Sagar Jha

CHAPTER

2

Vectors in R^n

1. Introduction

In XI and XII standards you have studied vectors, vector algebra, determinants and matrices to some extent. We are going to generalise these concepts of vectors in two and three dimensions further in this chapter. But before that we shall briefly take a review of what you have studied already. Then we shall study some properties of vectors in general form. Entities that we come across in daily life are classified in various ways - one of them is to classify them into two classes as follows : (i) entities that have only magnitude, (ii) entities that have magnitude as well as direction.

Scalars : Entities that have only magnitude are called **scalars.** They are completely described by numbers. For example, temperature, height, mass, etc. are scalars because they can be fully described by numbers. *e.g.*, Ahemdabad's temperature today is 23° C, Praveen's height is 150 cms, describe the events fully.

Vectors : Entities that have magnitudes as well as direction are called vectors. In order to describe them fully we need to know their magnitude as well as direction. For example, velocity, acceleration, force, etc. are vectors. *e.g.*, the velocity of the car is 80 km/hour in the direction of the east. We need to know the magnitude and the direction of the vector to describe it fully.

2. Geometric Representation of a Point or a Vector

A point and a vector have many things in common. Both of them can be represented by coordinates. We shall denote by coordinates a point and a vector. The interpretation will depend upon the context.

We know that a point on a line can be denoted by a single number, a point on a plane can be denoted by two numbers, a point in space can be denoted by three numbers. This is shown in the Fig. 2.1.

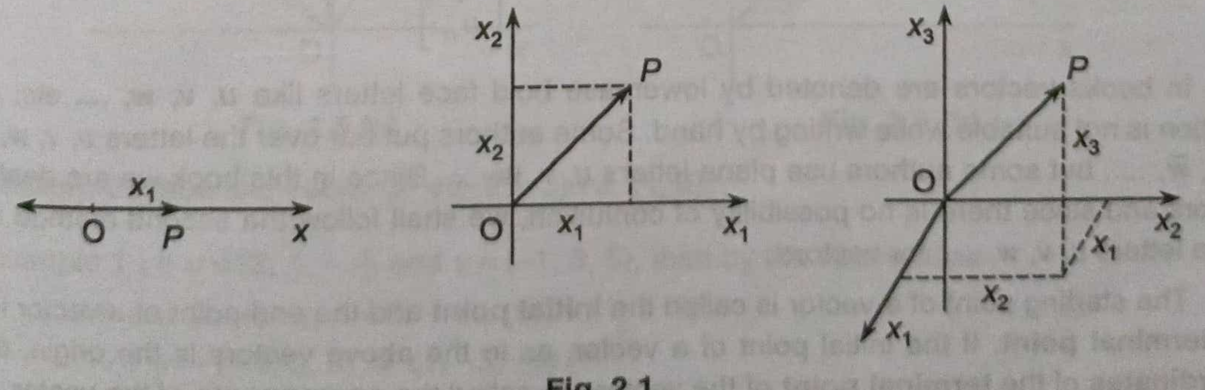


Fig. 2.1

We say that a single number (x_1) represents a point P (or a vector OP) in 1-space, a couple of two ordered numbers (x_1, x_2) represents a point P (or a vector OP) in 2-space, a triple of the three ordered numbers (x_1, x_2, x_3) represents a point P (or a vector OP) in 3-space. Although we cannot draw a picture as we go further but there is nothing that prevents us to say that a quadruple (x_1, x_2, x_3, x_4) represents a point P (on a vector OP) in 4-space, a quintuple in 5-space, a sextuple in 6-space, septuple in 7-space, octuple in 8-space, ... Generalising this idea, we say that an **n-tuple of numbers** $(x_1, x_2, x_3, \ldots, x_n)$ represents a point, or a vector in **n-space**. These spaces are alternatively also denoted by $R^1, R^2, R^3, R^4, \ldots, R^n$.

We shall denote a vector in n-space denoted by R^n by $u = (u_1, u_2, u_3, \ldots, u_n)$ and shall call $u_1, u_2, \ldots, u_n$ as its components.

Remark ....

A single point itself can be considered as zero space R^0.

3. Notation

Vectors are denoted in various ways. In 2-space we can denote a vector by the coordinates (x, y) of its end point or by using matrix notation by a row matrix $[x, y]$ or by a column matrix $\begin{bmatrix} x \\ y \end{bmatrix}$. We may also denote it by a single letter v or by the directed line segment $\overline{OP}$.

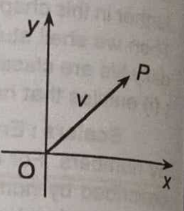


Fig. 2.2

In 3-space, in the same manner, we denote a vector by coordinates (x, y, z) or by a row matrix $[x, y, z]$ or by a column matrix $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$. All these natations have the same meaning and hence

$$v = \overline{OP} = (x, y, z) = [x, y, z] = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

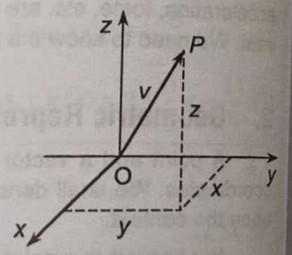


Fig. 2.3

In the same way a vector in R^n can be denoted by

$$u = (u_1, u_2, \ldots, u_n) = [u_1, u_2, \ldots, u_n] = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

In books vectors are denoted by lowercase bold face letters like **u, v, w, ...** etc. But this notation is not suitable while writing by hand. Some authors put bar over the letters $u, v, w, \ldots$ like $\bar{u}, \bar{v}, \bar{w}, \ldots$ but some authors use plane letters $u, v, w, \ldots$ Since in this book we are dealing with vectors and since there is no possibility of confusion, we shall follow the second method and use plane letters $u, v, w, \ldots$ for vectors.

The starting point of a vector is called the **initial point** and the end-point of a vector is called the **terminal point**. If the initial point of a vector, as in the above vectors is the origin, then the **coordinates** of the **terminal point** of the vector are called the **components** of the vector.

4. Vector Arithmetic

Addition, subtraction, the scalar multiplication of vectors in 2 and 3 space follow the following rules. If u, v, w are vectors and k, l are scalars then

(1) $u + v = v + u$	**(2)** $u + (v + w) = (u + v) + w$	**(3)** $u + 0 = 0 + u = u$
(4) $u + (-u) = 0$	**(5)** $k(u + v) = ku + kv$	**(6)** $(k + l)\,u = ku + lu$
(7) $k(lu) = (kl)\,u$	**(8)** $1 \cdot u = u$.	

Proof : Since a vector can be represented by ordered scalars or by line segments, we have two approaches to prove the above results - one is analytical and the other is geometrical.

We shall prove the first result by two methods in R^3 and leave the others for you as an exercise.

1. To prove $u + v = v + u$

Analytically : Let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$

Now,
$$\begin{aligned} u + v &= (u_1, u_2, u_3) + (v_1, v_2, v_3) \\ &= (u_1 + v_1) + (u_2 + v_2) + (u_3 + v_3) \\ &= (v_1 + u_1) + (v_2 + u_2) + (v_3 + u_3) \\ &= v + u \end{aligned}$$

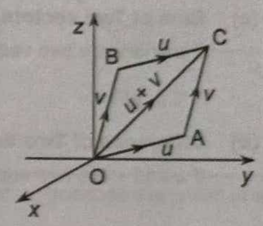


Fig. 2.4

Geometrically : Let $OA = u$ and $OB = v$. Then by completing the parallelogram $OACB$, we see that

$$OC = OA + AC = u + v \quad \text{and} \quad OC = OB + BC = u + v$$

$$\therefore \quad u + v = v + u$$

For example, let $u = (2, 3)$ and $v = (-3, 2)$, then

$$w = u + v = (2 + (-3), 3 + 2) = (-1, 5)$$

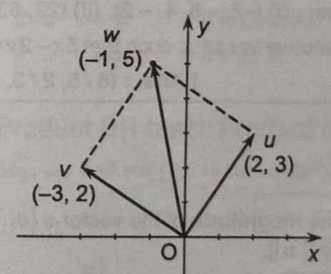


Fig. 2.5 (a)

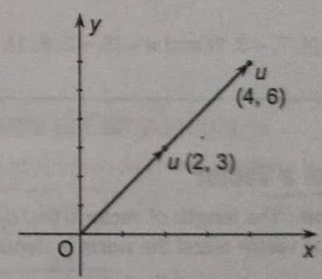


Fig. 2.5 (b)

Similarly, $2u$ means a vector $(2 \times 2, 2 \times 3)$ *i.e.* $(4, 6)$.

