Vector Spaces

1. Introduction

In chapter 2, we have learnt some properties of vectors in R^2, R^3, ... and also in R^n. We shall now consider the vector space generated by vectors in R^n.

We know that a point in two dimensional space is denoted by an ordered pair (x_1, x_2), in three dimensional space by an ordered triple (x_1, x_2, x_3). Similarly a point in four dimensional space is denoted by an ordered quadruple (x_1, x_2, x_3, x_4) in five dimensional space by an ordered quintuple $(x_1, x_2, x_3, x_4, x_5)$ and in n-dimensional space by an ordered n-tuple $(x_1, x_2, x_3, x_4, ..., x_n)$.

We know that a single real number can be used to denote a point on a real line. Conversely to every point on the real line there corresponds a real number. **The set of all points on the line is a one-dimensional space.** We also know that corresponding to any point on a plane (with a coordinate system) there is an ordered pair of real numbers (x, y) and corresponding to every ordered pair of real numbers (x, y), there is a point on the plane. **The set of all ordered pairs (x, y), $-\infty < x < \infty$, $-\infty < y < \infty$ is a two dimensional space.** Similarly, if (x, y, z) is an ordered triplet of real numbers, it corresponds to a point in three dimensional space or in R^3 and conversely for any point in R^3 there is an ordered triplet (x, y, z). Thus, one ordered triplet (x, y, z) gives us one point and **the set of all ordered triplets** (x, y, z) where $-\infty < x < \infty, -\infty < y < \infty, -\infty < z < \infty$ **is three dimensional space** consisting of all points in the space.

We also know that an ordered pair (x_1, y_1) or an ordered triplet (x_1, y_1, z_1) have two different geometrical interpretations. It may represent a point in R^2 or R^3 or it may represent a vector in R^2 or R^3.

Definition : If n is a positive integer then the sequence $(x_1, x_2, ..., x_n)$ of n real numbers, called an **ordered n-tuple represents** (a point or) a vector in n-dimensional space denoted by R^n and the set of **all** ordered n-tuples represents n-dimensional space denoted by R^n.

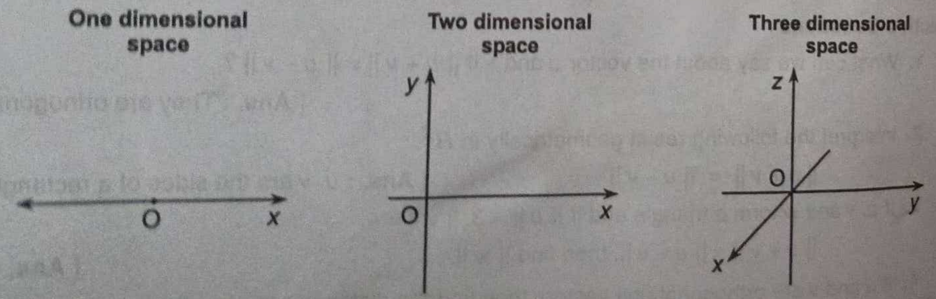

Fig. 3.1

2. Vector Space

Loosely speaking by space, we mean the set of all points which are represented in the same way. The set of the single point 0 is called the zero space, the set of all points on a line (with origin and a 'scale') is called 1-space, the set of points on a plane (with coordinate system defined) is called a 2-space, the set of points in three dimensional space (with a coordinate system) is called a 3-space. Although we are not familiar with spaces of higher dimension, examples of spaces of higher dimensions are not rare. Theory of relativity uses time as the fourth dimension. We can give another example as follows.

Suppose we want to discuss the expenses of constructing a building. We can then denote the cement used in the construction of the building by x_1, land by x_2, rubble by x_3, bricks by x_4, steel by x_5, labour by x_6, supervision by x_7, other expenses by x_8, then we are dealing with 8-space. Total expenditure then can be expressed in terms of these 8 variables.

Definition : Let V be a non-empty set of elements called **vectors** and k, m be scalars. If the set V satisfies the following axioms it is called a **vector space.**

3. Axioms of Vector Space

(a) **Closure Axioms**

C_1 : If u and v are in V then $u + v$ is in V (V is closed under addition).

C_2 : If k is any scalar then ku is in V. (V is closed under scalar multiplication).

(b) **Addition Axioms**

A_1 : $u + v = v + u$ [Commutativity]

A_2 : $(u + v) + w = u + (v + w)$ [Associativity]

A_3 : There is an element called **zero** in V such that $u + 0 = u$ for every u in V.
[Existence of additive identity]

A_4 : There is an element $-u$ in V called negative of u corresponding to every in V such that $u + (-u) = 0$. [Existence of additive inverse]

(c) **Scalar multiplication Axioms**

M_1 : $k(u + v) = ku + kv$ [Distributivity of scalar multiplication]

M_2 : $(k + l)\,u = ku + lu$ [Distributivity of scalars]

M_3 : $(kl)\,u = k\,(lu)$ [Associative law of scalars]

M_4 : There is an element 1 in V called **unity** such that $1 \cdot u = u$ for every u in V.
[Existence of multiplicative identity]

Remark ....

Vector spaces in which the scalar $k, m,$ ..... are real numbers are called **real vector spaces.** If the scalars $k, m,$ .... can be complex numbers then the vector space is called **complex vector space.**

Uniqueness Theorem : If V is a vector space then show that

(1) additive identity 0 is unique.

(2) additive inverse of a vector u is unique.

Proof : **(1)** If possible let there be two distinct additive identities 0 and 0'.

Then $u + 0 = u$ and $u + 0' = u$.

Putting $u = 0'$ in the first $0' + 0 = 0'$ and putting $u = 0$ in the second $0 + 0' = 0$.

Hence, $0' = 0$ *i.e.* the additive identity is unique.

(2) If possible let u_1 and u_2 be two additive inverses of u.

Then by definition of the inverse,

$u + u_1 = 0$ .... (1) and $u + u_2 = 0$ .... (2)

Adding u_2 to both sides of the first equation, we get

$\therefore\ u_2 + (u + u_1) = u_2 + 0 \qquad \therefore\ u_2 + (u + u_1) = u_2$ [By (A_3)]

$\therefore\ (u_2 + u) + u_1 = u_2 \qquad \therefore\ 0 + u_1 = u_2$ [By (2)]

$\therefore\ u_1 = u_2$ [By (A_3)]

4. Vector Space in General

We know that the above properties are satisfied by vectors in R^2 and R^3. We shall define the generalised vector space R^n in terms of eight properties given in (b) and (c) and two more called closure properties. These ten properties are called axioms and the elements which satisfy them define a vector space. We shall have 'vectors' of nature different from the vectors in R^2 and R^3 such as matrices or real valued functions. These are called vectors in the sense they satisfy the ten axioms of vectors.

5. Vector Space R^n

If V is the set of n-tuples of real numbers *i.e.* $x_1, x_2, \ldots, x_n$ are real numbers and if + is usual addition and ku is usual scalar multiplication *i.e.* $(kx_1, kx_2, \ldots, kx_n)$ then V is a vector space.

We denote this vector space by R^n.

Particular Cases

If $n = 1$ *i.e.* if V is the set of real numbers with usual addition and usual scalar multiplication then V is the vector space of real numbers or space of points on a line. We denote this space by R^1.

If $n = 2$, *i.e.* if V is the set of all ordered pairs of real numbers with usual addition and usual scalar multiplication, then V is the vector space of ordered pairs of real numbers or space of points on a plane. We denote it by R^2.

If $n = 3$, *i.e.* if V is the set of all ordered triplets of real numbers with usual addition and usual scalar multiplication, then V is the vector space of ordered triplets of real numbers or space of points in (usual three dimensional) space. We denote it by R^3.

Example 1 : Check whether the set of all pairs of real numbers of the form $(1, x)$ with operations $(1, y) + (1, y') = (1, y + y')$ and $k(1, y) = (1, ky)$ is a vector space. **(M.U. 2014)**

Sol. : (a) Closure

C_1 : If $u = (1, x)$, $v = (1, y)$ are two elements then

$u + v = (1, x) + (1, y) = (1, x + y)$

$\therefore$ V is closed under addition. [$\because$ *e.g.* $(1, 3) + (1, 5) = (1, 8)$]

C_2 : If $u = (1, x)$ then $ku = k(1, x) = (1, kx)$

$\therefore$ V is closed under multiplication. [e.g. If $u = (1, 3)$ then $4u = (1, 12)$]

(b) Addition

A_1 : $u + v = (1, x) + (1, y) = (1, x + y)$
$= (1, y + x) = (1, y) + (1, x)$
$= v + u$

A_2 : $(u + v) + w = [(1, x) + (1, y)] + (1, z)$
$= (1, x + y) + (1, z)$
$(u + v) + w = (1, (x + y) + z) = (1, x + (y + z))$
$= (1, x) + [(1, y + z)]$
$= (1 + x) + [(1 + y) + (1 + z)]$
$= u + (v + w)$

A_3 : $u + 0 = (1, x) + (1, 0) = (1, x + 0)$
$= (1, x) = u$

$0 = (1, 0)$ is zero for this space.

A_4 : $u + (-u) = (1, x) + (1, -x)$
$= (1, x - x) = (1, 0)$

$\therefore$ $(1, -x)$ is additive inverse $-u$ of $(1, x) = u$.

(c) Scalar Multiplication

M_1 : $k(u + v) = k[(1, x) + (1, y)]$
$= k(1, x + y) = (1, k(x + y))$
$= (1, kx + ky) = (1, kx) + (1, ky)$
$= k(1, x) + k(1, y) = ku + kv$

M_2 : $(k + l)u = (k + l)(1, x) = (1, (k + l)x)$
$= (1, kx + lx) = (1, kx) + (1, lx)$
$= k(1, x) + l(1, x)$
$= ku + lu$

M_3 : $(kl)u = (kl)(1, x)$
$= (1, klx) = k(1, lx)$
$= k[l(1, x)]$
$= k(lu)$

M_4 : $1 \cdot u = (1, 1 \cdot x)$
$= (1, x) = u$

Hence, 1 is the multiplicative identity for this space.

Since all axioms are satisfied V is vector space.

Example 2 : Let V be a set of positive real numbers with addition and scalar multiplication defined as $x + y = xy$ and $cx = x^c$, show that V is a vector space under this addition and scalar multiplication. **(M.U. 2014)**

Sol. : (a) Closure

C_1 : Since $x + y = xy$ and xy is a real number.

$\therefore$ V is closed under addition. [$\because$ *e.g.* $2 + 3 = 2 \times 3 = 6$ and 6 is real number]

C_2 : Since $cx = x^c$ and x^c is a real number V is closed under scalar multiplication.

[$\because$ *e.g.*, $c \cdot 3 = 3^c$, 3^c is a real numbers.]

(b) **Addition**

A_1 : $x + y = xy = yx = y + x$

A_2 : $(x + y) + z = xy + z = xyz$
$= x(yz) = x + (yz)$
$= x + (y + z)$

A_3 : $x + 1 = x1 = x$ for all x.

Here, 1 is zero for this operation of addition.

A_4 : $x + \frac{1}{x} = x\left(\frac{1}{x}\right) = 1.$

$\therefore$ $\frac{1}{x}$ is the negative of x because 1 is zero here.

(c) **Scalar Multiplication**

M_1 : $k(x + y) = k\,xy = (xy)^k$
$= x^k y^k = x^k + y^k$
$= kx + ky$

M_2 : $(k + l)\,x = x^{k+l} = x^k \cdot x^l$
$= x^k + x^l = kx + l\,x$

M_3 : $(kl)\,x = x^{kl} = (x^k)^l$
$= k(x^l) = k(l\,x)$

M_4 : $1 \cdot x = x^1 = x$

$\therefore$ $1 \cdot x = x$ for every x. Hence, V is a vector space.

Note ....

In this example 1 behaves as additive identity as well as multiplicative identity.

Example 3 : Let $V = \{ (x, y) \mid x, y \in R, y > 0 \}$. Let (a, b), $(c, d) \in V$ and $\alpha \in R$. Define $(a, b) + (c, d) = (a + c, b \cdot d)$ and $\alpha (a, b) = (\alpha a, b^{\alpha})$.

Examine whether V is a vector space.

Sol. : Left to you.

Example 4 : Consider the set V of all pairs of real numbers (x_1, x_2) and (y_1, y_2) such that their sum and scalar multiplication are defined as follows.

$$x + y = (x_1 + y_1, x_2 + y_2) \text{ and } kx = (kx, 0)$$

Examine whether V is a space.

Sol. : We leave it to you to verify other axioms except the last *viz.* multiplicative identity *i.e.*, M_4.

By data if $u = (3, 4)$ and $v = (2, 5)$ are two elements of V then

$$u + v = (3 + 2, 4 + 5) = (5, 9) \quad \text{and} \quad ku = k(3, 4) = (3k, 0)$$

Note that there does not exist the multiplicative identity.

We have $\quad 1 \cdot x = (x, 0) \neq x$

Hence, V is not a space.

Example 5 : Show that the set of real numbers (x, y) with operation

(i) $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $k(x, y) = (2kx, 2ky)$

(ii) $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (z_1 + z_2, y_1 + y_2, x_1 + x_2)$ and $k(x, y, z) = (kx, ky, kz)$

are not vector spaces in R^2 and R^3 respectively.

