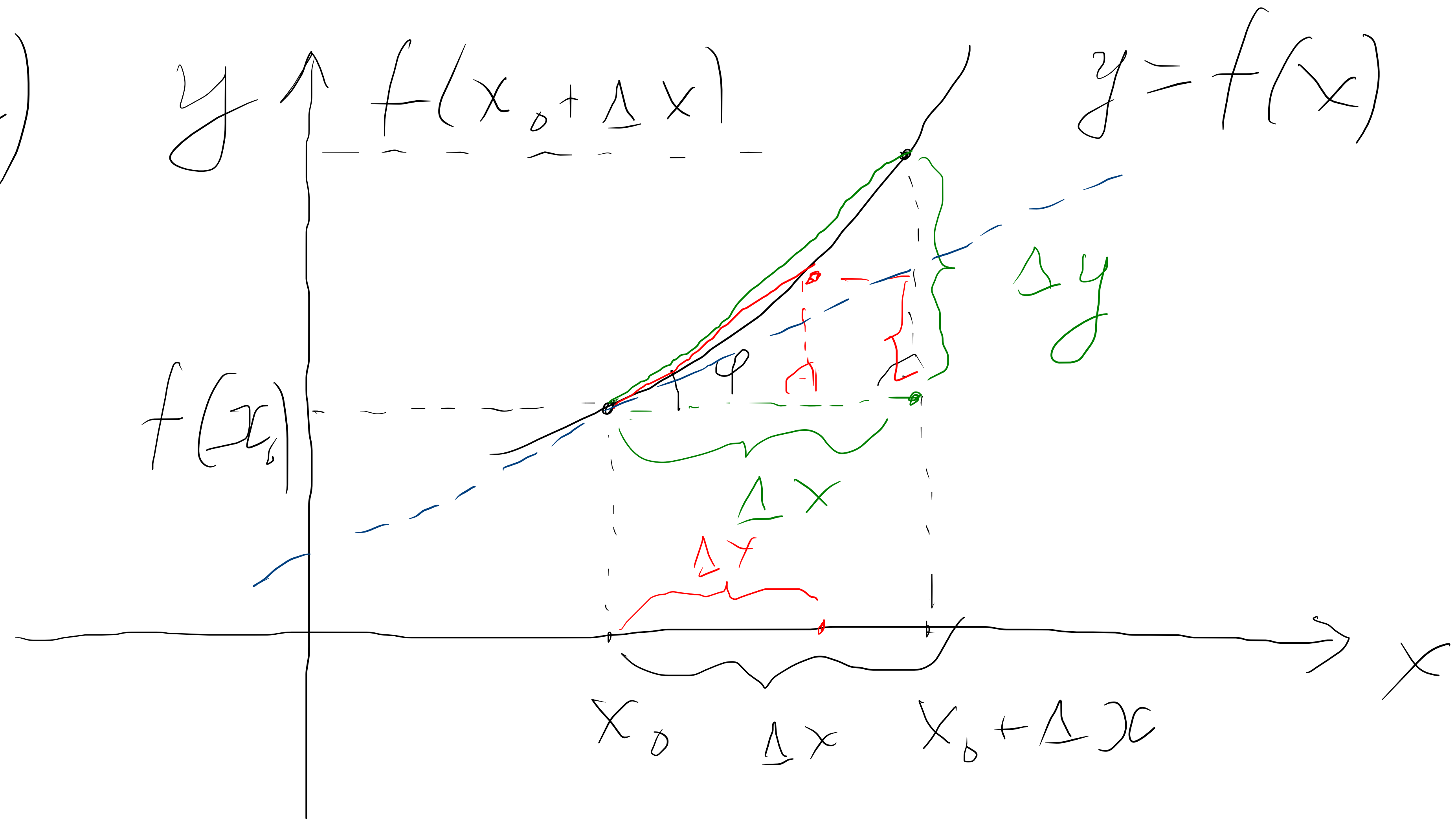
Производна на функция

$y = f(x)$



$x = x_0 + \Delta x$

нарастване

$= f(x_0 + \Delta x)$

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{x_0 + \Delta x - x_0}$$