Example 1 : If $u = (2, 1, -4)$ and $v = (-1, 3, 5)$, then by the above rules

$$\begin{aligned} u + v &= (2 + (-1), 1 + 3, -4 + (5)) = (1, 4, 1) \\ -v &= -(-1, 3, 5) = (1, -3, -5) \\ u - v &= u + (-v) = (2, 1, -4) + (1, -3, -5) = (3, -2, -9) \\ 2u &= 2(2, 1, -4) = (4, 2, -8) \\ -3v &= -3(-1, 3, 5) = (3, -9, -15) \end{aligned}$$

(a) Equality of Vectors

Definition : If $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ are two vectors in R^n then they are said to be **equal** if their components are equal

i.e., $$\boxed{u = v \text{ if } u_1 = v_1,\ u_2 = v_2, \ldots, u_n = v_n}$$

Zero vector : A vector whose all n-components are zero is called a zero vector and is denoted by O. Thus,

$$\boxed{O = (0, 0, \ldots, 0)}$$

(b) Scalar Multiple

If k is any scalar and u is vector then the scalar multiple of u, denoted by ku, is given by

$$\boxed{ku = (ku_1, ku_2, \ldots, ku_n)}$$

(c) Sum of Two vectors

If u and v are two vectors their sum, denoted by $u + v$ is defined by

$$\boxed{u + v = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)}$$

(d) Difference of Two Vectors

If u and v are two vectors then their difference, denoted by $u - v$ is defined by

$$\boxed{u - v = (u_1 - v_1, u_2 - v_2, \ldots, u_n - v_n)}$$

EXERCISE - I

1. If $u = (-3, 2, 1, 0)$ and $v = (4, 7, -3, 2)$, find (i) $u - v$, (ii) $2u + 7v$.
[Ans. : (i) $(-7, -5, 4, -2)$, (ii) $(22, 53, -19, 14)$]
2. If $v = (4, 7, -3, 2)$ and $w = (5, -2, 8, 1)$, find another vector x, such that $5x - 2v = 2(w - 5x)$.
[Ans. : $(6/5, 2/3, 2/3, 2/5)$]

5. Norm of a Vector

Definition : The **length** of vector $u(u_1, u_2)$ or the **magnitude** of the vector $u(u_1, u_2)$ is called the **norm** of the vector u and the **norm** is denoted by $\| u \|$.

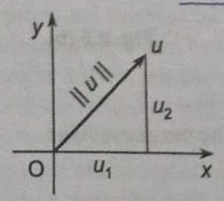


Fig. 2.6 (a)

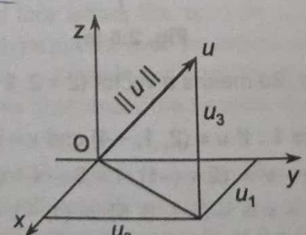


Fig. 2.6 (b)

By Pythagorean Theorem in R^2, $\boxed{\|u\| = \sqrt{u_1^2 + u_2^2}}$

In R^3, the **length** of the vector $u(u_1, u_2, u_3)$ or the **magnitude** of the vector *i.e.*, the **norm** of $u(u_1, u_2, u_3)$ is given by

$$\boxed{\|u\| = \sqrt{u_1^2 + u_2^2 + u_3^2}}$$

Generalising the above result, we define the norm of a vector $u = (u_1, u_2, \ldots, u_n)$ in R^n as

$$\boxed{\|u\| = \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2}}$$

6. Dot Product (Inner Product)

Definition : If u and v are any two non-zero vectors in 2 and 3 space and if θ is the angle between $(0 \le \theta \le \pi)$ them, the **dot product** or the **(Euclidean) inner product** of u and v denoted by $u \cdot v$ or by (u, v) is given by

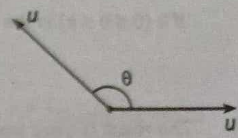


Fig. 2.7

$$\boxed{u \cdot v = \| u \| \| v \| \cos\theta}$$

Remark ....

The dot product or the inner product is also called the scalar product because the result is a pure number.

Corollaries : If u, v are two vectors in R^2 or R^3, then

(i) $u \cdot u = \| u \|^2$ or $\| u \| = \sqrt{u \cdot u}$

(ii) If $u \cdot v > 0$, θ is acute,

if $u \cdot v < 0$, θ is obtuse,

if $u \cdot v = 0$, $\theta = \pi/2$.

7. Dot Product OR Inner Product in Terms of Components

If $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are two vectors in R^2 then the **inner product** is given by

$$\boxed{u \cdot v = u_1 v_1 + u_2 v_2}$$

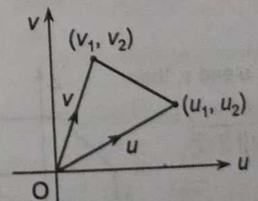


Fig. 2.8 (a)

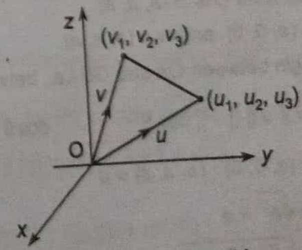


Fig. 2.8 (b)

If $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ are two vectors in R^3 then the **inner product** is given by

$$\boxed{u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3}$$

Example : If $u = (1, -2, 4)$, $v = (-2, 4, 6)$, $w = (-3, 4, -5)$, find $u \cdot v$, $v \cdot w$.

Sol. :
$$u \cdot v = (1)(-2) + (-2)(4) + (4)(6)$$
$$= -2 - 8 + 24 = 14$$
$$v \cdot w = (-2)(-3) + (4)(4) + (6)(-5)$$
$$= 6 + 16 - 30 = -8.$$

8. Angle Between Two Vectors

If θ $(0 \le \theta \le \pi)$ is the angle between two vectors u, v then the angle is given by

$$\boxed{\cos\theta = \frac{u \cdot v}{\|u\| \, \|v\|}}$$

The result is clear from § 6.

Example 1 : Find the angle between the vectors

(i) $u = (6, 2, 2)$, $v = (3, 0, -9)$; (ii) $u = (2, 1, -4)$, $v = (1, -2, 3)$

Sol. : If θ is the angle between two vectors u, v, then $\cos\theta = \dfrac{u \cdot v}{\|u\| \, \|v\|}$.

(i) Now, $\|u\| = \sqrt{36 + 4 + 4} = \sqrt{44}$; $\|v\| = \sqrt{9 + 0 + 81} = \sqrt{90}$

$u \cdot v = 18 + 0 - 18 = 0$

$\therefore \cos\theta = \dfrac{0}{\sqrt{44}\,\sqrt{90}} = 0$ $\therefore \theta = \dfrac{\pi}{2}$ $\therefore$ The vectors are orthogonal.

(ii) $\|u\| = \sqrt{4 + 1 + 16} = \sqrt{21}$; $\|v\| = \sqrt{1 + 4 + 9} = \sqrt{14}$

$u \cdot v = 2 - 2 - 12 = -12$

$\therefore \cos\theta = \dfrac{u \cdot v}{\|u\| \|v\|} = \dfrac{-12}{\sqrt{21}\,\sqrt{14}} = -\dfrac{12}{7\sqrt{6}}$ $\therefore$ θ is obtuse.

Example 2 : Find the angle between a diagonal of a cube and one of its edges.

Sol. : Without loss of generality, let one of the vertex of the cube of side a be $A(a, 0, 0)$. Then the end of the diagonal from O is $B(a, a, a)$.

Now, let $u = (a, 0, 0)$ and $v = (a, a, a)$.

If θ is the angle between OA and OB *i.e.*, between u and v, then

$u \cdot v = \| u \| \| v \| \cos\theta$ $\therefore \cos\theta = \dfrac{u \cdot v}{\|u\| \|v\|}$

Now, $u \cdot v = (a, 0, 0) \cdot (a, a, a) = a^2$

$\|u\| = \sqrt{a^2} = a$

$\|v\| = \sqrt{a^2 + a^2 + a^2} = \sqrt{3} \cdot a$

$\therefore \cos\theta = \dfrac{a^2}{a \cdot \sqrt{3}\, a} = \dfrac{1}{\sqrt{3}}$ $\therefore \theta = \cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$

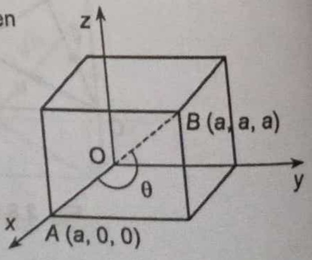


Fig. 2.9

Example 3 : Let R^4 have Euclidean inner product. Find the cosine of the angle between vectors $u = (4, 3, 1, -2)$ and $v = (-2, 1, 2, 3)$.

Sol. : We have

$$u \cdot v = (4, 3, 1, -2) \cdot (-2, 1, 2, 3)$$
$$= (4)(-2) + (3)(1) + (1)(2) + (-2)(3)$$
$$= -8 + 3 + 2 - 6 = -9$$

$\|u\| = \sqrt{16 + 9 + 1 + 4} = \sqrt{30}$; $\|v\| = \sqrt{4 + 1 + 4 + 9} = \sqrt{18}$

$$\therefore \cos\theta = \frac{u \cdot v}{\|u\| \|v\|} = \frac{-9}{\sqrt{30}\,\sqrt{18}} = -\frac{9}{\sqrt{3 \cdot 5 \cdot 2}\,\sqrt{9 \cdot 2}}$$
$$= -\frac{9}{3 \times 2\sqrt{15}} = -\frac{3}{2\sqrt{15}}.$$

9. Cauchy-Schwartz Inequality in R^n

If $u = (u_1, u_2, u_3, \ldots, u_n)$ and $v = (v_1, v_2, v_3, \ldots, v_n)$ are any two vectors in R^n, Cauchy-Schwartz Inequality states that

$$\boxed{|u \cdot v| \le \| u \| \| v \|}$$ **(M.U. 2015)**

Proof : The inequality can be proved very easily for vectors in R^2 and R^3.

