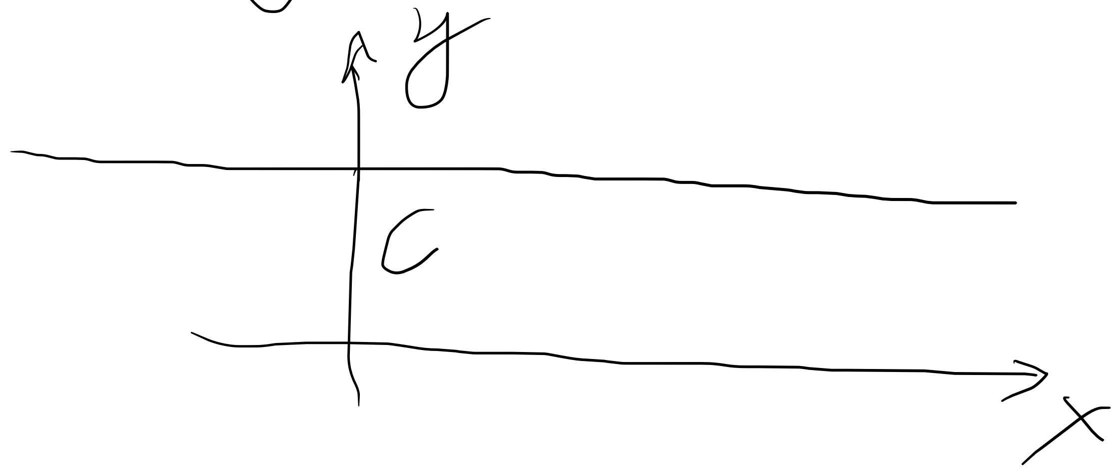
$y = f(x)$ $\quad f'(x) = 0$ реш. се нар. стац. точки



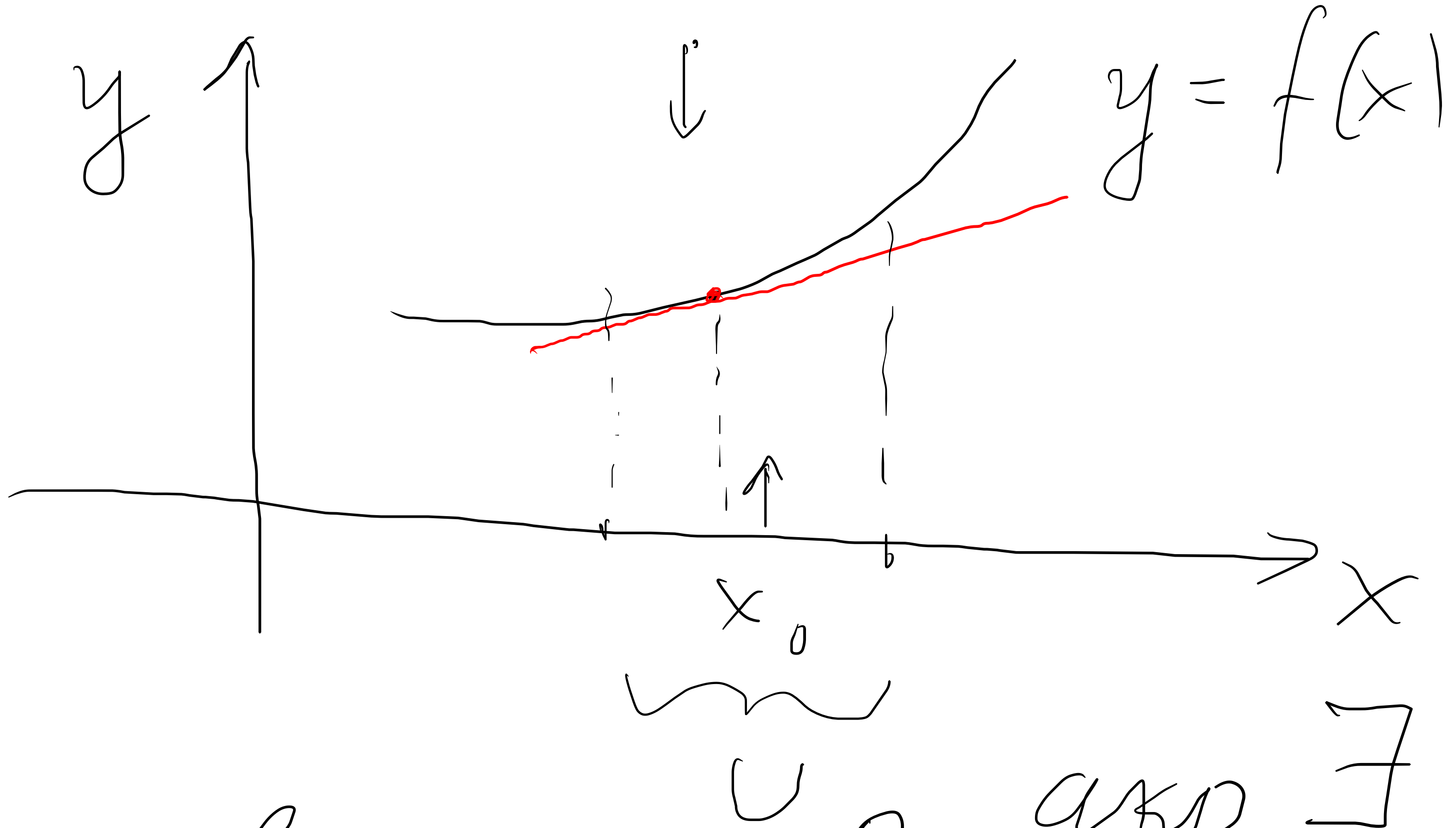
$y = C$ - конст. $\quad y' = 0$ $\quad x_0$ - стац. т.

$$f''(x_0) \begin{cases} > 0 & x_0 \text{ е лок. мин.} \\ < 0 & x_0 \text{ е лок. макс.} \\ = 0 & \text{необх. е допълн. изследване} \end{cases}$$

Изпъкнали и вдлъбнати функции.