$$tg\,\varphi = \frac{\Delta y}{\Delta x}$$

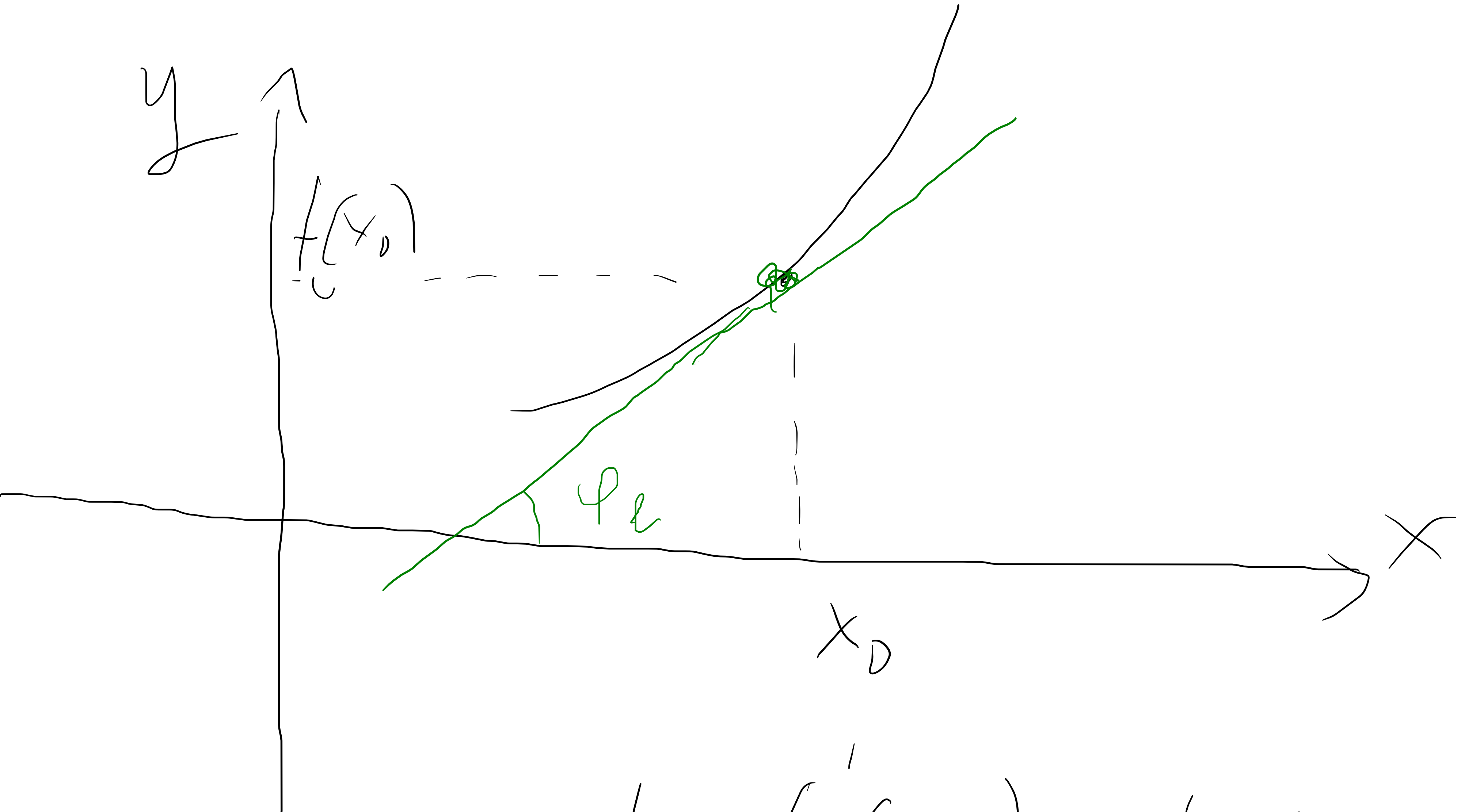
$\Delta x \to 0$

диференчно частно

крайни разлики

$$\exists?\ \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} =: f'(x_0) = \frac{df}{dx}$$