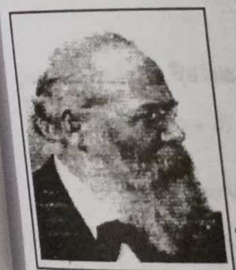
We know that if u, v are any two vectors in R^2 or R^3 then

$| u \cdot v | = \| u \| \| v \| \, | \cos\theta |$

But $| \cos\theta | \le 1$ $\therefore$ $| u \cdot v | \le \| u \| \| v \|$

Note that the equality bolds good when $u = v$.

Herman Amandus Schwartz (1843-1921)



He was considered as a leading mathematician in the first part of twentieth century in Germany. He was professor of mathematics at Berlin university. He was a devout teacher and took interest in teaching important and trivial facts with equal thoroughness. His techniques were very clever and influenced the work of other mathematicians. The reknowned Cauchy-Schwartz inequality was published in 1885.

Example 1 : Verify Cauchy-Schwartz inequality for the vectors $u = (2, 3, 0)$ and $v = (4, 2, 1)$.

Sol. : We have $\|u\| = \sqrt{4 + 9 + 0} = \sqrt{13}$ and $\|v\| = \sqrt{16 + 4 + 1} = \sqrt{21}$

$\therefore \|u\| \|v\| = \sqrt{13}\,\sqrt{21} = \sqrt{273}$

And $| u \cdot v | = | u_1 v_1 + u_2 v_2 + u_3 v_3 | = | (2)(4) + (3)(2) + (0)(1) |$

$= | 8 + 6 | = 14$

Since $14 < \sqrt{273}$ $[\because 14^2 = 196 < 273]$

$\therefore$ Cauchy-Schwartz inequality is verified.

Example 2 : Verify Cauchy-Schwartz inequality for the vectors (M.U. 2014, 15)

$$u = (-4, 2, 1) \text{ and } v = (8, -4, -2)$$

Sol. : We have $||u|| = \sqrt{16+4+1} = \sqrt{21}$ and $||v|| = \sqrt{64+16+4} = \sqrt{84}$

$\therefore\ ||u||\,||v|| = \sqrt{21}\sqrt{84} = \sqrt{21}\sqrt{21}\sqrt{4} = 21 \times 2 = 42$

And $|u \cdot v| = |u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4|$

$= |(-4)(8) + (2)(-4) + (1)(-2)|$

$= |-32 - 8 - 2| = 42$

$\therefore\ \|u\|\,\|v\| = |u \cdot v|$

By Cauchy-Schwartz inequality we should have $|u \cdot v| \le \|u\|\,\|v\|$.

Hence, Cauchy-Schwartz inequality holds good for the given vectors.

Example 3 : Verify Cauchy-Schwartz inequality for $u = (2, 3, 1)$ and $v = (3, 0, 4)$. Also find the angle between u and v. (M.U. 2016)

Sol. : (i) We have $||u|| = \sqrt{4+9+1} = \sqrt{14}$ and $||v|| = \sqrt{9+0+16} = \sqrt{25} = 5$

$\therefore\ ||u||\,||v|| = \sqrt{14}\sqrt{25} = 5\sqrt{14}$

And $|u \cdot v| = |u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4|$

$= |(2)(3) + (3)(0) + (1)(4)|$

$= |6 + 0 + 4| = |10| = 10$

Since $10 < 5\sqrt{14} = 2 < \sqrt{14}$, which is true, Cauchy-Schwartz inequality is verified.

(ii) Now angle θ between u and v is given by

$$\cos\theta = \frac{u \cdot v}{||u||\,||v||} = \frac{10}{5\sqrt{14}} = \frac{2}{\sqrt{14}}$$

$$\therefore\ \theta = \cos^{-1}\left(\frac{2}{\sqrt{14}}\right).$$

Example 4 : For real values of a, b and θ, show that $(a\cos\theta + b\sin\theta)^2 \le a^2 + b^2$ using Cauchy-Schwartz inequality. (M.U. 2017)

Sol. : Consider two vectors $u = (a, b)$ and $v = (\cos\theta, \sin\theta)$.

Then $u \cdot v = (a, b) \cdot (\cos\theta, \sin\theta) = a\cos\theta + b\sin\theta$

And $\|u\|^2 = a^2 + b^2$ and $\|v\|^2 = \cos^2\theta + \sin^2\theta = 1$

Now, by Cauchy-Schwartz inequality $|u \cdot v| \le \|u\|\,\|v\|$.

$\therefore\ |a\cos\theta + b\sin\theta| \le \sqrt{a^2 + b^2} \cdot 1$

Taking squares, we get $(a\cos\theta + b\sin\theta)^2 \le (a^2 + b^2)$.

Example 5 : If $x, y, z > 0$, show that $(x^2 + y^2 + z^2)^{1/2} \ge \frac{1}{13}(3x + 4y + 12z)$ using Cauchy-Schwartz inequality. (M.U. 2014)

Sol. : Consider two vectors $u = (x, y, z)$ and $v = (3, 4, 12)$.

Then $u \cdot v = (x, y, z) \cdot (3, 4, 12) = 3x + 4y + 12z$

And $\|u\|^2 = x^2 + y^2 + z^2$ and $\|v\|^2 = 3^2 + 4^2 + 12^2 = 169 = 13^2$

Now, by Cauchy-Schwartz inequality $|u \cdot v| \le \|u\|\,\|v\|$.

$\therefore\ (3x + 3y + 12z) \le \sqrt{x^2 + y^2 + z^2} \cdot \sqrt{13^2}$

$\therefore\ (3x + 3y + 12z) \le (x^2 + y^2 + z^2)^{1/2} \cdot 13$

$\therefore\ \frac{1}{13}(3x + 3y + 12z) \le (x^2 + y^2 + z^2)^{1/2}$

Example 6 : Using Cauchy-Schwartz inequality, prove that $\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \ge 6$.

Sol. : Consider two vectors $u = (\sqrt{a}, \sqrt{b}, \sqrt{c})$ and $v = \left(\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}, \frac{1}{\sqrt{c}}\right)$.

$$\therefore\ u \cdot v = (\sqrt{a}, \sqrt{b}, \sqrt{c}) \cdot \left(\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}, \frac{1}{\sqrt{c}}\right) = \sqrt{a} \cdot \frac{1}{\sqrt{a}} + \sqrt{b} \cdot \frac{1}{\sqrt{b}} + \sqrt{c} \cdot \frac{1}{\sqrt{c}}$$

$$= 1 + 1 + 1 = 3$$

Now, $\|u\|^2 = a + b + c$ and $\|v\|^2 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

By Cauchy-Schwartz inequality $(u \cdot v)^2 \le \|u\|^2\,\|v\|^2$.

$$\therefore\ 3^2 \le (a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

$$\le \frac{a+b+c}{a} + \frac{a+b+c}{b} + \frac{a+b+c}{c}$$

$$\le \frac{b+c}{a} + 1 + \frac{a+c}{b} + 1 + \frac{a+b}{c} + 1$$

$$\therefore\ 9 \le 3 + \left(\frac{a+b}{c} + \frac{b+c}{c} + \frac{c+a}{b}\right)$$

$$\therefore\ 6 \le \frac{a+b}{c} + \frac{b+c}{a} + \frac{c+a}{b}$$

Example 7 : If a, b, c are three positive real numbers, then using Cauchy-Schwartz inequality prove that $(a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \ge 3^2$.

Sol. : Consider two vectors $u = (\sqrt{a}, \sqrt{b}, \sqrt{c})$ and $v = \left(\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}, \frac{1}{\sqrt{c}}\right)$

then as seen above by Cauchy-Schwartz inequality,

$$3^2 \le (a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

Note ....

The result can be generalised further and we can show that

$$(a_1 + a_2 + \ldots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right) \ge n^2$$

The proof is left to you.

10. Relation Between Norm and Inner Product

If u, v are two vectors then

$$\boxed{u \cdot v = \frac{1}{4}\|u+v\|^2 - \frac{1}{4}\|u-v\|^2} \quad \text{.............. (A)}$$

Proof : By definition

$$\|u+v\|^2 = (u+v)\cdot(u+v) \quad \text{.............. (1)}$$
$$= \|u\|^2 + 2(u\cdot v) + \|v\|^2$$

And
$$\|u-v\|^2 = (u-v)\cdot(u-v) \quad \text{.............. (2)}$$
$$= \|u\|^2 - 2(u\cdot v) + \|v\|^2$$

By subtraction (2) from (1), we get

$$\|u+v\|^2 - \|u-v\|^2 = 4(u\cdot v)$$

Hence, the result.

By adding (1) and (2), we get another result

$$\boxed{\|u+v\|^2 + \|u-v\|^2 = 2[\|u\|^2 + \|v\|^2]}$$

11. Orthogonality

Definition : Two vectors u, v are said to be **orthogonal** if

$$\boxed{u\cdot v = 0}$$

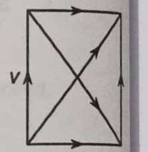


Fig. 2.10

Cor. 1 : From the result (A) proved, we get another condition for orthogonality.