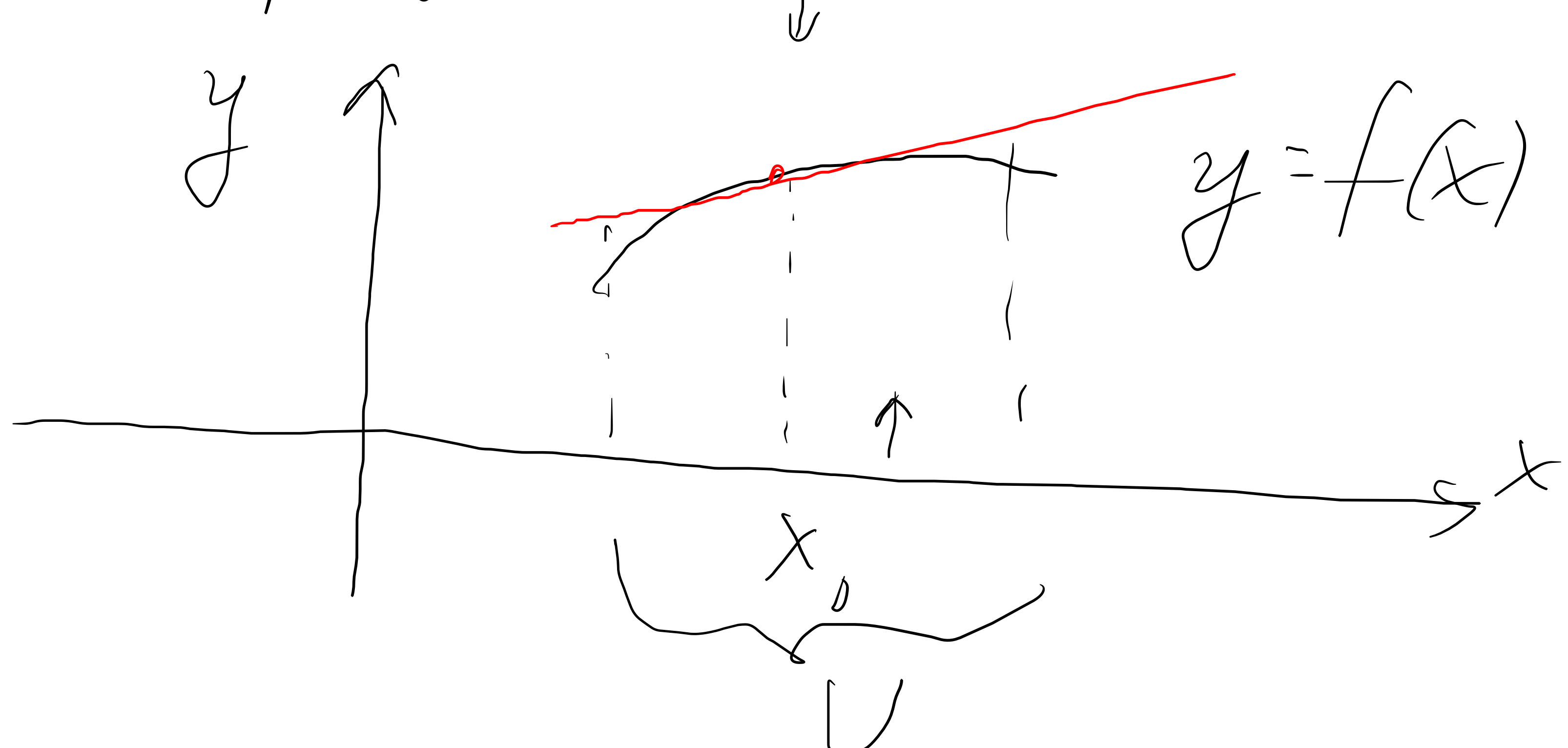
Инфлексни точки

<u>Def</u> Казваме, че $f(x)$ е изпъкнала в т. $x_0 \in D.O.$, ако $\exists$ околн. $U \ni x_0$, в която графиката на ф-цията лежи над графиката на допирателната в т. x_0



изпъкнала (отдолу) ≡ вдлъбната (отгоре)

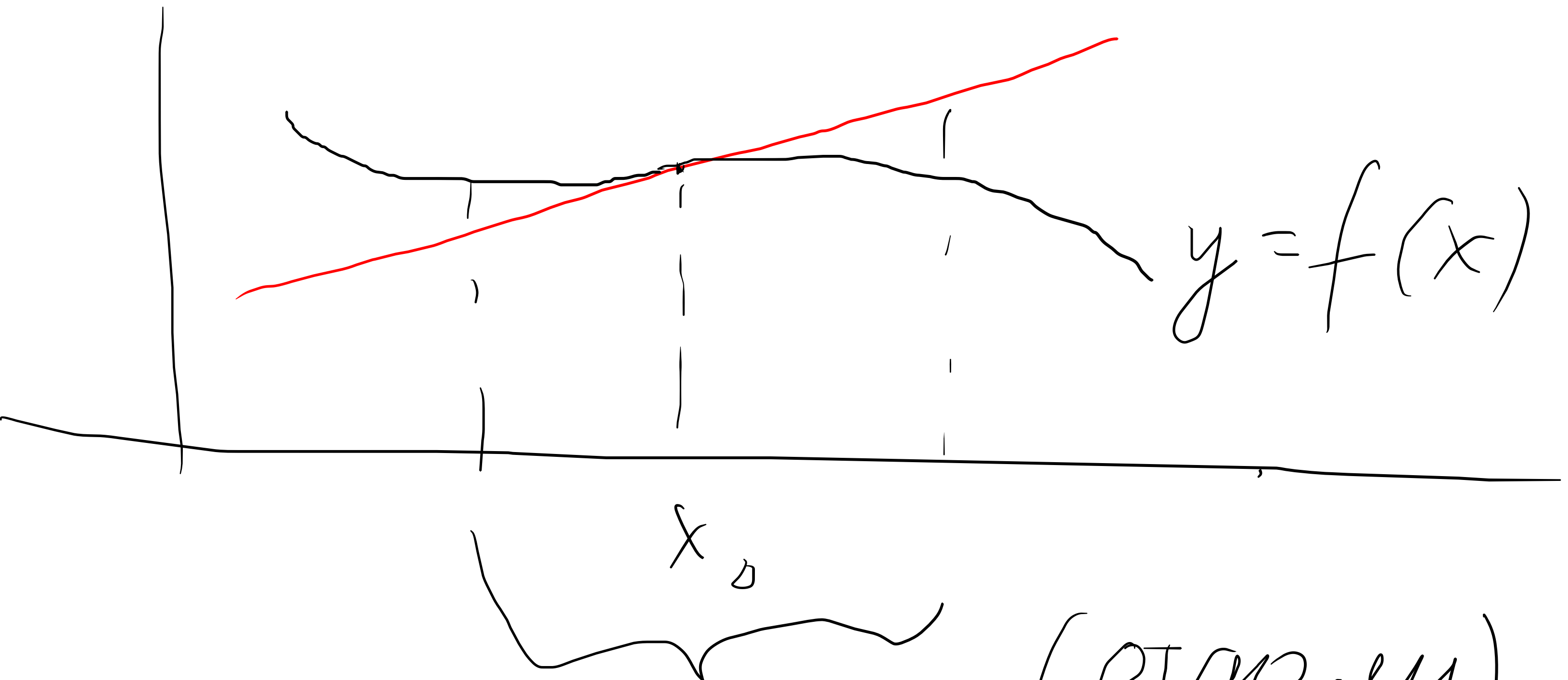
Def Казв., че $f(x)$ е вдлъбната в т. $x_0 \in D.O.$, ако $\exists$ околн. $U \ni x_0$ такава, че графиката на ф-цията лежи под графиката на тангентата.



