$$\frac{1}{2\sqrt{x}}$$

Правила за диференциране

$$y = x^{-\frac{1}{2}} = \frac{1}{\sqrt{x}}$$

$$y^{(n)} = ? \quad (y')' = y'' \quad y^{(0)} =: y$$

$$(y^{(n)})' = y^{(n+1)}$$

$$\frac{1}{\sqrt[3]{x}} = x^{-\frac{1}{3}}$$

$$y' = -\frac{1}{2} x^{-\frac{1}{2}-1} = -\frac{1}{2} x^{-\frac{3}{2}} = -\frac{1}{2\sqrt{x^3}}$$

$$dy = f'(x)dx$$

диференциали

$$d(dy) = d^2y = f''(x)dx^2$$

$$y'' = -\frac{1}{2}\left(-\frac{3}{2}\right)x^{-\frac{3}{2}-1} = \frac{1.3}{2^2}x^{-\frac{5}{2}}, \quad d^n y = d(d^{n-1}y) = f^{(n)}(x)dx^n \quad \text{П} \ \text{g-и}$$

$$y''' = \frac{1.3}{2^2}\left(-\frac{5}{2}\right)x^{-\frac{7}{2}} = -\frac{1.3.5}{2^3}x^{-\frac{7}{2}}$$

$$y^{(k)} = (-1)^k \frac{(2k-1)!!}{2^k} x^{-\frac{2k+1}{2}}$$

$$(2k-1)!!! = 1.5.9.13\ldots$$

$$(2k-1)!! = 1.3.5.7\ldots$$

$$(2k)!! \quad 2k-1$$

$$2.4.6$$

$$y^{(k)} = (-1)^k \frac{(2k-1)!!}{2^k} x^{-\frac{2k+1}{2}}$$

$$y^{(k+1)} = \left(y^{(k)}\right)' = \frac{(-1)^k (2k-1)!!}{2^k} \left(-\frac{2k+1}{2}\right) \cdot x^{-\frac{2k+1}{2}-1} =$$

$$= \frac{(-1)^{k+1}(2k+1)!!}{2^{k+1}} x^{-\frac{2k+3}{2}}$$

$$-\frac{2k+1}{2} - 1 = \frac{-(2k+1)-2}{2} = \frac{-2k-3}{2}$$

$$y^{(n)} = \frac{(-1)^n (2n-1)!!}{2^n} x^{-\frac{2n+1}{2}}$$

$$y = \sqrt{x}$$

$$d^n y = y^{(n)} dx^n$$

Формула на Нютон-Лайбниц

$$y = u(x)v(x) \qquad [u(x)v(x)]^{(n)} = \sum_{i=0}^{n} \binom{n}{i} u^{(n-i)} v^{(i)}$$

$\underline{\text{пр.}}$ $y = \underbrace{x^3}\underbrace{\ln x}$ $v(x) = x^3$; $v' = 3x^2$; $v'' = 6x$; $v''' = 6$;

$u(x) = \ln x$ $v^{(k)} = 0$

$k \geq 4$

$u' = \dfrac{1}{x}$; $u'' = -\dfrac{1}{x^2} = -x^{-2}$; $u''' = 2x^{-3}$; $u^{IV} = -2.3x^{-4}$; ...

$\overset{\shortparallel}{x^{-1}}$

$u^{(k)} = (-1)^{k+1}(k-1)!\,x^{-k} = \dfrac{(-1)^{k+1}(k-1)!}{x^k}$

$$(x^3 \ln x)^{(n)} = \binom{n}{0} u^{(n)} v^{(0)} + \binom{n}{1} u^{(n-1)} v' + \binom{n}{2} u^{(n-2)} v'' + \binom{n}{3} u^{(n-3)} v''' + 0 =$$

$$= \frac{(-1)^{n+1}(n-1)!}{x^n} x^3 + n\frac{(-1)^n (n-2)!}{x^{n-1}} 3x^2 + \frac{n(n-1)}{2} \frac{(-1)^{n-1}(n-3)!}{x^{n-2}} 6x + \frac{n(n-1)(n-2)}{6} \cdot$$