Sol. : (i) [Here we have if $u = (2, 3)$, $v = (6, 9)$ then

$$u + v = (2, 3) + (6, 9) = (2 + 6, 3 + 9) = (8, 12)$$

and $5u = 5(2, 3) = (20, 30)$

We shall consider the axiom M_3.

$$m(u) = m(x, y) = (2mx, 2my)$$

$$k(mu) = k(2mx, 2my) = (4kmx, 4kmy)$$

But $\quad (km)(u) = km(x, y) = (2kmx, 2kmy)$

Thus, $\quad k(mu) \neq km\, u.$

Further, we see that $\quad 1(x, y) = (2x, 2y) \neq (x, y)$

$\therefore$ There is no multiplicative identity.

Hence, this set is not a vector space.

(ii) For second space consider the axiom A_1.

[Here, we have if $u = (2, 3, 5)$, $v = (6, 9, 12)$, then

$$\begin{aligned} u + v &= (2, 3, 5) + (6, 9, 12) \\ &= (5 + 12, 3 + 9, 2 + 6) \\ &= (17, 12, 8) \;] \end{aligned}$$

We have
$$\begin{aligned} u + v &= (x_1, y_1, z_1) + (x_2, y_2, z_2) \\ &= (z_1 + z_2, y_1 + y_2, x_1 + x_2) \end{aligned}$$

$$\begin{aligned} \therefore \quad (u + v) + w &= (z_1 + z_2, y_1 + y_2, x_1 + x_2) + (x_3, y_3, z_3) \\ &= (x_1 + x_2 + z_3,\ y_1 + y_2 + y_3,\ z_1 + z_2 + x_3) \end{aligned}$$

And
$$\begin{aligned} v + w &= (x_2, y_2, z_2) + (x_3, y_3, z_3) \\ &= (z_2 + z_3, y_2 + y_3, x_2 + x_3) \\ u + (v + w) &= (x_1, y_1, z_1) + (z_2 + z_3, y_2 + y_3, x_2 + x_3) \\ &= (z_1 + x_2 + x_3,\ y_1 + y_2 + y_3,\ x_1 + z_2 + z_3) \end{aligned}$$

Thus, $u + (v + w) \neq (u + v) + w.$

Hence, this set is **not** a vector space.

[For example, if $u(2, 3, 6)$, $v = (3, -1, 2)$ and $w = (-2, 1, 0)$, then

$$\begin{aligned} u + v &= (2, 3, 5) + (3, -1, 2) \\ &= (5 + 2,\ 3 - 1,\ 2 + 3) = (7, 2, 5) \end{aligned}$$

$$(u+v)+w = (7, 2, 5) + (-2, 1, 0)$$
$$= (5+0, 2+1, 7-2) = (5, 3, 5)$$

Now, consider

$$v + w = (3, -1, 2) + (-2, 1, 0)$$
$$= (2+0, -1+1, 3-2) = (2, 0, 1)$$

and $\quad u + (v+w) = (2, 3, 5) + (2, 0, 1)$
$$= (5+1, 3+0, 2+2) = (6, 3, 4)$$

Thus, $(u+v)+w \neq u+(v+w)$.]

Example 6 : Let $V = R^2$ and define addition and scalar multiplication as follows :
Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$, then $u + v = (u_1 + v_1, u_2 + v_2)$ and $ku = (ku_1, u_2)$.
Prove that it is not a vector space.

Sol. : [Here we have if $u = (3, 5)$, $v = (1, 2)$ then
$$u + v = (3+1, 5+2) = (4, 7) \quad \text{and} \quad ku = k(3, 5) = (3k, 5) \text{]}$$

We leave it to you to verify, all axioms except the distributivity of scalar multiplication *i.e.*, M_2.
If $u = (u_1, u_2)$, then
$$ku = k(u_1, u_2) = (ku_1, u_2); \qquad mu = m(u_1, u_2) = (mu_1, u_2)$$
$$\therefore \quad ku + mu = (ku_1, u_2) + (mu_1, u_2) = ((k+m)u_1, 2u_2)$$
But $\quad (k+m)u = ((k+m)u_1, u_2)$
$$\therefore \quad (k+m)u \neq ku + mu. \qquad \text{Hence, } V \text{ is not a vector space.}$$
[For example let $u = (3, 5)$, then
$$2u = 2(3, 5) = (6, 5) \quad \text{and} \quad 4u = 4(3, 5) = (12, 5)$$
$$\therefore \quad 2u + 4u = (6, 5) + (12, 5) = (18, 10)$$
$$\therefore \quad (2+4)u = 6u = 6(3, 5) = (18, 5) \qquad \therefore \quad (2+4)u \neq 2u + 4u. \text{]}$$

Example 7 : Check whether $V = R^3$ is a vector space with respect to the operations
$$(u_1, u_2) + (v_1, v_2) = (u_1 + v_1 - 2, u_2 + v_2 - 3)$$
and $\quad k(u_1, u_2) = (ku_1 + 2k - 2, ku_2 - 3k + 3)$

Sol. : [Here, we have $\quad (2, 5) + (6, 7) = (2+6-2, 5+7-3) = (6, 9)$
and $\quad k(2, 5) = (k \cdot 2 + 2k - 2, k \cdot 5 - 3k + 3)$
$$= (4k - 2, 2k + 3) \text{]}$$

We leave it to you to verify all axioms except the distributivity of scalar multiplication *i.e.*, M_2.
If $u = (u_1, u_2)$, then
$$ku = (ku_1 + 2k - 2, ku_2 - 3k + 3) \qquad \text{and} \qquad lu = (lu_1 + 2l - 2, lu_2 - 3l + 3)$$
$$\therefore \quad ku + lu = [(k+l)u_1 + (2k + 2l) - 2 - 2 - 2, (k+l)u_2 - 3(k+l) + 3 + 3 - 3]$$
$$= [(k+l)u_1 + 2(k+l) - 6, (k+l)u_2 - 3(k+l) + 3]$$
and $\quad (k+l)u = (k+l)(u_1, u_2)$
$$= [(k+l)u_1 + 2(k+l) - 2, (k+l)u_2 - 3(k+l) + 3]$$
$$\therefore (k+l)u \neq ku + lu.$$
The axiom fails. Hence, V is not a vector space.

6. Vector Space of Real Valued Function F $(-\infty, \infty)$

Let V be the set of all real-valued functions defined for all x in $(-\infty, \infty)$. Let $f(x)$, $g(x)$ be two functions such that their sum is $f(x) + g(x)$. Let k be a scalar such that $(kf)(x) = kf(x)$.

In other words the value of the function $f + g$ at x is the sum of the values of f and g at x. The value of the function kf at x is k times the value of f at x.

The set of all such functions is a vector space V. It is denoted by $F(-\infty, \infty)$. The vector 0 is the function whose value for all x is 0. The graph of this function is the x-axis. The graph of $-f(x)$ is the reflection of $f(x)$ in the x-axis.

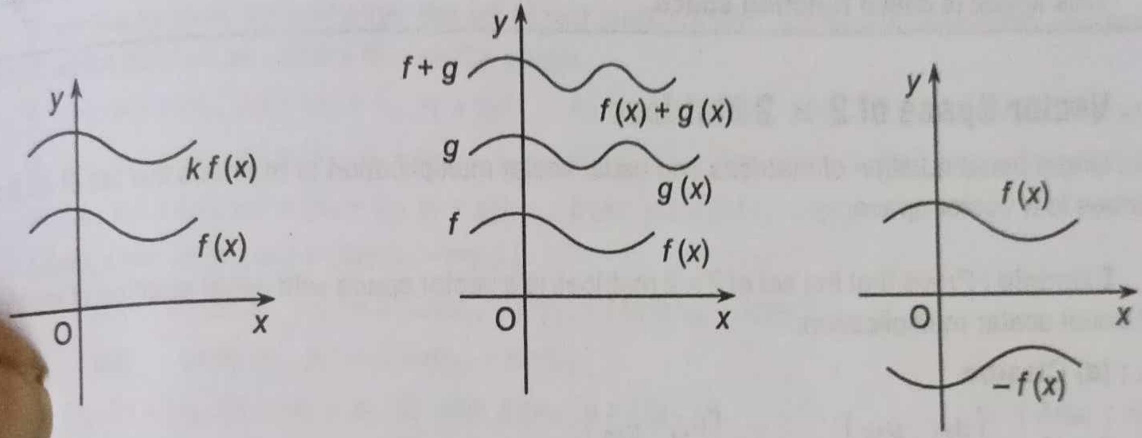

Fig. 3.2

Example 1 : Let $V = F(-\infty, \infty)$ be the set of all real valued functions defined on $(-\infty, \infty)$. For any f and g and for any scalar k, we define

(i) $f = g$ if and only if $f(x) = g(x)$ for all x.

(ii) $(f + g)(x) = f(x) + g(x)$. (iii) $(kf)(x) = kf(x)$.

Then V is a vector space.

Sol. : (a) Closure

C_1 : If f and g are real valued functions then $f + g$ is also a real valued function.

C_2 : If k is any scalar then kf also is a real valued function.

(b) Addition

A_1 :
$$(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x)$$
$$\therefore \quad f + g = g + f$$

A_2 :
$$(f+g)(x) + h(x) = f(x) + g(x) + h(x) = f(x) + (g+h)(x)$$
$$\therefore \quad (f+g) + h = f + (g+h)$$

A_3 : We define 0 as $0(x) = 0$. Then
$$(f+0)x = f(x) + 0(x) = f(x) + 0 = f(x)$$

A_4 : We define $(-f)(x) = -f(x)$

Now, $f(x) + (-f(x)) = f(x) - f(x) = 0$

$\therefore$ For every f there is $-f$.

(c) **Scalar Multiplication**

M_1 : $k[f(x)+g(x)] = k f(x) + k g(x)$ $\quad \therefore \; k(f+g) = kf + kg$

M_2 : $(k+l) f(x) = k f(x) + l f(x)$ $\quad \therefore \; (k+l) f = kf + lf$

M_3 : $(kl) f(x) = k[l f(x)]$ $\quad \therefore \; (kl) f = k(lf)$

M_4 : For any function $f(x)$, we define 1 such that $1 f(x) = f(x)$. $\quad \therefore \; 1 \cdot f = f$.

Thus, all axioms are satisfied.

Remark ....

This space is called **function space.**

7. Vector Space of 2 × 2 Matrices

Under usual addition of matrices and usual scalar multiplication of matrices the set of all 2×2 matrices is a vector space.

Example : Prove that the set of 2×2 matrices is a vector space with usual addition of matrices and usual scalar multiplication.

Sol. : (a) Closure

C_1 : If $u = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$ and $v = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$, then

$$u + v = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{bmatrix}$$

Since $u+v$ is a 2×2 matrix, $u+v$ is also is in V.

C_2 : If k is any scalar then

$$ku = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix} \qquad \therefore \; ku \text{ is in } V.$$

(b) **Addition Axioms**

Axioms A_1 and A_2 can be easily verified.

Axiom A_3 : If $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then $u + O = O + u = u$.

Since O is a 2×2 matrix axiom A_3 is satisfied.

Axiom A_4 : If we define $-u = \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix}$, then

$$u + (-u) = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus, the axiom A_4 is satisfied.

(c) **Scalar Multiplication Axioms**

Axioms M_1, M_2, M_3 can be similarly verified.

Axiom M_4 : We define $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Then, $I \cdot u = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$ for every u in V.

Hence, all the axioms are satisfied.

This space of 2×2 matrices is denoted by M_{22}. The above concept can be generalised. The set of all $m \times n$ matrices also satisfy the above axioms. This space is denoted by M_{mn}.

EXERCISE - I

Examine in each case whether the set of real numbers with operations of addition and scalar multiplication defined as follows is a vector space.

1. $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$; $k(x_1, y_1) = (3kx_1, 3ky_1)$ [**Ans. :** No. M_3 fails.]
2. $(x_1, y_1) + (x_2, y_2) = (x_1, y_1 + y_2)$; $k(x_1, y_1) = (kx_1, ky_1)$ [**Ans. :** No. M_2 fails]
3. $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$; $k(x_1, y_1) = (2kx_1, -ky_1)$

   [**Ans. :** No. $m(x_1, y_1) = (2mx_1, -my_1)$]

   and $k[m(x_1, y_1)] = k(2mx_1, -my_1) = (4km\,x_1, -kmy_1)$

   But $(km)(x_1, y_1) = (2kmx_1, -kmy_1)$]
4. $(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0)$ and $k(x_1, 0) = (kx_1, 0)$. [**Ans. :** Yes]
5. $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$; $k(x_1, y_1) = (kx_1, y_1)$ [**Ans. :** No. M_2 fails.]
6. $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$; $k(x_1, y_1) = (k^2 x_1, k^2 y_1)$ [**Ans. :** No. M_2 fails.]
7. $(x_1, y_1) + (x_2, y_2) = (|x_1 + x_2|, |y_1 + y_2|)$; $k(x_1, y_1) = (kx_1, ky_1)$

   [**Ans. :** No. A_2, A_3, A_4, M_1, M_2 fail.]
8. $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, 0)$; $k(x_1, y_1) = (kx_1, ky_1)$

   [**Ans. :** No. There is no additive identity.]
9. $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$; $k(x_1, y_1) = (|k|\, x_1, |k|\, y_1)$

   [**Ans. :** No. M_2 fails]
10. $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

    $k(x_1, y_1, z_1) = (k^2 x_1, 0, 0)$ [**Ans. :** No. M_2 fails]
11. $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

    $k(x_1, y_1, z_1) = (2kx_1, 2ky_1, 2kz_1)$ [**Ans. :** No. M_4 fails]
12. $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2 - 1, y_1 + y_2 - 1, z_1 + z_2 - 1)$

    $k(x_1, y_1, z_1) = (kx_1, ky_1, kz_1)$ [**Ans. :** No. M_1, M_2 fail]
13. $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

    $k(x_1, y_1, z_1) = (k^2 x_1, k^2 y_1, k^2 z_1)$ [**Ans. :** No. M_2 fails]
14. $(x_1, x_2, ..., x_n) + (y_1, y_2, ..., y_n) = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$

    $k(x_1, x_2, ..., x_n) = (3kx_1, 3kx_2, ..., 3kx_n)$ [**Ans. :** No, M_4 fails]
15. Examine whether the set of matrices of order 2×2 as defined below is a vector space.