вдлъбната (отдолу) ≡ изпъкнала (отгоре)

Def Казв., че т. $x_0 \in D.O.$ на $f(x)$ е инфлексна точка, ако $\exists\, U \ni x_0$ такава, че част от графиката на ф-цията лежи под

тангентата, а другата част над тангентата
(тангентата пресича графиката на ф-цията)



<u>Th</u> Ако $f''(x)=0$, то т. x е инфлексна т.

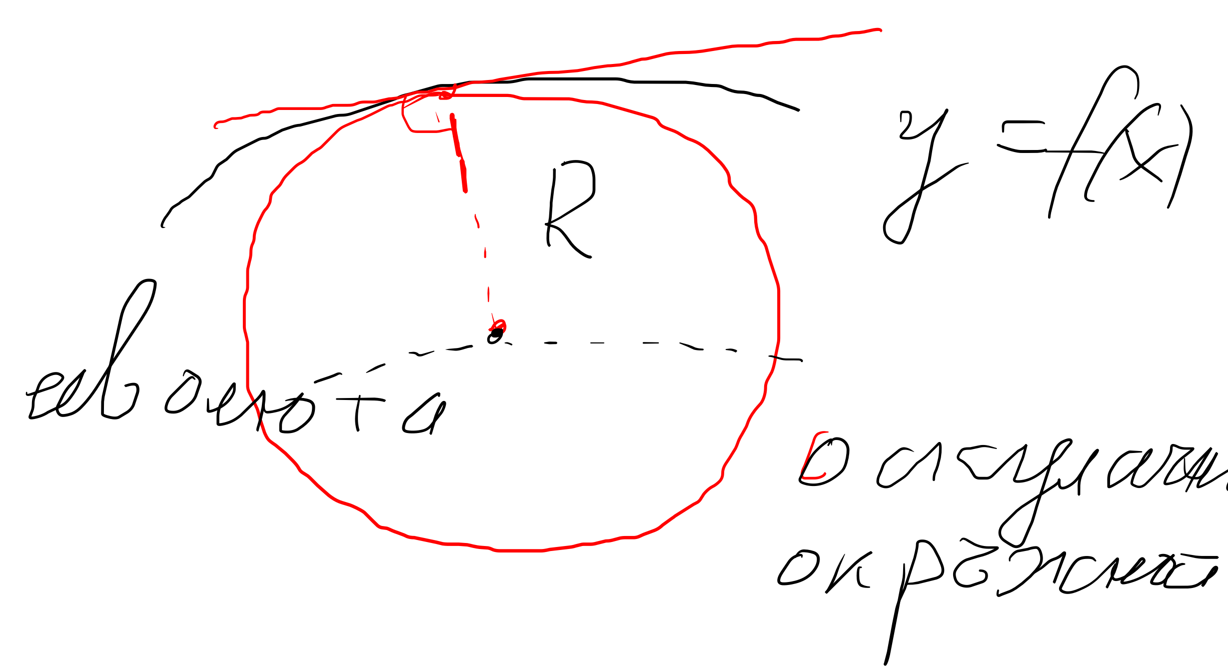
<u>ДУ</u> $f(x)$ да е изпъкнала (отдолу) в т. x е $f''(x)>0$

вдлъбната (отдолу) е $f''(x)<0$

кривина $\quad \varkappa = \dfrac{y''}{\sqrt{(1+y'^2)^3}} = \dfrac{y''}{(1+y'^2)^{\frac{3}{2}}}$

$\varkappa = \dfrac{1}{R} \qquad |\varkappa|$

$R>0 \quad R$ – радиус на кривината



пр. $y=(1+x^2)e^x$ $\quad y'=2xe^x+(1+x^2)e^x=$

$=e^x(x^2+2x+1)=\underbrace{e^x(x+1)^2}$ $\quad y'=0 \Longrightarrow x=-1$ стац. т.

$y''=e^x(x+1)^2+2e^x(x+1)=e^x\left[(x+1)^2+2(x+1)\right]=\underset{\neq 0}{e^x}(x+1)(x+3)$

$y''=0 \iff (x+1)(x+3)=0 \Longrightarrow \underline{x_1=-1}$ и $x_2=-3$

? $\quad$ инфл. т.

$y'(-1)=0$

$y''(-1)=0$

<u>Обобщена теорема за съществуване на локални екстремуми</u>

$y=f(x)$ и т. x_0 е стац. точка, т.е. $f'(x_0)=0$

и $f''(x_0)=\dots=f^{(n)}(x_0)=0$, и $f^{(n+1)}(x_0)\neq 0$.