$$\cdot \frac{(-1)^{n-2}(n-4)!}{x^{n-3}} 6$$

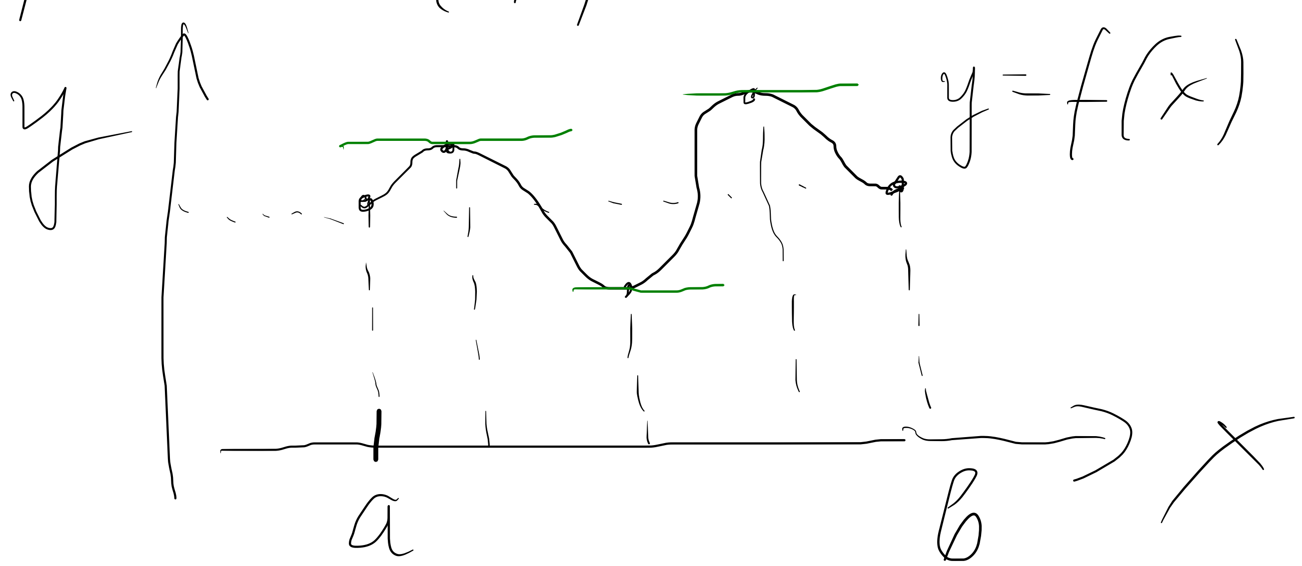
$$= \frac{(-1)^{n+1}(n-1)!}{x^n}\, x^3 + n\,\frac{(-1)^n (n-2)!}{x^{n-1}}\, 3x^2 + \frac{n(n-1)}{2}\,\frac{(-1)^{n-1}(n-3)!}{x^{n-2}}\, 6x + \frac{n(n-1)(n-2)}{6}\cdot\frac{(-1)^{n-2}(n-4)!}{x^{n-3}}\, 6 + 0 =$$

$$= \frac{1}{x^{n-3}}\left[(-1)^{n+1}(n-1)! + 3n(-1)^n(n-2)! + 3n(n-1)(-1)^{n-1}(n-3)! + n(n-1)(n-2)(-1)^{n-2}(n-4)!\right]$$

$$0! \overset{def}{=} 1$$

Теорема за крайните нараствания

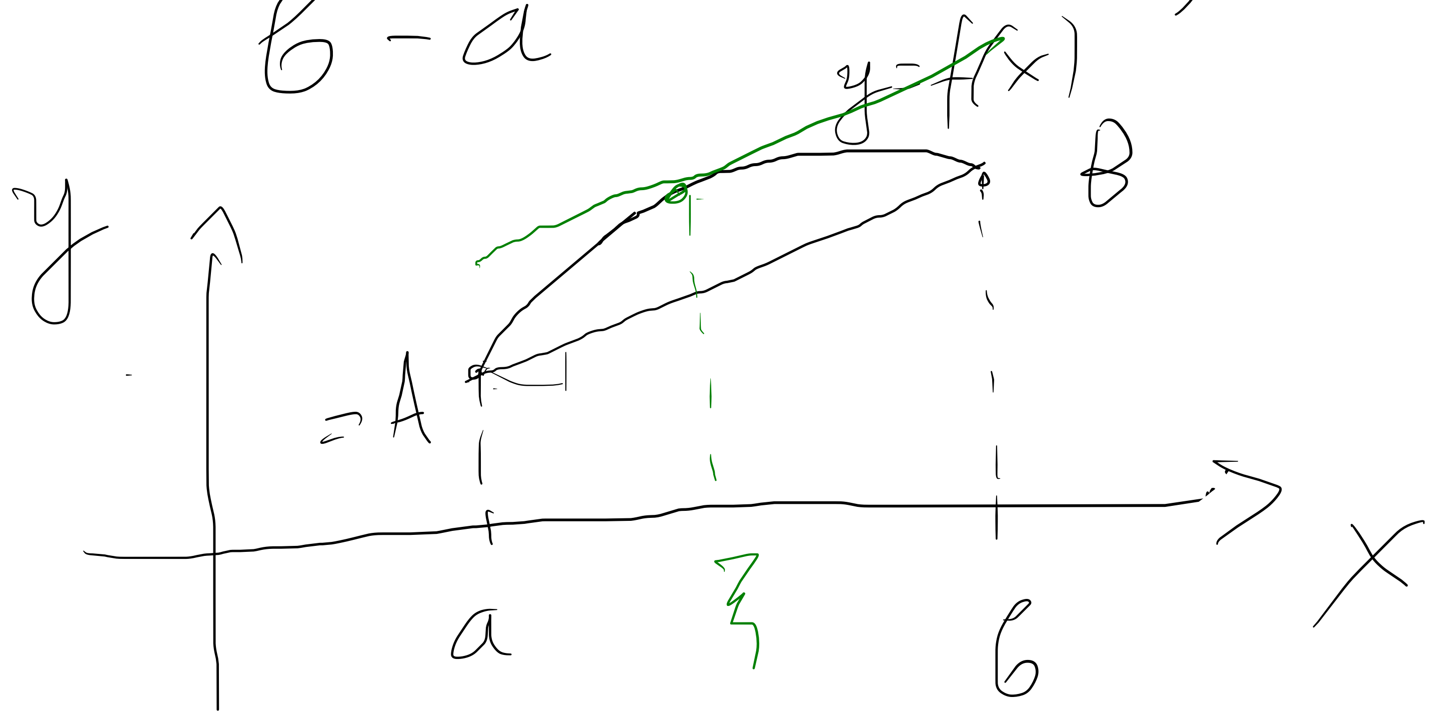
Th (Рол) $f(x)\in C_{[a,b]}$, $f(x)\in C^1_{(a,b)}$ и $f(a)=f(b)$. Тогава $\exists\, \xi\in(a,b) : f'(\xi)=0$.

