Интегрално смятане

Def

$f(x)$ и $F(x)$. Казваме, че $F(x)$ е примитивна на $f(x)$ в (a,b), ако $F'(x) = f(x)$, $F \in C^1_{(a,b)}$

Ако $F(x)$ е примитивна на $f(x)$ в (a,b), то $F(x)+C$ също е примит. $\left(F(x)+C\right)' = F'(x) = f(x)$

адитивна конст.

Th Ако $F_1(x)$ и $F_2(x)$ са примитивни на $f(x)$ в (a,b), то $F_1(x) - F_2(x) = C$

Def $\int f(x)\,dx = F(x)+C$ – неопределен интеграл

$\underbrace{F(x)+C}$ съвкупност от всички примитивни функции

Основни свойства

$f^{(n)}(x)$, $n \in \mathbb{N}$ $\quad f^{(0)}(x) =: f(x) \quad f^{(-1)}$

$d\int f(x)\,dx = f(x)$

$\int f(x)\,dx = \int F'(x)\,dx = \int dF(x) = F(x)+C$

$$\int [Af(x)+Bg(x)]\,dx = A\int f(x)\,dx + B\int g(x)\,dx$$ – линейност

$$y = f(x) \qquad y' = f'(x) = \frac{dy}{dx} \Rightarrow dy = f'(x)\,dx$$

$$\int f(x)\,dx = F(x) + C \qquad F'(x) = f(x) \Rightarrow \int f(u(x))\,du(x) = F(u(x)) + C$$

$$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + C$$

таблични интеграли

$$\int 0\,dx = C \ ; \ \int 1.dx = \int dx = x + C \qquad \underset{\substack{\alpha\in\mathbb{R}^1 \\ \alpha\neq -1}}{\int x^{\alpha}\,dx = \frac{x^{\alpha+1}}{\alpha+1} + C} \ ; \ \underset{\alpha=-1}{\int \frac{dx}{x} = \ln|x| + C}$$

$$\int e^x\,dx = e^x + C \Rightarrow \underset{a>0}{\int a^x\,dx} = \frac{1}{\ln a}\int e^{\overbrace{x\ln a}^{u}}\,d(\underbrace{x.\ln a}_{u}) = \frac{1}{\ln a}e^{x\ln a} = \frac{a^x}{\ln a} + C$$

$$\int e^u\,du = e^u$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \begin{cases} \arcsin x \\ -\arccos x \end{cases} \quad |x|<1 \quad ; \quad \int \frac{dx}{1+x^2} = \begin{cases} \operatorname{arctg} x \\ -\operatorname{arccotg} x \end{cases}$$

$$\int \frac{dx}{\sqrt{x^2 \pm 1}} = \ln\left| x + \sqrt{x^2 \pm 1} \right| + C \qquad \int \sin x \, dx = -\cos x + C$$

$$\begin{matrix} \operatorname{Arsh} x \\ \operatorname{Arch} x \end{matrix} \qquad \int \cos x \, dx = \sin x + C$$

<u>Извод</u>: Производната на $\forall$ елементарна ф-ция е елементарна ф-ция.

Интеграли на някои елемент. ф-ции не са елементарни ф-ции

$$\int e^{-x^2} dx \qquad \int \cos x^2 dx \text{ и } \int \sin x^2 dx \qquad Ln(x) = \int \frac{dx}{\ln x} \qquad Co(x) = \int \frac{\cos x}{x} dx \; ; \; Si(x) = \int \frac{\sin x}{x} dx$$

интеграл на Поасон; интеграли на Френел; интегрален логаритъм; интегрален косинус; интегрален синус

$$\int \sqrt{2px}\,dx = \sqrt{2p}\int \sqrt{x}\,dx = \sqrt{2p}\int x^{\frac{1}{2}}\,dx = \sqrt{2p}\,\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \sqrt{2p}\,\frac{2}{3}x^{\frac{3}{2}} + C =$$

$p > 0$

$$= \frac{2}{3}\sqrt{2p}\,\underbrace{\sqrt{x^3}}_{x\sqrt{x}} + C$$

$$;\quad \int \frac{dx}{x^2+7} = \frac{1}{7}\int \frac{dx}{1+\left(\frac{x}{\sqrt{7}}\right)^2} = \frac{1}{7}\sqrt{7}\int \frac{d\left(\frac{x}{\sqrt{7}}\right)}{1+\left(\frac{x}{\sqrt{7}}\right)^2} = \frac{1}{\sqrt{7}}\,\mathrm{arctg}\,\frac{x}{\sqrt{7}} + C$$

$$\int \frac{dx}{\sqrt{4+x^2}} = \frac{1\cdot 2}{2}\int \frac{d\left(\frac{x}{2}\right)}{\sqrt{\left(\frac{x}{2}\right)^2+1}} =$$

$$\frac{du}{1+u^2}$$

$$= \ln\left|\frac{x}{2} + \sqrt{\frac{x^2}{4}+1}\right| + C = \ln\left|\frac{x}{2} + \frac{1}{2}\sqrt{x^2+4}\right| + C$$

$$= \ln\left|\frac{x+\sqrt{x^2+4}}{2}\right| = \ln\left|x+\sqrt{x^2+4}\right| - \ln 2 + C$$