    (i) $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ with usual addition of matrices and scalar multiplication. [**Ans. :** Yes]

    (ii) $\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$ with usual addition of matrices and scalar multiplication. [**Ans. :** No. C_1 fails.]

(iii) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with usual matrix addition and $k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

[**Ans. :** No. There is no multiplicative identity.]

(iv) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$.

[**Ans. :** No. Not closed under addition.

e.g., if $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, then $|A| = 0$ and $|B| = 0$.

But $A + B = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$ and $|A + B| = -1 \neq 0$.]

(v) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$.

[**Ans. :** No. Not closed under addition.

If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$|A| \neq 0$, $|B| \neq 0$. But $|A + B| = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$.]

16. Prove that the set of all $m \times n$ matrices is a vector space under usual addition of matrices and scalar multiplication.

8. Subspaces

It is possible that one vector space is contained in another vector space. The planes passing through the origin form a vector space which is contained in R^3. (See Ex. 4, 5, page 3-12, 3-13) Similarly, $F[a, b]$ will be a subspace of $F(-\infty, \infty)$ (See § 6).

Definition : A subset W of a vector space V is called a **sub-space** of V if W is itself a vector space under the addition and scalar multiplication defined on V.

In other words, the subset W of V which satisfies all the ten axioms of V is called a subspace of V. However, in practice if V is a vector space and W is a subset of V, we need not verify all the ten axioms of vector space. All axioms **except c_1 and c_2** *i.e.* (1) for all u, v, $u + v$ is in w and (2) for any scalar k, k is in W, are inherited from V (See page 3-2). We need only to verify the axioms of **closure** (1) under **addition** and (2) **scalar multiplication** only. We state this as a theorem.

Theorem : If W is a non-empty subset of a vector space V then W is a subspace of V if

(i) for all u and v in W, $u + v$ also is in W and

(ii) for any scalar k and a vector u of W, ku is in W.

Example 1 : Show that a line through the origin in R^3 is a subspace of R^3.

Sol. : Consider a line through the origin in R^3.

If u and v are any two vectors (with origin as the initial point) on this line the $u + v$ is also a vector on this line. Thus, $u + v$ is closed under addition.

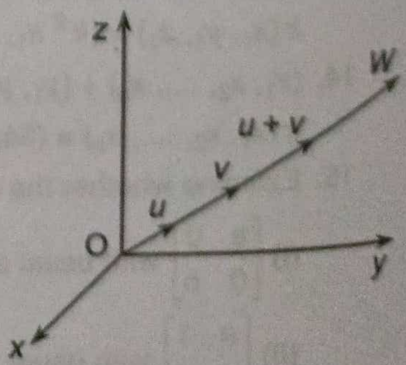

Fig. 3.3

Also if u is a vector on the line and k is any scalar then ku is also a vector on this line.

Thus, W is closed under addition and scalar multiplication.

Hence, a line through the origin is a subspace of R^3.

Remark ....

A line through the origin in R^2 is a subspace of R^2.

Using the result of Ex. 1 above, we get the following list of subspaces of R^2 and R^3.

Subspaces of R^2	Subspaces of R^3
1. $\{0\}$	**1.** $\{0\}$
2. Lines though the origin.	**2.** Lines through the origin.
	3. Planes through the origin.
3. R^2 itself	**4.** R^3 itself.

Example 2 : Show that $V = \{(x, y) \mid x = 3y\}$ is a subspace of R^2. State all possible subspaces of R^2. **(M.U. 2016)**

Sol. : Let $u = (x_1, y_1)$, $v = (x_2, y_2)$ be two elements of V.

Then $x_1 = 3y_1$ and $x_2 = 3y_2$

$\therefore\ x_1 + x_2 = 3(y_1 + y_2)$ $\quad\therefore\ u + v = (x_1 + x_2, y_1 + y_2)$ also is in V.

Again $ku = k(x_1, y_1) = (kx_1, ky_1)$

Since $x_1 = 3y_1$, $kx_1 = 3(ky_1)$. Hence, ku is in V.

Since V is closed under addition and scalar multiplication, V is a subspace of R^2.

If k is any real number then $V = \{(x, y) \mid y = kx\}$ gives all possible subspaces of R^2.

Geometrically the given set $V = \{(x, y) \mid x = 3y\}$ is a straight line passing through the origin and with slope $m = 1/3$. It is a subspace of the plane R^2. All lines passing through the origin give all possible subspaces of R^2. Every line through the origin is a sub-space of R^2.

Example 3 : If W is the set of all points (x, y) in R^2 such that $x \geq 0, y \geq 0$ then W is **not** a subspace of R^2.

Sol. : If (x_1, y_1) and (x_2, y_2) are any two points of W in the first quadrant, then

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2,\ y_1 + y_2)$$

is also a point of W in the first quadrant.

But $k(x, y) \neq (kx, ky)$ for all k.

For instance if $k = -2$,

$$(-2)(x, y) = (-2x, -2y)$$

and this point is **not** in the first quadrant.

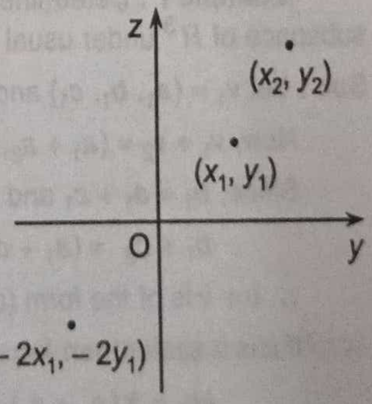

Fig. 3.4

Hence, W is not closed under scalar multiplication and hence is not a subspace of W.

Example 4 : Is $W = \{(a, 1, 1) \mid a \in R\}$ is a subspace of R^3. **(M.U. 2016)**

Sol. : Let $u = \{(a, 1, 1) \mid a \in R\}$ and $v = \{(b, 1, 1) \mid b \in R\}$ be two elements of W.

Then, $u + v = \{a, 1, 1\} + \{b, 1, 1\} = \{a + b, 2, 2\}$

But $u + v$ is not of form $\{ a, 1, 1 \mid a \in R\}$ and hence $\notin W$.

$\therefore$ W is not a subsapce of R^3.

Example 5 : Show that any plane through the origin is a sub-space of R^3. **(M.U. 2017)**

Sol. : Let the plane be $ax + by + cz = 0$.

If $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2)$ are any two points on the above plane, then $ax_1 + by_1 + cz_1 = 0$ and $ax_2 + by_2 + cz_2 = 0$.

$\therefore$ $a(x_1 + x_2) + b(y_1 + y_2) + c(z_1 + z_2) = 0$

$\therefore$ $(x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2)$ is also a point on the plane.

If k is any scalar then we have, $ku = (kx_1,\ ky_1,\ kz_1)$

$\therefore$ $akx_1 + bky_1 + ckz_1 = k(ax_1 + by_1 + cz_1) = k(0) = 0$

$\therefore$ (kx_1, ky_1, kz_1) also lies on the plane.

$\therefore$ Any plane passing through the origin is a subspace of R^3.

Example 6 : Show that the set of solution vectors of the equations

$x - 2y + 4z = 0,\ 2x - 4y + 8z = 0,\ 3x - 6y + 12z = 0$ is a subspace of R^3.

Sol. : We shall first solve these linear equations.

We have $\begin{bmatrix} 1 & -2 & 4 \\ 2 & -4 & 8 \\ 3 & -6 & 12 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ By $\begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix}\begin{bmatrix} 1 & -2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore$ $x - 2y + 4z = 0$.

Let $z = t$, $y = s$, then $x = 2s - 4t$ where s and t are parameters. For different values of s and t, we get different solutions.

The solutions are given by $x - 2y + 4z = 0$ which is a plane through the origin. This plane is a subspace of R^3 as seen in Ex. 5.

Example 7 : Determine whether the set of vectors of the form (a, b, c) where $b = a + c$ form a subspace of R^3 under usual addition and scalar multiplication. **(M.U. 2015, 16)**

Sol. : Let $v_1 = (a_1, b_1, c_1)$ and $v_2 = (a_2, b_2, c_2)$ be two vectors in R^3.

Now, $v_1 + v_2 = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$

Since, $b_1 = a_1 + c_1$ and $b_2 = a_2 + c_2$

$b_1 + b_2 = (a_1 + c_1) + (a_2 + c_2) = (a_1 + a_2) + (c_1 + c_2)$

$\therefore$ $u + v$ is of the form (α, β, γ) and $\beta = \alpha + \gamma$. $\therefore$ $u + v$ is in R^3.

If k is a scalar then $kv_1 = (ka_1, kb_1, kc_1)$ and since $b_1 = a_1 + c_1$.

$kb_1 = k(a_1 + c_1) = ka_1 + kc_1$

$\therefore$ ku is of the form (p, q, r) and $q = p + r$. $\therefore$ ku is in R^3.

Hence, the set of vectors of the form (a, b, c) is a subspace of R^3.

Example 8 : Check whether the following are subspaces of R^3.

(i) $W = \{(a, 0, 0) \mid a \in R\}$ (ii) $W = \{(x, y, z) \mid x^2 + y^2 + z^2 \le 1\}$

Sol. : (i) Let $v_1 = (a_1, 0, 0)$ and $v_2 = (a_2, 0, 0)$ be two vectors in R^3.

Now, $v_1 + v_2 = (a_1, 0, 0) + (a_2, 0, 0) = (a_1 + a_2, 0, 0)$

Since $a_1 + a_2 \in R$, $v_1 + v_2$ is in R^3.

If k is any scalar then $kv_1 = k(a_1, 0, 0) = (ka_1, 0, 0)$.

Hence, kv_1 is also in R^3.

Hence, $W = \{(a, 0, 0) \mid a \in R\}$ is a subspace of R^3.

(ii) Let $v_1 = (x_1, y_1, z_1)$ where $x_1^2 + y_1^2 + z_1^2 \leq 1$.

We consider the second condition of theorem, page 3-11.

If k is any scalar then $kv_1 = (kx_1, kx_2, kx_3)$.

But if $k > 1$ then $(kx_1)^2 + (kx_2)^2 + (kx_3)^2 = k^2 (x_1^2 + y_1^2 + z_1^2)$ is not ≤ 1.

Hence, V is not closed under scalar multiplication and hence **not** a subspace of R^3.

Example 9 : Let W be the set of 2×2 matrices of the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. Show that W is a subspace of space V of all 2×2 matrices.

Sol. : Let $P = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$, $R = \begin{bmatrix} r & 0 \\ 0 & s \end{bmatrix}$, then $P + R = \begin{bmatrix} a + r & 0 \\ 0 & q + s \end{bmatrix}$ $\therefore$ $P + R$ is in W.

And $kP = k\begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix} = \begin{bmatrix} kp & 0 \\ 0 & kq \end{bmatrix}$

$\therefore$ kP is in W. $\therefore$ W is a subspace of V.

Example 10 : If W is the set of all symmetric matrices of order $n \times n$, then W is a subspace of all $n \times n$ matrices V.

Sol. : (i) If u and v are two symmetric matrices then their sum $u + v$ is also symmetric and hence in V.

For example,
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{22} & b_{23} \\ b_{13} & b_{23} & b_{33} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{12} + b_{12} & a_{22} + b_{22} & a_{23} + b_{23} \\ a_{13} + b_{13} & a_{23} + b_{23} & a_{33} + b_{33} \end{bmatrix}$$

(ii) If k is a scalar then kA is also symmetric and hence is in V.

For example,
$$kA = \begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{12} & ka_{22} & ka_{23} \\ ka_{13} & ka_{23} & ka_{33} \end{bmatrix}$$

Hence, W is a subspace of V.

Example 11 : If W is the set of all $n \times n$ upper triangular matrices (lower triangular matrices) the W is a subspace of all $n \times n$ matrices.

Sol. : Left to you.

Example 12 : If W is the set of all diagonal matrices of order $n \times n$, then W is the subspace of all $n \times n$ matrices.

Sol. : Left to you.

Example 13 : Let W be the set of all functions of the form $p(x) = a_0 + a_1 x + \ldots + a_n x^n$ where n is any non-negative integer and where $a_0, a_1, \ldots, a_n$ are real numbers, then W is a subspace of all real valued functions *i.e.*, V.

Sol. : Let $p(x) = a_0 + a_1 x + \ldots + a_n x^n$ and $q(x) = b_0 + b_1 x + \ldots + b_n x^n$, then

(i) $p(x) + q(x) = (a_0 + b_0) x + (a_1 + b_1) x^2 + \ldots + (a_n + b_n) x^n$

$\therefore$ $p(x) + q(x)$ is of the above form.

$\therefore$ $p(x) + q(x)$ is in W.

(ii) $k\,p(x) = k[a_0 + a_1 x + \ldots + a_n x^n] = ka_0 + ka_1 x + \ldots + ka_n x^n$

$\therefore$ $k\,p(x)$ is of the above form. $\therefore$ $k\,p(x)$ is in W.

Hence, W is a subspace of V.

Example 14 : Sub spaces of continuous functions defined on $(-\infty, \infty)$.