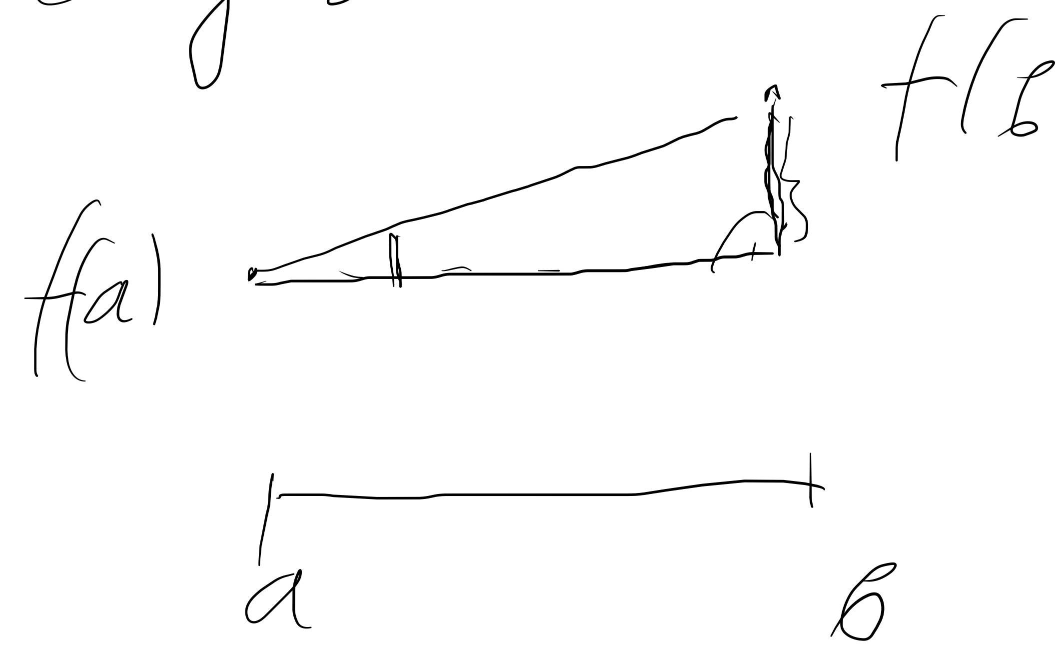


Th (Лагранж) $f(x)\in C_{[a,b]}$, $f(x)\in C^1_{(a,b)}$. Тогава $\exists\, \xi\in(a,b)$ и такава, че $f(b)-f(a)=(b-a)f'(\xi)$