б) интегр. ч/з вкарване под знака на диференциала

$$u = 5x - 2$$

$$\int \frac{dx}{\sqrt{5x-2}} = \frac{1}{5}\int (5x-2)^{-\frac{1}{2}}\,d(5x-2) = \frac{1}{5}\int u^{-\frac{1}{2}}\,du = \frac{1}{5}\,\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C =$$

$u = 5x - 2$

$$= \frac{1}{5}\int u^{-\frac{1}{2}}\,du = \frac{1}{5}\,\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C = \frac{2}{5}u^{\frac{1}{2}} + C = \frac{2}{5}\sqrt{5x-2} + C$$

$$\int \frac{x\,dx}{a^4+x^4} = \frac{1}{2}\int \frac{dx^2}{a^4+x^4} = \frac{1}{2}\int \frac{du}{a^4+u^2} = \frac{1\cdot a^2}{2a^4}\int \frac{d\left(\frac{u}{a^2}\right)}{1+\left(\frac{u}{a^2}\right)^2} = \frac{1}{2a^2}\,\mathrm{arctg}\,\frac{u}{a^2} + C =$$

$a \neq 0$

$$x\,dx = \frac{1}{2}\,dx^2 \qquad u = x^2 \qquad = \frac{1}{2a^2}\,\mathrm{arctg}\left(\frac{x}{a}\right)^2 + C$$

$$\int \frac{dx}{\sin x} = \frac{1\cdot 2}{2}\int \frac{d\left(\frac{x}{2}\right)}{\sin\frac{x}{2}\cos\frac{x}{2}} = \int \frac{\frac{d\left(\frac{x}{2}\right)}{\cos^2\frac{x}{2}}}{\mathrm{tg}\,\frac{x}{2}} = \int \frac{d\left(\mathrm{tg}\,\frac{x}{2}\right)}{\mathrm{tg}\,\frac{x}{2}} = \ln\left|\mathrm{tg}\,\frac{x}{2}\right| + C$$

$$\cos^2\frac{x}{2}$$

$$\int \frac{du}{u} \qquad (\mathrm{tg}\,x)' = \frac{1}{\cos^2 x}$$

$$\int \frac{dx}{\cos x} = \int \frac{dx}{\sin\left(x+\frac{\pi}{2}\right)}$$

$$\int tg\sqrt{x}\,\frac{dx}{\sqrt{x}} = \int \frac{tg\sqrt{x}}{\sqrt{x}}\,dx = 2\int tg\sqrt{x}\,d\sqrt{x} = 2\int tg\,u\,du = 2\int \frac{\sin u}{\cos u}\,du =$$

$$\frac{1}{\sqrt{x}}\,dx = x^{-\frac{1}{2}}dx = \frac{d\,x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = 2\,dx^{\frac{1}{2}} = 2d\sqrt{x} \qquad u = \sqrt{x}$$

$$= -2\int \frac{d\cos u}{\cos u} = -2\ln|\cos u| + C = -2\ln|\cos\sqrt{x}| + C = \ln(\cos\sqrt{x})^{-2} + C$$

$$= \ln\frac{1}{\cos^2\sqrt{x}} + C = -\ln\cos^2\sqrt{x} + C$$

в) смяна на променливите

$$x = \varphi(t) \qquad dx = d\varphi(t) = \varphi' dt$$

$$\int f(x)\,dx = \int f(\varphi(t))\,\varphi'(t)\,dt$$

$$\int x\sqrt{x-1}\,dx = \quad \text{субст. } t = \sqrt{x-1} \Rightarrow x - 1 = t^2 \Rightarrow x = t^2 + 1 \Rightarrow dx = 2t\,dt$$

$$= 2\int (t^2+1)\,t^2\,dt = 2\int (t^4 + t^2)\,dt = 2\left(\frac{t^5}{5} + \frac{t^3}{3}\right) + C = \frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{2}{3}(x-1)^{\frac{3}{2}} + C$$