Sol. : We have seen in § 6, page 3-8 that the set of all real valued functions denoted by $F(-\infty, \infty)$ is a vector space.

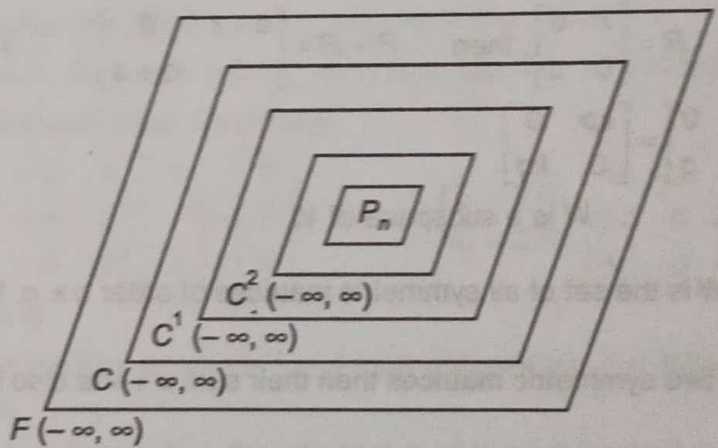

Fig. 3.5

Let $C(-\infty, \infty)$ denote the set of all continuous functions. If f and g are continuous then $f + g$ is also continuous and kf where k is any scalar is also continuous. Thus $C(-\infty, \infty)$ is a subspace of $F(-\infty, \infty)$.

Let $C^1(-\infty, \infty)$ denote the set of all functions for which first order derivatives exist and are continuous. Then $C^1(-\infty, \infty)$ is a subspace of $F(-\infty, \infty)$. But since a differentiable function is also a continuous function, $C^1(-\infty, \infty)$ is a subspace of $C(-\infty, \infty)$. Similarly, we can show that $C^2(-\infty, \infty)$ *i.e.* the functions having continuous second order derivatives is a subspace of $C^1(-\infty, \infty)$. Thus, we get a hierarchy of differentiable functions $C^1(-\infty, \infty)$, $C^2(-\infty, \infty)$, ... $C^m(-\infty, \infty)$. If we have set of functions which have continuous derivatives of all orders, we denote the subspace by $C^\infty(-\infty, \infty)$.

The hierarchy is shown diagrammatically in the Fig. 3.5. Further, we know that the set of **polynomials** P_n has continuous derivative of all orders. Hence, P_n is a subspace of $C^\infty(-\infty, \infty)$.

If instead of taking the open interval $(-\infty, \infty)$, we take closed interval $[a, b]$ then we get the corresponding subspaces $C[a, b]$, $C^m[a, b]$ and $C^\infty[a, b]$. Similarly, if we take open interval (a, b) then the spaces can be denoted by $C(a, b)$, $C^m(a, b)$ and $C^\infty(a, b)$.

Example 15 : Determine whether the set V of all functions f such that $f(0) = 0$ is a subspace of $F(-\infty, \infty)$.

Sol. : If f and g belonging to V are two functions such $f(0) = 0$ and $g(0) = 0$, then $f(0) + g(0) = 0$.

Hence, $f + g$ belong to V.

If $f(0) = 0$, then $k\, f(0) = 0$.

Hence, $k\, f$ belongs to V.

$\therefore$ V is a subset of $F(-\infty, \infty)$.

Example 16 : Let V be the vector space of all real valued functions defined on $[0, 1]$. Let W be the set of all real valued functions in V such that $f(c) > 0$ where c is a fixed number in $[0, 1]$. Examine whether W is a subspace of V.

Sol. : If f and g are two functions such that $f(c) > 0$ and $g(c) > 0$, then $f(c) + g(c) > 0$ and hence $f(x) + g(x)$ also is in V.

But if $k < 0$, then $k\, f(c) < 0$ as $f(c) > 0$.

Hence, $k\, f(x)$ is not in W. $\therefore$ W is **not** a subspace of V.

Example 17 : State only one axiom that fails to hold for each of the following sets W to be subspaces of the respective real vector spaces V with standard operations.

(1) $W = \{(x, y) \mid x^2 = y^2\}$, $V = R^2$

(2) $W = \{(x, y) \mid xy \geq 0\}$, $V = R^2$

(3) $W = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$, $V = R^3$.

(4) $W = \{f \mid f(x) \leq 0 \text{ for all } x\}$, $V = F(-\infty, \infty)$.

Sol. : Since W is to be subspace of V the two axioms of closure must be satisfied (i) W must be closed under addition, (ii) W must be closed under scalar multiplication.

(1) If $u = (x_1, y_1)$ and $v = (x_2, y_2)$, then $u + v = \{(x_1 + x_2), (y_1 + y_2)\}$.

If $x_1^2 = y_1^2$ and $x_2^2 = y_2^2$, then $(x_1 + x_2)^2 \neq (y_1 + y_2)^2$.

W is not closed under addition.

(e.g., if $u = (2, 2)$, $v = (4, -4)$, then $u + v = (2 + 4, 2 - 4)$.

But $(2 + 4)^2 \neq (2 - 4)^2$.)

(2) If $u = (x_1, y_1)$ and $v = (x_2, y_2)$, then $u + v = (x_1 + x_2, y_1 + y_2)$.

But $(x_1 + x_2)(y_1 + y_2) \not\geq 0$ for all x_1, x_2, y_1, y_2.

(e.g., if $u = (2, 3)$, $v = (-6, -1)$, then both u, v are in W because

$2 \times 3 = 6 > 0$ and $(-6) \times (-1) = 6 > 0$

But $u + v = (2 - 6, 3 - 1) = (-4, 2)$ is not in V because $-4 \times 2 = -8 \not> 0$.)

(3) If $u = (x_1, y_1, z_1)$, then $x_1^2 + y_1^2 + z_1^2 = 1$.

$ku = (kx_1, ky_1, kz_1)$

But $k^2 x_1^2 + k^2 x_2^2 + k^2 z_1^2 = k^2 (x_1^2 + y_1^2 + z_1^2) \neq 1$ if $k \neq 1$.

(4) If $f(x) \in W$, then $k\, f(x) \notin W$ if $k \leq 0$.

e.g. if $f(x) = -x^2$ and $k = -3$, then $3x^2 \not\leq 0$.

Hence, $f(x)$ is not a subspace of W.

EXERCISE - II

1. Check whether the following are subspaces of R^3.
   (i) $W = \{ (a, 0, 0) \mid a \in R \}$
   (ii) $W = \{ (a, b, c) \mid b = a + c,\ a, b, c \in R \}$ **[Ans. :** Yes]
2. Check whether the following are subspaces of R^3.
   (i) $W = \{ (a, 1, 1) \mid a \in R \}$ **(M.U. 2014, 16)**
   (ii) $W = \{ (a, b, c) \mid b = a + c + 1;\ a, b, c \in R \}$ **[Ans. :** No]
3. If $V = R^3$ is the vector space of ordered triplets of real numbers with usual addition and scalar multiplication.

   Determine which of the following are subspaces of V.

   (1) $W = \{ (x, y, z) \mid x = 1, z = 1 \}$ (2) $W = \{ (x, y, z) \mid x + y + z = 3 \}$
   (3) $W = \{ (x, y, z) \mid x^2 - y^2 = 0 \}$ (4) $W = \{ (x, y, z) \mid y \geq 0 \}$
   (5) $W = \{ (x, y, z) \mid x + y = 1 \}$ (6) $W = \{ (x, y, z) \mid x - y = 0 \text{ or } y - z = 0 \}$
   (7) $W = \{ (x, y, z) \mid x, y, z \leq 0 \}$ (8) $W = \{ (x, y, z) \mid y \text{ is an integer} \}$

[Ans. : **(1)** No. if $u = (1, 2, 1)$ and $v = (1, 3, 1)$, then $u + v = (2, 5, 2)$. But $x \neq 1,\ z \neq 1,\ u + v \notin W$.
(2) No. **(3)** No. **(4)** No. **(5)** No. **(6)** ~~No~~ Y. **(7)** No. **(8)** No.]

4. Check whether the following are subspaces of M_{22}.

   (1) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, a, b, c, d are integers. (2) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $|A| = 0$

   (3) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, a, b, c, d are rationals.

[Ans. : **(1)** No. If $k = 1/2$, then kA does not have integral elements.

(2) No. If $A = \begin{bmatrix} 2 & 6 \\ 3 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$, then $|A| = 0$, $|B| = 0$.

But $A + B = \begin{bmatrix} 6 & 8 \\ 5 & 10 \end{bmatrix}$ and $|A + B| = 60 - 40 = 20 \neq 0$.

(3) No. kA is not matrix of rational numbers if k is irrational.]

5. Determine whether the set V of triplets (x, y, z) of real numbers such that $2x + 3y + 4z = 0$ is a subspace of R^3. **[Ans. :** Yes]
6. Determine whether the set V of triplets (x, y, z) of real numbers such that either $x = 0$ or $z = 0$ is a subspace of R^3. **[Ans. :** No.]
7. Determine whether the set $V = \{ (x, y, z) \mid x = 1, y = 0 \text{ or } z = 0 \}$ is a subset of R^3.

   [Ans. : No. $(1, y, 0) + (1, 0, z) = (2, y, z) \notin V$]
8. If V is the function space with usual addition of functions and scalar multiplication in R, check whether W is a subspace of V if

   (1) $W = \{ f \mid f(1/2) > 0 \}$ (2) $W = \{ f \mid f(1) = 1 + f(2) \}$
   (3) $W = \{ f \mid f(x) \leq 0,\ [-1, 1] \}$

[**Ans.** : (1) No. $k f(1/2) \not> 0$, if $k < 0$.

(2) No. $f(1) = 1 + f(2)$, $g(1) = 1 + g(2)$

$f(1) + g(1) = 2 + f(2) + g(2) \quad \therefore \; f + g \notin W$.

(3) No. If $k < 0$, $k f(x) \leq 0 \quad \therefore \; k f \notin W$.]

9. Check whether the following set of V polynomials is a subspace of the set of polynomials P_3.

$a_0 + a_1 x + a_2 x^2 + a_3 x^3$ where a_0, a_1, a_2, a_3 are integers.

[**Ans.** : No. If k is irrational then $ku \notin P_3$.]

10. Determine whether the set V of all functions f such that $f(1/2) = 0$ is a subspace of $F(-\infty, \infty)$.

[**Ans.** : Yes]

9. Linear Combination of Vectors

Definition : A vector W is called a linear combination of the vectors $v_1, v_2, \ldots, v_n$ if it can be expressed in the form

$$W = k_1 v_1 + k_2 v_2 + \ldots + k_n v_n \quad \text{where } k_1, k_2, \ldots, k_n \text{ are scalars not all zero.}$$

Example 1 : (a) Let $v_1 = (1, 0)$ and $v_2 = (0, 1)$ be two vectors in R^2. Show that every vector in R^2 is a linear combination of v_1, v_2. Express $(-2, 3)$ as a linear combination of v_1, v_2.

(b) Let $v_1 = (1, 0, 0)$, $v_2 = (0, 1, 0)$ and $v_3 = (0, 0, 1)$ be three vectors in R^3. Show that every vector in R^3 is a linear combination of v_1, v_2, v_3. Express $(2, -3, 5)$ in terms of v_1, v_2, v_3.

(M.U. 2016)

Sol. : (a) If $W = (w_1, w_2)$ is any arbitrary vector in R^2 then W can be expressed as

$$W = a v_1 + b v_2 \qquad i.e. \qquad (w_1, w_2) = a(1, 0) + b(0, 1).$$

Hence, every vector in R^2 can be expressed as a linear combination of v_1 and v_2.

It is easy to see that $(-2, 3) = (-2)(1, 0) + 3(0, 1)$.

(b) If $W = (w_1, w_2, w_3)$ is any arbitrary vector in R^3, when W can be expressed as

$$W = a v_1 + b v_2 + c v_3 ; \quad (w_1, w_2, w_3) = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1)$$

Hence, every vector in R^3 can be expressed as a linear combination of v_1, v_2, v_3.

It is easy to see that $(2, -3, 5) = 2(1, 0, 0) - 3(0, 1, 0) + 5(0, 0, 1)$.

Example 2 : Every vector in R^3 is a linear combination of unit vectors i, j, k. **(M.U. 2016)**

Sol. : If $u = (a, b, c)$, then we can write

$$u = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1) = ai + bj + ck$$

where $i = (1, 0, 0)$, $j = (0, 1, 0)$ and $k = (0, 0, 1)$.

Remark ....

Every vector x in R^1 can be expressed as a linear combination of 1 as $1 \cdot x$. Every vector (x, y) in R^2 can be expressed as a linear combination of $(1, 0)$, $(0, 1)$ and $x(1, 0) + y(0, 1)$. Every vector (x, y, z) in R^3 can be expressed as a linear combination of $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ as $x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1)$. Every vector $(x_1, x_2, \ldots, x_n)$ in R^n can be expressed as a linear combination of $(1, 0, \ldots, 0)$, $(0, 1, \ldots, 0)$, ..., $(0, 0, \ldots, 1)$ as $x_1(1, 0, 0, \ldots, 0) + x_2(0, 1, 0, \ldots, 0) + \ldots + x_n(0, 0, \ldots, 1)$.

Example 3 : If $u = (1, 2, 2)$, $v = (3, 4, 6)$, then prove that $w = (5, 8, 10)$ is a linear combination of u and v but $w = (6, 7, -4)$ is **not** a linear combination of u and v.