Ако $f(b)=f(a) \Rightarrow f'(\xi)=0 \Rightarrow$ Th (Рол) е частен случай

$$\frac{f(b)-f(a)}{b-a}=f'(\xi)$$



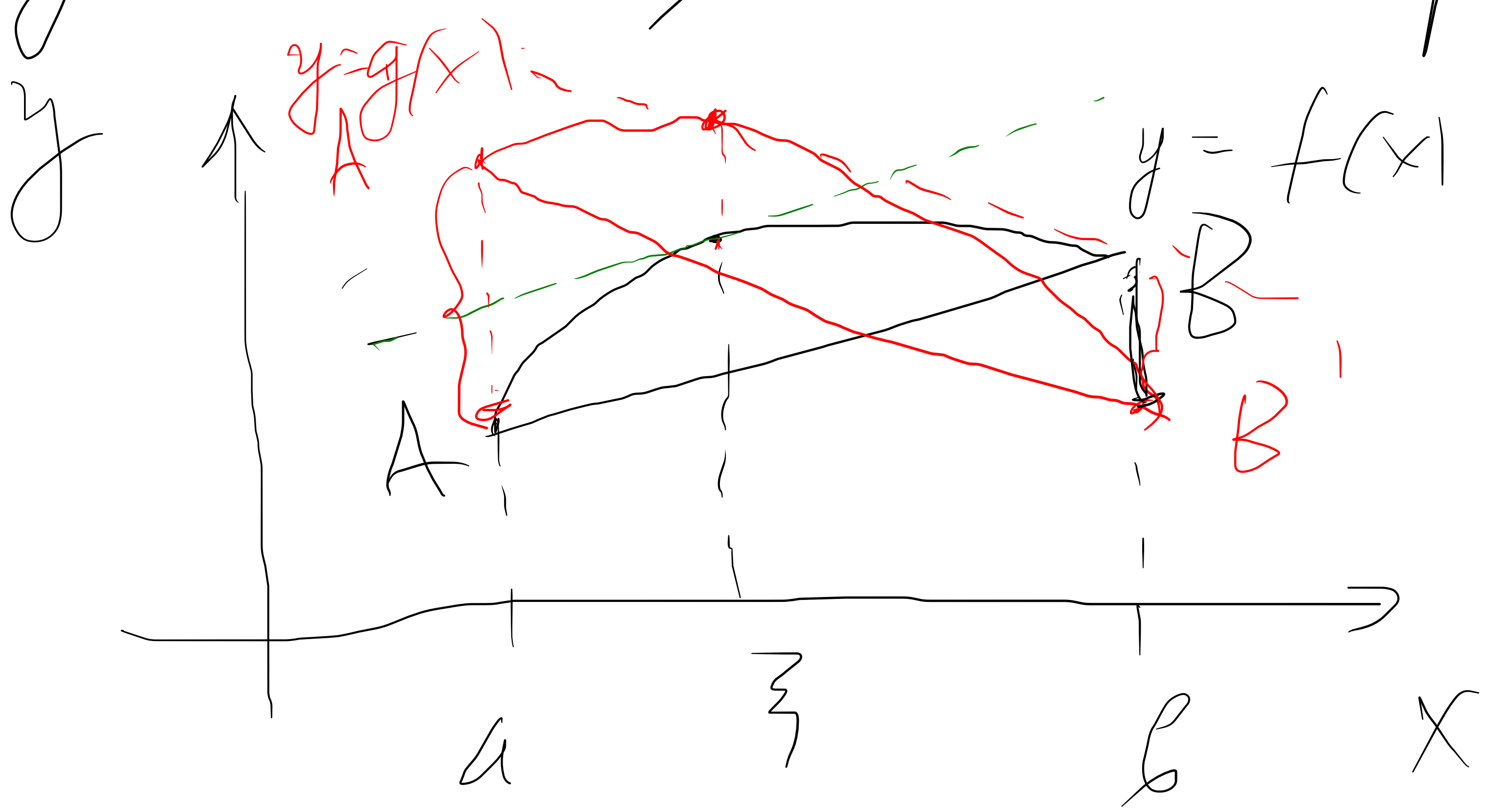


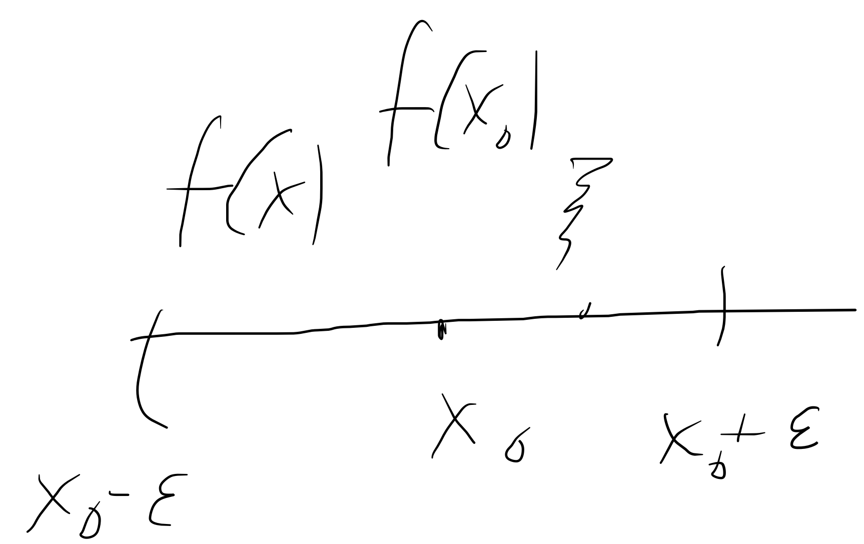
$A(a, f(a))$ $B(b, f(b))$

Th (Коши) $f(x), g(x) \in C_{[a,b]}$ и $f(x), g(x) \in C^1_{(a,b)}$ и $g'(x) \neq 0$ в (a,b)

Тогава $\exists\, \xi \in (a,b)$: $\dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f'(\xi)}{g'(\xi)}$

Ако $g(x) = x \Rightarrow$ Th на Лагранж.





Формула на Тейлор

$f(x) \in C^{n+1}_{(x_0-\varepsilon,\, x+\varepsilon)}$ $f(x_0)$ $\quad f(x) = f(x_0) + \dfrac{f'(x_0)}{1!}(x-x_0) + \dfrac{f''(x_0)}{2!}(x-x_0)^2 + \ldots + \dfrac{f^{(n)}(x_0)}{n!}(x-x_0)^n + R_n(x)$

$$R_n(x) = \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(\xi) \quad \xi = x_0 + \theta(x-x_0)$$
$$0 < \theta < 1$$

Ако $x_0 \equiv 0$: $f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots +$

формула на Маклорен $\frac{f^{(n)}(0)}{n!}x^n + R_n$

$$R_n = \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(\xi), \quad \xi = \theta x$$

За полиноми $R_n(x) \equiv 0$ от степен $\underline{n}$

<u>пр.</u> $f(x) = x^5 - 2x^4 + x^3 - x^2 + 2x - 1$

Тейлор в околн. на т. $x_0 = 1$ (по степените на $x-1$)
МакЛорен

$$f' = 5x^4 - 8x^3 + 3x^2 - 2x + 2 \qquad f(1) = 1 - 2 + \cancel{1} - \cancel{1} + 2 - \cancel{1} = 0 \; ; \; f(0) = -1$$

