COMP1215 - Sets

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1 What is a set?

Definition 1.1. Set An **unordered** collection of **objects** with **no duplicates**.

1.1 Defining a set

To Define a set, we use the following syntax:

$$Let\ X = \{1, 2, 3\}$$

This defines a set called X with elements 0, 2 and 3.

1.2 Properties of a set

Note, that a set **doesn't specify the order**, so:

$$X = \{3, 1, 2\}$$

The set also **doesn't have duplicates**, so they can be simply ignored:

$$Let\ Y = \{apple, apple, 1, 2, 2, 3, 3, pi, \{3.12\}\}$$

$$\therefore Y = \{apple, 1, 2, 3, pi, \{3.12\}$$

Notice how I used different types of data types in the above example. This is because a set is a collection of **objects**, therefore it can include anything.

1.3 Cardinality

For finite sets, the cardinality of a set is simply its size:

Example 1.1.

$$Let\ X = \{1, 2, 3, 5\}$$

$$|X| = 4$$

1.4 Membership

Given a set:

$$Let\ X = \{...\}$$

If an element x is included within X:

$$x \in X$$

If it is not included:

$$x \notin X$$

Example 1.2.

$$Let\ X = \{1, 2, 3, 4\}$$
$$1 \in X$$
$$0 \notin X$$

1.5 Subsets

Definition 1.2. Subset A set A is a subset of set B if all elements of A are included within B.

To explain, given a set:

$$Let\ B = \{...\}$$

If another set A is a subset of B:

$$A \subseteq B$$

This also means that B is a superset of A, which is just another way of writing it:

$$B \supseteq A$$

Example 1.3.

$$Let\ X = \{1, 2, 3\}$$
$$\{1\} \subseteq X$$
$$\{1, 3\} \subseteq X$$
$$\{1, 3, 4\} \not\subseteq X$$

1.6 Proper subset

A proper subset works exactly the same as a subset, except for it **cannot contain all elements of the superset**.

If set A is a proper subset of B:

$$A \subset B$$

Example 1.4.

$$Let\ X = \{1, 2, 3\}$$
$$\{1, 2\} \subset X$$
$$\{1, 2, 3\} \not\subset X$$

1.7 Existing sets

Symbol	Name	Description
$\mathbb{R}$	Real	any real number
$\mathbb{N}$	Natural	integer > 0
$\mathbb{Z}$	Integer	whole number
2	Binary Set	$2 = \{0, 1\}$

2 Ordered Pairs

Unline sets, ordered pairs can only contain two elements and their order matters.

$$(1, 0) \neq (0, 1)$$

Example 2.1.

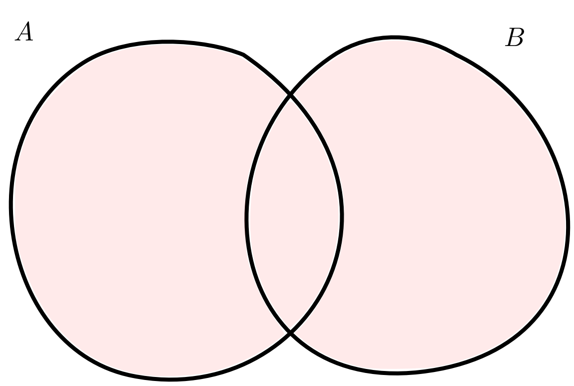
$$\textit{Let } X = \{(0, 1), (1, 1)\}$$

Note that ordered pairs can also contain duplicates.

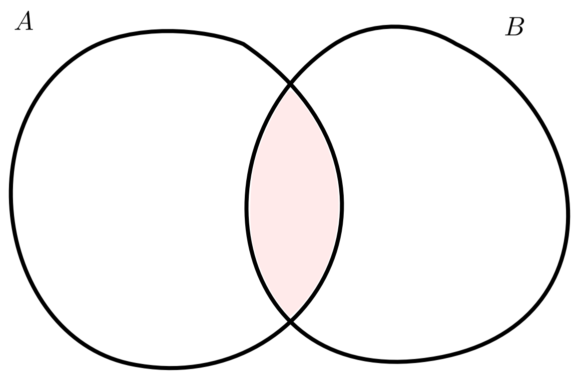
3 Set Operations

Definition 3.1. Set Operation Takes two sets and returns a single set.