If u and v are two orthogonal vectors, then from (A), we get

$$\|u+v\| = \|u-v\| \qquad (\because\ u\cdot v = 0)$$

In R^2 this means, if u and v are the sides of a rectangle then the diagonals of a rectangle are equal.

Cor. 2 : If u and v are orthogonal vectors then the distance between them is given by

$$\|u-v\|^2 = \|u\|^2 + \|v\|^2$$

This is so because

$$\|u-v\|^2 = (u-v)\cdot(u-v)$$
$$= \|u\|^2 - 2(u\cdot v) + \|v\|^2$$
$$= \|u\|^2 + \|v\|^2 \qquad [\because\ u\cdot v = 0]$$

12. Euclidean Inner Product

We can generalise the concept of inner product given in § 7 to R^n.

Definition : If $u = (u_1, u_2, ..., u_n)$ and $v = (v_1, v_2, ..., v_n)$ are any two vectors in R^n then the **(Euclidean) inner product** of u, v denoted by $u\cdot v$ is defined by

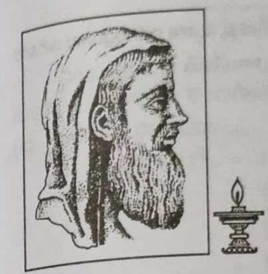
$$\boxed{u\cdot v = u_1v_1 + u_2v_2 + ... + u_nv_n}$$

Remark ....

Inner product $u\cdot v$ is also denoted by $\langle u, v\rangle$.

Cor. : Two vectors are orthogonal if $\boxed{u\cdot v = 0}$ .... (3)

Euclid (300?-275? B.C.)



He taught at Alexandria and founded the Alexandrian School of Mathematics. Euclid is called **"the father of geometry"**. When the king of Alexandria (Egypt) Ptolemy I asked Euclid if there is an easy way to learn geometry, he replied "there is no royal road to geometry". Euclid wrote thirteen books on algebra and geometry mostly compiling the works of earlier mathematicians but had made also significant contributions of his own. Algebra and geometry that is taught at school level all over the world is based on his books. It is said that his books are read and studied only next to the Bible. More than 2000 editions of his book are published in various languages since the first print in 1482. Unfortunately very little is known about this great mathematician.

Example 1 : Find the inner product of $u = (-2, 4, 0, -3)$, $v = (3, 4, 6, -4)$.

Sol. :
$$u\cdot v = (-2)(3) + (4)(4) + (0)(6) + (-3)(-4)$$
$$= -6 + 16 + 12 = 22$$

Example 2 : If R^3 has Euclidean inner product and $u = (4, -3, 2)$, $v = (5, 4, -4)$, show that u and v are orthogonal.

Sol. : u, v are orthogonal if $u\cdot v = 0$.

Now,
$$u\cdot v = 4\times 5 + (-3)\times 4 + (2)\times(-4)$$
$$= 20 - 12 - 8 = 0$$

Hence, u and v are orthogonal.

Example 3 : Let R^3 have the Euclidean inner product. Find the values of k for which $u = (k, k, 1)$ and $v = (k, 5, 6)$ are orthogonal.

Sol. : Two vectors are orthogonal if $u\cdot v = 0$.

$$\therefore\ (k, k, 1)\cdot(k, 5, 6) = 0 \qquad \therefore\ k^2 + 5k + 6 = 0$$
$$\therefore\ (k+2)(k+3) = 0$$

Hence, $k = -2$ or -3.

Many of the results with which we are familiar in 2-space and 3-space are true in R^n also. For this reason R^n is often referred to as **Euclidean n-space.**

Example 4 : Check whether the following vectors are orthogonal

(i) $u = (-2, 3, 4)$, $v = (2, 4, -2)$

(ii) $u = (-4, 6, -10, 1)$, $v = (2, 1, -2, 9)$

Sol. : (I) We have

$$\langle u, v\rangle = u \cdot v = (-2, 3, 4) \cdot (2, 4, -2) = -4 + 12 - 8 = 0$$

Hence, u, v are orthogonal.

(II) $$\langle u, v\rangle = u \cdot v = (-4, 6, -10, 1) \cdot (2, 1, -2, 9) = -8 + 6 + 20 + 9 = 27 \neq 0$$

Hence, u, v are not orthogonal.

Example 5 : If R^3 has the Euclidean inner product, find k such that u, v are orthogonal where

(a) $u = (k, k, 2)$, $v = (k, -5, 3)$; **(b)** $u = (2, 1, 3)$, $v = (1, 7, k)$

Sol. : Since u, v are orthogonal $\langle u, v\rangle = 0$.

(a) $$u \cdot v = (k, k, 2) \cdot (k, -5, 3) = k(k) + k(-5) + (2)\,3 = 0$$

$\therefore\ k^2 - 5k + 6 = 0 \quad \therefore\ (k-3)(k-2) = 0$

$\therefore\ k = 3, 2$

(b) $$u \cdot v = (2, 1, 3) \cdot (1, 7, k) = 2(1) + 1(7) + 3(k) = 0$$

$\therefore\ 3k + 9 = 0 \quad \therefore\ k = -3.$

13. Properties of Euclidean Inner Product

If $u = (u_1, u_2, \ldots, u_n)$, $v = (v_1, v_2, \ldots, v_n)$ and $w = (w_1, w_2, \ldots, w_n)$ are vectors in R^n and k is a scalar, then

(1) $u \cdot v = v \cdot u$	(2) $(u + v) \cdot w = u \cdot w + v \cdot w$
(3) $(ku) \cdot v = k(u \cdot v)$	(4) $v \cdot v \geq 0$ if $v \neq 0$ and $v \cdot v = 0$ if and only if $v = 0$

14. Euclidean Norm (or Euclidean Length)

If $u = (u_1, u_2, \ldots, u_n)$ is a vector in R^n then the **Euclidean norm (or Euclidean length)** of u is defined by

$$||u|| = (u \cdot u)^{1/2} = \sqrt{(u_1 + u_2 + \ldots + u_n) \cdot (u_1 + u_2 + \ldots + u_n)}$$

$$\boxed{||u|| = \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2}}$$

For example, if $u = (2, 1, 3, 0, 1)$, then

$$||u|| = \sqrt{4 + 1 + 9 + 0 + 1} = \sqrt{15}$$

15. Properties of Norm in R^n

If u and v are any two vectors in R^n and k is a scalar, then

(1) $\| u \| \geq 0$	**(2)** $\| u \| = 0$ **if and only if** $u = 0$
(3) $\| ku \| = \vert k \vert\, \| u \|$	**(4)** $\| u + v \| \leq \| u \| + \| v \|$ **(Triangular Inequality)**

Proof : (1) By definition,

$$||u|| = \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2}$$

Clearly, $\| u \| \geq 0$.

(2) And $\| u \| = 0$ if $u_1 = u_2 = \ldots = u_n = 0$

(3) If $u = (u_1, u_2, \ldots, u_n)$ then $ku = (ku_1, ku_2, \ldots, ku_n)$

$$\therefore\ ||ku|| = \sqrt{(ku_1)^2 + (ku_2)^2 + \ldots + (ku_n)^2} = \sqrt{k^2 (u_1^2 + u_2^2 + \ldots + u_n^2)} = |k|\,||u||$$

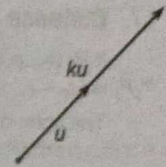


Fig. 2.11

This property states that if a vector u is multiplied by a scalar then the length of the vector is multiplied by the scalar $|k|$.

(4) Now, by definition

$$\begin{aligned}
\| u + v \|^2 &= (u + v) \cdot (u + v) \\
&= (u \cdot u) + 2(u \cdot v) + (v \cdot v) \\
&= \| u \|^2 + 2(u \cdot v) + \| v \|^2 \\
&\leq \| u \|^2 + 2|u \cdot v| + \| v \|^2 && \text{(Property of absolute value)} \\
&\leq \| u \|^2 + 2\| u \|\,\| v \| + \| v \|^2 && \text{(Cauchy-Schwartz Inequality)} \\
&\leq (\| u \| + \| v \|)^2
\end{aligned}$$

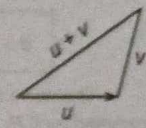


Fig. 2.11

This property is known as **triangular inequality**. It generalises the famous result in Euclidean geometry that in a triangle a side of a triangle is smaller than or equal to the sum of the other two sides.

16. Unit Vector

A vector whose norm is equal to one is called a **unit vector**. It is denoted by $\hat{u}$. It is easy to see that a unit vector is in the direction of the vector u and is obtained by dividing the vector u by its norm $\| u \|$. Thus, unit vector in the direction u is

$$\boxed{\hat{u} = \frac{1}{||u||}\, u}$$

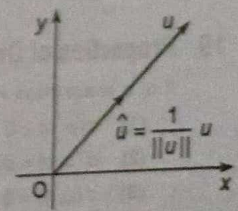


Fig. 2.13

For example, the vectors

$(1/\sqrt{2}, 1/\sqrt{2})$, $(\sqrt{2/3}, 1/\sqrt{3})$,

$(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$, $(1/2, 1/\sqrt{2}, 1/2)$,

$(1/\sqrt{14}, 2/\sqrt{14}, 3/\sqrt{14})$

are unit vectors. You can write some more unit vectors.