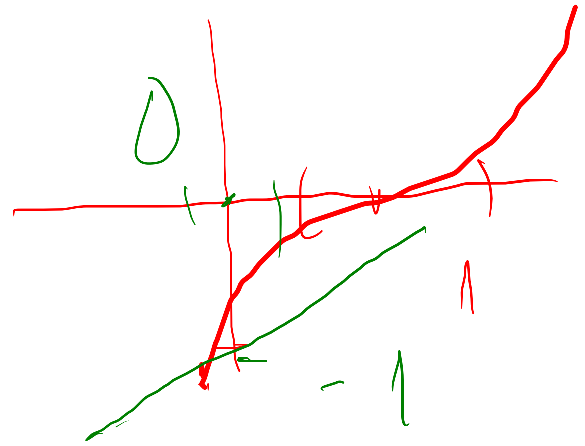
$$f'' = 20x^3 - 24x^2 + 6x - 2 \qquad f'(1) = 5 - 8 + 3 \cancel{-2+2} = 0 \; ; \; f'(0) = 2$$

$$f''' = 60x^2 - 48x + 6 \qquad f''(1) = 20 - 24 + 6 - 2 = 0 \; ; \; f''(0) = -2$$

$$f^{IV} = 120x - 48 \qquad f'''(1) = 60 - 48 + 6 = 18 \; ; \; f'''(0) = 6$$

$$f^{V} = 120 \qquad f^{IV}(1) = 120 - 48 = 72 \; ; \; f^{IV}(0) = -48$$

$$f^{V}(1) = 120 \; ; \; f^{V}(0) = 120$$



$$f^{(n)} = 0$$

$$n \geq 6$$

$$x^5 - 2x^4 + x^3 - x^2 + 2x - 1 = \begin{cases} 0 + \frac{x-1}{1!}\,0 + 0\,\frac{(x-1)^2}{2!} + 18\,\frac{(x-1)^3}{3!} + 72\,\frac{(x-1)^4}{4!} + 120\,\frac{(x-1)^5}{5!} \\ -1 + 2\,\frac{x}{1!} - 2\,\frac{x^2}{2!} + 6\,\frac{x^3}{3!} - 48\,\frac{x^4}{4!} + 120\,\frac{x^5}{5!} \end{cases}$$

$$= \begin{cases} 3(x-1)^3 + 3(x-1)^4 + (x-1)^5 \\ \boxed{-1 + 2x} - x^2 + x^3 - 2x^4 + x^5 \end{cases}$$

$$|x - x_0| < \varepsilon \ll 1 \qquad x \to x_0$$

$$|x| \ll 1 \qquad n \in \mathbb{N} \qquad |x - x_0|^n \ll 1$$

пр.: $f(x) = \operatorname{sh} x$ за $x \in [0, 1]$ Маклорен до VI член

$f' = \operatorname{ch} x$ $\quad f(0) = 0$

$f'' = \operatorname{sh} x$ $\quad f'(0) = 1$

$f''' = \operatorname{ch} x$ $\quad f''(0) = 0$

$f^{IV} = \operatorname{sh} x$ $\quad f'''(0) = 1$

$f^{V} = \operatorname{ch} x$ $\quad f^{IV}(0) = 0$

$f^{VI} = \operatorname{sh} x$ $\quad f^{V}(0) = 1$

$f^{VII} = \operatorname{ch} x$ $\quad f^{VI}(0) = 0$

$f^{VII}(\xi) = \operatorname{ch} \xi$

$0 < \xi < 1$

$$\operatorname{sh} x = 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 + \frac{1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \frac{0}{6!}x^6 + R_6$$

$$R_6 = \frac{x^7}{7!}\operatorname{ch} \xi$$

$$\operatorname{sh} x = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!}\operatorname{ch} \xi$$

$f^{VII}(\xi) = ch\,\xi$ $\quad sh\,x = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!}ch\,\xi$

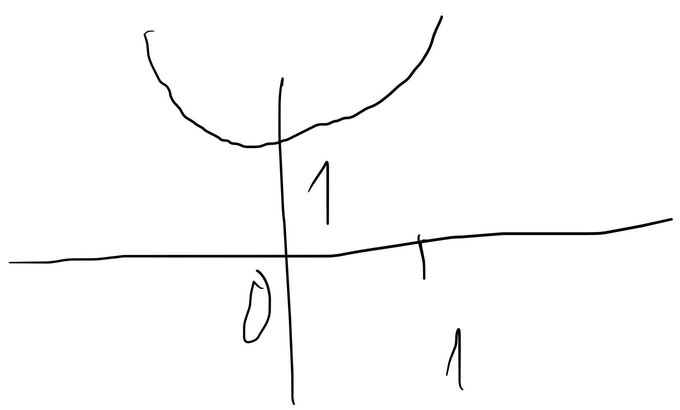
$0 < \xi < 1$

$$\left|\frac{x^7}{7!}ch\,\xi\right| \le \frac{ch\,\xi}{7!} \le \frac{ch\,1}{7!} \approx \frac{10^0}{10^3} = 10^{-3}$$

$ch\,x = \frac{e^x + e^{-x}}{2}$ $\quad 0 \le \xi \le 1$ $\quad 0 \le x^7 \le 1$ $\quad 7! = 2\cdot3\cdot4\cdot5\cdot6\cdot7$

