е) рац. ф-ции $\displaystyle\int \frac{P_m(x)}{Q_n(x)}\,dx = \sum$ елемент. дроби

$$\int \frac{dx}{x^4-1} = \qquad \frac{1}{x^4-1} = \frac{1}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1} =$$

$$1 = A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x^2-1)$$

$x=1:\quad 1 = 4A \Rightarrow A = \frac{1}{4}$

$x=-1\quad 1 = -4B \Rightarrow B = -\frac{1}{4}$

$$= \frac{1}{4}\frac{1}{x-1} - \frac{1}{4}\frac{1}{x+1} - \frac{1}{2}\frac{1}{x^2+1}$$

$x=i\quad 1+i0 = (Ci+D)(-1-1) = -2D - 2Ci \Rightarrow -2D = 1 \Rightarrow D = -\frac{1}{2}$

$-2C = 0 \Rightarrow C = 0$

$$\frac{1}{4}\int\frac{d(x-1)}{x-1} - \frac{1}{4}\int\frac{d(x+1)}{x+1} - \frac{1}{2}\int\frac{dx}{x^2+1} = \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| - \frac{1}{2}\operatorname{arctg} x + C$$

$$\frac{1}{4}\ln|x-1|-\frac{1}{4}\ln|x+1|-\frac{1}{2}\operatorname{arctg}x+C=\ln\sqrt[4]{\frac{x-1}{x+1}}-\frac{1}{2}\operatorname{arctg}x+C$$

ж) интегриране на някои класове ирац. и трансцендентни ф-ции

— $\int f\left(x,\left(\frac{ax+b}{cx+d}\right)^{\frac{p_1}{q_1}},\ldots,\left(\frac{ax+b}{cx+d}\right)^{\frac{p_n}{q_n}}\right)dx$ $\quad t^{\text{НОК}(q_1,q_2,\ldots,q_n)}=\frac{ax+b}{cx+d}$

$\begin{vmatrix} a & b \\ c & d \end{vmatrix}\neq 0$ дробно-линейна ирационалност субстит.

$$\int\sqrt[3]{\frac{x+1}{x-1}}\,dx=\qquad\left(\frac{x+1}{x-1}\right)^{\frac{1}{3}}\qquad t^3=\frac{x+1}{x-1}\Rightarrow t^3(x-1)=x+1$$

$$(t^3-1)x=t^3+1\Rightarrow x=\frac{t^3+1}{t^3-1}$$

$$dx=\frac{3t^2(\cancel{t^3}-1)-3t^2(\cancel{t^3}+1)}{(t^3-1)^2}dt=\frac{-6t^2}{(t^3-1)^2}dt\qquad\Bigg|\ -6\int\frac{t^3\,dt}{(t^3-1)^2}$$

$$-6\int \frac{t^3\,dt}{(t^3-1)^2} =$$

$$\frac{t^3}{(t-1)^2(t^2+t+1)^2} = \frac{A}{t-1} + \frac{B}{(t-1)^2} + \frac{Ct+D}{t^2+t+1} + \frac{Et+F}{(t^2+t+1)^2}$$

$$t^3 = A(t-1)(t^2+t+1)^2 + B(t^2+t+1)^2 + (Ct+D)(t-1)^2(t^2+t+1) + (Et+F)(t-1)^2$$

$t=1:\quad 1 = 9B \Rightarrow B = \frac{1}{9}$

$$\frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

$$Ct + \tfrac{1}{2} + D - \tfrac{1}{2}$$

$$= -6A\int \frac{d(t-1)}{t-1} - 6B\int \frac{d(t-1)}{(t-1)^2} - 6\int \frac{Ct+D}{t^2+t+1}\,dt - 6\int \frac{Et+F}{(t^2+t+1)^2} =$$

$$= -6A\ln|t-1| + 6B\frac{1}{t-1} - \frac{6C}{2}\int \frac{t+\frac{D}{C}}{\left(t+\frac{1}{2}\right)^2+\frac{3}{4}}\,dt - 6\int \frac{Et+F}{\left[\left(t+\frac{1}{2}\right)^2+\frac{3}{4}\right]^2}\,dt$$

$$-6A\ln\left|\sqrt[3]{\frac{x+1}{x-1}} - 1\right| - 6B\frac{1}{\sqrt[3]{\frac{x+1}{x-1}}-1} - 3C\int \frac{t+\frac{1}{2}+\frac{D}{C}-\frac{1}{2}}{\left(t+\frac{1}{2}\right)^2+\frac{3}{4}}\,d\left(t+\tfrac{1}{2}\right)$$