For example, if $u = (2, 4, -5)$, then $||u|| = \sqrt{4 + 16 + 25} = 3\sqrt{5}$.

The unit vector in the direction of u denoted by $\hat{u}$ is obtained by dividing each component by the norm. Thus, the unit vector

$$\hat{u} = \left(2/3\sqrt{5},\ 4/3\sqrt{5},\ -5/3\sqrt{5}\right)$$

17. Distance Between Two Points P_1 and P_2

If $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are two points in R^2 then the vector P_1P_2 is $(x_2 - x_1, y_2 - y_1)$.

The norm of P_1P_2 denoted by d is given by

$$\boxed{d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

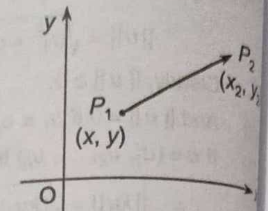


Fig. 2.14

If $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are two points in R^3 then the vector P_1P_2 is $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$ and the norm of P_1P_2 denoted by d is given by

$$\boxed{d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

The norm of the vector P_1P_2 is also called the distance between the points P_1 and P_2 and it is denoted by $\| P_1 - P_2 \|$. Thus,

$$\boxed{\| P_1 - P_2 \| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}}$$

18. Euclidean Distance in R^n

Definition : The Euclidean distance between two points $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ is defined by

$$\boxed{d(u, v) = \| u - v \| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \ldots + (u_n - v_n)^2}}$$

Example : If $u = (2, 4, -3, 6)$ and $v = (-3, 2, -1, 4)$ then

$$\|u\| = \sqrt{4 + 16 + 9 + 36} = \sqrt{65}$$

$$d(u, v) = \sqrt{(2+3)^2 + (4-2)^2 + (-3+1)^2 + (6-4)^2} = \sqrt{25 + 4 + 4 + 4} = \sqrt{37}$$

19. Properties of Distance in R^n

If u, v, w are three vectors in R^n and k is a scalar then

(1) $d(u, v) \geq 0$

(2) $d(u, v) = d(v, u)$

(3) $d(u, v) = 0$ if and only if $u = v$

(4) $d(u, v) \leq d(u, w) + d(w, u)$ [Triangular Inequality]

Proof : (1) By definition,

$$d(u, v) = \| u - v \| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \ldots + (u_n - v_n)^2}$$

Clearly $d(u, v) \geq 0$

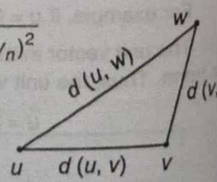


Fig. 2.15

(2) and $d(u, v) = 0$ if and only if $u_1 = v_1, u_2 = v_2, \ldots, u_n = v_n$

i.e. if $u = v$.

(3) By definition

$$\begin{aligned} d(u, v) &= \| u - v \| = \| (u - w) + (w - v) \| \\ &\leq \| u - w \| + \| w - v \| \quad \text{[By (4) of § 15, page 2-12]} \\ &\leq d(u, w) + d(w, v) \end{aligned}$$

This property is also called triangular inequality as it generalises the Euclidean result that the shortest distance between two points is along the straight.

Example 1 : If $u = (3, 4, -2)$, $v = (4, -2, 1)$ and $w = (1, -3, 4)$, find

(a) $\| u \|$, (b) $\frac{1}{\|v\|} v$, (c) $\| 2u - 3v + 4w \|$

Sol. : (a) $\|u\| = \sqrt{9 + 16 + 4} = \sqrt{29}$

(b) $\|v\| = \sqrt{16 + 4 + 1} = \sqrt{21}$ $\therefore$ $\frac{1}{\|v\|} v = \left(\frac{4}{\sqrt{21}}, \frac{-2}{\sqrt{21}}, \frac{1}{\sqrt{21}} \right)$

(c)
$$\begin{aligned} 2u - 3v + 4w &= 2(3, 4, -2) - 3(4, -2, 1) + 4(1, -3, 4) \\ &= (6, 8, -4) - (12, -6, 3) + (4, -12, 16) \\ &= (6 - 12 + 4,\ 8 + 6 - 12,\ -4 - 3 + 16) \\ &= (-2, 2, 9) \end{aligned}$$

$$\therefore \| 2u - 3v + 4w \| = \sqrt{4 + 4 + 81} = \sqrt{89}.$$

Example 2 : If $u = (2, 3, -1, 4)$ and $v = (-1, 2, 1, 5)$, find $\| u \|$, $\| v \|$, $\frac{1}{\|v\|} v$ and $d(u, v)$.

Sol. : By definitions, we get

$$\|u\| = \sqrt{4 + 9 + 1 + 16} = \sqrt{30}; \qquad \|v\| = \sqrt{1 + 4 + 1 + 25} = \sqrt{31}$$

$$\frac{1}{\|v\|} v = \left(-\frac{1}{\sqrt{31}}, \frac{2}{\sqrt{31}}, \frac{1}{\sqrt{31}}, \frac{5}{\sqrt{31}} \right)$$

$$d(u, v) = \sqrt{(2+1)^2 + (3-2)^2 + (-1-1)^2 + (4-5)^2} = \sqrt{9 + 1 + 4 + 1} = \sqrt{15}$$

Example 3 : If $\| u + v \| = 6$ and $\| u - v \| = 4$, find $u \cdot v$.

Sol. : We have by § 10, page 2-9

$$u \cdot v = \frac{1}{4}\|u + v\|^2 - \frac{1}{4}\|u - v\|^2$$

Hence, $u \cdot v = \frac{1}{4}[36 - 16] = 5.$

Example 4 : Find the Euclidean norms of the following vectors

(i) $(1, -2, 2)$ (ii) $(3, -4, 0, 12)$

Sol. : By definition

(i) $\|u\| = \sqrt{u_1^2 + u_2^2 + u_3^2} = \sqrt{1 + 4 + 4} = \sqrt{9} = 3$

(ii) $\|u\| = \sqrt{u_1^2 + u_2^2 + u_3^2 + u_4^2} = \sqrt{9 + 16 + 0 + 144} = \sqrt{169} = 13$

Example 5 : If v is $(3, -4, 12)$, find all scalars k such that $\| kv \| = 3$.

Sol. : $\|v\| = \sqrt{9+16+144} = 13$

$\therefore \ \pm k \cdot 13 = 3 \quad \therefore \ k = \pm 3/13$

EXERCISE - II

1. Find the norms of (a) $u = (4, 3)$, (b) $u = (2, 3, 5)$, (c) $u = (3, 3, 3)$, (d) $u = (5, 0, 0)$.
   [**Ans. : (a)** 5, **(b)** $\sqrt{38}$, **(c)** $3\sqrt{3}$, **(d)** 5]
2. Find the distance between the points (a) $(5, 6), (2, -3)$; (b) $(2, 3, -1), (6, -3, -2)$
   [**Ans. : (a)** $3\sqrt{10}$, **(b)** $\sqrt{53}$]
3. If $u = (2, 2, -3)$, $v = (4, 3, -1)$, find
   (a) $\frac{1}{\|u\|} u$, (b) $\| 2u + 3v \|$, (c) $\frac{1}{\|u - v\|}(u - v)$.
   [**Ans. : (a)** $\left(\frac{2}{\sqrt{17}}, \frac{2}{\sqrt{17}}, \frac{-3}{\sqrt{17}}\right)$, **(b)** $\sqrt{406}$, **(c)** $\left(-\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right)$.]
4. If $\| u + v \| = 7$ and $\| u - v \| = 3$, find $u \cdot v$.
   [**Ans. : 10**]

20. Pythagorean Theorem in R^n

$\|u+v\| \to$ dist betn u, v

condition If u, v are orthogonal vectors in R^n with Euclidean product then

$$\Rightarrow \boxed{\| u+v \|^2 = \| u \|^2 + \| v \|^2}$$

Proof : By definition

$$\| u + v \|^2 = (u + v) \cdot (u + v)$$
$$= (u \cdot u) + 2\,(u \cdot v) + (v \cdot v)^2$$
$$= \| u \|^2 + 2\,(u \cdot v) + \| v \|^2$$

But since u and v are orthogonal $u \cdot v = 0$.

$$\therefore \quad \| u + v \|^2 = \| u \|^2 + \| v \|^2$$

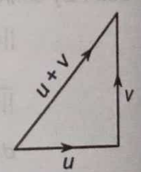


Fig. 2.16

Generalisation

If $u_1, u_2, \ldots, u_n$ are pairwise orthogonal vectors in R^n with the Euclidean inner product then

$$\| u_1 + u_2 + \ldots + u_n \|^2 = \| u_1 \|^2 + \| u_2 \|^2 + \ldots + \| u_n \|^2$$

Pythagoras of Samos (495B.C. - 570B.C.)

Pythagoras was a great Greek mathematician, philosopher and founder of the religious movement called Pythagoreanism. He made significant contributions to philosophy and religious teaching. He is best known for Pythagorean Theorem. His disciples believed that everything was related to numbers and mathematics; the ultimate reality was unknown. He was a great orator influencing the audience to abandon luxurious and corrupt ways of life and to live purer systems

which he introduced. Plato was greatly influenced by Pythagoras. **"Platonem ferunt didicisse Pythagorea omina"** (Plato learned all things Phythagorean). Bertrand Russel in his "A History of Western Philosophy" says that Pythagoras should be considered as the most influential of all western philosophers.