$0 \le x \le 1$ $\quad ch\,0 = 1$ $\quad 720\cdot 7 \approx 5000$

$ch\,\xi \ge 1$ $\quad \approx 5\cdot 10^3$



$\frac{e + \frac{1}{e}}{2}$ $\quad \frac{2{,}7 + \frac{1}{2{,}7}}{2}$

$$\left|\frac{x^n}{n!}ch\,\xi\right| \le \frac{ch\,\xi}{n!} < \varepsilon$$

$\frac{n!}{ch\,1} > \frac{1}{\varepsilon} \Rightarrow$

$n! > \frac{ch\,1}{\varepsilon}$

$\Downarrow$

n

пр. $f(x) = (1+x)^{\alpha}$

$\alpha \in \mathbb{R}^1$ $\alpha = n \in \mathbb{N}$

$x > -1$

$f' = \alpha(1+x)^{\alpha-1}$

$f'' = \alpha(\alpha-1)(1+x)^{\alpha-2}$

$\vdots$

$f^{(n)} = \alpha(\alpha-1)\dots(\alpha-n+1)(1+x)^{\alpha-n}$

$\Rightarrow f^{(n)}(0) = \alpha(\alpha-1)\dots(\alpha-n+1)$

$$(1+x)^{\alpha} = 1 + \frac{\alpha}{1!}x + \frac{\alpha(\alpha-1)}{2!}x^2 + \dots + \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}x^n + R_n$$

$$R_n = \frac{\alpha(\alpha-1)\dots(\alpha-n+1)(\alpha-n)}{(n+1)!}x^{n+1}$$

Ако $\alpha = n \Rightarrow \alpha - n = n - n = 0$

$\Downarrow$

$R_n = 0$

Монотонни функции.

Екстремуми на ф-ция

Def $f(x)$: $\forall x_1 \geq x_2 \Rightarrow f(x_1) \underset{\leq}{\geq} f(x_2)$ – монот. раст. (монот. нам.)

$\Rightarrow$ монотонни ф-ции

th Ако $f'(x) > 0 \Rightarrow$ то $f(x)$ е растяща $f \nearrow$

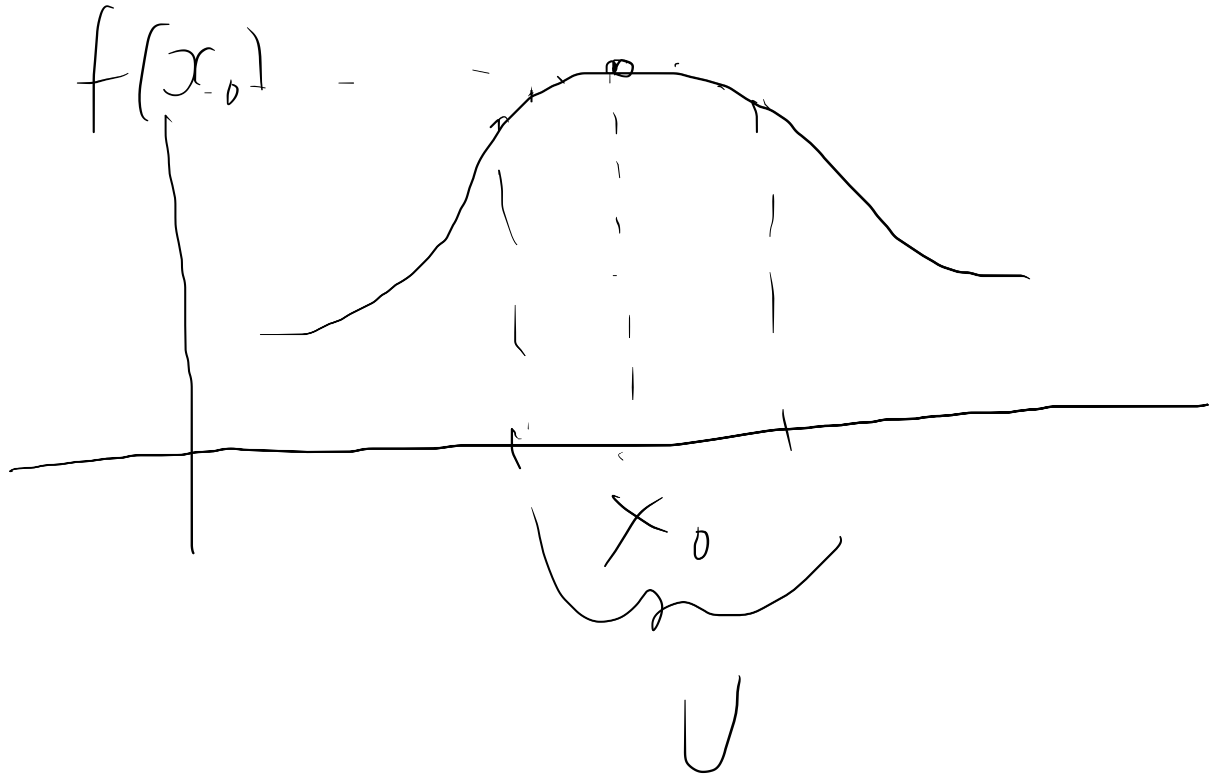
$f'(x) < 0 \Rightarrow$ то $f(x)$ е намаляв. $f \searrow$