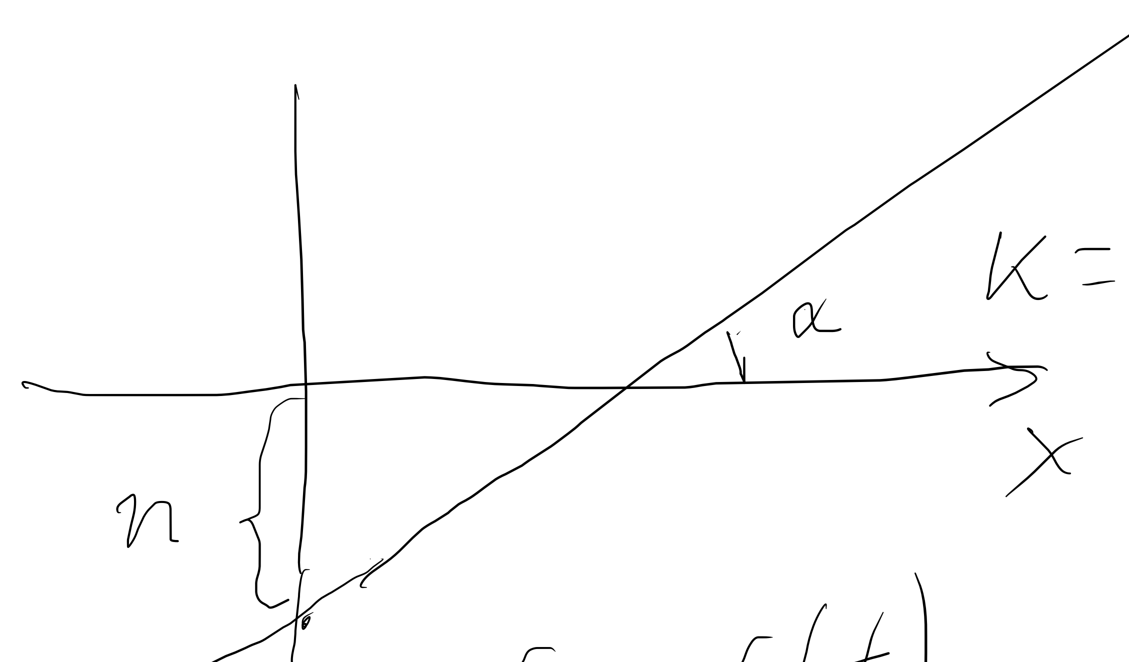
допирателна (тангента) към графиката на $y = f(x)$ в т. x_0



$y = f(x)$ – декартово у-ние

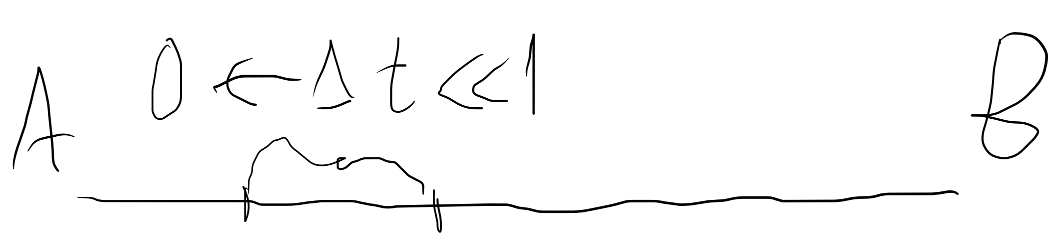
$y = kx + n$

$y' = f'(x_0) = tg\,\varphi_e$ – наклон на $f(x)$ в т. x_0



$k = tg\,\alpha$

закон за пътя



мом. скор.