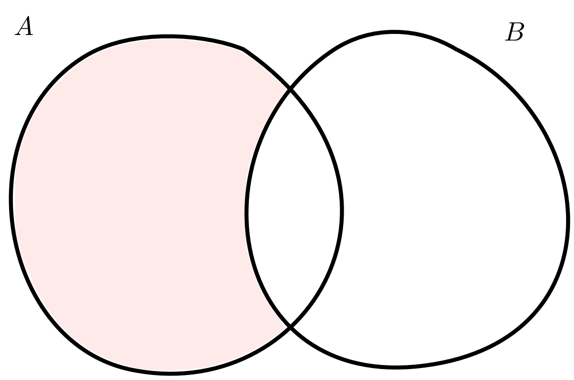
3.1 Union: $A \cup B$



3.2 Intersection: $A \cap B$



3.3 Difference: $A - B$



3.4 Cartesian Product: $A \times B$

Gives a 2D matrix representation of two sets, using a set of pairs.

$$A \times B = \{(a, b) \mid a \in A \; and \; b \in B\}$$

	b_1	b_2	...	b_m
a_1	(a_1, b_1)	(a_1, b_2)	...	(a_1, b_m)
a_2	(a_2, b_1)	(a_2, b_2)	...	(a_2, b_m)
...	...	...	...	...
a_n	(a_n, b_1)	(a_n, b_2)	...	(a_n, b_m)

Example 3.1.

$$Let\ A = \{a, b\}$$

$$Let\ B = \{1, 2, 3\}$$

$$\therefore A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$$

	1	2	3
a	(a,1)	(a,2)	(a,3)
b	(b,1)	(b,2)	(b,3)

3.5 Sum / Disjoint Union: $A + B$

Disjoint union takes two sets, A and B and unions them such that all elements from both sets are included:

$$X + Y = \{(x, 0) \mid x \in X\} \cup \{(y, 1) \mid y \in Y\}$$

Example 3.2.

$$Let\ X = \{1, 2, 3\}$$

$$Let\ Y = \{2, 3, 4\}$$

Notice that the union skips the duplicates:

$$X \cup Y = \{1, 2, 3, 4\}$$

However the disjoint union keeps them:

$$X + Y = \{(1, 0), (2, 0), (3, 0), (2, 1), (3, 1), (4, 1)\}$$

4 Relations

A relation from set X to set Y is **some set** of **pairs from their cartesian product**, so:

$$R \subseteq X \times Y$$

4.1 Number of relations

The number of relations is simply the number of subsets in the cartesian product.
The number of items in the cartesian product of sets X and Y is $|X| \times |Y|$.
So the number of relations must be:

$$2^{|X| \times |Y|}$$

4.2 Identity Relation: I_X

An identity relation is an example of a relation that maps a value of a set to the same value.
We use the notation I_X where X is the set that the relation is on.

Example 4.1.

$$Let\ X = \{1, 2, 3\}$$
$$\therefore I_X : X \to X\ (more\ on\ this\ syntax\ later)$$
$$\therefore I_X = \{(1, 1), (2, 2), (3, 3)\}$$

If you imagine the cartesian product as a 2D table matrix, then the identity relation is simply the diagonal:

	x_1	x_2	...	x_m
x_1	(x_1, x_1)	(x_1, x_2)	...	(x_1, x_m)
x_2	(x_2, x_1)	(x_2, x_2)	...	(x_2, x_m)
...	...	...	...	...
x_n	(x_n, x_1)	(x_n, x_2)	...	(x_n, x_m)

4.3 Equivalence relations

An equivalence relation is a special relation that satisfies the following conditions:

1. **reflexivity**: $\forall x \in X, (x, x) \in \sim$
2. **symmetry**: $\forall x, y \in X, if\ (x, y)\ \in \sim\ then\ (y, x) \in \sim$
3. **transitivity**: $x, y, z \in X,\ for\ (x, y) \in \sim\ and\ (y, z)\ then\ (x, z) \in \sim$

Example 4.2.

$$Let\ A = \{1, 2, 3\}$$

Then a valid equivalence relation, R, can defined as follows:

$$\{(1, 1), (2, 2), (3, 3), (1, 3), (3, 1)\}$$

Of course, there are many other equivalence relations that can be defined, as long as they follow the given 3 rules.

4.3.1 Equivalence classes

An equivalence class i a set of elements a given element is equivalent to.
Given a set X and element a such that $a \in X$:

$$[a] = \{x \mid x \in X\ and\ (x, a) \in \sim\}$$

Example 4.3.

$$Let\ X = \{1, 2, 3\}$$

$$Let\ \sim = \{(1, 1), (2, 2), (3, 3), (1, 3), (3, 1)\}$$

$$\therefore [1] = \{1, 3\}, [2] = \{2\}, [3] = \{3, 1\}$$

4.3.2 Quotients

Definition 4.1. Quotient The set of equivalence classes for each element in the given set.

A quotient of X with respect to   is usually denoted as:

$$X/\sim = \{[x] \mid x \in X\}$$

4.4 Functions as relations

A function (function graph to be specific, but we will cover that later) is simply a relation (called **f**) that satisfies the following conditions:

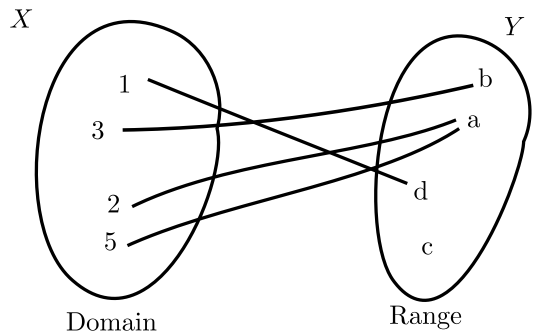
1. $\forall x \in X$, there exists $y \in Y$ such that $(x, y) \in f$
2. if $(x, y) \in f$ and $(x, z) \in f$ then $y = z$

The following points can be easily simplied:

1. function is defined on the entire domain
2. function can only have 1 output for a given input

5 Functions

A function maps from one set of values to a different set. The same x-value in a set cannot map to two different y-values, but two x-values can map to the same y-value.



