incrNat x) (incrInt x) $ reflP (incrInt x)
:: PEq (Nat → Int) {incrNat} {incrInt}
```

PEq: Type-indexed equality

```
data PEq :: * → * where
  XEq :: f:(a → b) → g:(a → b)
         → (x:a → PEq b {f x} {g x})
         → PEq (a → b) {f} {g}

  BEq :: AEq a ⇒ x:a → y:a → { x ≡ y }
        → PEq a {x} {y}

  CEq :: x:a → y:a → ctx:(a → b) → PEq a {x} {y}
        → PEq b {ctx x} {ctx y}
```

e.g.,

```
XEq incrNat incrInt $ \x →
     BEq (incrNat x) (incrInt x) $ reflP (incrInt x)
:: PEq (Nat → Int) {incrNat} {incrInt}
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PEq: Type-indexed equality

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```

e.g.,

```
CEq incrNat incrInt map $
    XEq incrNat incrInt $ \x →
          BEq (incrNat x) (incrInt x) $ reflP (incrInt x)
:: PEq ([Nat] → [Int]) {map incrNat} {map incrInt}
```

PEq: Type-indexed equality

```
data PEq :: * → * where
  XEq :: f:(a → b) → g:(a → b)
         → (x:a → PEq b {f x} {g x})
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e.g.,

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CEq incrNat incrInt map $
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  CEq :: x:a → y:a → ctx:(a → b) → PEq a {x} {y}
       → PEq b {ctx x} {ctx y}

\xs → CEq (map incrNat) (map incrInt) (\f → f xs) $
         CEq incrNat incrInt map $
            XEq incrNat incrInt $ \x →
                BEq (incrNat x) (incrInt x) $ reflP (incrInt x)
:: xs:[Nat] → PEq ([Int]) {map incrNat xs} {map incrInt xs}
```

Can I convert PEq back to SMT equality?

```
\xs → CEq (map incrNat) (map incrInt) (\f → f xs) $
          CEq incrNat incrInt map $
             XEq incrNat incrInt $ \x →
                 BEq (incrNat x) (incrInt x) $ reflP (incrInt x)
:: xs:[Nat] → PEq ([Int]) {map incrNat xs} {map incrInt xs}
```

Can I convert PEq back to SMT equality?

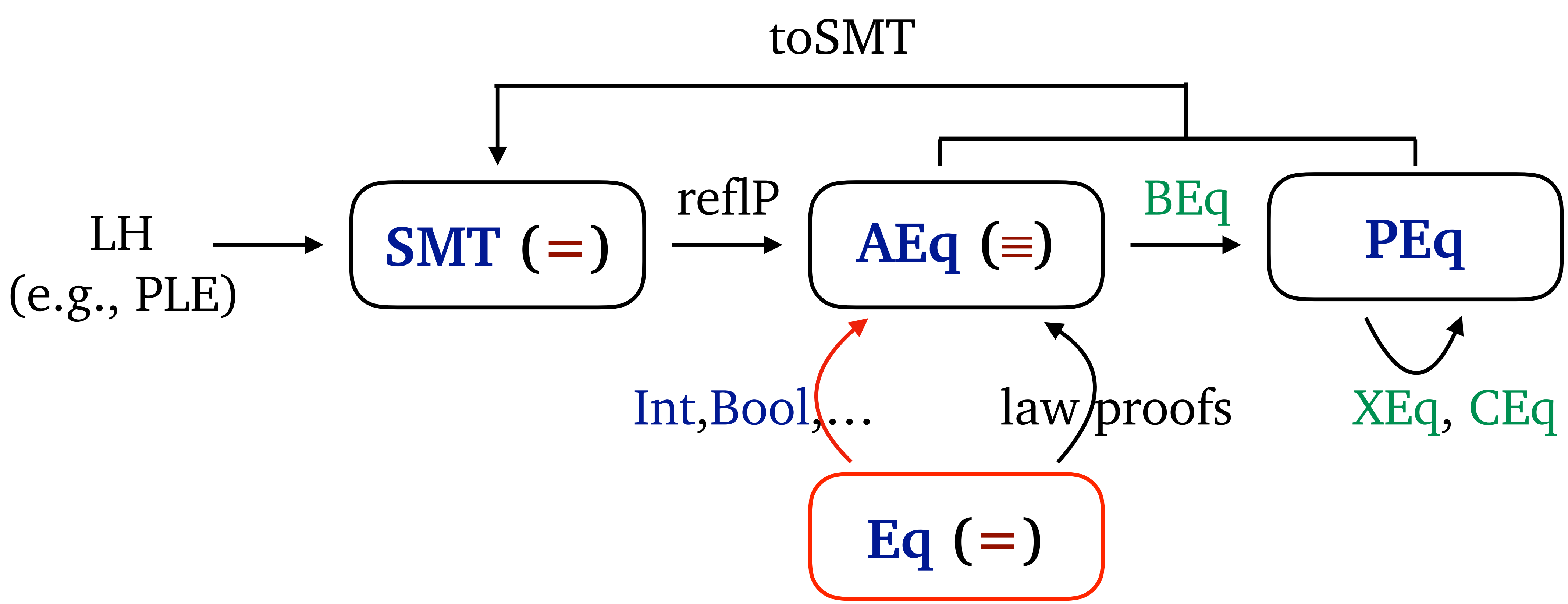
We defined a class for this conversion

```
class AEq a ⇒ SMTEq a where
  toSMT :: x:a → y:a → PEq a {x} {y} → {x = y}
```

which comes with a default instance

```
assume instance AEq a ⇒ SMTEq a where
  toSMT _ _ _ = ()
```

Can I convert PEq back to SMT equality?



* the more equalities, the more consistent the system

Is PEq expressive?

We used PEq to prove

Monoid laws for Endofuctors (64 LoC)

Monad & Monoid laws for Reader (243 LoC)

```
monadLeftIdentity :: a:a → f:(a → Reader r b) → PEq (Reader r b) {bind (pure a) f} {f a}
monadRightIdentity :: m:(Reader r a) → PEq (Reader r a) {bind m pure} {m}
monadAssociativity :: m:(Reader r a) → f:(a → Reader r b) → g:(b → Reader r c)
                    → PEq (Reader r c) {bind (bind m f) g} {bind m (kleisli f g)}
```

Is PEq expressive?

We used PEq to prove
Monoid laws for Endofuctors (64 LoC)
Monad & Monoid laws for Reader (243 LoC)

Is PEq an equivalence?

Paper-and-pencil proofs for toy core calculus
Equivalence properties proofs within Liquid Haskell

PEq is reflexive by “classy induction”

```
-- (1) Property Definition by Refined typeclass
class Reflexivity a where
  refl :: x:a → PEq a {x} {x}


-- (2) Base case (AEq types)
instance AEq a ⇒ Reflexivity a where

  refl a = BEq a a (reflP a)


-- (3) Inductive case (function types)
instance Reflexivity b ⇒ Reflexivity (a → b) where

  refl f = XEq f f (\a → refl (f a))
```

Functional Extensionality for Refinement Types

Problem: Naive axiom forgets the domain type
which leads to inconsistencies, i.e., proves false

Solution: Type Indexed Equality
which is verbose, but expressive and consistent

Draft: https://nikivazou.github.io/static/equality-PLDI21.pdf **Thanks!**