Assignment - 4

(1) Given:

$\bar{\alpha} \oplus \bar{\beta} = \bar{\alpha} - \bar{\beta}$

$c\bar{\alpha} = -c\bar{\alpha}$

1) Association of Field:

The set given here is $(R^n, \oplus, \circ)$

⇒ Closure:

let $a, b \in R$

then $a \oplus b = a - b \in R$

So closure is satisfied

⇒ commutativity:-

$a \circ b = -a \times b$

$b \circ a = -b \times a$

$\because$ $a \times b = b \times a$, $-a \times b = -b \times a$

but $a \oplus b = a - b \neq b \oplus a = b - a$

so commutativity is violated.

It does not have a field associated with it.

) Commutativity of $\oplus$:-

$\bar{\alpha} \oplus \bar{\beta} = \bar{\alpha} - \bar{\beta} \quad \forall \bar{\alpha}, \bar{\beta} \in R^n$

But $\bar{\beta} \oplus \bar{\alpha} = \bar{\beta} - \bar{\alpha} \neq \bar{\alpha} - \bar{\beta}$

$\bar{\alpha} \oplus \bar{\beta} \neq \bar{\beta} \oplus \bar{\alpha}$

So commutativity is not followed.

3) Associativity of $\oplus$:-

$(\bar{\alpha} \oplus \bar{\beta}) \oplus \bar{\gamma} = \bar{\alpha} - \bar{\beta} - \bar{\gamma}$

$\bar{\alpha} \oplus (\bar{\beta} \oplus \bar{\gamma}) = \bar{\alpha} - (\bar{\beta} - \bar{\gamma})$
$= \bar{\alpha} - \bar{\beta} + \bar{\gamma}$

$\bar{\alpha} - \bar{\beta} - \bar{\gamma} \neq \bar{\alpha} - \bar{\beta} + \bar{\gamma}$

So associativity is also not followed

4) Existence of Identity/Zero vector :-

$\cancel{\bar{\alpha} \oplus \bar{\beta} = \bar{\alpha}}$
$\cancel{\bar{\alpha} - \bar{\beta} = \bar{\alpha}}$
$\cancel{\bar{\beta} = \bar{0}}$

$\bar{\alpha} \oplus \bar{\beta} \neq \bar{\beta} \oplus \bar{\alpha}$
So we cannot take $\bar{\alpha} + \bar{0} = \bar{0} \oplus \bar{\alpha} = \bar{\alpha}$

Therefore, identity does not exist under $\oplus$ opn.

5) Existence of negative vector :-

$\cancel{\bar{\alpha} \oplus \bar{\beta} = \bar{0}}$
$\cancel{\bar{\alpha} - \bar{\beta} = \bar{0}}$
$\cancel{\bar{\beta} = \bar{\alpha}}$

Since identity does not exist inverse also does not exist

~~Therefore the inverse of any vector in this space is the vector itself.~~

Existence of $c \in \mathbb{R}$ s.t. $c \circ \bar{\alpha} = \bar{\alpha}$:-

$c \circ \bar{\alpha} = \bar{\alpha}$
$-c \times \bar{\alpha} = \bar{\alpha}$
clearly $c = -1$

There exist $c \in F$ s.t product of c and $\bar{\alpha}$ is $\bar{\alpha}$ itself.

7) Let $c_1, c_2 \in R$

$c_1(\bar{\alpha} \oplus \bar{\beta}) = -c_1(\bar{\alpha} \oplus \bar{\beta})$

$= -c_1(\bar{\alpha} - \bar{\beta})$

$= -c_1\bar{\alpha} + c_1\bar{\beta}$

Finding $c_1\bar{\alpha} \oplus c_1\bar{\beta}$

$= -c_1\bar{\alpha} - (-c_1\bar{\beta})$

$= -c_1\bar{\alpha} + c_1\bar{\beta}$

$c_1(\bar{\alpha} \oplus \bar{\beta}) = c_1\bar{\alpha} \oplus c_1\bar{\beta}$

Distributive property is verified.

* $-c_1 \circ \bar{\alpha} = -(-c_1\bar{\alpha})$

$= c_1\bar{\alpha}$

Finding $-(c_1\bar{\alpha})$,

$\Rightarrow -(-c_1\bar{\alpha}) = c_1\bar{\alpha}$

$\Rightarrow (-c_1)\cdot\bar{\alpha} = -(c_1\bar{\alpha})$

$(c_1 \oplus c_2)\cdot\bar{\alpha} = -(c_1 \oplus c_2)\bar{\alpha}$

$= \cancel{c_1\bar{\alpha}}$

$= -(c_1 - c_2)\bar{\alpha}$

$= -c_1\bar{\alpha} + c_2\bar{\alpha}$

Finding $c_1\bar{\alpha} \oplus c_2\bar{\alpha}$

$= -c_1\bar{\alpha} \oplus -c_2\bar{\alpha}$

$= -c_1\bar{\alpha} - (-c_2\bar{\alpha})$

$= -c_1\bar{\alpha} + c_2\bar{\alpha}$

$\Rightarrow (c_1 \oplus c_2)\bar{\alpha} = c_1\bar{\alpha} \oplus c_2\bar{\alpha}$

(2) <u>Given</u>

f be the complex valued function such that $\forall t \in \mathbb{R}$, $f(-t) = \overline{f(t)}$ and V being the set of all such functions f.