$u = t+\frac{1}{2}$ $\quad \int \frac{u\,du}{u^2+\frac{3}{4}} \quad \int \frac{du}{u^2+\frac{3}{4}}$

$$\int \frac{\sqrt{x}-1}{\sqrt[3]{x}+1}\,dx =$$

$a=1 \quad b=c=0$

$d=1$

$x^{\frac{1}{2}}, \; x^{\frac{1}{3}} \qquad t^6 = x$

$dx = 6t^5\,dt$

$$= \int \frac{t^{\frac{6}{2}}-1}{t^{\frac{6}{3}}+1}\, 6t^5\,dt = 6\int \frac{t^3-1}{t^2+1}\, t^5\,dt$$

$$\frac{t^8-t^5}{t^2+1} = t^6 - t^4 - t^3 + t^2 + t - 1 + \frac{1-t}{t^2+1}$$

$$\begin{array}{l|l}
-\; t^8+0t^7+0t^6-t^5+0t^4+0t^3+0t^2+0t+0 & t^2+0t+1 \\
\quad t^8+0t^7+t^6 & t^6-t^4-t^3+t^2+t-1 \\
\hline
\quad -t^6-t^5+0t^4 & \\
\quad -t^6+0t^5-t^4 & \\
\hline
\quad -t^5+t^4+0t^3 & \\
\quad -t^5-0t^4-t^3 & \\
\hline
\quad t^4+t^3+0t^2 & \\
\quad t^4+0t^3+t^2 & \\
\hline
\quad t^3-t^2+0t & \\
\quad t^3+0t+t & \\
\hline
\quad -t^2-t+0 & \\
\quad -t^2+0-1 & \\
\hline
\quad -t+1 &
\end{array}$$

$$= 6\int t^6\,dt - 6\int t^4\,dt - 6\int t^3\,dt + 6\int t^2\,dt + 6\int t\,dt - 6\int dt + 6\int \frac{1-t}{t^2+1}\,dt$$

$$= \frac{6t^7}{7} - \frac{6t^5}{5} - \frac{6t^4}{4} + \frac{6t^3}{3} + \frac{6t^2}{2} - 6t + 6\,\mathrm{arctg}\,t - 3\int \frac{d(t^2+1)}{t^2+1}$$

$$t = \sqrt[6]{x} = x^{\frac{1}{6}}$$

$$\ln|t^2+1| + C$$

$$= 6\int \frac{t^3-1}{t^2+1}\,t^5\,dt$$

$$\frac{t^8 - t^5}{t^2+1} = t^6 - t^4 - t^3 + t^2 + t - 1 + \frac{1-t}{t^2+1}$$

— $\int f(x, \sqrt{ax^2+bx+c})\,dx$

квадратична
ирационалност
(Ойлерови интеграли)

Субституции на Ойлер

1) $b^2 - 4ac < 0$ и $a > 0$

$x_1, x_2 \in \mathbb{C}$ или $x_1 = x_2$

$$\sqrt{ax^2+bx+c} = t + x\sqrt{a}$$

2) $D>0$ и $x_1, x_2 \in \mathbb{R}$, $x_1 \neq x_2$ $\quad \sqrt{ax^2+bx+c} = \left(x - \begin{cases} x_1 \\ x_2 \end{cases}\right) t$

3) $c>0$ $\quad \sqrt{ax^2+bx+c} = xt + \sqrt{c}$

$$\int \frac{dx}{\sqrt{x^2+x+1}} = \qquad \sqrt{x^2+x+1} = t + x\sqrt{1} = t + x \;\Big|^2 \Rightarrow t = \sqrt{x^2+x+1} - x$$

$$\cancel{x^2} + x + 1 = (t+x)^2 = t^2 + 2tx + \cancel{x^2}$$

$$\frac{-1 \pm \sqrt{1-4}}{2} \qquad x(1-2t) = t^2 - 1 \Rightarrow x = \frac{t^2-1}{1-2t}$$

$$dx = \frac{2t(1-2t) + 2(t^2-1)}{(1-2t)^2}\,dt = \frac{-2t^2+2t-2}{(1-2t)^2}\,dt$$

$$\sqrt{x^2+x+1} = t + \frac{t^2-1}{1-2t} = \frac{t - 2t^2 + t^2 - 1}{1-2t} = \frac{-t^2+t-1}{1-2t}$$