5.1 Range vs Codomain

Definition 5.1. Range. All of Y values that are mapped to by X.

This can also be written as follows:

$$\{f(x) \mid x \in X\}$$

5.2 Declaring functions

In discrete mathematics, a funciton can only have 1 input and 1 output.

Example 5.1. Let's see an example of a function declaration.
A function called **f** that takes any integer, multiplies it by two and returns it would be defined as such:

$$f : \mathbb{Z} \to \mathbb{Z}$$

$$f(x) = x \times 2$$

We need to define what set the domain and codomain are before we write the equation for the function.

Example 5.2. A function that takes more than 1 input, for example a function **g** that multiplies two real numbers together would have to take a pair from the cartesian product of the two real sets:

$$g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

$$g(a, b) = a \times b$$

And if a function took more than two arguments, it would look somewhat like this:

$$h : (\mathbb{R} \times \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$$

$$h(a, b, c) = a \times b + c$$

Note, that $h(a, b, c)$ is just a nicer way of writing $h((a, b), c)$ and this is actually what is happening under the hood.

5.3 Number of functions between two sets

5.4 Composing functions

Definition 5.2. Composition. Taking output from one function and feeding it to another one

$$g(f(x)) = (f; g)(x) = (f \circ g)(x)$$

5.5 Injective, surjective and bijective

Functions can have their own properties. For example, a function can be injective, surjective, or both.

5.5.1 Injective

In an injective function, no 2 of the same X values give the same Y value.

$$\forall x, x' \in X,\ if\ f(x) = f(x')\ then\ x = x'$$

5.5.2 Surjective

In a surjective function, all of the codomain is being mapped to by the function. This means that **range = codomain**.

$$let\ f : X \to Y$$

$$\forall y \in Y,\ there\ exists\ x \in X\ such\ that\ f(x) = y$$

5.5.3 Bijective

A function is bijective if it's both injective and surjective.
This is also known as **1-1 correspondence**.

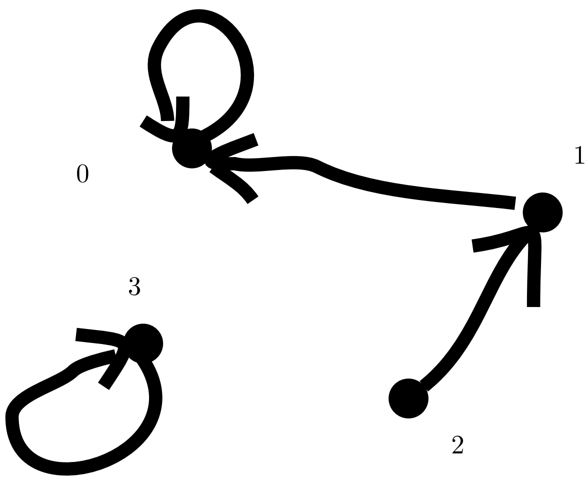
5.6 Isomorphism theorem

Theorem 1. All bijective functions have an inverse.

5.7 Graph of a function

Definition 5.3. function graph Set of ordered pairs relating the inputs to the outputs of the function.

Given a graph, $F = \{(0,0),(1,0),(2,1),(3,3)\}$, we can represent it with a diagraph:



5.8 Function spaces

Definition 5.4. function space. A set of functions.

Given sets X and Y, the set of all functions from X to Y is defined by:

$$Y^X$$

This allows us to define functions like this one:

$$f : X \times Y^X \to Y$$

$$f(x, g) = g(x + 4)$$

6 Powersets

Definition 6.1. Powerset of X. Set of all subsets of X.

Given a set X, we define the powerset with the following:

$$P(X)$$

Example 6.1.

$$P(\{0,1\}) = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$$

The subset $\{1,0\}$ is the same as $\{0,1\}$, therefore it's not included.

Example 6.2.

$$P(\emptyset) = \{\emptyset\}$$

Example 6.3.

$$PP(\emptyset) = \{\emptyset, \{\emptyset\}\}$$

6.1 Cardinality of powersets

The number of subsets of set X is given by:

$$2^{|X|}$$

6.2 Partitions

Given a set X and subsets $X_0, X_1, ..., X_n$, if:

$$X_0 \cup X_1 \cup ... \cup X_n = X$$

Then the given set of subsets is called a partition.

Theorem 2. If is an equivalence relation on X, then $X/\sim$ is a partition of X.