$$y''' = e^x(x^2+4x+3) + e^x(2x+4) =$$

$$= e^x(x^2+6x+7)$$

$$e^x(x+1)(x+3) = y''$$

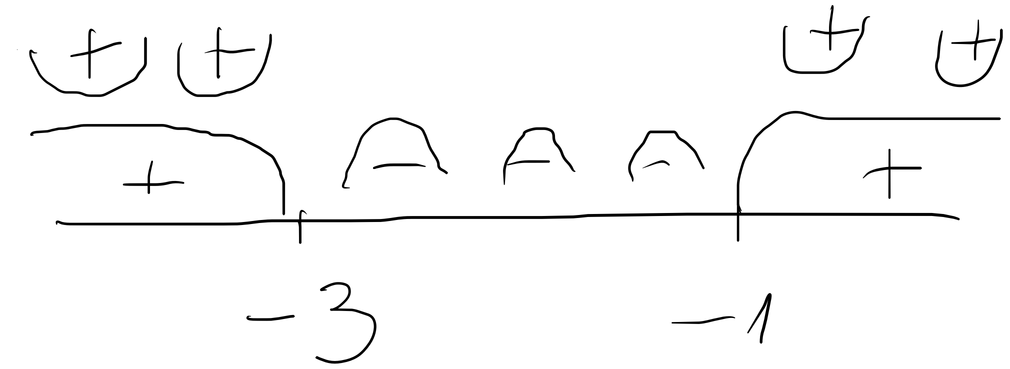
$$\Uparrow$$

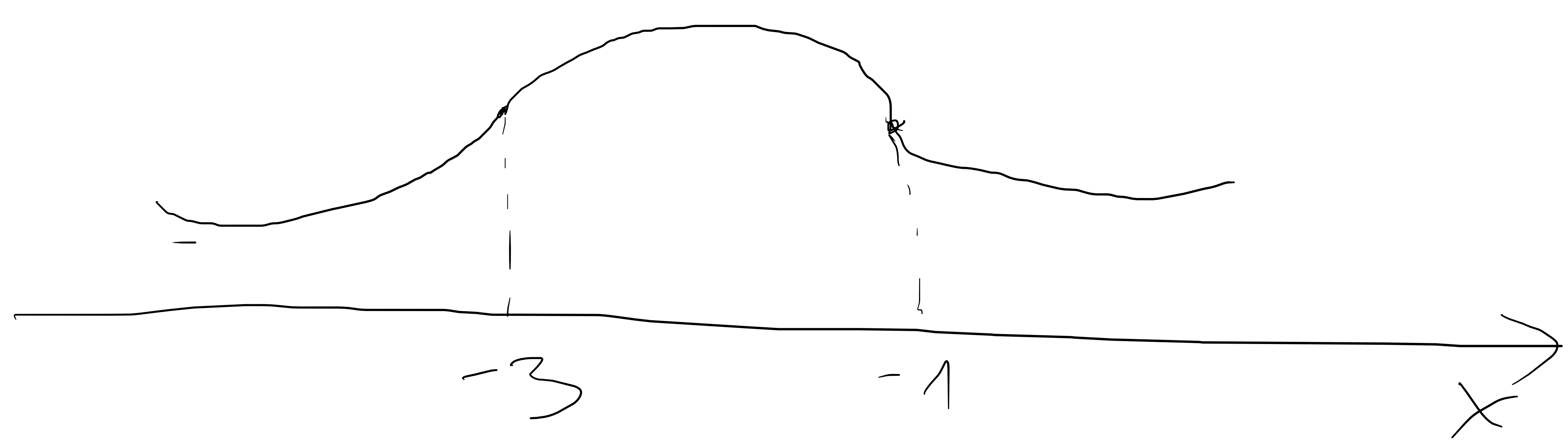
$$e^x(x^2+4x+3)$$

$$y'''(-1) = e^{-1}(1-6+7) \neq 0$$

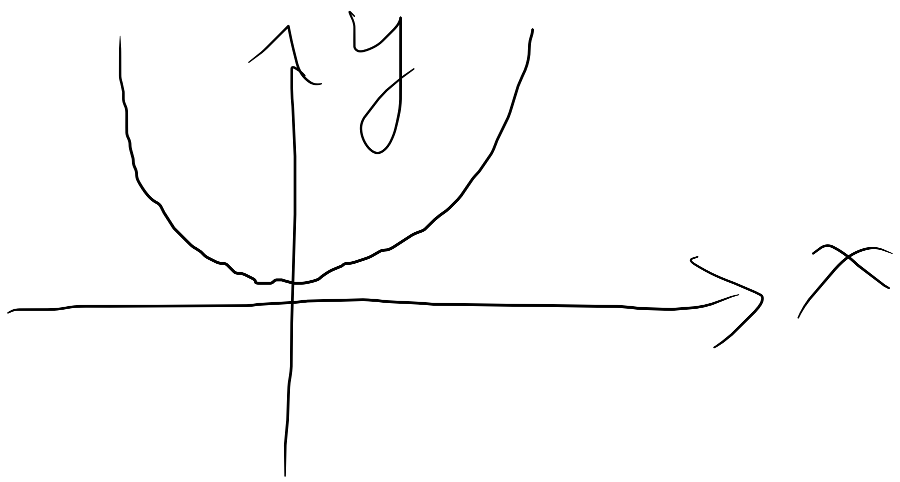
$\Longrightarrow$ $x_1 = -1$ е инфлексна т.

$$y'' > 0 \iff (x+1)(x+3) > 0$$





пр. 1 $\quad y = x^2$ парабола



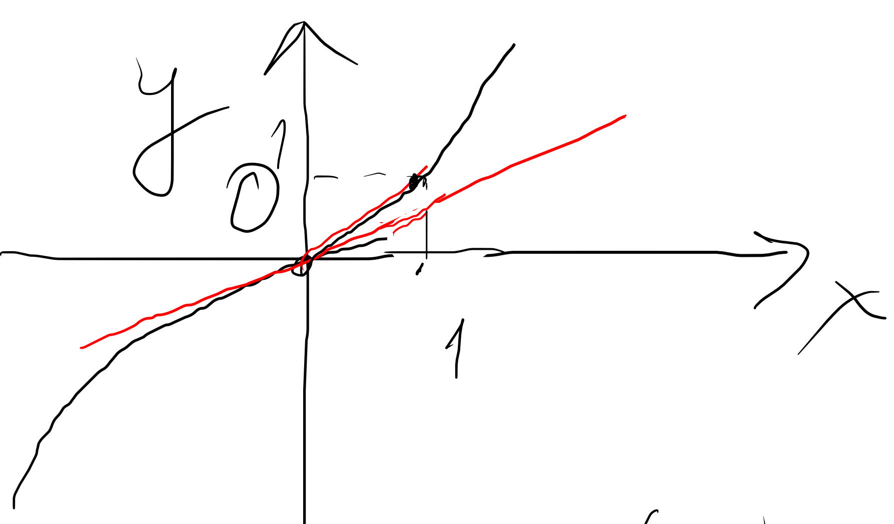
$y' = 2x = 0 \Rightarrow x = 0$ стац. т.

$y'' = 2 > 0$ за $\forall x$

$y''(0) > 0 \Rightarrow x = 0$ е лок. мин.

$$\varkappa = \frac{6x}{\sqrt{(1+9x^4)^3}} \xrightarrow[x\to\infty]{} 0$$

пр. 2 $\quad y = x^3$ (кубична парабола)



$y' = 3x^2 = 0 \Rightarrow x = 0$ стац. т. $\Rightarrow$ инфл. т.

