More formal statistical calculations:

$E(\cdot)$: Expected value $\rightarrow$ think average of everything in population

$$E(x_i) = \mu = \frac{1}{N}\sum_{i=1}^{N} x_i$$

$$E\left[(x_i-\mu)^2\right] = \sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i-\mu)^2$$

$$\hookrightarrow E\left[(x_i-\mu)^2\right] = E\left[(x_i^2 - 2x_i\mu + \mu^2)\right]$$

$$= E(x_i^2) - 2\mu E(x_i) + \mu^2$$

$$= E(x_i^2) - 2\mu^2 + \mu^2$$

$$= E(x_i^2) - \mu^2 = \sigma^2 \quad \text{OR} \quad E(x_i^2) = \sigma^2 + \mu^2$$

What if we had $\hat{\sigma}^2 = \frac{1}{n}\Sigma(x_i-\bar{x})^2$ NOT $s^2 = \frac{1}{n-1}\Sigma(x_i-\bar{x})^2$??

$$E(\hat{\sigma}^2) = \frac{1}{n}\sum_{i=1}^{n} E\left[(x_i-\bar{x})^2\right] = n \cdot \frac{1}{n} E\left[(x_i-\bar{x})^2\right]$$

$$= E\left[(x_i-\bar{x})^2\right]$$

$$= E(x_i^2) - 2E(x_i \times \bar{x}) + E(\bar{x}^2)$$

BY FUN ALGEBRA

$$= (\sigma^2+\mu^2) - 2\left(\frac{n-1}{n}\mu^2 + \frac{1}{n}(\sigma^2+\mu^2)\right)$$

$$+ \left(\frac{n^2-n}{n^2}\mu^2 + \frac{n}{n^2}(\sigma^2+\mu^2)\right) \quad \text{EWW!!}$$

BUT IT SIMPLIFIES

$$= \frac{n-1}{n}\sigma^2 \neq \sigma^2 \quad \text{UH OH!}$$

Luckily! $s^2 = \frac{n}{n-1}\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2$

$$E(s^2) = \frac{n}{n-1}E(\hat{\sigma}^2) = \frac{\cancel{n}}{\cancel{n-1}} \times \frac{\cancel{n-1}}{\cancel{n}} \times \sigma^2 = \sigma^2$$

YAY !!!