$$= 2\int \frac{\frac{-t^2+t-1}{(1-2t)^2}\,dt}{\frac{-t^2+t-1}{1-2t}} = 2\int \frac{\cancel{1-2t}}{(1-2t)^{\cancel{2}}}\,dt = 2\int \frac{dt}{1-2t} = -\int \frac{d(1-2t)}{1-2t} = -\ln|1-2t| + C$$

$$=-\ln|1-2t|+C=-\ln\left|1-2\sqrt{x^2+x+1}+2x\right|+C \qquad t=\frac{\sqrt{x^2+x+1}-1}{x}$$

$$c=1>0 \qquad \sqrt{x^2+x+1}=xt+\sqrt{1}=xt+1 \Rightarrow x^2+x+\cancel{1}=(xt+1)^2=$$

$$=x^2t^2+2xt+\cancel{1}$$

$$x+1=xt^2+2t \Rightarrow x(1-t^2)=2t-1$$

$$x=\frac{2t-1}{1-t^2}\Rightarrow dx=\frac{2(1-t^2)+2t(2t-1)}{(1-t^2)^2}dt=\frac{2t^2-2t+2}{(1-t^2)^2}dt$$

$$\sqrt{x^2+x+1}=\frac{2t-1}{1-t^2}t+1=\frac{2t^2-t+1-t^2}{1-t^2}=\frac{t^2-t+1}{1-t^2} \qquad \frac{1}{1-t^2}=\frac{A}{1-t}+\frac{B}{1+t}$$

$$\int\frac{dx}{\sqrt{x^2+x+1}}=2\int\frac{\frac{t^2-t+1}{(1-t^2)^2}dt}{\frac{t^2-t+1}{1-t^2}}=2\int\frac{dt}{1-t^2}= \qquad (1-t)(1+t)$$

$$1=A+At+B-Bt$$

$$A=\frac{1}{2}=B \qquad =\int\frac{dt}{1-t}+\int\frac{dt}{1+t}=-\ln|1-t|+\ln|1+t|+C= \qquad \begin{array}{l|l} t: & A-B=0\\ t^0 & A+B=1\end{array}$$

$$1-t^2$$

$$= \int \frac{dt}{1-t} + \int \frac{dt}{1+t} = -\ln|1-t| + \ln|1+t| + C = \ln\left|\frac{1+t}{1-t}\right| + C = \ln\left|\frac{\cancel{1} + \frac{\sqrt{x^2+x+1}}{x} \cancel{-1}}{1 - \frac{\sqrt{x^2+x+1}}{x} \cancel{+1}}\right| + C$$

$$\int \frac{dx}{(x+4)\sqrt{x^2+3x-4}} = \int \frac{\frac{-10t}{\cancel{(t^2-1)^2}}\,dt}{\frac{5t^2}{\cancel{t^2-1}}\cdot\frac{5\cancel{t}}{\cancel{t^2-1}}} = -\frac{2}{5}\int\frac{dt}{t^2} = +\frac{2}{5t} + C = +\frac{2}{5}\frac{1}{\sqrt{\frac{x+4}{x-1}}} + C = +\frac{2}{5}\sqrt{\frac{x-1}{x+4}}$$

$$\frac{-3 \pm \sqrt{9+16}}{2} = \frac{-3\pm 5}{2} = -4;\ 1$$

$$\left(\frac{t^2+4}{t^2-1} - 1\right)t = \frac{\cancel{t^2}+4\cancel{-t^2}+1}{t^2-1}\,t = \frac{5\cdot t}{t^2-1}$$

$$\sqrt{x^2+3x-4} = (x-1)t \Rightarrow \cancel{(x-1)}(x+4) = (x-1)^{\cancel{2}}t^2 \Rightarrow x+4 = t^2(x-1)$$

$$(x-1)(x+4)$$

$$x(1-t^2) = -4 - t^2 \Rightarrow x = \frac{t^2+4}{t^2-1} \Rightarrow dx = \frac{2t(\cancel{t^2}-1) - 2t(\cancel{t^2}+4)}{(t^2-1)^2}\,dt$$

$$x+4 = \frac{t^2+4}{t^2-1} + 4 = \frac{t^2+\cancel{4}+4t^2\cancel{-4}}{t^2-1} = \frac{5t^2}{t^2-1}$$

$$= \frac{-10t\,dt}{(t^2-1)^2}$$

– диференциален бином $\int x^m(a+bx^n)^p\,dx$

3 субституции на Чебишев   1) $p\in\mathbb{Z} \Rightarrow$ частен случай на дробно-лин. ирационалност