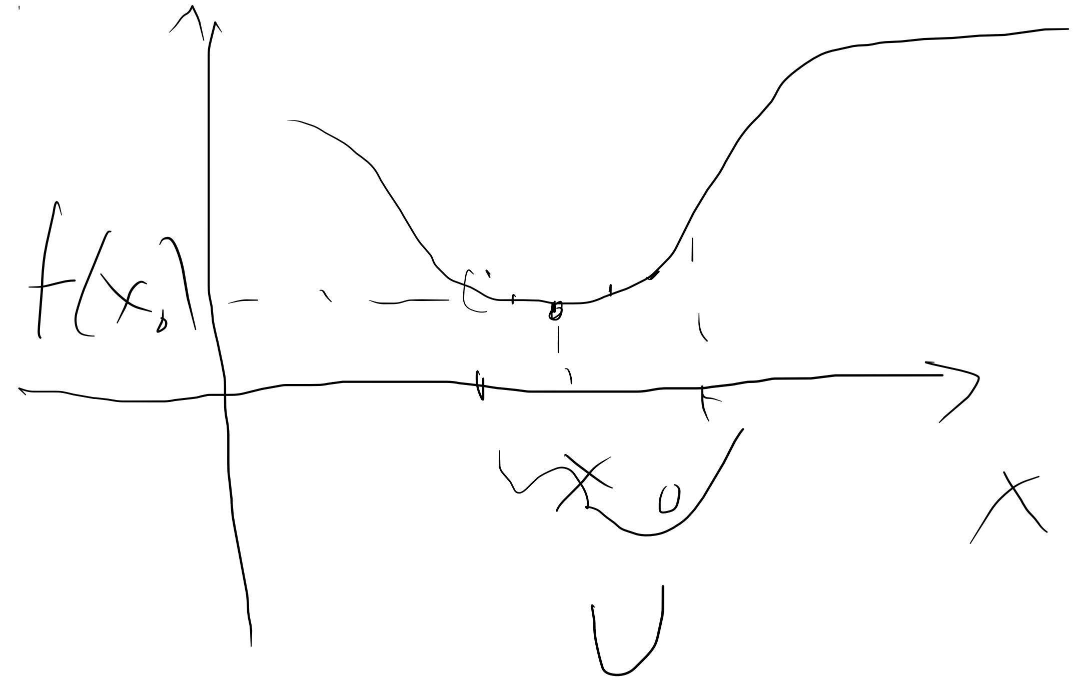
Def Ако $f'(x) = 0$, то т. $\underline{x}$ се нар. стационарна точка

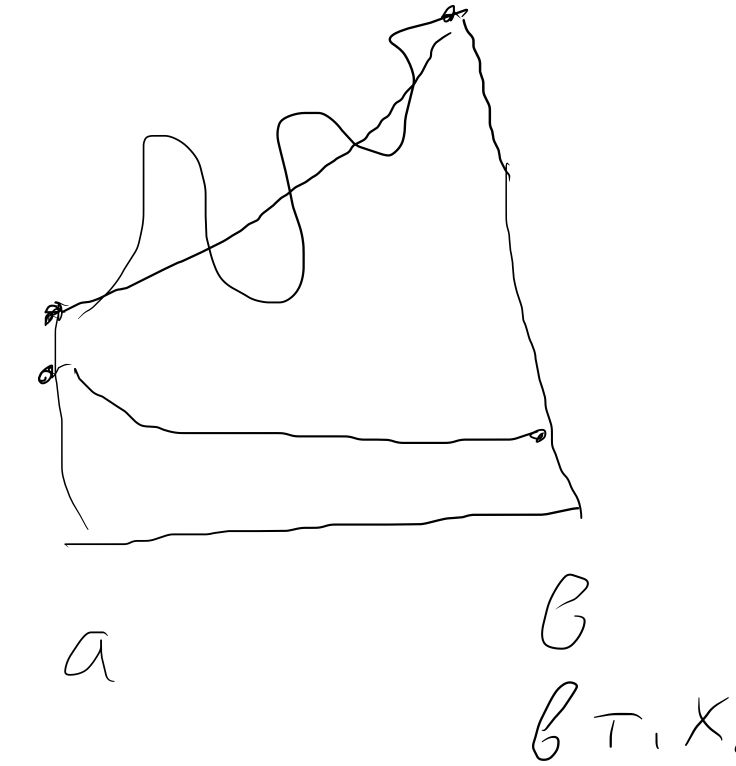
Def Казв., че ф-цията $f(x)$ има локален максимум в т. $x_0 \in D.O.$, ако $\exists\, U \ni x_0$: $f(x) \leq f(x_0)$.

Def ———————— " ———————— минимум в т. $x_0 \in D.O.$, ако $\exists\, U \ni x_0$: $f(x) \geq f(x_0)$