$v_{ср.} = \frac{|AB|}{t}$

$s = s(t)$

$s'(t) = v(t)$

Таблични производни

$y = x^n,\ n \in \mathbb{N}\ (n \in \mathbb{R});\ y' = n x^{n-1}$ $\qquad y = x \Rightarrow y' = 1$

$y = e^x;\ y' = e^x$

$y = \ln x;\ y' = \dfrac{1}{x}$

$y = \sin x;\ y' = \cos x$

$y = \cos x;\ y' = -\sin x$

$y = \operatorname{tg} x;\ y' = \dfrac{1}{\cos^2 x}$

$y = \operatorname{cotg} x;\ y' = \dfrac{1}{-\sin^2 x}$

$y = \operatorname{sh} x;\ y' = \operatorname{ch} x$

$y = \operatorname{ch} x;\ y' = \operatorname{sh} x$

$y = \operatorname{Arsh} x;\ y' = \dfrac{1}{\sqrt{x^2+1}}$

$y = \operatorname{Arch} x;\ y' = \dfrac{1}{\sqrt{x^2-1}}\quad |x| > 1$

$y = \operatorname{Arth} x;\ y' = \dfrac{1}{1-x^2},\ |x| < 1$

$y = \operatorname{Arcth} x;\ y' = -\dfrac{1}{x^2-1},\ |x| > 1$

$y = \operatorname{th} x;\ y' = \dfrac{1}{\operatorname{ch}^2 x}$ $\quad|\quad$ $y = \operatorname{cth} x;\ y' = \dfrac{1}{-\operatorname{sh}^2 x}$

правила за диференциране

$$[u(x) \pm v(x)]' = u'(x) \pm v'(x)$$

$$[u(x)v(x)]' = u'(x)v(x) + u(x)v'(x)$$

$$\prod_{i=1}^{n} u_i(x) \quad \left[\prod_{i=1}^{n} u_i(x)\right]' = u_1' u_2 \dots u_n + u_1 u_2' u_3 \dots u_n + u_1 u_2 u_3' u_4 \dots u_n +$$

$$+ \dots + u_1 \dots u_{n-1} u_n'$$

$$\left[\frac{u(x)}{v(x)}\right]' = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$$

$v(x) \neq 0$

$y = f(x) + C \quad y = C$

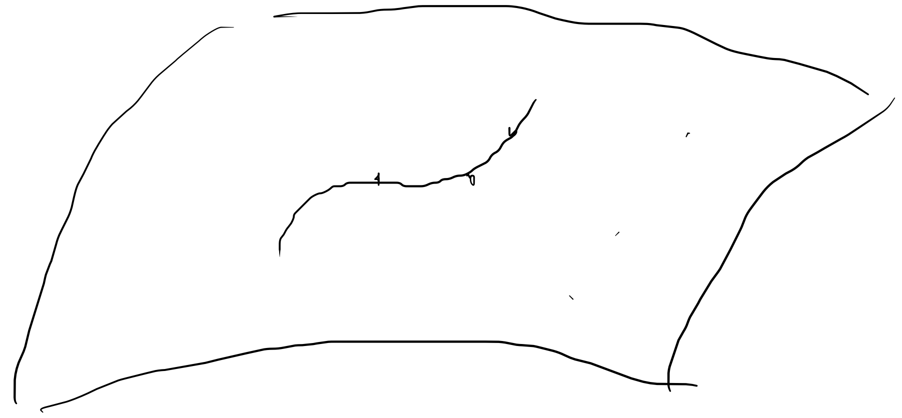
$y' = 0$

$y' = f' + C' = f'$

$y = Cf(x) \quad ; \; y' = Cf'(x) \quad ; \quad y = f(x), \; x = \varphi(t)$

$y = f(\varphi(t))$ - съставна ф-ция

$y_t' = f_x' \varphi_t'$



Логаритмична производна

e^x $\quad a^x$ – показат. ф-ция

$a>0$

$y=a^x$, $\quad y'=(a^x)'=$ $\qquad a^x = e^{x\ln a}$

$$=(e^{x\ln a})' = e^{x\ln a}\cdot(x\ln a)' = e^{x\ln a}\ln a = a^x \ln a$$

$y=[u(x)]^{v(x)}$ – сложна показателна ф-ция $\quad y'=?$

$$\ln y = v(x)\ln u(x) \,\Big|\, \frac{d}{dx} \qquad \frac{y'}{y} = v'\ln u + \frac{v}{u}u'$$

$$y' = y\left(v'\ln u + \frac{u'v}{u}\right) = u^v\left(v'\ln u+\frac{u'v}{u}\right)$$