<u>To Prove</u> $(f+g)(t) = f(t) + g(t)$

$(cf)(t) = c(f(t))$ being the operation,

V is a vector space over $\mathbb{R}$.

Let $f(t) = it$

$f(-t) = \overline{f(t)} = \overline{it}$

$\overline{it} = -it$ [$\because$ Real part's conjugate is itself]

$f(-t) = -it$

$\Rightarrow f(t) + f(-t) = 0$

$= (cf)(t) = i(ct)$

$= c \times it$

$c(f(t)) = c \times (it)$

$= cit$

$c f(t) = (cf)(t)$

1) <u>Closure:</u>

$f(t) \in F,\ g(t) \in F,\ (f+g)(t) = f(t) + g(t) \in F$

Let us check $(f+g)(-t)$

$(f+g)(-t) = f(-t) + g(-t)$

$= f^*(t) + g^*(t)$

$= (f+g)^*(t)$

$(f+g)(-t) \in R$

Closure is satisfied

2) <u>Closure under scalar multiplication:</u>

For $c \in R$, $f \in V$, $(cf)(t) = c[f(t)]$ is still in V for any $c \in R$

$(cf)(-t) = c[f(-t)] = c(f^*(t))$

$= (cf)^* t$

3) <u>Commutativity:</u>

$(f+g)(t) = f(t) + g(t)$

$(g+f)(t) = g(t) + f(t)$

$\forall\ f(t) + g(t) = g(t) + f(t)$

$(f+g)(t) = (g+f)(t)$

4) ~~(f+g)~~ <u>Associativity:</u>

$(f+g)(t) + h(t) = f(t) + g(t) + h(t)$

Taking, $f(t) + (g+h)(t) = f(t) + g(t) + h(t)$

Clearly $(f+g)(t) + h(t) = f(t) + (g+h)(t)$

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5) Identity:

$$f(t) + g(t) = f(t)$$
$$g(t) = 0$$

There exists a function $g(t)$ such that its value is 0, $\forall\ t \in R$

Therefore identity exists

6) Inverse:

$$f(t) + g(t) = \text{identity} = 0$$
$$f(t) + g(t) = 0$$
$$g(t) = -f(t)$$

We can say that for all $f \in V$ there will be a function $g \in V$ s.t $f(t) = -g(t)$
So inverse also exists.

7) Distributivity:

$$c(f+g)(t) = [c(f+g)](t)$$
$$= c[f(t) + g(t)]$$
$$= (cf)(t) + (cg)(t)$$
$$= c(f(t)) + c(g(t))$$

Therefore distributivity is satisfied.

8)
* $c(d(f(t))) = d(c(f(t))) = (cdf)(t) = cd(f(t))$

$\because$ V satisfies all conditions of a vector space
V is a vector space

(3) To prove: Non-empty subset W of vector V is a subspace of V if and only if, for each $\bar{\alpha}, \bar{\beta} \in W$, $c \in F$; $c\bar{\alpha} + \bar{\beta} \in W$.

* Proving $c\bar{\alpha} + \bar{\beta} \in W$ when W is subspace

Since W is a subspace, $\bar{0} \in W$.
Take $c = 0$, $\bar{\beta} = \bar{0}$, then we get,
$\Rightarrow c\bar{\alpha} + \bar{\beta} = \bar{0} \in W$

Say vector $\bar{\alpha} \in W$,
Taking $c = -1$ and $\bar{\beta} = \bar{0}$
$\Rightarrow c\bar{\alpha} + \bar{\beta} = -\bar{\alpha} + \bar{0} = -\bar{\alpha}$
* we are able to express $\bar{0}, -\bar{\alpha}$ in terms of $c\bar{\alpha} + \bar{\beta}$
* when $c = 1$, $\bar{\alpha} + \bar{\beta}$ also belongs to W when $\bar{\alpha}, \bar{\beta} \in W$
* when $\bar{\beta} = \bar{0}$, $c\bar{\alpha} \in W$ $\forall c \in F$ when $\bar{\alpha} \in W$

From the above observations we can conclude that $c\bar{\alpha} + \bar{\beta} \in W$ when W is a subspace of V.

* Proving converse:

* $c\bar{\alpha} + \bar{\beta} \in W$, we need to show that W is a subspace.
* when $c = 0$ and $\bar{\beta} = \bar{0}$,
$\Rightarrow c\bar{\alpha} + \bar{\beta} = \bar{0}$
$\bar{0} \in W$

$\Rightarrow$ There exists an identity in W.

* when $c=-1$, and $\beta=\bar{0}$, for every $\bar{\alpha}\in W$, there exist $c\bar{\alpha}+\bar{\beta}=-\bar{\alpha}$ such that $\bar{\alpha}+(c\bar{\alpha}+\bar{\beta})=\bar{\alpha}+(-\bar{\alpha})$ $=\bar{0}$

$\Rightarrow$ There exist an inverse for each $\bar{\alpha}\in W$.

* when $c=1$, for every $\bar{\alpha},\bar{\beta}\in W$, $\bar{\alpha}+\bar{\beta}\in W$, therefore closure is also satisfied.

* Since W satisfies all conditions of a vector space, it is a vector subspace.