$$\int \sqrt{a^2-x^2}\,dx =$$

$a>0$

Тригонометр. субституция

$$x = a\begin{cases}\sin t\\ \cos t\end{cases}$$

$$\frac{x = a\sin t}{dx = a\cos t\,dt}$$

$$= a\int\sqrt{a^2-a^2\sin^2 t}\,\cos t\,dt = a^2\int\sqrt{\underbrace{1-\sin^2 t}_{\cos t}}\,\cos t\,dt =$$

$$= a^2\int\cos^2 t\,dt = \frac{a^2}{2}\int(1+\cos 2t)\,dt = \frac{a^2}{2}\left(t + \frac{1}{2}\int\cos 2t\,d2t\right) = \frac{a^2}{2}\left(t + \frac{\sin 2t}{2}\right) + C =$$

$$\left.\begin{aligned}\cos^2\alpha &= \frac{1+\cos 2\alpha}{2}\\ \sin^2\alpha &= \frac{1-\cos 2\alpha}{2}\end{aligned}\right\}$$ формули за понижаване на степента

$$t = \arcsin\frac{x}{a}$$

$$= \frac{a^2}{2}\left(\arcsin\frac{x}{a} + 2\sin\arcsin\frac{x}{a}\cdot\cos\arcsin\frac{x}{a}\right) + C =$$

$$\sqrt{1-\sin^2\arcsin\frac{x}{a}}$$

$$= \frac{a^2}{2}\left(\arcsin\frac{x}{a} + 2\frac{x}{a}\sqrt{1-\left(\frac{x}{a}\right)^2}\right) + C$$

2) интегриране по части $u(x), v(x)$: $\int u(x)\,dv(x) = u(x)v(x) - \int v(x)\,du(x)$

формула за интегриране по части

$$\int \frac{x\,dx}{\sin^2 x} = -\int x\,d\cot g\,x = -\left(x\cot g\,x - \int \cot g\,x\,dx\right) =$$

$$= -x\cot g\,x + \int \frac{\cos x}{\sin x}\,dx = -x\cot g\,x + \int \frac{d\sin x}{\sin x} = -x\cot g\,x + \ln|\sin x| + c$$

$$I = \int \sqrt{a^2-x^2}\,dx = x\sqrt{a^2-x^2} - \int x\,d\sqrt{a^2-x^2} = x\sqrt{a^2-x^2} - \int \frac{a^2-a^2-x^2}{\sqrt{a^2-x^2}}\,dx =$$

$$d(a^2-x^2)^{\frac{1}{2}} = \frac{1}{\cancel{2}}(a^2-x^2)^{\frac{1}{2}-1}\cdot(-\cancel{2}x)\,dx = \frac{-x}{\sqrt{a^2-x^2}}\,dx$$

$$= x\sqrt{a^2-x^2} - \int \frac{a^2-x^2-a^2}{\sqrt{a^2-x^2}}\,dx = x\sqrt{a^2-x^2} - \underbrace{\int \sqrt{a^2-x^2}\,dx}_{I} + a^2\int \frac{dx}{\sqrt{a^2-x^2}}$$

$$2I = x\sqrt{a^2-x^2} + \frac{a^2\,\cancel{a}}{\cancel{a}}\int \frac{d\left(\frac{x}{a}\right)}{\sqrt{1-\left(\frac{x}{a}\right)^2}} \Rightarrow 2I = x\sqrt{a^2-x^2} + a^2\arcsin\frac{x}{a}$$

$$2I = x\sqrt{a^2-x^2} + a^2 \arcsin\frac{x}{a} \Rightarrow I = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C$$

$$I_n = \int\frac{1\,dx}{(a^2+x^2)^n} = \frac{1}{a^2}\int\frac{x^2+a^2-x^2}{(a^2+x^2)^n}dx = \frac{1}{a^2}\int\frac{dx}{(a^2+x^2)^{n-1}} - \frac{1}{a}\int\frac{x.x\,dx}{(a^2+x^2)^n} =$$

$a>0$
$n\in\mathbb{N}$

$$n=1 \quad I_1 = \int\frac{dx}{a^2+x^2} = \frac{1a}{a^2}\int\frac{d\left(\frac{x}{a}\right)}{1+\left(\frac{x}{a}\right)^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a}$$

$$= \frac{1}{a^2}I_{n-1} - \frac{1}{2a}\int\frac{x\,d(x^2+a^2)}{(a^2+x^2)^n} = \frac{1}{a^2}I_{n-1} + \frac{1}{2a(n-1)}\int x\,d\frac{1}{(a^2+x^2)^{n-1}} =$$

$$x\,u^{-n}d\underset{x^2+a^2}{u} = x\frac{d\,u^{-n+1}}{-n+1} = \frac{1}{1-n}x\,d\left(\frac{1}{u^{n-1}}\right)$$