$y'' = 6x \quad y'(0) = y''(0) = 0$

$y''' = 6$ за $\forall x \Rightarrow y'''(0) = 6 \neq 0$

$$\frac{x^1}{x^{\frac{3}{2}}}$$

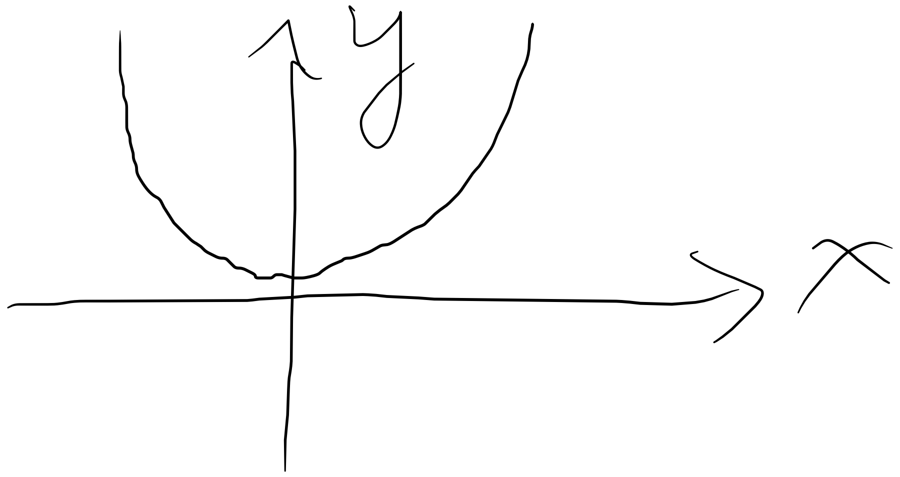
$$\varkappa(1) = \frac{6}{\sqrt{(1+9)^3}} = \frac{6}{10^{\frac{3}{2}}}$$

пр. 1 $y = x^2$ парабола

$y' = 2x = 0 \Rightarrow x = 0$ стац. т.

$y'' = 2 > 0$ за $\forall x$

$y''(0) > 0 \Rightarrow x = 0$ е лок. мин.



$$\varkappa = \frac{6x}{\sqrt{(1+9x^4)^3}} \xrightarrow[x \to \infty]{} 0$$

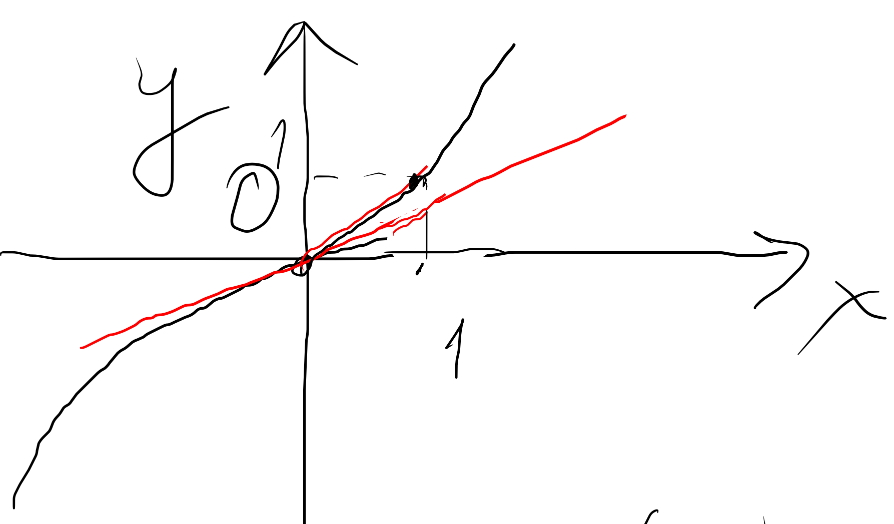
пр. 2 $y = x^3$ (кубична парабола)

$y' = 3x^2 = 0 \Rightarrow x = 0$ стац. т. $\Rightarrow$ инфл. т.

$y'' = 6x$ $y'(0) = y''(0) = 0$

$y''' = 6$ за $\forall x \Rightarrow y'''(0) = 6 \neq 0$

$$\frac{x^1}{x^{\frac{3}{2}}}$$



$$\varkappa(1) = \frac{6}{\sqrt{(1+9)^3}} = \frac{6}{10^{\frac{3}{2}}}$$

пр. 1 $\quad y = x^2$ парабола

$y' = 2x = 0 \Rightarrow x = 0$ стац. т.

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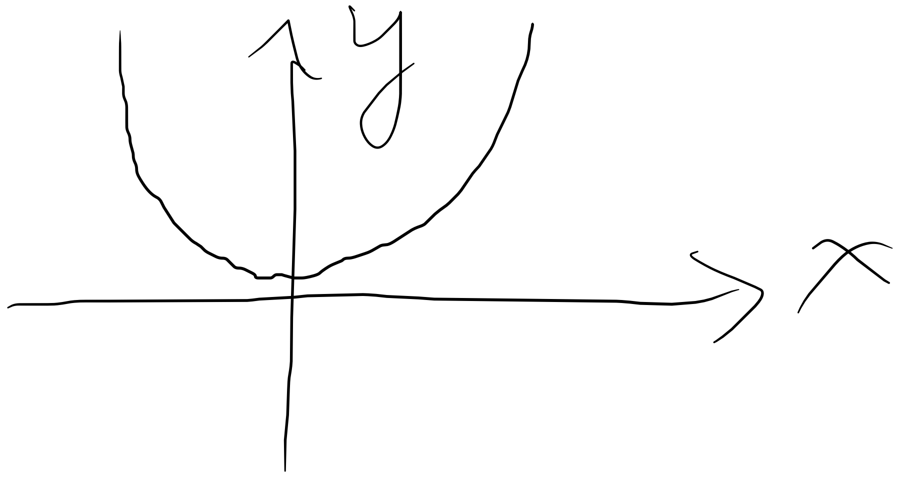


пр. 2 $\quad y = x^3$ (кубична парабола)