Sol. : (i) If w is a linear combination of u and v, we should have $w = k_1 u + k_2 v$ for some k_1, k_2.

Now, $(5, 8, 10) = k_1 (1, 2, 2) + k_2 (3, 4, 6)$ .... (1)

gives $(5, 8, 10) = (k_1 + 3k_2,\ 2k_1 + 4k_2,\ 2k_1 + 6k_2)$

$\therefore\ 5 = k_1 + 3k_2, \quad 8 = 2k_1 + 4k_2, \quad 10 = 2k_1 + 6k_2.$

We solve these equations for k_1 and k_2.

$$\therefore \quad \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 10 \end{bmatrix}$$

$$\text{By} \quad \begin{matrix} R_2 - 2R_1 \\ R_3 - R_2 \end{matrix} \begin{bmatrix} 1 & 3 \\ 0 & -2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 2 \end{bmatrix}$$

$$\text{By} \quad \begin{matrix} R_3 + R_2 \\ -R_2 \end{matrix} \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}$$

$\therefore\ k_1 + 3k_2 = 5$ and $2k_2 = 2 \quad \therefore\ k_2 = 1,\ k_1 = 2.$

Hence, from (1), we get $(5, 8, 10) = 2(1, 2, 2) + 1(3, 4, 6)$.

$\therefore$ W is a linear combination of u and v.

(ii) If $w = (6, 7, -4)$ is a linear combination of v_1, v_2, we should have,

$$w = k_1 v_1 + k_2 v_2$$

$$(6, 7, -4) = k_1 (1, 2, 2) + k_2 (3, 4, 6)$$

$$= (k_1 + 3k_2,\ 2k_1 + 4k_2,\ 2k_1 + 6k_2)$$

$$6 = k_1 + 3k_2, \quad 7 = 2k_1 + 4k_2, \quad -4 = 2k_1 + 6k_2.$$

We solve these equations for k_1 and k_2.

$$\therefore \quad \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ -4 \end{bmatrix}$$

$$\text{By} \quad \begin{matrix} R_2 - 2R_1 \\ R_3 - R_2 \end{matrix} \begin{bmatrix} 1 & 3 \\ 0 & -2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \\ -11 \end{bmatrix}$$

$$\text{By} \quad \begin{matrix} R_3 + R_2 \\ -R_2 \end{matrix} \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ -16 \end{bmatrix}$$

$\therefore\ k_1 + 3k_2 = 6,\ \ 2k_2 = 5$ and $0 = -16$. ×

The last equality gives an absurd result $0 = -16$.

Hence, the equations are inconsistent.

The vector $(6, 7, -4)$ cannot be expressed as a linear combination of v_1, v_2.

Example 4 : Which of the following is a linear combination of $v_1 = (0, -2, 2)$, $v_2 = (1, 3, -1)$?

(a) (2, 2, 2), (b) (3, 1, 5), (c) (0, 4, 5), (d) (0, 0, 0).

Sol. : (a) Let $w = (2, 2, 2)$. We have to find if possible, k_1, k_2 such that

$$w = k_1 v_1 + k_2 v_2$$

$$\therefore \; (2, 2, 2) = k_1 (0, -2, 2) + k_2 (1, 3, -1) \qquad \text{.......... (1)}$$

$$\therefore \; (0k_1, + k_2, -2k_1 + 3k_2, 2k_1 - k_2) = (2, 2, 2).$$

$$\therefore \quad \begin{bmatrix} 0 & 1 \\ -2 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

$$\text{By} \quad R_{31} \begin{bmatrix} 2 & -1 \\ -2 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

$$\text{By} \quad R_2 + R_1 \begin{bmatrix} 2 & -1 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$$

$\therefore \; 2k_1 - k_2 = 2, \quad 2k_2 = 4, \; k_2 = 2 \qquad \therefore \; 2k_1 = 4 \qquad \therefore \; k_1 = 2$

From (1), we get $w = 2v_1 + 2v_2$.

$$\therefore \; (2, 2, 2) = 2(0, -2, 2) + 2(1, 3, -1)$$

(b), (c), (d) is left to you.

Example 5 : Express the following as a linear combination of

$v_1 = (2, 1, 4)$, $v_2 = (1, -1, 3)$ and $v_3 = (3, 2, 5)$.

(a) $(-9, -7, -15)$, (b) (6, 11, 6), **(M.U. 2016)** (c) (0, 0, 0), (d) (7, 8, 9).

Sol. : (a) Let $w = (-9, -7, -15)$.

We have to find k_1, k_2, k_3 such that

$$w = k_1 v_1 + k_2 v_2 + k_3 v_3$$

$$\therefore \; (-9, -7, -15) = k_1 (2, 1, 4) + k_2 (1, -1, 3) + k_3 (3, 2, 5) \qquad \text{.......... (1)}$$

$$\therefore \quad \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 4 & 3 & 5 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} -9 \\ -7 \\ -15 \end{bmatrix}$$

$$\text{By} \quad R_{21} \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 3 \\ 4 & 3 & 5 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} -7 \\ -9 \\ -15 \end{bmatrix}$$

$$\text{By} \quad \begin{matrix} R_2 - 2R_1 \\ R_3 - 2R_2 \end{matrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} -7 \\ 5 \\ 3 \end{bmatrix}$$

$$\text{By} \quad R_3 - R_2 \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -1 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} -7 \\ 5 \\ -2 \end{bmatrix}$$

$\therefore\ k_1 - k_2 + 2k_3 = -7, \quad 3k_2 - k_3 = 5$

$-2k_2 = -2 \qquad \therefore\ k_2 = 1,\ k_3 = -2$

$\therefore\ k_1 = 1 + 4 - 7 = -2$

$\therefore\ w = -2v_1 + v_2 - 2v_3.$

$(-9, -7, -15) = -2(2, 1, 4) + (1, -1, 3) - 2(3, 2, 5)$

(You can check the above equality)

(b) (c) (d) left to you.

Example 6 : Are the vectors $v_1 = [1, 3, 4, 2]$, $v_2 = [3, -5, 2, 6]$, $v_3 = [2, -1, 3, 4]$ linearly dependent ? If so, express X_1 as a linear combination of the others.

Sol. : Consider the matrix equation.

$$k_1 v_1 + k_2 v_2 + k_3 v_3 = 0. \qquad \text{..........(i)}$$

$$k_1[1, 3, 4, 2] + k_2[3, -5, 2, 6] + k_3[2, -1, 3, 4] = [0, 0, 0, 0]$$

$$\therefore\ k_1 + 3k_2 + 2k_3 = 0,$$
$$3k_1 - 5k_2 - k_3 = 0,$$
$$4k_1 + 2k_2 + 3k_3 = 0,$$
$$2k_1 + 6k_2 + 4k_3 = 0 \qquad \text{which can be written in matrix form as}$$

$$\therefore \begin{bmatrix} 1 & 3 & 2 \\ 3 & -5 & -1 \\ 4 & 2 & 3 \\ 2 & 6 & 4 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \quad \begin{matrix} R_2 - 3R_1 \\ R_3 - 4R_1 \\ R_4 - 2R_1 \end{matrix} \begin{bmatrix} 1 & 3 & 2 \\ 0 & -14 & -7 \\ 0 & -10 & -5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \quad \begin{matrix} (-1/7)R_2 \\ (-1/5)R_3 \end{matrix} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \quad R_3 - R_2 \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$k_1 + 3k_2 + 2k_3 = 0,\ 2k_2 + k_3 = 0.$$

If we put $k_3 = -2t$, we get $k_2 = t$ and $k_1 = t$.

Now, from (i), we get

$\therefore\ t v_1 + t v_2 - 2t v_3 = 0 \quad \therefore\ v_1 + v_2 - 2v_3 = 0 \quad \therefore\ 2v_3 = v_1 + v_2$

Since, k_1, k_2, k_3 are not all zero, the vectors are linearly dependent and $v_1 = -v_2 + 2v_3$.

Example 7 : Determine the linear dependence or independence of vectors (2, − 1, 3, 2), (1, 3, 4, 2) and (3, − 5, 2, 2). Find the relation between them if dependent.

Sol. : Consider the matrix equation

$$k_1 v_1 + k_2 v_2 + k_3 v_3 = 0$$

$$\therefore \; k_1 [2, -1, 3, 2] + k_2 [1, 3, 4, 2] + k_3 [3, -5, 2, 2] = [0, 0, 0, 0] \qquad \text{.......... (i)}$$

$$\therefore \; 2k_1 + k_2 + 3k_3 = 0$$

$$-k_1 + 3k_2 - 5k_3 = 0$$

$$3k_1 + 4k_2 + 2k_3 = 0$$

$$2k_1 + 2k_2 + 2k_3 = 0$$

which can be written as

$$\therefore \begin{bmatrix} 2 & 1 & 3 \\ -1 & 3 & -5 \\ 3 & 4 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $$\xrightarrow{R_{12}} \begin{bmatrix} -1 & 3 & -5 \\ 2 & 1 & 3 \\ 3 & 4 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $$\xrightarrow{(-1) R_1} \begin{bmatrix} 1 & -3 & 5 \\ 2 & 1 & 3 \\ 3 & 4 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $$\xrightarrow[R_4 - 2R_1]{\substack{R_2 - 2R_1 \\ R_3 - 3R_1}} \begin{bmatrix} 1 & -3 & 5 \\ 0 & 7 & -7 \\ 0 & 13 & -13 \\ 0 & 8 & -8 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $$\xrightarrow[(1/8) R_4]{\substack{(1/7) R_2 \\ (1/13) R_3}} \begin{bmatrix} 1 & -3 & 5 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $$\xrightarrow[R_4 - R_3]{R_3 - R_2} \begin{bmatrix} 1 & -3 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \; k_1 - 3k_2 + 5k_3 = 0, \; k_2 - k_3 = 0.$$

If we put $k_3 = t$, we get

$$k_2 = t \text{ and } k_1 = -2t.$$

Now, from (i) we get,

$$\therefore \; -2t\, v_1 + t\, v_2 + t\, v_3 = 0 \qquad \therefore \; 2\, v_1 - v_2 - v_3 = 0$$

Since k_1, k_2, k_3 are not all zero, the vectors are linearly dependent and $2v_1 = v_2 + v_3$.

Example 8 : Show that the vectors v_1, v_2, v_3 are linearly independent and vector X_4 depends upon them where $v_1 = (1, 2, 4)$, $v_2 = (2, -1, 3)$, $v_3 = (0, 1, 2)$, $v_4 = (-3, 7, 2)$.

Sol. : Consider the matrix equation

$$k_1 v_1 + k_2 v_2 + k_3 v_3 = 0$$

$$\therefore \quad k_1 [1, 2, 4] + k_2 [2, -1, 3] + k_3 [0, 1, 2] = [0, 0, 0]$$

$$\therefore \quad k_1 + 2k_2 + 0k_3 = 0$$

$$2k_1 - k_2 + k_3 = 0$$

$$4k_1 + 3k_2 + 2k_3 = 0$$

$$\therefore \quad \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 1 \\ 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \quad \xrightarrow{\substack{R_2 - 2R_1 \\ R_3 - 4R_1}} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -5 & 1 \\ 0 & -5 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \quad \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -5 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad k_1 + 2k_2 = 0, \; -5k_2 + k_3 = 0, \; k_3 = 0$$

$$\therefore \quad k_3 = 0, \; k_2 = 0, \; k_1 = 0.$$

$\therefore$ v_1, v_2, v_3 are linearly independent.

Now, consider the matrix equation

$$k_1 v_1 + k_2 v_2 + k_3 v_3 + k_4 v_4 = 0 \qquad \text{.........(i)}$$

$$\therefore \quad k_1 [1, 2, 4] + k_2 [2, -1, 3] + k_3 [0, 1, 2] + k_4 [-3, 7, 2] = [0, 0, 0]$$

$$\therefore \quad k_1 + 2k_2 + 0k_3 - 3k_4 = 0$$

$$2k_1 - k_2 + k_3 + 7k_4 = 0$$

$$4k_1 + 3k_2 + 2k_3 + 2k_4 = 0$$

$$\therefore \quad \begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \quad \xrightarrow{\substack{R_2 - 2R_1 \\ R_3 - 4R_1}} \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & -5 & 2 & 14 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \quad \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad k_1 + 2k_2 - 3k_4 = 0$$
$$-5k_2 + k_3 + 13k_4 = 0$$
$$k_3 + k_4 = 0$$

Let $k_4 = t \qquad \therefore \; k_3 = -t$

$$\therefore \; -5k_2 - t + 13t = 0 \qquad \therefore \; k_2 = \frac{12}{5}t$$

$$\therefore \; k_1 + \frac{24}{5}t - 3t = 0 \qquad \therefore \; k_1 = -\frac{9}{5}t$$

Putting the values of k_1, k_2, k_3, k_4 in (i), we get

$$-\frac{9}{5}t\,v_1 + \frac{12}{5}t\,v_2 - t\,v_3 + t\,v_4 = 0$$

$$\therefore \; \frac{9}{5}v_1 - \frac{12}{5}v_2 + v_3 - v_4 = 0 \qquad \therefore \; 9v_1 - 12v_2 + 5v_3 - 5v_4 = 0$$

Hence, v_1, v_2, v_3, v_4 are linearly dependent and $v_4 = \frac{9}{5}v_1 - \frac{12}{5}v_2 + v_3$.

Example 9 : Show that the rows of the matrix $\begin{bmatrix} 1 & 0 & -5 & 6 \\ 3 & -2 & 1 & 2 \\ 5 & -2 & -9 & 14 \\ 4 & -2 & -4 & 8 \end{bmatrix}$ are linearly dependent and express any row as a linear combination of other rows.