Example 1 : If u and v are the orthogonal unit vectors, find the distance between u and v.

Sol. : Since u and v are orthogonal vectors by Pythagoras Theorem

$$\| u + v \|^2 = \| u \|^2 + \| v \|^2$$

But u and v are unit vectors.

$\therefore \quad \| u \| = 1, \quad \| v \| = 1$

$\therefore \quad \| u + v \|^2 = 1 + 1 = 2 \quad \therefore \ \| u + v \| = \sqrt{2}$

But $\| u + v \|$ is the distance between u, v

$\therefore \quad d(u,v) = \sqrt{2}$

Example 2 : Verify Pythagorean theorem for $u = (4, 0, 2, 3, -1)$, $v = (3, 2, 3, -2, 12)$.

Sol. : By Pythagorean Theorem if u and v are orthogonal vectors then

$$\| u \|^2 + \| v \|^2 = \| u + v \|^2$$

Now, $u \cdot v = (4)(3) + (0)(2) + (2)(3) + (3)(-2) + (-1)(12)$
$= 12 + 0 + 6 - 6 - 12 = 0$

$\therefore$ u, v are orthogonal.

Now, $\| u \|^2 = u_1^2 + u_2^2 + u_3^2 + u_4^2 + u_5^2$
$= 16 + 0 + 4 + 9 + 1 = 30$

$\| v \|^2 = 9 + 4 + 9 + 4 + 144 = 170$

Now, $u + v = (4 + 3, 0 + 2, 2 + 3, 3 - 2, -1 + 12) = \underline{(7, 2, 5, 1, 11)}$

$\therefore \quad \| u + v \|^2 = 49 + 4 + 25 + 1 + 121 = 200$

and $\| u \|^2 + \| v \|^2 = 30 + 170 = 200$

$\therefore \quad \| u + v \|^2 = \| u \|^2 + \| v \|^2$

$\therefore$ Thus, the theorem is verified.

Example 3 : Verify Pythagorean theorem for the vectors

$$u = (3, 0, 1, 0, 4, -1) \text{ and } v = (-2, 5, 0, 2, -3, -18)$$

Sol. : By Pythagorean Theorem, if u, v are orthogonal vectors, then

$$\| u + v \|^2 = \| u \|^2 + \| v \|^2.$$

Now $u \cdot v = (3)(-2) + (0)(5) + (1)(0) + (0)(2) + (4)(-3) + (-1)(-18)$
$= -6 - 12 + 18 = 0$

The vectors u, v are orthogonal.

Now, $\| u \|^2 = u_1^2 + u_2^2 + u_3^2 + u_4^2 + u_5^2$
$= 9 + 0 + 1 + 0 + 16 + 1 = 27.$

$\| v \|^2 = v_1^2 + v_2^2 + v_3^2 + v_4^2 + v_5^2$
$= (4 + 25 + 0 + 4 + 9 + 324) = 366$

Now, $\quad u + v = (3-2, 0+5, 1+0, 0+2, 4-3, -1-18)$

$\quad\quad = (1, 5, 1, 2, 1, -19)$

$\therefore \quad \| u+v \|^2 = (1+25+1+4+1+361) = 393$

And $\quad \| u \|^2 + \| v \|^2 = 27 + 366 = 393$

$\therefore \quad \| u+v \|^2 = \| u \|^2 + \| v \|^2.$

Thus, the theorem is verified.

EXERCISE - III

Verify whether the Pythagorean Theorem is applicable for

1. $u = (3, 4),\ v = (-4, 3)$
2. $u = (-3, -5, -2),\ v = (1, 2, -3)$
3. $u = (0, 1, 3),\ v = (1, -3, 1)$
4. $u = (1, 0, 2, -4),\ v = (0, 3, 4, 2)$

[**Ans. : (1)** Yes, **(2)** No, **(3)** Yes, **(4)** Yes]

Example 1 : Show that $u = (a, b)$ and $v = (-b, a)$ are orthogonal. Find two vectors that are orthogonal to $(2, -5)$. Generalise your result.

Sol. : For the first part see that

$$u \cdot v = (a, b) \cdot (-b, a) = -ab + ba = 0$$

$\therefore$ u, v are orthogonal.

From this we see that in R^2, a vector orthogonal to $u\,(a, b)$ is obtained by interchanging their positions and changing the sign of one of them.

$\therefore$ Vectors orthogonal to $(2, -5)$ are $(5, 2)$ and $(-5, -2)$ *i.e.* $(5k, 2k)$ for any scalar k.

Example 2 : Find two vectors in R^2 with Euclidean norm 1 where Euclidean inner product with $(3, -1)$ is zero.

Sol. : Let the required vector be $u = (u_1, u_2)$.

Then by data $\sqrt{u_1^2 + u_2^2} = 1$ and $(u_1, u_2) \cdot (3, -1) = (0, 0)$

$\therefore \quad 3u_1 - u_2 = 0 \qquad \therefore \quad u_2 = 3u_1$

But $\quad u_1^2 + u_2^2 = 1 \qquad \therefore \quad u_1^2 + 9u_1^2 = 1$

$\therefore \quad 10\,u_1^2 = 1 \qquad \therefore \quad u_1 = \pm \dfrac{1}{\sqrt{10}} \qquad \therefore \quad u_2 = 3u_1 = \pm \dfrac{3}{\sqrt{10}}.$

Hence, the two vectors are $\left(\dfrac{1}{\sqrt{10}}, \dfrac{3}{\sqrt{10}}\right)$ and $\left(-\dfrac{1}{\sqrt{10}}, -\dfrac{3}{\sqrt{10}}\right)$.

Example 3 : Find a vector in R^2 with norm 1 and whose inner product with $(3, -4)$ is zero.

Sol. : Let the required vector be $u = (u_1, u_2)$.

By data its norm = 1.

$\therefore \quad \sqrt{u_1^2 + u_2^2} = 1 \qquad \therefore \quad u_1^2 + u_2^2 = 1 \qquad$ .... (1)

Let $v = (3, -4)$.

By data $u \cdot v = 0 \qquad \therefore \quad (u_1, u_2) \cdot (3, -4) = 0$

$\therefore \quad 3u_1 - 4u_2 = 0 \qquad \therefore \quad u_1 = \dfrac{4}{3}\, u_2.$

Putting these values in (1), we get

$\dfrac{16}{9}u_2^2 + u_2^2 = 1 \qquad \therefore \quad \dfrac{25\,u_2^2}{9} = 1 \qquad \therefore \quad u_2 = \pm\dfrac{3}{5}$

But $\quad u_1 = \dfrac{4}{3}u_2 \qquad \therefore \quad u_1 = \pm\dfrac{4}{5}$

Hence, the required vectors are $\left(\dfrac{4}{5}, \dfrac{3}{5}\right), \left(-\dfrac{4}{5}, -\dfrac{3}{5}\right)$.

Example 4 : Find a unit vector in R^3 orthogonal to both $u = (1, 0, 1)$ and $v = (0, 1, 1)$.

Sol. : Let $x = (x_1, x_2, x_3)$ be the vector orthogonal to both u and v. Then

$\quad x \cdot u = (x_1, x_2, x_3) \cdot (1, 0, 1) = x_1 + x_3 = 0$

and $\quad x \cdot v = (x_1, x_2, x_3) \cdot (0, 1, 1) = x_2 + x_3 = 0$

$\therefore \quad x_1 = -x_3 = x_2 \qquad \therefore \quad x = (x_1, x_1, -x_1)$

$\therefore \quad \| x \| = \sqrt{x_1^2 + x_1^2 + x_1^2} = \pm\sqrt{3} \cdot x_1$

$\therefore$ Unit vector orthogonal to both $= \dfrac{(x_1, x_1, -x_1)}{\pm\sqrt{3} \cdot x_1} = \pm\dfrac{(1, 1, -1)}{\sqrt{3}}.$

Example 5 : Find a vector orthogonal to both $u = (-6, 4, 2)$, $v = (3, 1, 5)$. **(M.U. 2014, 15, 17)**

Sol. : Let $x = (x_1, x_2, x_3)$ be the vector orthogonal to both u and v. Then

$\quad x \cdot u = (x_1, x_2, x_3) \cdot (-6, 4, 2) = 0 \qquad \therefore \quad -6x_1 + 4x_2 + 2x_3 = 0$

and $\quad x \cdot v = (x_1, x_2, x_3) \cdot (3, 1, 5) = 0 \qquad \therefore \quad 3x_1 + x_2 + 5x_3 = 0$

By Cramer's rule, (See note at the bottom.)

$$\frac{x_1}{20-2} = \frac{-x_2}{-30-6} = \frac{x_3}{-6-12}$$

$\therefore \quad \dfrac{x_1}{18} = -\dfrac{x_2}{-36} = \dfrac{x_3}{-18} \qquad \therefore \quad \dfrac{x_1}{1} = \dfrac{x_2}{2} = -\dfrac{x_3}{1}$

Hence, $(1, 2, -1)$ is orthogonal to both vectors.