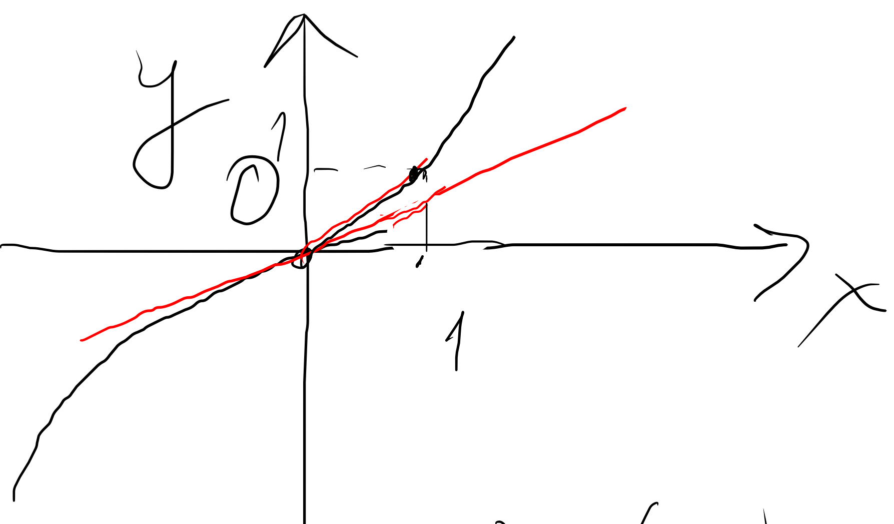
$$\varkappa = \frac{6x}{\sqrt{(1+9x^4)^3}} \xrightarrow[x \to \infty]{} 0$$

$y' = 3x^2 = 0 \Rightarrow x = 0$ стац. т. $\Rightarrow$ инфл. т.

$y'' = 6x \quad y'(0) = y''(0) = 0$

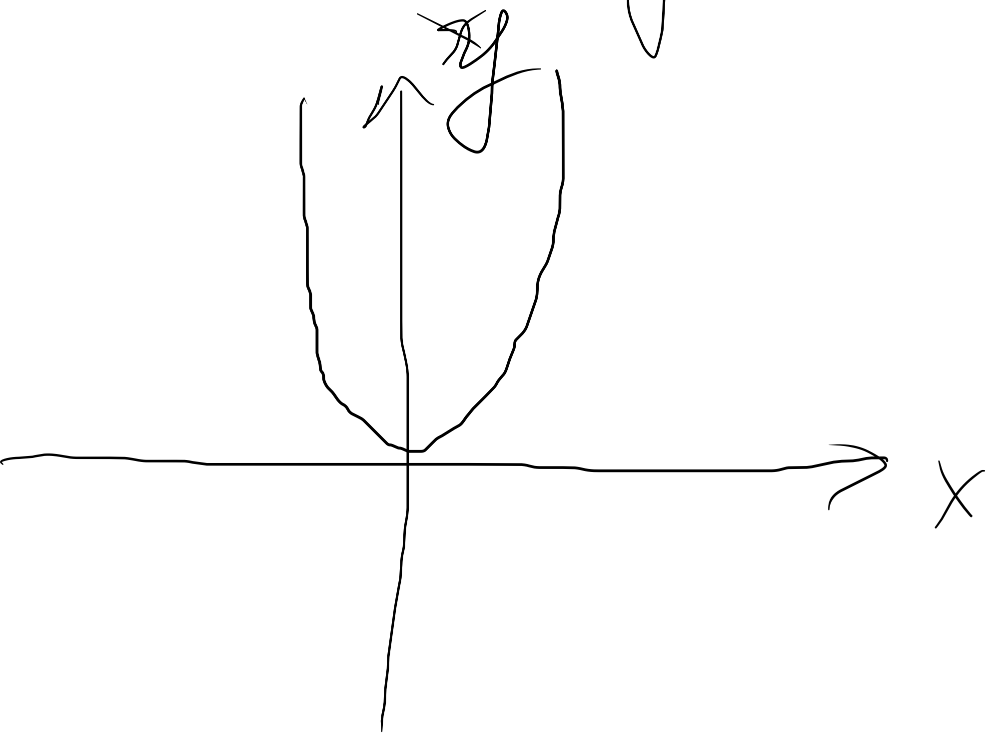
$y''' = 6$ за $\forall x \Rightarrow y'''(0) = 6 \neq 0$

$$\frac{x^1}{x^{\frac{3}{2}}}$$



$$\varkappa(1) = \frac{6}{\sqrt{(1+9)^3}} = \frac{6}{10^{\frac{3}{2}}}$$

пр. 3 $y = x^4$ биквадратна парабола



$y' = 4x^3 = 0$ $x = 0$ – стац. т.

$y'' = 12x^2$ $y'(0) = y''(0) = 0$

$y''' = 24x$ $y'(0) = y''(0) = y'''(0) = 0$

$y^{IV} = 24 \Rightarrow y^{IV}(0) \neq 0$

$$\underset{\displaystyle 24 > 0}{\overset{\shortparallel}{}}$$

$\Downarrow$

$x = 0$ е лок. мин.

$y = -x^4$

$y' = -4x^3$

$y'' = -12x^2$

$y''' = -24x$

$y^{IV} = -24 < 0$ $x = 0$ е лок. макс.

Тогава, ако $n = 2k+1$ (нечетно число), то

x_0 е лок. макс., когато $f^{(n+1)}(x_0) < 0$

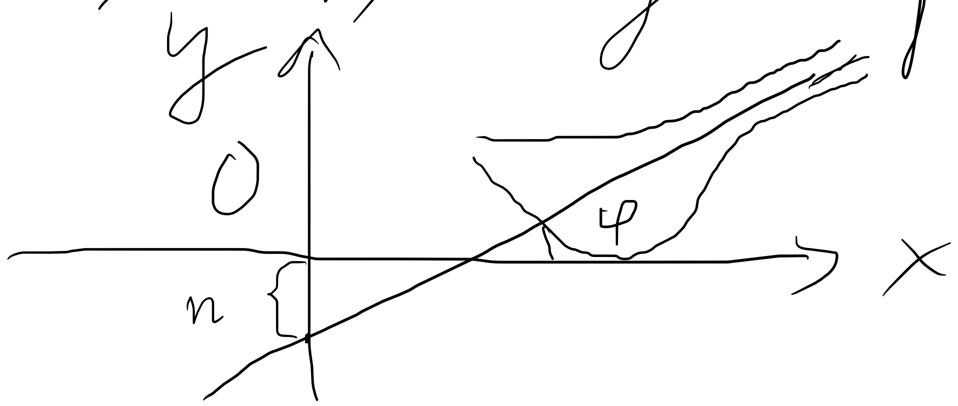
x_0 е лок. мин., когато $f^{(n+1)}(x_0) > 0$.

Ако $n = 2k$ (четно число), то т. x_0 е инфлексна точка.

