COMP1215 - Linear Algebra

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1 Vectors

Think of vectors as ordered lists of numbers.
They can be used to represent numerical data about certain objects.
We use square brackets to denote vectors, as such:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{1}$$

Example 1.1. For example, we can represent a person that is a male and 18 years old as such:

$$\begin{pmatrix} 18 \\ 1 \end{pmatrix} \tag{2}$$

Notice how we cannot express the male as "M" or "Male", we need to code it into a number, so let's just assume Female=0, Male=1. This is a 2 Dimensional vector, however vectors of any size can be used.

1.1 Vector Operations

1.1.1 Vector and scalar multiplication

To multiply a vector by a scalar (a whole number / constant), we simply multiply all of the terms by the given scalar:

$$x \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ax \\ bx \end{pmatrix} \tag{3}$$

1.1.2 Adding two vectors

You can only add vectors if their size is the same:

$$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a+x \\ b+y \end{pmatrix} \tag{4}$$

1.1.3 Dot product

To get the dot product, multiply the columns together and add them up.
The dot product of two vectors, a and a is defined by:

$$a \cdot b = (a_1 \times b_1) + (a_2 \times b_2) + ... + (a_n \times b_n)$$

Example 1.2.

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 3 \end{pmatrix} = (2 \times 4) + (1 \times 3) = 11 \tag{5}$$

2 Matrices

A matrix is similar to a vector, but it is defined by numbers in two dimensions.

Example 2.1.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{6}$$

However, although a vector represents a point in space, a matrix is used to represent a translation of multiple points.

2.1 Multiplying matrices by vectors

Example 2.2.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} a \\ c \end{bmatrix} + y \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix} \tag{7}$$

Example 2.3.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + y \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + z \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} a_{11}x + a_{12}y + a_{13}z \\ a_{21}x + a_{22}y + a_{23}z \\ a_{31}x + a_{32}y + a_{33}z \end{bmatrix} \tag{8}$$

2.2 Multiplying matrices together

In order to multiply a matrix by another matrix, we need to use the dot product.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 58 & \\ & \end{bmatrix}$$

As you can see, we took the first row and the 1nd column and performed the cross product.

$$1 \times 7 + 2 \times 9 + 3 \times 11 = 7 + 18 + 33 = 58$$

Now, we can take stay on the 1st row and the 2nd column and perform the same operation.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ & \end{bmatrix}$$

Hopefully, you can see how this carries on, by moving into the 2nd row now and starting with the 1st column again. You should end up with something this:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}$$

2.3 Row replacement as vector multiplication

Vector multiplication can be used to extract a row from a matrix.

To do that, we create a vector called E_{rs}. This is just a convention that is used to define a vector for row replacement that contains 0s and 1s.
The $(i, j)^{th}$ element of E_{rs} places row s where row r used to be and sets the rest to 0s.
The matrix $(E_{rs})_{ij}$ can be defined with:

$$(E_{rs})_{ij} = \delta_{ri}\delta_{sj} \ where \ \delta_{pq} = \begin{cases} 1 & p = q \\ 0 & otherwise \end{cases}$$

todo: example/explaination, explain indexing(ij notation)

Example 2.4.

$$let\ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{9}$$

To extract the 3^{rd} row and put it in the 2^{nd} row, we can multiply it by the following matrix:

$$E_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{10}$$

TODO: multiply to prove

2.4 Row interchange

To interchange (swap) row i with row j, we write:

$$\rho_i \leftrightarrow \rho_j$$

Rows can also be interchanged by multiplying them by a 2D vector consisting of 1s and 0s as such:

Example 2.5. To interchange row 2 and row 5:

$$p_2 \leftrightarrow p_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 3 \\ 4 \\ 2 \\ 6 \end{bmatrix} \tag{11}$$

3 Span of a set of vectors

We say that a vector w is in the span of a set of vectors V, if we can use the vectors V to construct w.

Example 3.1. Consider the following vectors as the span:

$$\text{let V} = \{ \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \end{bmatrix} \}$$

Now, if a vector can be created by scaling both of the given vectors and adding the result together, then the vector can be considered in the span of V.
Let's consider a vector $w = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$. It can be rewritten as:

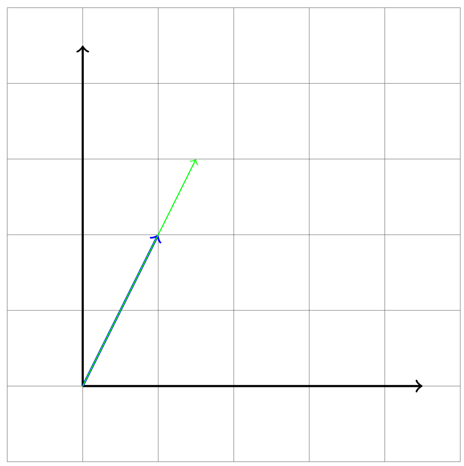
$$\frac{5}{2} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + \frac{15}{4} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

Which shows that w is in the span of V as it can be represented as sum of vectors in V scaled.