Example 6 : Find all vectors in R^3 of Euclidean norm 1 that are orthogonal to the vectors $u_1 = (1, 1, 1)$ and $u_2 = (1, 1, 0)$.

Sol. : Let the required vector be (v_1, v_2, v_3). Since it is orthogonal to u_1 and u_2, we have

$\quad u_1 \cdot v = (1, 1, 1) \cdot (v_1, v_2, v_3) = v_1 + v_2 + v_3 = 0$

and $\quad u_2 \cdot v = (1, 1, 0) \cdot (v_1, v_2, v_3) = v_1 + v_2 + 0 = 0.$

$\therefore$ By Cramer's Rule,

$$\frac{v_1}{0-1} = -\frac{v_2}{0-1} = \frac{v_3}{1-1} = k \text{ say} \qquad \text{.............. (1)}$$

$\therefore \quad v_1 = -k,\ v_2 = k,\ v_3 = 0$

Since the norm of v_1 is 1,

$\quad v_1^2 + v_2^2 + v_3^2 = 1 \qquad \therefore \quad k^2 + k^2 + 0 = 1$

$\therefore \quad 2k^2 = 1 \qquad \therefore \quad k = \pm\dfrac{1}{\sqrt{2}}.$

Taking $k = \frac{1}{\sqrt{2}}$, we have from (1) $\quad v_1 = -\frac{1}{\sqrt{2}},\ v_2 = \frac{1}{\sqrt{2}},\ v_3 = 0.$

Taking $k = -\frac{1}{\sqrt{2}}$, we have from (1) $\quad v_1 = \frac{1}{\sqrt{2}},\ v_2 = -\frac{1}{\sqrt{2}},\ v_3 = 0.$

Example 7 : Find a unit vector orthogonal to both (1, 1, 0) and (0, 1, 1). **(M.U. 2015, 16)**

Sol. : Let the required vector be (v_1, v_2, v_3).

Since it is orthogonal to both the given vectors u_1 and u_2.

$$u_1 \cdot v = (1, 1, 0) \cdot (v_1, v_2, v_3) = v_1 + v_2 + 0 = 0$$
$$u_2 \cdot v = (0, 1, 1) \cdot (v_1, v_2, v_3) = 0 + v_2 + v_3 = 0$$

$\therefore$ By Cramer's Rule,

$$\frac{v_1}{1-0} = -\frac{v_2}{1-0} = \frac{v_3}{1-0} = k \text{ say}$$

$\therefore\ v_1 = k,\ v_2 = -k,\ v_3 = k.$

Since the norm is 1,

$$v_1^2 + v_2^2 + v_3^2 = 1 \quad \therefore\ k^2 + k^2 + k^2 = 1 \quad \therefore\ k = \pm\frac{1}{\sqrt{3}}.$$

Taking $k = \frac{1}{\sqrt{3}}$, $\quad v_1 = \frac{1}{\sqrt{3}},\ v_2 = -\frac{1}{\sqrt{3}},\ v_3 = \frac{1}{\sqrt{3}}.$

Taking $k = -\frac{1}{\sqrt{3}}$, $\quad v_1 = -\frac{1}{\sqrt{3}},\ v_2 = \frac{1}{\sqrt{3}},\ v_3 = -\frac{1}{\sqrt{3}}.$

Example 8 : Find the unit vector in R^3 orthogonal to both $u = (1, 0, 1)$ and $v = (0, 1, 1)$. **(M.U. 2016)**

Sol. : Let the required vector be $w = (w_1, w_2, w_3)$.

Since, it is orthogonal to both the given vectors u and v,

$$u \cdot w = (1, 0, 1) \cdot (w_1, w_2, w_3) = w_1 + 0 + w_3 = 0$$

and $\quad v \cdot w = (0, 1, 1) \cdot (w_1, w_2, w_3) = 0 + w_2 + w_3 = 0$

By Cramer's rule,

$$\frac{w_1}{0-1} = -\frac{w_2}{1-0} = \frac{w_3}{1-0} = k \text{ say}$$

$\therefore\ w_1 = -k,\ w_2 = -k,\ w_3 = k.$

Since, the norm of $w = 1$,

$$w_1^2 + w_2^2 + w_3^2 = 1 \quad \therefore\ k^2 + k^2 + k^2 = 1 \quad \therefore\ k = \pm\frac{1}{\sqrt{3}}.$$

Taking $k = \frac{1}{\sqrt{3}}$, $\quad w_1 = -\frac{1}{\sqrt{3}},\ w_2 = -\frac{1}{\sqrt{3}},\ w_3 = \frac{1}{\sqrt{3}}$

Taking $k = -\frac{1}{\sqrt{3}}$, $\quad w_1 = \frac{1}{\sqrt{3}},\ w_2 = \frac{1}{\sqrt{3}},\ w_3 = -\frac{1}{\sqrt{3}}.$

$$\therefore\ w = \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \text{ or } w = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right).$$

Note ....

Cramer's Rule : We know that the roots of the equations

$a_1x + b_1y = d_1\ ;\quad a_2x + b_2x = d_2\quad$ are given by

$$x = \frac{d_1 b_2 - d_2 b_1}{a_1 b_2 - a_2 b_1} = \frac{\begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{D_x}{D};\quad y = \frac{a_1 d_2 - a_2 d_1}{a_1 b_2 - a_2 b_1} = \frac{\begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{D_y}{D}$$

This is known as Cramer's rule. The rule is highly convenient to solve simultaneous equations in two and three unknonws.

The rule states that if

(i) D is the determinant of the coefficients.

(ii) D_x = What D becomes when the coefficients of x are replaced by d's.

(iii) D_y = What D becomes when the coefficients of y are replaced by d's then

$$x = \frac{D_x}{D},\quad y = \frac{D_y}{D}.$$

Remark ....

Before applying Cramer's rule it is necessary to see that the constant terms are transferred on the right hand side.

Three Unknonws : If we are given

$$a_1\,x + b_1\,y + c_1\,z = d_1$$
$$a_2\,x + b_2\,y + c_2\,z = d_2$$
$$a_3\,x + b_3\,y + c_3\,z = d_3,$$

then $x = \frac{D_x}{D},\quad y = \frac{D_y}{D},\quad z = \frac{D_z}{D}$

where, $$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

i.e. the determinant of the coefficients of x, y, z.

$$D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

i.e. the determinant obtained from D by replacing a's by d's.

$$D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

i.e. the determinant obtained from D by replacing b's by d's.

$$D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

i.e. the determinant obtained by D by replacing c's by d's.

Modification : If we have $a_1 x + b_1 y + c_1 z = 0$

$a_2 x + b_2 y + c_2 z = 0$

Then, by eliminating y first and then eliminating x, we get

$$\frac{x}{b_1 c_2 - b_2 c_1} = -\frac{y}{a_1 c_2 - a_2 c_1} = \frac{z}{a_1 b_2 - a_2 b_1}.$$

Gabriel Cramer (1704 - 1752)

Cramer was a Swiss Mathematician. He has earned a well deserved place for spreading mathematical ideas all over the world. Cramer travelled extensively and met many leading mathematicians of his day.

His most widely known work "Introduction a'l'analyse des lignes courbes alge'briques" was a study of classification of curves. He is known for the rule of solving simultaneous equations. He also wrote on philosophy of law and government, and the history of mathematics. Cramer received numerous honours for his activities.

EXERCISE - IV

1. Find the Euclidean norms of the following vectors
   (i) (3, – 4) ; (ii) (6, 8) ; (iii) (1, 2, – 2) ; (iv) (2, 3, 0, – 6)
   [**Ans. : (i) 5, (ii) 10, (iii) 3, (iv) 7.**]
2. Find the Euclidean norms of the following vectors.
   (i) (1, – 2, – 2) ; (ii) (3, – 4, 0, – 12) ; (iii) (2, 1, – 1, 3, – 4)
   [**Ans. : (i) 3, (ii) 13, (iii) $\sqrt{31}$.**]
3. Find $u \cdot v$, where (i) $u = (2, 1, 3)$, $v = (-3, 2, 1)$
   (ii) $u = (-2, 2, 3)$, $v = (1, 7, -4)$
   [**Ans. : (i) – 1, (ii) 0**]
4. Show that the Euclidean norm of $\left(\frac{1}{\|u\|}\right)u$ where u is a non-zero vector in R^n is unity.
5. Find the values of k such that $\| kv \| = 3$ where $v = (2, -3, 0, 6)$. [**Ans. : ± 3/7**]
6. Find the Euclidean distance between the vectors u and v.
   (i) $u = (2, -1, 3)$, $v = (2, 5, -5)$
   (ii) $u = (2, 6, 3, 3, -1)$, $v = (-5, 2, 2, -2, 2)$
   [**Ans. : (i) 10, (ii) 10**]
7. Find the Euclidean distance between the following vectors.
   (i) $u = (2, 3, 1)$, $v = (3, -1, 2)$
   (ii) $u = (2, 1, -1, 2)$, $v = (2, -3, 1, 2)$
   [**Ans. : (i) $3\sqrt{2}$, (ii) $2\sqrt{5}$.**]
8. Find the angle between the following vectors.
   (i) $u = (2, -1, 1)$, $v = (1, 1, 2)$
   (ii) $u = (6, 2, 4)$, $v = (2, 0, -3)$
   [**Ans. : (i) θ = 60°, (ii) 90°**]

