COMP1215 - Combinatorics

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Contents

1 Inclusion and Exclusion

1. $|A \cup B| = |A| + |B| - |A \cap B|$

2 Pigeonhole principle

If $|A| > |B|$ (size of A > size of B), then every function from A to B maps at least 2 distinct elements of A to the same element of B.
In other words:

$$\text{For } f : A \to B\ \exists x_1, x_2 \in A, x_1 \neq x_2 \text{ such that } f(x_1) = f(x_2)$$

Example 2.1.

$$A = \{x \mid x \in \mathbb{Z} \text{ } and \text{ } |x| \leq 5\}$$
$$B = \{x \mid x \in \mathbb{N} \text{ } and \text{ } x \leq 5\}$$

As you can see, $|A| > |B|$, so all valid functions between A and B must map at least one pair of x values to the same y value.

$$f : A \to B$$
$$f(x) = x^2$$
$$f(-5) = 25$$
$$f(5) = 25$$

Example 2.2. A human has a maximum of approx 200,000 hairs on his head. Prove that at least 2 people in London have exactly the same number of hairs on their head.

$$Let\ A = \{x \mid x \in \mathbb{N}\ and\ x \leq 8.9 \times 10^6\}$$

$$Let\ B = \{x \mid x \in \mathbb{N}\ and\ x \leq 200,000\}$$

$|A| > |B|$ therefore a function $f : A \to B$ must map at least 2 values in A to the same B value.

3 Fibonacci series and recursion

$$|A| = \text{cardinality of set } A = \#\{a|a \in A\}$$

3.1 Recurrence relation

$$f_n = f_{n-1} + f_{n-2}$$
$$\begin{pmatrix} f_n \\ f_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_{n-1} \\ f_{n-2} \end{pmatrix}$$

3.2 Inductive proofs and recursion

$\forall n : P(n)$

1. **Base Case:** First, prove $P(1)$ is true
2. **Induction Step**
   Assume $P(0), P(1), ..., P(n)$ are all true (Induction hypothesis)
   Use the assumption above to prove $P(n+1)$
   If $P(n+1) = True$, proven

Example 3.1. Prove that the sum of all natural numbers adds up to $\frac{n(n+1)}{2}$ using a proof by induction.
So we need to prove $0 + 1 + 2 + ... + n = \frac{n(n+1)}{2}$. Base case proof:

$$Let\ n = 1 \therefore 0 + 1 = \frac{1(2)}{2} = 1 \therefore \text{works for } n = 1$$

Now, we assume $n = k$ holds (Induction Hypothesis):

$$1 + 2 + ... + k = \frac{k(k+1)}{2}$$

And we need to show $n = k + 1$ works:

$$1 + 2 + ... + k + (k+1) = (1 + 2 + ... + k) + (k+1) = \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{k(k+1) + 2(k+1)}{2} = \frac{k^2 + 3k + 1}{2} = \frac{(k+1)(k+2)}{2} \quad \square$$

3.2.1 Base Case

This case proves that the property holds for the first item. It doesn't have to begin with $n = 0$, it often starts with $n = 1$ but it can start with any number within the given set.

3.3 Fibonacci Q-Matrix

The Q matrix is defined by:

$$Q = Q^1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

The n^{th} value is given by:

$$Q^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$$