пр. $y=\sqrt[x]{x} = x^{\frac{1}{x}}$ $\qquad \ln y = \frac{\ln x}{x} \,\Big|\, \frac{d}{dx}$

$$\ln y = \frac{\ln x}{x} \Big| \frac{d}{dx} \qquad \frac{y'}{y} = \frac{\frac{1}{\cancel{x}}\cdot \cancel{x} - \ln x}{x^2}$$

$$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

$$y' = y\left(\frac{1-\ln x}{x^2}\right) = x^{\frac{1}{x}-2}(1-\ln x) \qquad \left(\frac{1}{x^2}\right)'$$

$$y = \left(1+\frac{1}{x}\right)^x \to e \qquad y' = ? \qquad \ln y = x \ln\left(1+\frac{1}{x}\right) \Big| \frac{d}{dx}$$

$$x \to \infty \qquad \frac{y'}{y} = \ln\left(1+\frac{1}{x}\right) + x \frac{-\frac{1}{x^2}}{1+\frac{1}{x}} = \qquad \left(\frac{1}{x^n}\right)'$$

$$\left(x^{\frac{n}{m}}\right)' = \frac{n}{m} x^{\frac{n}{m}-1} \qquad \left(\sqrt[m]{x^n}\right)' \qquad \left(x^{-n}\right)' = -n x^{-n-1}$$

$$= \ln\left(1+\frac{1}{x}\right) - \frac{\frac{1}{x}}{\frac{x+1}{x}} = \ln\left(1+\frac{1}{x}\right) - \frac{1}{x+1} \Rightarrow y' = \left(1+\frac{1}{x}\right)^x \left[\ln\left(1+\frac{1}{x}\right) - \frac{1}{x+1}\right]$$

3 начина за задаване на ф-ция

явно задаване $y = f(x)$ - декартово у-ние

$f(x) - y = 0$

$\underbrace{}_{F(x,y)}$

неявно задаване $F(x, y) = 0$

$y = y(x)$

параметрично задаване $\begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases}$ $\alpha \le t \le \beta$

$t = \psi^{-1}(y)$

$\begin{cases} x = t \\ y = f(t) \end{cases}$

$F(x, y(x)) = 0 \quad \Big| \frac{d}{dx}$

$y = f(x)$

$y' = f' = \frac{dy}{dx}$

$F'_x + \underset{\neq 0}{F'_y} y' = 0 \Rightarrow y' = -\frac{F'_x}{F'_y}$; $y' = \frac{dy}{dx} = \frac{\psi' \cancel{dt}}{\varphi' \cancel{dt}} = \frac{\psi'}{\varphi'}$

$\Rightarrow \underbrace{dy}_{} = f' dx$

диференциал на ф-цията $f(x)$

пр.

$$\begin{cases} x = \dfrac{\cos^3 t}{\sqrt{\cos 2t}} \\ y = \dfrac{\sin^3 t}{\sqrt{\cos 2t}} \end{cases}$$

$$\frac{dx}{dt} = \frac{-3\cos^2 t \sin t \sqrt{\cos 2t} + \dfrac{\cos^3 t \sin 2t}{\sqrt{\cos 2t}}}{\cos 2t}$$