Example 3.2. Given the previous example, you're probably wondering, when would a vector not be in the span of two other vectors? Let's consider the following span:

$$\text{let V} = \{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}1.5\\3\end{bmatrix}\}$$

The problem with the given span, is that two two vectors are multiples of each other.



As you can seee, the two vectors overlap, and the only way for a vector to be in their span is to be on the same line. Here are a few examples:

$$\begin{bmatrix}-3\\-6\end{bmatrix} \text{ is in the span of } V$$

$$\begin{bmatrix}0\\0\end{bmatrix} \text{ is in the span of } V$$

$$\begin{bmatrix}1\\3\end{bmatrix} \text{ is not in the span of } V$$

3.1 Linear Independence

Like shown in the previous example, sometimes having 2 vectors in the span doesn't mean that the span will be 2 dimensional. If a vector is a scalar of another one, this means that they are lineary dependent.

Example 3.3.

$$\begin{bmatrix}4\\4\end{bmatrix} \text{ is linearly dependent to } \begin{bmatrix}-2\\-2\end{bmatrix} \text{ but is independent from } \begin{bmatrix}3\\2\end{bmatrix}$$

A set of vectors is **linearly independent** iff none of the vectors in the set can be expressed as a sum of scalars of the others.

3.2 Spans in 3 dimensions

Spans in 3 dimensions work in the exact same way given that only 2 vectors are given in V. The span simply creates a 2D place in the 3D space which defines the span.
Adding a 3rd vector that is not within the span of the other 2, will allow for any point in the 3 dimensional space to be within the span.
This 3 dimensional aspect is relatively hard to explain on paper, I recommend you watch this video if you don't understand this concept.

4 Gaussian Elimination

Let's consider a system of 3 simultaneous equations with 3 unknowns:

$$3x_1 + 2x_2 + x_3 = 39$$

$$2x_1 + 3x_2 + x_3 = 34$$

$$x_1 + 2x_2 + 3x_3 = 26$$

Now, to find the value of x_1, x_2 and x_3, we can use what's called **Gaussian Elimination**.
Firstly, let's agree on notation. Let ρ_n be the n_{th} equation, so ρ_1 would be the first equation ($3x_1 + ... = 39$) and so on.
Firstly, we can replace ρ_3 with $\rho_1 - 3\rho_3$ to get rid of x_1

$$-4x_2 - 8x_3 = -39$$

Now, we can replace ρ_2 with $\rho_1 - \frac{3}{2}\rho_2$ to get rid of x_1

$$-\frac{5}{2}x_2 - \frac{1}{2}x_3 = -12$$

The final step is replacing ρ_3 with $-\frac{8}{5}\rho_2 + \rho_3$ to get rid of x_2

$$-\frac{36}{5}x_3 = -\frac{99}{5}$$

This leaves us with:

$$3x_1 + 2x_2 + x_3 = 39$$
$$-\frac{5}{2}x_2 - \frac{1}{2}x_3 = -12$$
$$-\frac{36}{5}x_3 = -\frac{99}{5}$$

The equations can be represented as an **Augmented matrix** as such:

$$\left(\begin{array}{ccc|c} 3 & 2 & 1 & 39 \\ 0 & \frac{5}{2} & -\frac{1}{2} & -12 \\ 0 & 0 & -36 & -99 \end{array}\right)$$

This can now be solved by back substitution, but in order to make it easier, we can divide all the rows by the first entry.

$$\left(\begin{array}{ccc|c} 1 & \frac{2}{3} & \frac{1}{3} & 13 \\ 0 & 1 & \frac{1}{5} & \frac{24}{5} \\ 0 & 0 & 1 & \frac{11}{4} \end{array}\right)$$

4.1 Echelon form

The form above is called **echelon form**. In order for an augmented matrix to be in echelon form, it needs to satisfy the following rules:

1. All leading entries in each row are equal to 1
2. If a column contains a leading entry, then all entries below have to equal 0
3. In any 2 consecutive non-zero entries, the leading entry in the upper row occurs to the left of the leading entry in the lower row.

Example 4.1. Here is another example of a valid echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 4.2. Sometimes, an equation might not have a solution. Take $3x_1 + 2x_2 = 1$ as an example. This is the equivalent of the above equation in row echelon form:

$$\left(\begin{array}{cc|c} 1 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 \end{array}\right)$$

Which brings us to the following conclusion

In row echelon form,
if there exists at least one row full of 0s
there are **infinitely many solutions**

In order to obtain a solution in row echelon form, each row has to have a pivot (non zero leading value).