2) $p\notin\mathbb{Z}$, но $\frac{m+1}{n}\in\mathbb{Z}$ $a+bx^n=t^s$, s е знаменателя на p

3) p и $\frac{m+1}{n}\notin\mathbb{Z}$, но $\frac{m+1}{n}+p\in\mathbb{Z}$ $ax^{-n}+b=t^s$, s —//—

$$\int\frac{dx}{x^2(2+x^3)^{\frac{5}{3}}}=\int x^{-2}(2+x^3)^{-\frac{5}{3}}\,dx=\int x^{-2}x^{-5}(\underbrace{2x^{-3}+1}_{t^3})^{-\frac{5}{3}}dx$$

$$\sqrt[3]{(2+x^3)^5}$$

$m=-2$ $\frac{m+1}{n}=\frac{-2+1}{3}=-\frac{1}{3}$

$n=3$

$p=-\frac{5}{3}$ $\frac{m+1}{n}+p=-\frac{1}{3}-\frac{5}{3}=-\frac{6}{3}=-2$

$2x^{-3}+1=t^3$ – III субст.

$$2x^{-3}=t^3-1\Rightarrow x^{-3}=\frac{t^3-1}{2}\Big|^{-\frac{1}{3}}\Rightarrow x=\left(\frac{t^3-1}{2}\right)^{-\frac{1}{3}}$$

$2x^{-3}+1=t^3$ — III суба.

$2x^{-3}=t^3-1 \Rightarrow x^{-3}=\frac{t^3-1}{2} \Big|^{\wedge -\frac{1}{3}} \Rightarrow x=\left(\frac{t^3-1}{2}\right)^{-\frac{1}{3}} \Big|^{\wedge -7} \Rightarrow x^{-7}=\left(\frac{t^3-1}{2}\right)^{\frac{7}{3}}$

$dx=-\frac{1}{\cancel{3}}\left(\frac{t^3-1}{2}\right)^{-\frac{4}{3}}\frac{\cancel{3}t^2}{2}dt=-\left(\frac{t^3-1}{2}\right)^{-\frac{4}{3}}\frac{t^2}{2}dt$

$\int x^{-7}(2x^{-3}+1)^{-\frac{5}{3}}dx=-\int\left(\frac{t^3-1}{2}\right)^{\frac{7}{3}}(t^3)^{-\frac{5}{3}}\left(\frac{t^3-1}{2}\right)^{-\frac{4}{3}}\frac{t^2}{2}dt=$

$=-\frac{1}{2^{\frac{7}{3}}\cdot 2^{-\frac{4}{3}}\cdot 2}\int (t^3-1)^{\frac{7}{3}-\frac{4}{3}}t^{-5+2}dt=-\frac{1}{2^2}\int t^{-3}(t^3-1)dt=-\frac{1}{4}\int\frac{t^3-1}{t^3}dt=$

$2^{\frac{3}{3}}\cdot 2$

$=-\frac{1}{4}\int\frac{dt}{t}-\frac{1}{4}\int t^{-3}dt=-\frac{1}{4}\ln|t|-\frac{1}{4}\frac{t^{-3+1}}{-3+1}+C=$

$t=(2x^{-3}+1)^{\frac{1}{3}}=\sqrt[3]{\frac{2}{x^3}+1}=\frac{\sqrt[3]{2+x^3}}{x}$

$=-\frac{1}{4}\ln\left|\frac{\sqrt[3]{2+x^3}}{x}\right|+\frac{1}{8}\frac{x^2}{\sqrt[3]{(2+x^3)^2}}+C$

$$\int \frac{\sqrt{1+\sqrt{x}}}{x}dx = \int x^{-1}\big(\underbrace{1+x^{\frac{1}{2}}}_{t^2}\big)^{\frac{1}{2}}dx = 4\int (t^2-1)^{-2}(t^2)^{\frac{1}{2}}t(t^2-1)\,dt =$$

$$t=\sqrt{1+\sqrt{x}}$$

$m=-1$

$n=\frac{1}{2}$

$p=\frac{1}{2}$

$$\frac{m+1}{n} = \frac{-1+1}{\frac{1}{2}} = 0$$

$$1+x^{\frac{1}{2}} = t^2 \Rightarrow x^{\frac{1}{2}} = t^2-1 \;\Big|^{\wedge 2} \Rightarrow x = (t^2-1)^2 \;\Big|^{\wedge\{-1\}}$$