$y'(x) = ?$

$$\left[(\cos 2t)^{\frac{1}{2}}\right]' = -\frac{1}{\cancel{2}}(\cos 2t)^{-\frac{1}{2}} \sin 2t \cdot \cancel{2}$$

$$= -\frac{\sin 2t}{\sqrt{\cos 2t}}$$

$$\frac{1}{\sqrt{x}} \qquad \frac{1}{2\sqrt{x}}$$

$$= \frac{-3\cos^2 t \sin t \cos 2t + \cos^3 t \sin 2t}{(\cos 2t)^{\frac{3}{2}}} = \varphi'$$

$$\frac{dy}{dt} = \frac{3\sin^2 t \cos t \sqrt{\cos 2t} - \sin^3 t \dfrac{-\cancel{2}\sin 2t}{\cancel{2}\sqrt{\cos 2t}}}{\cos 2t} = \underset{\psi'}{\frac{3\sin^2 t \cos t \cos 2t + \sin^3 t \sin 2t}{(\cos 2t)^{\frac{3}{2}}}}$$

$$\frac{-3\cos^2 t\sin t\cos 2t+\cos^3 t\sin 2t}{(\cos 2t)^{\frac{3}{2}}}=\varphi'$$

$$\psi'_t=\frac{3\sin^2 t\cos t\sqrt{\cos 2t}-\sin^3 t\,\frac{-2\sin 2t}{2\sqrt{\cos 2t}}}{\cos 2t}=\frac{3\sin^2 t\cos t\cos 2t+\sin^3 t\sin 2t}{(\cos 2t)^{\frac{3}{2}}}$$

$$y'_{(x)}=\frac{\psi'_{(t)}}{\varphi'_{(t)}}=\frac{3\sin^2 t\cos t\cos 2t+\sin^3 t\sin 2t}{-3\cos^2 t\sin t\cos 2t+\cos^3 t\sin 2t}$$

Производни и диференциали от по-висок ред

$y=f(x)$ $\qquad y'=f'(x)\Rightarrow dy=f'(x)dx$ $\qquad (y')'=y''=\frac{d^2y}{dx^2}\Rightarrow$

$\frac{dy}{dx}$ $\qquad d^2y=f''(x)dx^2$ $\qquad d(dy)$

$d^ny=f^{(n)}(x)dx^n$ $\qquad$ втори диференциал np. $y = \ln(1+x)$ $y' = \frac{1}{1+x}$; $y'' = -(1+x)^{-2} = -\frac{1}{(1+x)^2}$

$y^{(n)} = ?$ $(1+x)^{-1}$ $y''' = 2(1+x)^{-3}$

$d^n y = ?$

$y^{IV} = -2.3(1+x)^{-4}$

Don. ,ze $y^{(k)} = (-1)^{k+1} \frac{(k-1)!}{(1+x)^k}$. $y^{(k+1)} = \left(y^{(k)}\right)' =$

$k > 4$ $(1+x)^{-k}$

$= (-1)^{k+1} (k-1)! (-k)(1+x)^{-k-1} = \frac{(-1)^{k+2} k!}{(1+x)^{k+1}}$

$\Rightarrow y^{(n)} = (-1)^{n+1} \frac{(n-1)!}{(1+x)^n}$ $d^n y = y^{(n)} dx^n = (-1)^{n+1} \frac{(n-1)!}{(1+x)^n} dx^n$

<u>пр. 2</u>

$y = \frac{1+x}{1-x}$; $y' = \frac{1-\cancel{x}+(1+\cancel{x})}{(1-x)^2} = \frac{2}{(1-x)^2} = 2(1-x)^{-2}$

$y^{(n)} = ?$

$d^n y = ?$

$y'' = +2 \cdot 2(1-x)^{-3} = 4(1-x)^{-3} = 2 \cdot 2(1-x)^{-3}$

$y''' = +2 \cdot 2 \cdot 3(1-x)^{-4}$

Доп., че $y^{(k)} = 2\frac{k!}{(1-x)^{k+1}} = 2k!(1-x)^{-(k+1)}$

$y^{(k+1)} = (y^{(k)})' = 2k![-(k+1)](1-x)^{-(k+1)-1} \cdot (-1) =$

$= 2(k+1)!\,(1-x)^{-(k+2)} = \frac{2(k+1)!}{(1-x)^{k+2}} \Longrightarrow y^{(n)} = \frac{2n!}{(1-x)^{n+1}}$