$\Rightarrow$ локални екстремуми



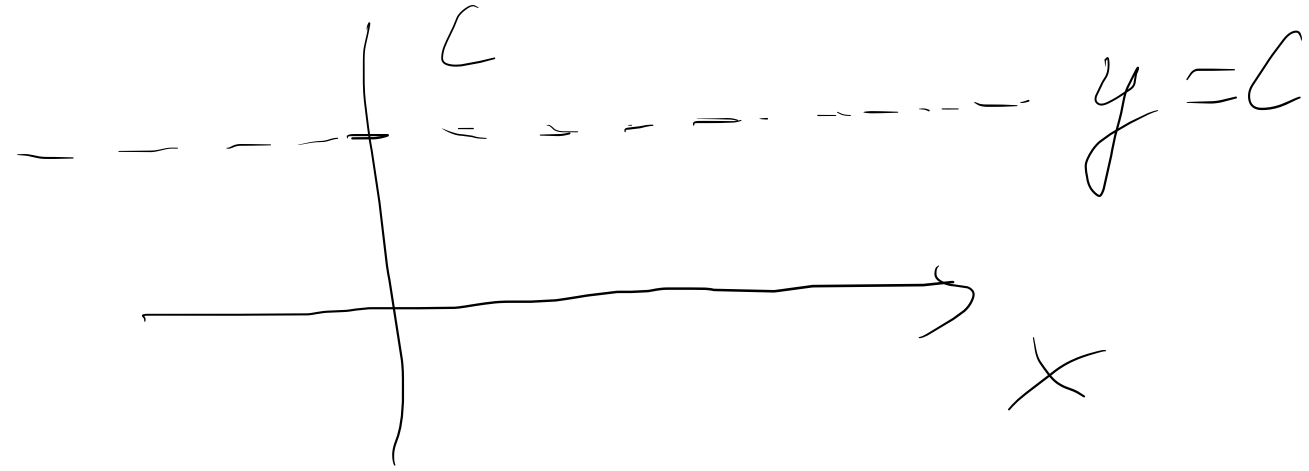




Th (Ферма) НУ ф-цията $f(x)$ да има лок. екстремум, е $f'(x_0) = 0$

$y = C$

$y' = 0$



Th НДУ ф-цията $f(x)$ да има лок. екстремум в т. $x_0 \in D_f$ е:

1. Т. x_0 да е стационарна точка, т.е. $f'(x_0) = 0$
2. Ако $f''(x_0) > 0$ — x_0 е лок. минимум
   < 0 — x_0 е лок. максимум

Ако $f''(x_0)=0$, то необх. е допълнително изследв.,

пр.: $y = x^2 e^{-x}$ $\quad y' = 2xe^{-x} - x^2 e^{-x} = \underbrace{e^{-x}}_{\neq 0}\underbrace{(2x-x^2)}_{} = 0$

квазиполином

$x^2 - 2x = 0 \Longrightarrow x(x-2) = 0$ $\quad x_1 = 0$, $x_2 = 2$ стац. т.

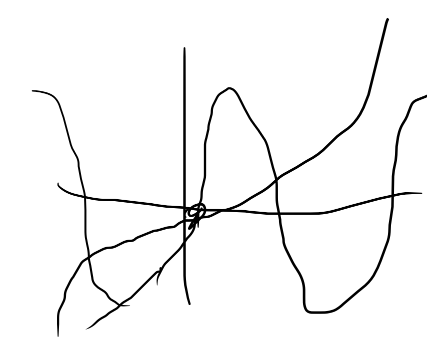
$y' = e^{-x}(2x - x^2)$ $\quad y'' = -e^{-x}(2x - x^2) + e^{-x}(2 - 2x) =$

$= e^{-x}(-2x + x^2 \cancel{+2x} \cancel{-2x}) = e^{-x}(x^2 - 2x)$

$y''(0) = e^0(0 - 0) = 0$; $y''(2) = e^{-2}(4 - 4) = 0$

пр.: $y = \operatorname{ch} x + \cos x$ $\quad y' = \operatorname{sh} x - \sin x = 0$

$x = 0$ е стац. т. $\quad \operatorname{sh} x = \sin x$



$y'' = \operatorname{ch} x - \cos x$ $\quad y''(0) = 1 - 1 = 0$

пр. 3 $y = 2x^2 - \ln x$ $D.O \Rightarrow \{x : x > 0\}$

$y' = 4x - \frac{1}{x} = \frac{4x^2 - 1}{x} = 0$ $x_{1,2} = \pm\frac{1}{2} \Rightarrow x = \frac{1}{2}$

$\cancel{x} \neq 0$ стац. т.

$y'' = 4 + \frac{1}{x^2}$ $y''\left(\frac{1}{2}\right) = 4 + \frac{1}{\frac{1}{4}} = 4 + 4 = 8 > 0 \Rightarrow$

$x = \frac{1}{2}$ е лок. мин.

$y_{min} = y\left(\frac{1}{2}\right) = 2\frac{1}{4} - \ln\frac{1}{2} = \frac{1}{2} - (\underbrace{\ln 1}_{0} - \ln 2) = \frac{1}{2} + \ln 2$

$\lim\limits_{x \to 0} (\cot g\, x)^{\frac{1}{\ln x}}$ $[\infty^0]$ $y = (\cot g\, x)^{\frac{1}{\ln x}} \mid \ln$ $\frac{\infty}{\infty}$

$\ln y = \frac{\ln \cot g\, x}{\ln x}$ $\lim\limits_{x \to 0} \ln y = \ln \lim\limits_{x \to 0} y = \lim\limits_{x \to 0} \frac{\ln \cot g\, x}{\ln x}$

$$\lim_{x\to 0} \ln y = \ln \lim_{x\to 0} y = \lim_{x\to 0} \frac{\ln \operatorname{cotg} x}{\ln x} = -\lim_{x\to 0} \frac{\frac{1}{\operatorname{cotg} x}\,\frac{1}{\sin^2 x}}{\frac{1}{x}} =$$

$$= -2\lim_{x\to 0} \frac{x}{\sin 2x} = -\lim_{x\to 0} \frac{2x}{\sin 2x} \to 1$$

$$= -1$$

$$\lim_{x\to 0} y = e^{-1} = \frac{1}{e}$$

$$\ln \lim_{x\to 0} y = -1$$