Sol. : Let the row vectors of the given matrix be denoted by v_1, v_2, v_3, v_4. Now, consider the matrix equation

$$k_1\,v_1 + k_2\,v_2 + k_3\,v_3 + k_4\,v_4 = 0 \qquad \text{.....................} \; \text{(i)}$$

$$\therefore \; k_1\,[1, 0, -5, 6] + k_2\,[3, -2, 1, 2] + k_3\,[5, -2, -9, 14] + k_4\,[4, -2, -4, 8] = [0, 0, 0, 0]$$

$$\therefore \; k_1 + 3k_2 + 5k_3 + 4k_4 = 0, \qquad 0 - 2k_2 - 2k_3 - 2k_4 = 0$$
$$-5k_1 + k_2 - 9k_3 - 4k_4 = 0, \qquad 6k_1 + 2k_2 + 14k_3 + 8k_4 = 0$$

$$\therefore \quad \begin{bmatrix} 1 & 3 & 5 & 4 \\ 0 & -2 & -2 & -2 \\ -5 & 1 & -9 & -4 \\ 6 & 2 & 14 & 8 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \quad \begin{matrix} R_3 + 5R_1 \\ R_4 - 6R_1 \end{matrix} \begin{bmatrix} 1 & 3 & 5 & 4 \\ 0 & -2 & -2 & -2 \\ 0 & 16 & 16 & 16 \\ 0 & -16 & -16 & -16 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \quad \begin{matrix} -\frac{1}{2}R_2 \\ R_3 + 8R_2 \\ R_4 + R_3 \end{matrix} \begin{bmatrix} 1 & 3 & 5 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \; k_1 + 3k_2 + 5k_3 + 4k_4 = 0 \quad \text{and} \quad k_2 + k_3 + k_4 = 0$$

Let $k_4 = t,\ k_3 = s$, then $k_2 = -s - t$.

$\therefore\ k_1 = -3k_2 - 5k_3 - 4k_4 = 3s + 3t - 5s - 4t = -2s - t.$

Hence, from (i), we get

$$(-2s - t)\,v_1 + (-s - t)\,v_2 + s\,v_3 + t\,v_4 = 0$$

$$\therefore\ s\,v_3 = (2s + t)\,v_1 + (s + t)\,v_2 - t\,v_4 \qquad \text{.......... (ii)}$$

This expresses v_3 in terms of v_1, v_2, v_4.

Note ....

You can verify the validity of (ii) by putting the values of v_1, v_2, v_3, v_4.

Example 10 : Express $p(x) = 6 + 11x + 6x^2$, as a linear combination of the following :

$$p_1 = 2 + x + 4x^2,\ p_2 = 1 - x + 3x^2,\ p_3 = 3 + 2x + 5x^2.$$

Sol. : Let $p = k_1 p_1 + k_2 p_2 + k_3 p_3$

$$\therefore\ 6 + 11x + 6x^2 = k_1(2 + x + 4x^2) + k_2(1 - x + 3x^2) + k_3(3 + 2x + 5x^2)$$

$$= (2k_1 + k_2 + 3k_3) + (k_1 - k_2 + 2k_3)\,x + (4k_1 + 3k_2 + 5k_3)\,x^2 \qquad \text{.......... (i)}$$

We have to solve the equations

$$\therefore\ 2k_1 + k_2 + 3k_3 = 6$$
$$k_1 - k_2 + 2k_3 = 11$$
$$4k_1 + 3k_2 + 5k_3 = 6$$

Taking the second equation first, which amounts to interchange of two rows, we solve the following system.

$$\therefore\ \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 3 \\ 4 & 3 & 5 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 6 \\ 6 \end{bmatrix}$$

$$\text{By} \quad \begin{matrix} R_2 - 2R_1 \\ R_4 - 4R_1 \end{matrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -1 \\ 0 & 7 & -3 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 11 \\ -16 \\ -38 \end{bmatrix}$$

$$\text{By} \quad R_3 - 3R_2 \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -1 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 11 \\ -16 \\ 10 \end{bmatrix}$$

$\therefore\ k_1 - k_2 + 2k_3 = 11,\ 3k_2 - k_3 = -16$ and $-2k_2 = 10.$

$\therefore\ k_2 = -5 \quad \therefore\ k_3 = 1,\ k_1 = 4$

Putting these values in (i), we get

$$6 + 11x + 6x^2 = 4(2 + x + 4x^2) - 5(1 - x + 3x^2) + (3 + 2x + 5x^2)$$

Example 11 : Express $p(x) = 7 + 8x + 9x^2$ as linear combination of

$$p_1 = 2 + x + 4x^2,\ p_2 = 1 - x + 3x^2,\ p_3 = 2 + x + 5x^2.$$

Sol. : Let $p = k_1 p_1 + k_2 p_2 + k_3 p_3$

$$7 + 8x + 9x^2 = k_1(2 + x + 4x^2) + k_2(1 - x + 3x^2) + k_3(2 + x + 5x^2) \qquad \text{.......... (1)}$$

$$= (2k_1 + k_2 + 2k_3) + (k_1 - k_2 + k_3)\,x + (4k_1 + 3k_2 + 5k_3)\,x^2$$

We have to solve the equations

$$2k_1 + k_2 + 2k_3 = 7$$
$$k_1 - k_2 + k_3 = 8$$
$$4k_1 + 3k_2 + 5k_3 = 9$$

Taking the second equation first, we solve the following system,

$$\therefore\ \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 2 \\ 4 & 3 & 5 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 9 \end{bmatrix}$$

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$$\text{By} \quad \begin{matrix} R_2 - 2R_1 \\ R_3 - 2R_2 \end{matrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -9 \\ -5 \end{bmatrix}$$

$$\text{By} \quad R_{23} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \\ -9 \end{bmatrix}$$

$\therefore\ k_1 - k_2 + k_3 = 8,\quad k_2 + k_3 = -5$

$3k_2 = -9 \quad \therefore\ k_2 = -3$

$\therefore\ k_3 = -2, \quad \therefore\ k_1 = 8 + k_2 - k_3 \quad \therefore\ k_1 = 7$

Hence, we get, from (1)

$$7 + 8x + 9x^2 = 7(2 + x + 4x^2) - 3(1 - x + 3x^2) - 2(2 + x + 5x^2)$$

EXERCISE - III

1. Which of the following is a linear combination of $v_1 = (-2, 1, 0),\ v_2 = (1, 2, 4)$

   (i) $w = (-1, 3, 4)$, (ii) $w = (-1, 8, 12)$, (iii) $w = (5, 5, 12)$.

   [Ans. : (i) $w = v_1 + v_2$, **(ii)** $w = 2v_1 + 3v_2$, **(iii)** $w = -v_1 + 3v_2$. **]**

2. Show that $w = (9, 2, 7)$ is a linear combination of the vectors $u = (1, 2, -1)$ and $v = (6, 4, 2)$ in R^2. **[Ans. :** $w = -3u + 2v$ **]**

3. Express the following as a linear combination of

   $v_1 = (-2, 1, 3),\ v_2 = (3, 1, -1),\ v_3 = (-1, -2, 1)$

   (a) $w = (-6, -2, 5)$ (b) $w = (0, 0, 3)$ (c) $w = (5, 0, -1)$

   [Ans. : (a) $w = v_1 - v_2 + v_3$, **(b)** $w = v_1 + v_2 + v_3$, **(c)** $w = 0v_1 + 2v_2 + v_3$. **]**

4. Express the following as a linear combination of $p_1 = 1 - x + 2x^2,\ p_2 = 3 + x - 2x^2$

   (a) $7 + x - 2x^2$, (b) $-1 - 3x + 6x^2$

   (c) 4 (d) $2 + 2x - 4x^2$

   [Ans. : (a) $p = p_1 + 2p_2$, **(b)** $p = 2p_1 - p_2$, **(c)** $p = p_1 + p_2$, **(d)** $p = p_2 - p_1$. **]**

5. Express the following as linear combination of

   $p_1 = 2 + x + 4x^2,\ p_2 = 1 - x + 3x^2,\ p_3 = 3 + 2x + 5x^2$

   (a) $-9 - 7x - 15x^2$, (b) $7 + 8x + 9x^2$.

   [Ans. : (a) $-9 - 7x - 15x^2 = -2p_1 + p_2 - 2p_3$; **(b)** $7 + 8x + 9x^2 = 0p_1 - 2p_2 + 3p_3$ **]**

6. Which of the following are linear combinations of

$$A=\begin{bmatrix}1&1\\0&2\end{bmatrix},\quad B=\begin{bmatrix}2&0\\1&-1\end{bmatrix},\quad C=\begin{bmatrix}1&1\\-1&1\end{bmatrix}$$

(a) $D=\begin{bmatrix}4&2\\0&2\end{bmatrix}$ (b) $D=\begin{bmatrix}0&2\\-2&4\end{bmatrix}$ (c) $D=\begin{bmatrix}-2&0\\0&2\end{bmatrix}$

[**Ans. : (a)** $D=A+B+C$, **(b)** $D=A-B+C$, **(c)** $D=A-B-C$.]

7. Find k such that $(1, 1, k)$ will be a linear combination of $(3, 0, -2)$ and $(2, -1, -5)$ in R^3.

[**Ans. :** 3]

10. Space Spanned by A Vector

Definition : Let V be a given vector space and $S = \{ v_1, v_2, ..., v_r \}$ be a set of vectors of V.

Let W be the subspace of V consisting of all linear combinations of $v_1, v_2, ..., v_r$ of S. Then W is called the **space spanned by** S. We also say that the vectors $v_1, v_2, ..., v_n$ span W. (To span means to furnish with something that extends or stretches over. Loosely speaking to span means to generate, to create). This is denoted by

$$W = \text{span } (S) \quad \text{or} \quad W = \text{span } (v_1, v_2, ..., v_n).$$

Using the relation of linear combination of vectors $v_1, v_2, ..., v_n$ we can define the space spanned by S as follows.

Definition : Let V be a given vector space and $S = \{ v_1, v_2, ..., v_r \}$ be a set of vectors of V. Then the set of all linear combinations of the vectors $v_1, v_2, ..., v_r$ is called a the span of the set S and is denoted by $L(S)$, or by Span (S) or Span $(v_1, v_2, ..., v_r)$.

Thus, Span $(S) = \{ k_1 v_1 + k_2 v_2 + ... + k_r v_r \mid k_1, k_2, ..., k_r$ are scalars $\}$

If $k_1 = 1, k_2 = k_3 = ... = k_r = 0$, Span $(S) = v_1$. Similarly, Span $(S) = v_2$.

$\therefore$ $v_1, v_2, ..., v_r$ are the elements of Span (S).

Example 1 : Space spanned by one vector in R^2

If v is a non-zero vector in R^2 (or R^3) with initial point at the origin then the set of all scalar multiples of v *i.e.* kv span the line determined by v.

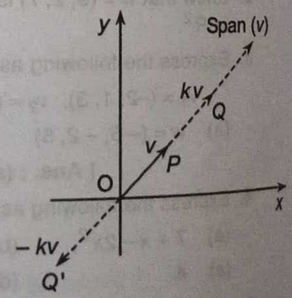


Fig. 3.6

Sol. : If v is the vector OP, then for any value of k, the vector kv is a vector like OQ or OQ'.

Thus, for different values of k, we get different vectors like OQ, OQ', ... Any vector on this line can be expressed as a multiple of v (*i.e.* OP). In this sense v spans the entire **space of this line.**

Thus, W = Span (v) is a line through the origin containing the vector v, W is a subspace of R^2 (or R^3).

Example 2 : Space spanned by two vectors in R^2

If v_1 and v_2 are two non-colinear vectors with initial points at the origin in R^2, then find the span $\{v_1, v_2\}$.

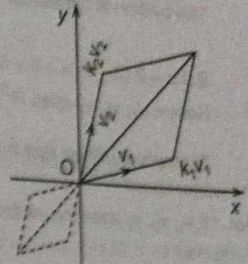


Fig. 3.7

Sol. : If v_1 and v_2 are two non-colinear vectors with the initial points at the origin in R^2 then $k_1 v_1 + k_2 v_2$ is the diagonal of the parallelogram. For different values of k_1 and k_2, we get different parallelograms and hence different diagonals. But all these diagonals are in the plane of the vectors v_1 and v_2.

Any vector in the plane can be expressed in the form $k_1 v_1 + k_2 v_2$ if v_1, v_2 are not colinear. In this sense v_1 and v_2 which are not colinear span the plane.

The span (v_1, v_2) is the plane in R^2 determined by the vectors v_1 and v_2.

Example 3 : Space spanned by two vectors in R^3

If we have two vectors v_1 and v_2 in R^3 which are not colinear considering a parallelogram as above, we see that $k_1 v_1 + k_2 v_2$ spans the plane in which the vectors v_1 and v_2 lie.

Example 4 : Determine whether

(i) $v_1 = (2, -1, 3), v_2 = (4, 1, 2), v_3 = (8, -1, 8)$

(ii) $v_1 = (2, 2, 2), v_2 = (0, 0, 3), v_3 = (0, 1, 1)$

span a vector space in R^3.

Sol. : (i) To determine whether v_1, v_2, v_3 span vector space R^3 or not, we have to check whether any vector in R^3 can be expressed as a linear combination of v_1, v_2, v_3.

Let $w = (w_1, w_2, w_3)$ be any vector in R^3.