9. Which of the following sets of vectors are orthogonal with Euclidean inner product.
   (a) (i) (0, 1), (3, 0) (ii) $(1/\sqrt{3}, -1/\sqrt{3})$, $(1/\sqrt{3}, 1/\sqrt{3})$
   (iii) (0, 0), (1, 1) (iv) (– 1/2, – 1/2), (1/2, 1/2) in R^2 [**Ans. : All except (iv)**]
   (b) (i) $\left(\frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$
   (ii) $\left(\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{2}}, \frac{1}{3}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, 0\right)$
   (iii) $\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right), \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)$
   (iv) $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1\right), \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{2}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$
   [**Ans. : All except (iv)**]
10. Check whether the following vectors are orthogonal
    (i) $u = (-2, 3, 4)$, $v = (3, -2, 3)$
    (ii) $u = (2, 4, -3, 5)$, $v = (3, 2, 3, -1)$
    (iii) $u = (2, -1, 4, 2, 6)$, $v = (3, 2, 1, 3, -2)$ [**Ans. : (i) Yes, (ii) Yes, (ii) No**]
11. For which values of k are the following vectors orthogonal
    (i) $u = (k, k, 4)$, $v = (k, -7, 3)$
    (ii) $u = (k, -6, k, 8)$, $v = (k, k, -4, 2)$ [**Ans. : (i) $k = 4$, $k = 3$; (ii) $k = 8$, $k = 2$**]
12. Find the value of k, if $u = (2, 1, 3)$ and $v = (4, 7, k)$ are orthogonal. [**Ans. : $k = -5$**]
13. Find two vectors of unit norm which are orthogonal to (3, 4).
    [**Ans. : $\left(\frac{4}{5}, -\frac{3}{5}\right), \left(-\frac{4}{5}, \frac{3}{5}\right)$**]
14. Find two vectors orthogonal to $u = (a, 0, b)$. [**Ans. : (i) $(-b, 0, a)$; (ii) $(b, 0, -a)$**]
15. Find the Euclidean inner product $u \cdot v$ where
    (i) $u = (3, 1, 4, -2)$, $v = (2, 2, 0, 1)$
    (ii) $u = (1, -2, 3, 2)$, $v = (2, 1, 3, 0)$ [**Ans. : (i) 6, (ii) 9**]
16. Find two vectors in R^2 with Euclidean norm 1 whose Euclidean inner product with (3, – 4) is zero.
    [**Ans. : $\left(\frac{4}{5}, \frac{3}{5}\right), \left(-\frac{4}{5}, -\frac{3}{5}\right)$**]
17. If $\| u + v \| = 5$ and $\| u - v \| = 1$, find $u \cdot v$. [**Ans. : 6**]
18. Check whether
    (i) $u = (-1, 3, 2)$, $v = (4, 2, -1)$
    (ii) $u = (-4, 6, -10, 1)$, $v = (2, 2, -2, 9)$ are orthogonal. [**Ans. : (i) Yes, (ii) No**]
19. Verify the Cauchy-Schwartz inequality
    (i) $u = (4, 5)$, $v = (1, 3)$
    (ii) $u = (2, 1, -3)$, $v = (3, 4, -2)$
    (iii) $u = (2, 1, 3, -1)$, $v = (1, -2, 1, 2)$

21. Orthogonal Projection of a Vector u

Sometimes we need to 'decompose' a vectors u into sum of two vectors, one in the direction of a given non-zero vector a and the other in the direction perpendicular to the vector a.

Suppose we have u and a is as shown in the Fig. 2.17 (a). We now find a from the tip of u to a. Now, Now, drop a perpendicular from the tip of u to a. Now, find one component w_1 from the initial point of u to the foot of the perpendicular. We also find another vector component w_2 from the foot of the perpendicular to the tip of the vector u.

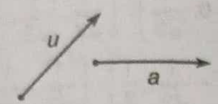


Fig. 2.17 (a)

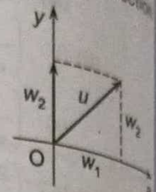


Fig. 2.17 (b)

Thus, $w_1 + w_2 = u$.

The vector component w_1 in the direction of a is called **the orthogonal projection of u on a** or **the vector component of u along a.** It is denoted by $\text{proj}_a\, u$. The vector component w_2 in the direction perpendicular to a is called the vector component of u **orthogonal** to a. Since $w_2 = u - w_1$ it can be denoted by $u - \text{proj}_a\, u$.

Vector component of u along a i.e. the projection of u on a is given by

$$\Rightarrow \boxed{\text{proj}_a\, u = \frac{u \cdot a}{\|a\|^2}\, a}$$

and the vector component of u orthogonal to a is given by

$$\Rightarrow \boxed{u - \text{proj}_a\, u = u - \frac{u \cdot a}{\|a\|^2}\, a}$$

Proof : Let as above $w_1 = \text{proj}_a\, u$ and $w_2 = u - \text{proj}_a\, u$ be the two components of u. Since w_1 is in the direction of a, it must be some scalar multiple of a. Hence, we can write $w_1 = ka$.

Since $u = w_1 + w_2$, we get

$$u \cdot a = (ka + w_2) \cdot a = ka \cdot a + w_2 \cdot a$$

But $a \cdot a = \| a \|^2$ and $w_2 \cdot a = 0$ because w_2 is perpendicular to a.

$$\therefore \quad u \cdot a = k\,\|a\|^2 \quad \therefore \quad k = \frac{u \cdot a}{\|a\|^2}$$

But $\text{proj}_a\, u = w_1 = ka$.

$$\therefore \quad \boxed{\text{proj}_a\, u = \frac{u \cdot a}{\|a\|^2}\, a}$$

Cor. : If the vector u is perpendicular to the direction of a, it is easy to see that $w_1 = 0$ i.e. projection of u on a is zero and $w_2 = u$.

Example 1 : Find the projection of $u = (1, -2, 3)$ along $v = (1, 2, 1)$ in R^3.

Sol. : We have $\quad u \cdot v = (1)(1) + (-2)(2) + (3)(1)$

$$= 1 - 4 + 3 = 0$$

This means u and v are perpendicular to each other.

$\therefore$ The projection of u along v is zero.

Example 2 : Find the projection of $u = (3, 1, 3)$ along and perpendicular to $v = (4, -2, 2)$.

Sol. : We have $\quad u \cdot v = (3)(4) + (1)(-2) + (3)(2) = 16$

$$\| v \|^2 = 16 + 4 + 4 = 24$$

$$\therefore \quad \text{proj}_v\, u = \frac{u \cdot v}{\|v\|^2}\, v = \frac{16}{24}(4, -2, 2) = \frac{2}{3}(4, -2, 2) = \left(\frac{8}{3}, -\frac{4}{3}, \frac{4}{3}\right)$$

$$u - \text{proj}_v\, u = (3, 1, 3) - \left(\frac{8}{3}, -\frac{4}{3}, \frac{4}{3}\right) = \left(\frac{1}{3}, \frac{7}{3}, \frac{5}{3}\right)$$

22. Magnitudes of the Components of u along and perpendicular to a

Since a makes an angle θ with u the magnitudes of the projection of u along a and perpendicular to a are $u \cos\theta$ and $u \sin\theta$ if u denotes the magnitude of u.

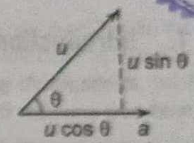


Fig. 2.18

EXERCISE - V

1. Find the orthogonal projection u on v where $u = (3, 1, -7)$ and $v = (1, 0, 5)$. Also find the vector component of u orthogonal to v.

   [**Ans. :** $\left(-\frac{16}{13}, 0, -\frac{80}{13}\right), \left(\frac{55}{13}, 1, -\frac{11}{13}\right)$]

2. Find $\| \text{proj}_v\, u \|$ where $u = (3, 0, 4)$ and $v = (2, 3, 3)$. [**Ans. :** $\frac{9\sqrt{2}}{\sqrt{11}}$]

3. Find six non-zero vectors perpendicular to $(2, 4, -1)$.

   [**Ans. :** $(4, -2, 0), (-4, 2, 0), (0, 1, 4), (0, -1, -4), (1, 0, 2), (-1, 0, -2)$]

4. Find a unit vector orthogonal to both $u = (1, 0, 1)$ and $v = (1, 1, 0)$.

   [**Ans. :** $\pm\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$]

EXERCISE - VI

Objective Questions

1. What can we say about the vector u and v if $\| u + v \| = \| u - v \|$?

   [**Ans. :** They are orthogonal.]

2. Interpret the following result geometrically in R^2.

   $\| u + v \| = \| u - v \|$ [**Ans. :** u, v are the sides of a rectangle.]

3. If u, v and w form a triangle and if $\| u \| = 3$, $\| v \| = 4$,

   $\| u + v \| = \| u - v \|$, then find $\| w \|$. [**Ans. :** 5]

4. If u and v are orthogonal unit vectors then find the distance between them. [**Ans. :** $\sqrt{2}$]

☆☆☆