$$\text{труба}. \quad dx = 2(t^2-1)2t\,dt = 4t(t^2-1)dt$$

$$x^{-1} = (t^2-1)^{-2}$$

$$= 4\int t^2(t^2-1)^{-1}dt = 4\int \frac{t^2-1+1}{t^2-1}dt = 4\int dt + 4\int \frac{dt}{t^2-1} =$$

$$\frac{1}{t^2-1} = \frac{A}{t-1} + \frac{B}{t+1} \Rightarrow 1 = At + A + Bt - B$$

$$= 4t + 2\int \frac{dt}{t-1} - 2\int \frac{dt}{t+1} =$$

$A+B=0$

$A-B=1$

$$\Rightarrow A=\frac{1}{2}\,;\; B=-\frac{1}{2}$$

$$= 4t + 2\ln|t-1| - 2\ln|t+1| + C = 4t + \ln\left(\frac{t-1}{t+1}\right)^2 + C$$

$$= 4\sqrt{1+\sqrt{x}} + \ln\left(\frac{\sqrt{1+\sqrt{x}}-1}{\sqrt{1+\sqrt{x}}+1}\right)^2 + C$$

3) интегриране на трансцендентни ф-ции

$\int R(\sin x, \cos x)\,dx$ — рационална ф-ция на sin и cos

$$\sin x = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}} = \frac{2\operatorname{tg}\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}} = \frac{2t}{1+t^2}$$

$$\operatorname{tg}\frac{x}{2} = t \qquad x = 2\operatorname{arctg} t \Rightarrow dx = \frac{2\,dt}{1+t^2}$$

универсална тригоном. субст.

$$\cos x = \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}$$

$$\int\frac{dx}{\sin x} = \int\frac{\frac{2\,dt}{1+t^2}}{\frac{2t}{1+t^2}} = \int\frac{dt}{t} = \ln|t| + C = \ln\left|\operatorname{tg}\frac{x}{2}\right| + C$$

алтернативи на унив. тригоном. субституция

1. $R(-\sin x, -\cos x) = R(\sin x, \cos x)$ $\quad \operatorname{tg} x = t$ – субст.

$x = \operatorname{arctg} t \qquad dx = \frac{dt}{1+t^2} \qquad \sin x = \frac{\sin x}{\sqrt{\cos^2 x + \sin^2 x}} = \frac{\operatorname{tg} x}{\sqrt{1+\operatorname{tg}^2 x}} = \frac{t}{\sqrt{1+t^2}}$

$\cos x = \frac{\cos x}{\sqrt{\cos^2 x + \sin^2 x}} = \frac{1}{\sqrt{1+t^2}}$

$$\int \frac{dx}{\sin x + \cos x} =$$

$$= \int \frac{\frac{2dt}{1+t^2}}{\frac{2t+1-t^2}{1+t^2}} =$$

$u = t-1$

$$= -2\int \frac{d(t-1)}{t^2-2t-1} = -2\int \frac{du}{u^2-(\sqrt{2})^2} =$$

$(t-1)^2 - 2$

$1 \pm \sqrt{1+1}$

2) $R(-\sin x, \cos x) = -R(\sin x, \cos x)$

$\cos x = t$

3) $R(\sin x, -\cos x) = -R(\sin x, \cos x)$

$\sin x = t$

$x = \arcsin t \qquad dx = \pm \frac{dt}{\sqrt{1-t^2}}$

$$= -2A\int \frac{du}{u-\sqrt{2}} - 2B\int \frac{du}{u+\sqrt{2}} = -2A \ln|t-1-\sqrt{2}| - 2B \ln|t-1+\sqrt{2}| + C$$

$$\int \frac{dx}{3\sin^2 x + 5\cos^2 x} = \int \frac{\frac{dx}{\cos^2 x}}{3tg^2 x + 5} = \int \frac{d\,tg\,x}{3tg^2 x + 5} = \int \frac{dt}{5+3t^2} =$$

$$tg\,x = t$$

$$= \frac{1}{5}\sqrt{\frac{5}{3}}\int \frac{d\left(t\sqrt{\frac{3}{5}}\right)}{1+\left(t\sqrt{\frac{3}{5}}\right)^2} = \frac{1}{\sqrt{15}}\, arctg\, t\sqrt{\frac{3}{5}} + C = \frac{1}{\sqrt{15}}\, arctg\left(tg\,x\,\frac{\sqrt{15}}{5}\right) + C$$

$$-\int \frac{\frac{dx}{\sin^2 x}}{3+5\,cotg^2 x} = -\int \frac{d\,cotg\,x}{3+5\,cotg^2 x} = -\int \frac{dt}{3+5t^2}$$

$$cotg\,x = t$$