We try to find k_1, k_2, k_3 such that

$$w = k_1 v_1 + k_2 v_2 + k_3 v_3 \quad \text{.........(1)}$$

$$\therefore \quad (w_1, w_2, w_3) = k_1 (2, -1, 3) + k_2 (4, 1, 2) + k_3 (8, -1, 8)$$
$$= (2k_1 + 4k_2 + 8k_3, \ -k_1 + k_2 - k_3, \ 3k_1 + 2k_2 + 8k_3)$$

$\therefore$ $w_1 = 2k_1 + 4k_2 + 8k_3$, $w_2 = -k_1 + k_2 - k_3$, $w_3 = 3k_1 + 2k_2 + 8k_3$

The system is consistent if $\begin{vmatrix}2&4&8\\-1&1&-1\\3&2&8\end{vmatrix} \neq 0$

Now, $\Delta = 2(8+2) - 4(-8+3) + 8(-2-3) = 20 + 20 - 40 = 0$.

Hence, the system is not consistent. There are no constants k_1, k_2, k_3 satisfying (1).

Hence, v_1, v_2, v_3 do not span R^3.

(ii) Let $w = (w_1, w_2, w_3)$ be any vector in R^3.

We try to find k_1, k_2, k_3 such that $w = k_1 v_1 + k_2 v_2 + k_3 v_3$.

$$\therefore \quad (w_1, w_2, w_3) = k_1 (2, 2, 2) + k_2 (0, 0, 3) + k_3 (0, 1, 1)$$
$$= (2k_1 + 0 + 0, \ 2k_1 + 0 + k_3, \ 2k_1 + 3k_2 + k_3)$$

$\therefore$ $w_1 = 2k_1 + 0k_2 + 0k_3$, $w_2 = 2k_1 + 0k_2 + k_3$, $w_3 = 2k_1 + 3k_2 + k_3$

The system is consistent if $\begin{vmatrix} 2 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 3 & 1 \end{vmatrix} \neq 0$

But $\Delta = 2(0-3) = -6 \neq 0$. $\therefore$ The system is consistent.

Hence, v_1, v_2, v_3 span R^3.

Example 5 : Show that the vectors $v_1 = (1, 0, 1)$, $v_2 = (2, 1, 4)$ and $v_3 = (1, 1, 3)$ do not span the vector space R^3.

Sol. : If v_1, v_2, v_3 span the vector space R^3, then any vector in R^3 should be expressible in terms of v_1, v_2, v_3.

Let $w = (w_1, w_2, w_3)$ be a vector in R^3. We must be able to find k_1, k_2, k_3 such that

$$w = k_1 v_1 + k_2 v_2 + k_3 v_3 \quad \text{.......... (1)}$$

$$\therefore \quad (w_1, w_2, w_3) = k_1(1, 0, 1) + k_2(2, 1, 4) + k_3(1, 1, 3)$$

$$= (k_1 + 2k_2 + k_3,\ 0 + k_2 + k_3,\ k_1 + 4k_2 + 3k_3)$$

$$\therefore \quad w_1 = k_1 + 2k_2 + k_3, \quad w_2 = 0k_1 + k_2 + k_3, \quad w_3 = k_1 + 4k_2 + 3k_3.$$

This system is consistent if $\begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 4 & 3 \end{vmatrix} \neq 0$

But $\Delta = (3-4) - 2(0-1) + 1(0-1) = -1 + 2 - 1 = 0$.

The equations are not consistent. This means there do not exist k_1, k_2, k_3 satisfying (1) which means no vector in R^3 can be expressed as a linear combination of v_1, v_2, v_3.

Hence, v_1, v_2, v_3 do not span R^3.

Example 6 : Determine whether the following vectors span P_2. (P_2 means vector space consisting of all polynomials of second order.)

$$p_1 = 1 - x + 2x^2, \quad p_2 = 5 - x + 4x^2, \quad p_3 = -2 - 2x + 2x^2.$$

Sol. : If p_1, p_2, p_3 span the entire vector space P_2 then any vector in P_2 must be expressible in terms of p_1, p_2, p_3.

If $p = b_1 + b_2 x + b_3 x^3$ is any vector in P_2, we must be able to find k_1, k_2, k_3 such that

$$p = k_1 p_1 + k_2 p_2 + k_3 p_3$$

i.e. $(b_1, b_2, b_3) = k_1(1, -1, 2) + k_2(5, -1, 4) + k_3(-2, -2, 2)$

$$= (k_1 + 5k_2 - 2k_3,\ -k_1 - k_2 - 2k_3,\ 2k_1 + 4k_2 + 2k_3)$$

$$\therefore \quad b_1 = k_1 + 5k_2 - 2k_3, \quad b_2 = -k_1 - k_2 - 2k_3, \quad b_3 = 2k_1 + 4k_2 + 2k_3.$$

The system is consistent if $\begin{vmatrix} 1 & 5 & -2 \\ -1 & -1 & -2 \\ 2 & 4 & 2 \end{vmatrix} \neq 0$.

But, $\Delta = 1(-2+8) - 5(-2+4) - 2(-4+2) = 6 - 10 + 4 = 0$

Hence, the equations are not consistent. This means, p_1, p_2, p_3 do not span P_2.

Example 7 : Let $f = \cos^2 x$, $g = \sin^2 x$. Which of the following lie in the space spanned by f and g.

(a) $\cos 2x$, (b) $\sin 2x$, (c) 1, (d) $\sin x$, (e) 0.

Sol. : (i) We have $\cos 2x = \cos^2 x - \sin^2 x = f - g$

(ii) $1 = \cos^2 x + \sin^2 x = f + g$

(iii) $0 = 0 \cos^2 x + 0 \sin^2 x = 0f + 0g$

Hence, these three are in the space spanned by f and g, the remaining are not.

Example 8 : Vector space spanned by 1, x, ..., x^n

Sol. : Let P_n denote the vector space consisting of all polynomials of degree n or less, of the form

$$p(x) = a_0 + a_1 x + \ldots + a_n x^n$$

Since each polynomial p is a linear combination of the functions $1, x, \ldots, x^n$. We can say that the polynomials $1, x, \ldots, x^n$ span the vector space P_n. We denote this as

$$P_n = \text{span}(1, x, \ldots, x^n).$$

EXERCISE - IV

1. Check whether the given set of vectors span a subspace of R^2
   (i) (2, 3), (– 4, – 6), (ii) (1, 4), (– 3, – 12), (iii) (2, 5), (4, 10), (6, 11).
   [Ans. : (i) Yes, (ii) Yes, (iii) No **]**

2. Check whether the given set of vectors span a subspace of R^3
   (i) (1, 1, 1), (0, 1, 1), (0, 1, 1); (ii) (2, 1, 0), (0, 3, – 4), (1, –1, 2)
   (iii) (1, 2, 3), (0, 0, 1), (0, 1, 2); (iv) (3, 1, 4), (2, – 3, 5), (5, – 2, 9), (1, 4, – 1)
   (v) (1, 2, 6), (3, 4, 1), (4, 3, 1), (3, 3, 1)
   [Ans. : (i) Yes, (ii) Yes, (iii) No, (iv) No, (v) No **]**

11. Linear Independence of Vectors

We now consider linear dependence or independence of vectors.

Linear Dependence

Definition : A set or r-vectors $v_1, v_2, v_3, \ldots, v_r$ is said to be **linearly dependent** if there exist r numbers $k_1, k_2, k_3, \ldots, k_r$ **not all zero** such that

$$k_1 v_1 + k_2 v_2 + k_3 v_3 + \ldots + k_r v_r = 0. \quad \text{.................... (A)}$$

Linear Independence

Definition : A set of r-vectors $v_1, v_2, v_3, \ldots, v_r$ is said to be **linearly independent** if every relation of the type

$$k_1 v_1 + k_2 v_2 + k_3 v_3 + \ldots + k_r v_r = 0 \quad \text{implies that } k_1 = k_2 = k_3 = \ldots = k_r = 0$$

The following two results are clear :

1. If a set of vectors is linearly dependent then at least one member can be expressed as a linear combination of the remaining vectors. For, in (A) if $k_1 \neq 0$ then transposing $k_1 v_1$ on the r.h.s. and then dividing by $-k_1$ throughout we can have.

$$v_1 = I_2 v_2 + I_3 v_3 + \ldots + I_r v_r$$

2. If a set of vectors is linearly independent then no member of the set can be expressed as a linear combination of the other members.

Example 1 : Show by inspection that the set of vectors $u_1 = (-1, 3, 5)$, $u_2 = (4, -12, -20)$ is linearly dependent.

Sol. : It is easy to see that

$$4u_1 + u_2 = (4-4,\ 12-12,\ 20-20) = 0 \qquad i.e. \qquad u_2 = -4u_1$$

Hence, u_1 and u_2 are linearly dependent.

Example 2 : Show that $p_1 = 1 - 2x$, $p_2 = 2 - 3x + x^2$, $p_3 = 3 - 5x + x^2$ are linearly dependent.

Sol. : It is easy to see that

$$p_1 + p_2 - p_3 = (1 + 2 - 3,\ -2x - 3x + 5x,\ x^2 - x^2) = 0$$

$$\therefore \quad p_3 = p_1 + p_2$$

Hence, p_1, p_2, p_3 are linearly dependent.

Example 3 : Examine whether the vectors

$$v_1 = [3, 1, 1],\ v_2 = [2, 0, -1],\ v_3 = [4, 2, 1] \quad \text{are linearly independent}$$

Sol. : Consider the matrix equation

$$k_1v_1 + k_2v_2 + k_3v_3 = 0$$

$$\therefore \quad k_1[3, 1, 1] + k_2[2, 0, -1] + k_3[4, 2, 1] = [0, 0, 0]$$

$$\therefore \quad 3k_1 + 2k_2 + 4k_3 = 0,\ k_1 + 0k_2 + 2k_3 = 0,\ k_1 - k_2 + k_3 = 0$$

which can be written as

$$\therefore \quad \begin{bmatrix} 3 & 2 & 4 \\ 1 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \quad R_{13} \quad \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 2 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \quad \begin{matrix} R_2 - R_1 \\ R_3 - 3R_1 \end{matrix} \quad \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \quad R_3 - R_2 \quad \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad k_1 - k_2 + k_3 = 0,\ k_2 + k_3 = 0,\ 4k_2 = 0$$

$$\therefore \quad k_2 = 0,\ k_3 = 0 \quad \therefore \quad k_1 = 0$$

Since, all k_1, k_2, k_3 are zero, the vectors are linearly independent.

Example 4 : For what value of λ the following vectors are linearly dependent?

$$\left(\lambda, -\frac{1}{2}, -\frac{1}{2}\right), \left(-\frac{1}{2}, \lambda, -\frac{1}{2}\right), \left(-\frac{1}{2}, -\frac{1}{2}, \lambda\right)$$

Sol. : Consider the matrix equation $\quad k_1 v_1 + k_2 v_2 + k_3 v_3 = 0$

$$\therefore \quad k_1\left[\lambda, -\frac{1}{2}, -\frac{1}{2}\right] + k_2\left[-\frac{1}{2}, \lambda, -\frac{1}{2}\right] + k_3\left[-\frac{1}{2}, -\frac{1}{2}, \lambda\right] = 0$$

$$\therefore \quad \lambda k_1 - \frac{1}{2}k_2 - \frac{1}{2}k_3 = 0; \quad -\frac{1}{2}k_1 + \lambda k_2 - \frac{1}{2}k_3 = 0; \quad -\frac{1}{2}k_1 - \frac{1}{2}k_2 + \lambda k_3 = 0$$

$$\therefore \quad \begin{bmatrix} \lambda & -1/2 & -1/2 \\ -1/2 & \lambda & -1/2 \\ -1/2 & -1/2 & \lambda \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \quad \begin{matrix} R_2 - R_1 \\ R_3 - R_2 \end{matrix} \begin{bmatrix} \lambda & -1/2 & -1/2 \\ -\lambda-(1/2) & \lambda+(1/2) & 0 \\ 0 & -(1/2)-\lambda & \lambda+(1/2) \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad \lambda k_1 - \frac{1}{2}k_2 - \frac{1}{2}k_3 - 0 \qquad \text{.......... (i)}$$

$$-\left(\lambda + \frac{1}{2}\right)k_1 + \left(\lambda + \frac{1}{2}\right)k_2 = 0 \qquad \text{.......... (ii)}$$

$$\therefore \quad -\left(\lambda + \frac{1}{2}\right)(k_1 - k_2) = 0$$

$$-\left(\lambda + \frac{1}{2}\right)k_2 + \left(\lambda + \frac{1}{2}\right)k_3 = 0 \qquad \text{.......... (iii)}$$

$$\therefore \quad -\left(\lambda + \frac{1}{2}\right)(k_2 - k_3) = 0$$

If $\lambda \neq -\frac{1}{2}$ then from (ii) and (iii), we get $k_1 = k_2$, $k_2 = k_3$.

Then from (i), $\quad \lambda k_1 - \frac{1}{2}k_1 - \frac{1}{2}k_1 = 0 \quad \therefore \quad (\lambda - 1)k_1 = 0$

If $\lambda \neq 1$, $k_1 = 0$. This means if $\lambda = 1/2$ or $\lambda = 1$, k_1, k_2, k_3 are not zero and the vectors are dependent.

Example 5 : Show that $S = \{1 - t - t^2, \ -2 + 3t + 2t^3, \ 1 + t + 5t^3\}$ is linearly independent in P_3. (P_3 means the set of all polynomials of degree 3 *i.e.*, of the form $a_0 + a_1 x + a_2 x^2 + a_3 x^3$.)