Асимптоти

$x \to \pm\infty$

$y = kx + n$ – декартово у-ние k – наклон $= \operatorname{tg}\varphi$



$$k = \lim_{x \to \pm\infty} \frac{f(x)}{x}$$ Ако $\nexists\, k$

$$n = \lim_{x \to \pm\infty} [f(x) - kx]$$

$k = 0 \Rightarrow$ хоризонтална асим

$\varphi \to \frac{\pi}{2} \Rightarrow tg\varphi = k \to \infty$

$y = kx + n$

$\infty x + n$

вертикална асимпт.

пр: $y = \frac{5x}{x-3}$ $x \neq 3$

$$\lim_{x \to 3^+} y = \lim_{\varepsilon \to 0} \frac{5(3+\varepsilon)}{\cancel{3}+\varepsilon-\cancel{3}} = \frac{15}{+0} = \infty$$

$x = 3+\varepsilon$

$$\lim_{x \to 3^-} y = \lim_{\varepsilon \to 0} \frac{5(3-\varepsilon)}{\cancel{3}-\varepsilon-\cancel{3}} = \frac{15}{-0} = -\infty$$

$x = 3-\varepsilon$

$\varepsilon > 0$

$x = 3$ е верт. асимпт.

$\forall x \neq 3$ $-\infty, \infty$

$$k = \lim_{x\to\pm\infty} \frac{5x}{(x-3)x} = \lim_{x\to\pm\infty} \frac{\frac{5}{x}}{1-\frac{3}{x}} = 0$$

$$n = \lim_{x\to\pm\infty}\left(\frac{5x}{x-3} - 0x\right) = \lim_{x\to\pm\infty} \frac{5x}{x-3} = \frac{5}{1} = 5 \Rightarrow y = 5$$

пр.: $y = \sqrt[3]{x+2}$ $\quad k = \lim_{x\to\pm\infty} \frac{y(x)}{x} = \lim_{x\to\pm\infty} \frac{\sqrt[3]{x+2}}{x} = 0$ $\quad \frac{x^{\frac{1}{3}}}{x}$

$\forall x$

$$n = \lim_{x\to\pm\infty} \sqrt[3]{x+2} = \pm\infty \Rightarrow \text{асимптоти няма}$$

$$y' = \left[(x+2)^{\frac{1}{3}}\right]' = \frac{1}{3}(x+2)^{-\frac{2}{3}} = \frac{1}{3}\,\frac{1}{\sqrt[3]{(x+2)^2}} \neq 0 \quad x \neq -2 \Rightarrow y'_{x\to -2} \to \infty$$ – вертик. тангента

$$y' = \left[(x+2)^{\frac{1}{3}}\right]' = \underbrace{\frac{1}{3}(x+2)^{-\frac{2}{3}}} = \underbrace{\frac{1}{3}\frac{1}{\sqrt[3]{(x+2)^2}}} \neq 0 \quad x \neq -2$$

$\Rightarrow y'_{x \to -2} \to \infty$ – вертик. тангента

$y' > 0$ за $\forall\, x$ $\quad y \nearrow$

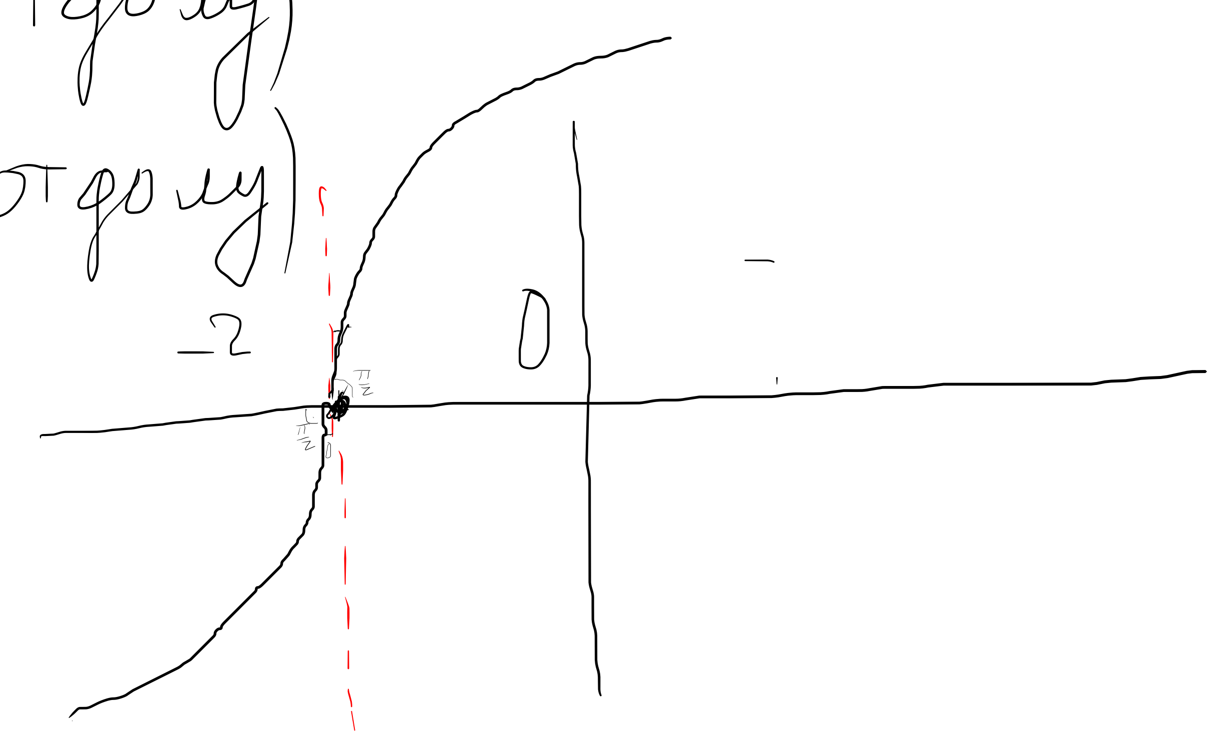
$$y'' = -\frac{1\cdot 2}{3^2}(x+2)^{-\frac{5}{3}} = -\frac{2}{9}\frac{1}{\sqrt[3]{(x+2)^5}} \neq 0$$

$\Rightarrow$ няма инфл. т.