$d^n y = \frac{2n!}{(1-x)^{n+1}} dx^n$

Th (за диференциране на обратна ф-ция)

Нека $y=f(x)$ е монотонна и непрекъсната в околн. на т. $x \in D.O$, диференцируема и $f'(x) \neq 0$. Тогава $\exists$ $x=f^{-1}(y)$, обратна на $y=f(x)$ и $[f^{-1}(y)]' = \frac{1}{f'(x)}$.

<u>пр. 1</u> $y=\sin x$ $\quad (\arcsin y)' = \frac{1}{(\sin x)'} = \frac{1}{\cos x} = \frac{1}{\sqrt{1-\sin^2 x}} = \frac{1}{\sqrt{1-y^2}}$

$\cos x = \sqrt{1-\sin^2 x}$

$y=\arcsin x$; $\quad y' = \frac{1}{\sqrt{1-x^2}}$ $\qquad |x|<1$

$|y| \leq \frac{\pi}{2}$

$y=\arccos x$; $\quad y' = -\frac{1}{\sqrt{1-x^2}}$

<u>пр. 2</u> $y=\operatorname{tg} x$ $\quad (\operatorname{arctg} y)' = \frac{1}{(\operatorname{tg} x)'} = \frac{1}{\frac{1}{\cos^2 x}} = \cos^2 x = \frac{1}{1+\operatorname{tg}^2 x}$

$\cos^2 x = \frac{\cos^2 x}{1} = \frac{\cos^2 x}{\cos^2 x + \sin^2 x} = \frac{1}{1+\operatorname{tg}^2 x} \qquad = \frac{1}{1+y^2}$

$y = \operatorname{arcctg} x \; ; \; y' = -\dfrac{1}{1+x^2}$

Пр: 3 $\quad y = a^x \qquad \log_a y = x$

$$(\log_a y)' = \frac{1}{(a^x)'} = \frac{1}{(e^{x\ln a})'} = \frac{1}{e^{x\ln a}\cdot \ln a} = \frac{1}{a^x \ln a} = \frac{1}{y \ln a}$$

$y = e^x \; ; \; y' = e^x$

$y = a^x \; ; \; y' = a^x \ln a$

$y = \ln x \; ; \; y' = \dfrac{1}{x}$

$y = \log_a x \; ; \; y' = \dfrac{1}{x \ln a} \qquad a = e$

$F(x,y)=0$ $\quad \dfrac{d}{dx}$ $\quad \underbrace{F'_x + \underset{\neq 0}{F'_y}\, y' = 0}_{\frac{d}{dx}} \Rightarrow y' = -\dfrac{F'_x}{F'_y}$

$y=y(x)$

$\underbrace{F''_{xx} + F''_{xy}y' + (F''_{yx}+F''_{yy}y')y' + \underset{\neq 0}{F'_y}\, y'' = 0}_{\frac{d}{dx}} \Rightarrow y'' = ?$

$y''(x) = [y'(x)]' =$

$\dfrac{d}{dx}$ $\quad y=y(x) \quad y'(x)=?$

$= \dfrac{d}{dx}\left[\dfrac{\varphi'(t)}{\psi'(t)}\right] = \dfrac{d}{dt}\underbrace{\left[\dfrac{\varphi'(t)}{\psi'(t)}\right]}\dfrac{dt}{dx} =$

$$\begin{cases} x=\psi(t) \\ y=\varphi(t) \end{cases} \qquad y'(x) = \frac{\varphi'(t)}{\psi'(t)}$$

$= \dfrac{\varphi''\psi' - \varphi'\psi''}{\psi'^2}\dfrac{1}{\frac{dx}{dt}}\psi' = \dfrac{\varphi''\psi' - \varphi'\psi''}{\psi'^3}$ $\quad \dfrac{d}{dx}(\quad)$