$$= \frac{1}{a^2}I_{n-1} + \frac{1}{2a(n-1)}\left[\frac{x}{(a^2+x^2)^{n-1}} - \int\frac{dx}{(a^2+x^2)^{n-1}}\right] \Rightarrow I_n = \left(\frac{1}{a^2} - \frac{1}{2a(n-1)}\right)I_{n-1} +$$

$$+ \frac{1}{2a(n-1)}\frac{x}{(a^2+x^2)^{n-1}}$$

$$I=\int e^{ax}\sin bx\,dx=\frac{1}{a}\int \sin bx\,de^{ax}=\frac{1}{a}\Big(e^{ax}\sin bx-b\overbrace{\int e^{ax}\cos bx\,dx}^{J}\Big)$$

$$J=\int e^{ax}\cos bx\,dx=\frac{1}{a}\int \cos bx\,de^{ax}=\frac{1}{a}\Big(e^{ax}\cos bx+b\underbrace{\int e^{ax}\sin bx\,dx}_{I}\Big)$$

$$\left|\begin{array}{l} I+\frac{b}{a}J=\frac{1}{a}e^{ax}\sin bx \\ -\frac{b}{a}I+J=\frac{1}{a}e^{ax}\cos bx \end{array}\right. \qquad \left|\begin{array}{llll} aI+bJ=e^{ax}\sin bx & b & I & a \\ -bI+aJ=e^{ax}\cos bx & a\ + & & b\ - \end{array}\right.$$

$$(b^2+a^2)J=e^{ax}(b\sin bx+a\cos bx)\Longrightarrow J=\frac{e^{ax}}{a^2+b^2}(b\sin bx+a\cos bx)+C$$

д) отделяне на точен квадрат

$$\int\frac{dx}{3x^2-x+1}$$

$$3x^2-x+1=(x\sqrt{3})^2+2x\sqrt{3}\left(-\frac{1}{2\sqrt{3}}\right)+\left(-\frac{1}{2\sqrt{3}}\right)^2-$$
$$-\left(-\frac{1}{2\sqrt{3}}\right)^2+1=\left(x\sqrt{3}-\frac{\sqrt{3}}{6}\right)^2+1-\frac{\sqrt{3}}{6}$$

$$3x^2-x+1=\left(x\sqrt{3}\right)^2+2x\sqrt{3}\left(-\frac{1}{2\sqrt{3}}\right)+\left(-\frac{1}{2\sqrt{3}}\right)^2-$$

$$-\left(-\frac{1}{2\sqrt{3}}\right)^2+1=\left(x\sqrt{3}-\frac{\sqrt{3}}{6}\right)^2+1-\left(\frac{\sqrt{3}}{6}\right)^2=\left(x\sqrt{3}-\frac{\sqrt{3}}{6}\right)^2+1-\frac{1}{12}=\left(x\sqrt{3}-\frac{\sqrt{3}}{6}\right)^2+\left(\frac{\sqrt{11}}{2\sqrt{3}}\right)^2$$

$$\int\frac{dx}{3x^2-x+1}=\frac{1}{\sqrt{3}}\int\frac{d\left(x\sqrt{3}-\frac{\sqrt{3}}{6}\right)}{\left(x\sqrt{3}-\frac{\sqrt{3}}{6}\right)^2+\left(\frac{\sqrt{11}}{2\sqrt{3}}\right)^2}=\frac{1}{\sqrt{3}}\int\frac{du}{u^2+\left(\frac{\sqrt{33}}{6}\right)^2}=\frac{1}{\sqrt{3}}\int\frac{du}{u^2+\frac{11}{12}}=$$

$$u=x\sqrt{3}-\frac{\sqrt{3}}{6}$$

$$=\frac{1}{\sqrt{3}}\frac{12}{11}\frac{6}{\sqrt{33}}\int\frac{d\left(\frac{u}{\frac{\sqrt{33}}{6}}\right)}{1+\left(\frac{u}{\frac{\sqrt{33}}{6}}\right)^2}=\frac{12\cdot6}{11\underset{3\sqrt{11}}{\sqrt{99}}}\,arctg\,\frac{6u}{\sqrt{33}}+C$$

$$=\frac{12\cdot\cancel{6}^{2}}{11\cdot\cancel{3}\sqrt{11}}=\frac{24}{11\sqrt{11}}\,arctg\,\frac{6x\sqrt{3}-\sqrt{3}}{\sqrt{33}}+C=\frac{24}{11\sqrt{11}}\,arctg\,\frac{6x-1}{\sqrt{11}}+C$$