$x = -2 \quad y'' \to \infty$

$x < -2 \quad y'' > 0$ изпъкн. (отдолу)

$x > -2 \quad y'' < 0$ вдлъбн. (отдолу)



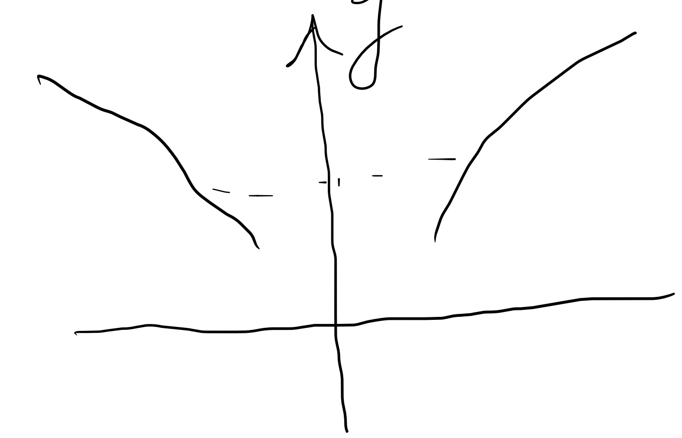
$$\lim_{x \to -2} y'' = \pm\infty$$

пр. $\quad y = \frac{x^2}{\sqrt{x^2-1}}$

$y(-x) = y(x)$ четна

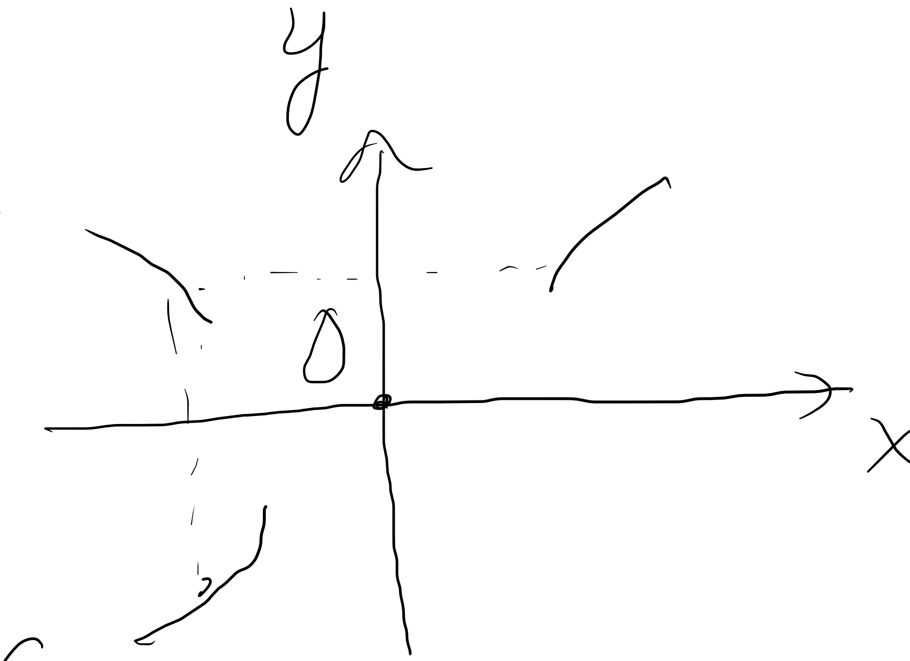
$y = f(x)$

Ако $f(-x) = f(x)$, то y е четна ф-ция



x^{2k}

Ако $f(-x) = -f(x)$, то y е нечетна ф-ция



x^{2k+1}

Ако $f(x+T) = f(x)$, то $f(x)$ е период. с

$\min T > 0 \Rightarrow \min T$ - период   период $\min T$

$T = 2k\pi$ $(x, x+T)$ $\left(-\frac{T}{2}, \frac{T}{2}\right)$ $\left(0, \frac{T}{2}\right)$

$k = 1$

$y = \frac{x^2}{\sqrt{x^2-1}}$ $\qquad y(-x) = y(x)$

четна

$x \neq \pm 1$

$D' = \{ x : x \geq 0,\ x \neq 1 \}$

$x > 1$

$y = x^2 (x^2-1)^{-\frac{1}{2}}$

$y' = 2x(x^2-1)^{-\frac{1}{2}} + x^2\left(-\frac{1}{\cancel{2}}\right)(x^2-1)^{-\frac{3}{2}}\cancel{2}x$

$= \underbrace{2x(x^2-1)}_{x^2-1}{}^{-\frac{1}{2}} - x^3(x^2-1)^{-\frac{3}{2}} =$

$= \frac{2x}{\sqrt{x^2-1}} - \frac{x^3}{\sqrt{(x^2-1)^3}} = \frac{2x^3 - x^3 - 2x}{\underbrace{\sqrt{(x^2-1)^3}}} =$

$= \frac{x(x^2-2)}{\sqrt{(x^2-1)^3}} = y'$

$y' = 0 \iff x(x^2-2) = 0 \implies \cancel{x_1 = 0}$

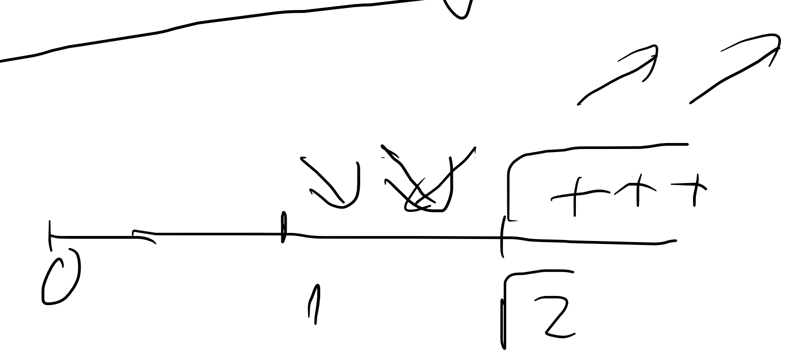
$x_{2,3} = \pm\sqrt{2}$

$1 < \sqrt{2}$ стац. т.

$y' > 0 \qquad \frac{x(x^2-2)}{\sqrt{(x^2-1)^3}} > 0$

$\sqrt{(x^2-1)^3} > 0$

$\cancel{x}(x^2-2) > 0$



$$y=\frac{x^2}{\sqrt{x^2-1}} \qquad \lim_{x\to 1}\frac{x}{\sqrt{x^2-1}}=\frac{1}{0}=\infty \Rightarrow x=1 \text{ е верт. асимпт.}$$

Д.О. $x>1$

$$k=\lim_{x\to\infty}\frac{\cancel{x}}{\cancel{x}\sqrt{x^2-1}}=0$$

$$n=\lim_{x\to\infty}\frac{x}{\sqrt{x^2-1}}=\lim_{x\to\infty}\frac{1}{\sqrt{1-\frac{1}{x^2}}}=1 \Rightarrow y=1$$

дясна асимп