Sol. : Let $v_1 = 1 - t - t^2 + 0t^3; \quad v_2 = -2 + 3t + 0\,t^2 + 2\,t^3; \quad v_3 = 1 + t + 0\,t^2 + 5\,t^3$

Consider the matrix equation

$$k_1 v_1 + k_2 v_2 + k_3 v_3 = 0$$

$$\therefore \quad k_1[1, -1, -1, 0] + k_2[-2, 3, 0, 2] + k_3[1, 1, 0, 5] = 0$$

$$\therefore \quad k_1 - 2k_2 + k_3 = 0, \qquad -k_1 + 3k_2 + k_3 = 0,$$

$$-k_1 + 0\,k_2 + 0\,k_3 = 0, \qquad 0k_1 + 2k_2 + 5k_3 = 0$$

$$\therefore \quad \begin{bmatrix} 1 & -2 & 1 \\ -1 & 3 & 1 \\ -1 & 0 & 0 \\ 0 & 2 & 5 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2 + R_1$ $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \\ -1 & 0 & 0 \\ 0 & 2 & 5 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

By $R_4 - 2R_2$ $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore \ k_1 - 2k_2 + k_3 = 0;$ $\quad k_2 + 2k_3 = 0$

$\therefore \ -k_1 = 0, \ k_3 = 0$ $\quad \therefore \ k_2 = 0$

Hence, S is linearly independent in P_3.

12. Linear Independence of Matrices

We prove the linear independence of matrices using the same technique as above.

Example 1 : Determine whether $A = \begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}$ are linearly independent in $M_{2,2}$.

Sol. : Consider the equation $\quad k_1 A + k_2 B + k_3 C = 0$

$$\therefore \ k_1 \begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix} + k_2 \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} + k_3 \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\therefore \ \begin{bmatrix} 4k_1 + k_2 + 0k_3 & 0k_1 - k_2 + 2k_3 \\ -2k_1 + 2k_2 + k_3 & -2k_1 + 3k_2 + 4k_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Equating the corresponding elements, we get,

$$4k_1 + k_2 + 0k_3 = 0$$
$$0k_1 - k_2 + 2k_3 = 0$$
$$-2k_1 + 2k_2 + k_3 = 0$$
$$-2k_1 + 3k_2 + 4k_3 = 0$$

$$\begin{bmatrix} 4 & 1 & 0 \\ 0 & -1 & 2 \\ -2 & 2 & 1 \\ -2 & 3 & 4 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_{1,4}$ $\begin{bmatrix} -2 & 3 & 4 \\ 0 & -1 & 2 \\ -2 & 2 & 1 \\ 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

By $-\frac{1}{2}R_1$ $\begin{bmatrix} 1 & -3/2 & -2 \\ 0 & -1 & 2 \\ -2 & 2 & 1 \\ 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$\text{By}\quad \begin{matrix} \\ R_3+2R_1 \\ R_4+2R_3 \\ \\ \end{matrix}\begin{bmatrix} 1 & -3/2 & -2 \\ 0 & -1 & 2 \\ 0 & -1 & -3 \\ 0 & 5 & 2 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By}\quad \begin{matrix} \\ R_3-R_2 \\ R_4+5R_3 \\ \\ \end{matrix}\begin{bmatrix} 1 & -3/2 & -2 \\ 0 & -1 & 2 \\ 0 & 0 & -5 \\ 0 & 0 & -13 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

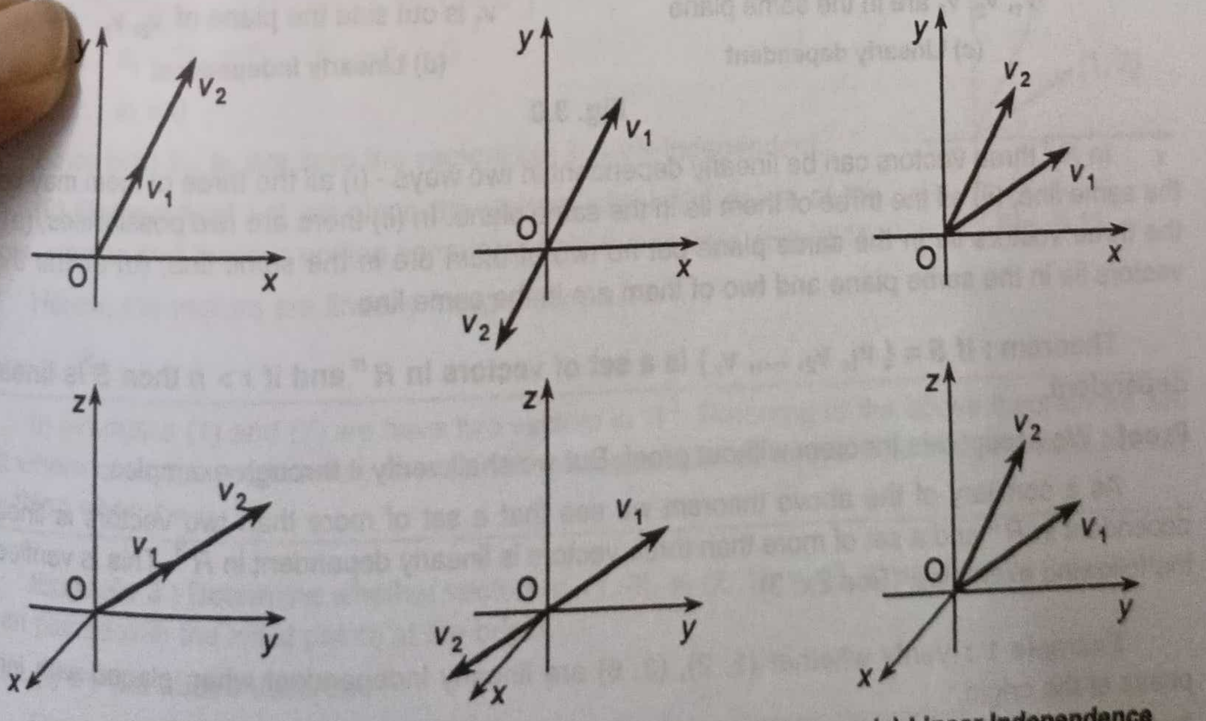
$$\therefore\ k_1 - \frac{3}{2}k_2 - 2k_3 = 0; \qquad -k_2 + 2k_3 = 0$$

$$\therefore\ -5\,k_3 = 0, \qquad -13k_3 = 0 \qquad \therefore\ k_3 = 0,\ k_2 = 0,\ k_1 = 0$$

$\therefore$ The matrices A, B, C are linearly independent.

13. Geometric Interpretation of Linear Independence

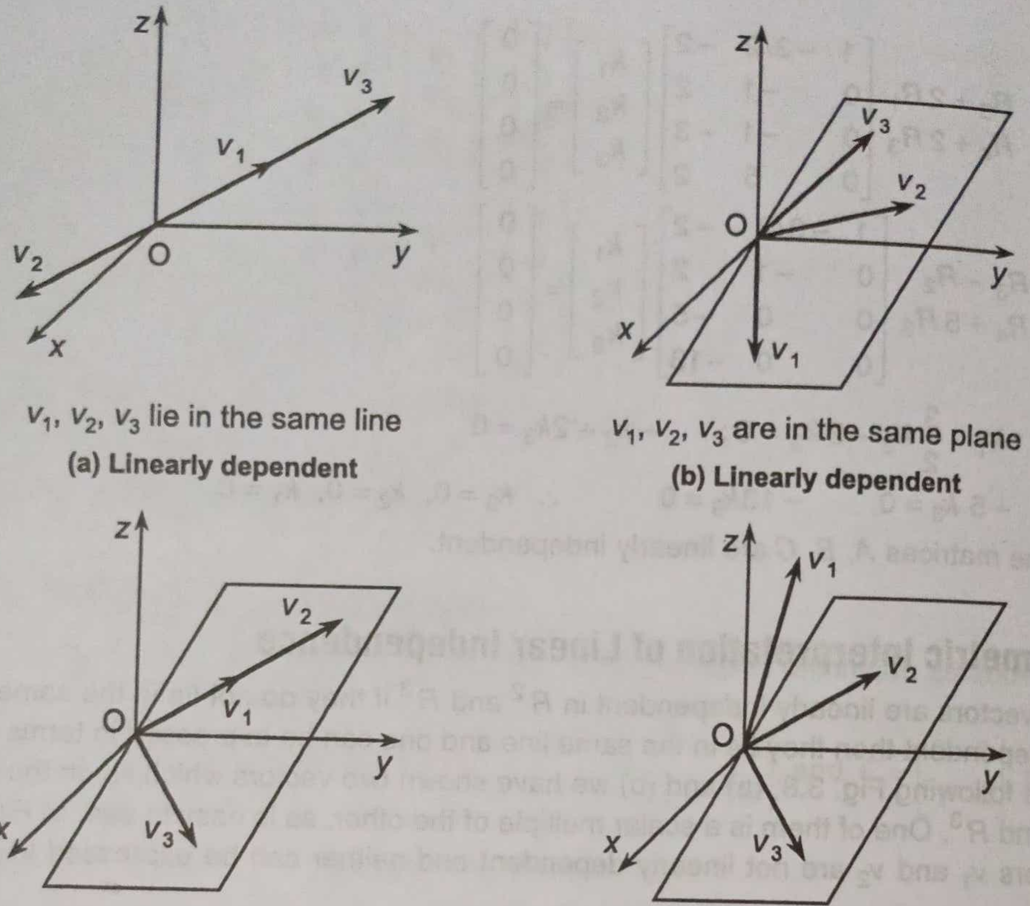
(a) Two vectors are linearly independent in R^2 and R^3 if they do not lie in the same line. If they are dependent then they lie in the same line and one can be expressed in terms of the other. In the following Fig. 3.8, (a) and (b) we have shown two vectors which lie on the same line in R^2 and R^3. One of them is a scalar multiple of the other, as is easy to see. In Fig. 3.8 (c) the vectors v_1 and v_2 are not linearly dependent and neither can be expressed in terms of the other.



(a) Linear dependence (b) Linear dependence (c) Linear Independence

Fig. 3.8

(b) In R^3 three vectors are linearly independent if and only if the vectors do not lie in the same plane when they are placed with their initial points at the origin. In other words, if three vectors in R^3 are linearly independent then they do not lie in the same plane. Below we have shown three linearly dependent vectors in Fig. 3.9 (a) (b) and (c) and linearly independent vectors in Fig. 3.9 (d).



v_1, v_2, v_3 lie in the same line
(a) Linearly dependent

v_1, v_2, v_3 are in the same plane
(b) Linearly dependent

v_1, v_2, v_3 are in the same plane
(c) Linearly dependent

v_1 is out side the plane of v_2, v_3
(d) Linearly Independent

Fig. 3.9

In R^3 three vectors can be linearly dependent in two ways - (i) all the three of them may lie in the same line, (ii) all the three of them lie in the same plane. In (ii) there are two possibilities, (a) all the three vectors lie in the same plane but no two of them are in the same line, (b) all the three vectors lie in the same plane and two of them are in the same line.

Theorem : If $S = \{v_1, v_2, \ldots, v_r\}$ is a set of vectors in R^n and if $r > n$ then S is linearly dependent.

Proof : We accept this theorem without proof. But we shall verify it through examples.

As a corollary of the above theorem we see that a set of more than two vectors is linearly dependent in R^2 and a set of more than three vectors is linearly dependent in R^3. This is verified in the following examples. (See Ex. 3)

Example 1 : Verify whether (1, 2), (3, 6) are linearly independent when placed with initial points at the origin.

Sol. : We shall use both analytical and geometric methods.

(i) Analytical : Let the vectors be $v_1 = (1, 2)$ and $v_2 = (3, 6)$ and consider

$$k_1 v_1 + k_2 v_2 = 0 \qquad \text{.........(1)}$$

$$\therefore \quad k_1 (1, 2) + k_2 (3, 6) = 0$$

$$\therefore \quad \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{By } R_2 - 2R_1 \quad \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore \quad k_1 + 3k_2 = 0 \quad \therefore \quad k_1 = -3k_2$

Hence, the vectors are dependent.

If we put $k_2 = t$, then $k_1 = -3t$.

Now, from (1), we get,

$$-3t\, v_1 + t\, v_2 = 0 \qquad \therefore \quad 3v_1 - v_2 = 0$$

$\therefore \quad v_2 = 3\, v_1$ which is clear even by a close look.

Hence, the vectors we **not** linearly independent.

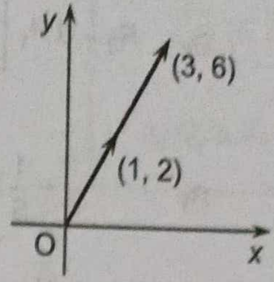


Fig. 3.10

(ii) Geometrical : If we place the vectors with the initial points at the origin, we see that the two vectors lie on the same line. (Fig. 3.10)

$\therefore$ They are **not** independent.

Example 2 : Verify whether (1, 2), (2, 3) are linearly dependent when placed with the initial points at the origin.

Sol. : (i) Analytical : Let the vectors be $v_1 = (1, 2)$ and $v_2 = (2, 3)$ and consider

$$k_1\, v_1 + k_2\, v_2 = 0$$

$$k_1\,(1, 2) + k_2\,(2, 3) = 0$$

$$\therefore \quad \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By $\quad R_2 - 2R_1 \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore \quad k_1 + 2k_2 = 0 \quad \therefore \quad -k_2 = 0$

$\therefore \quad k_1 = 0$

Since both k_1, k_2 are zero the vectors are linearly independent.

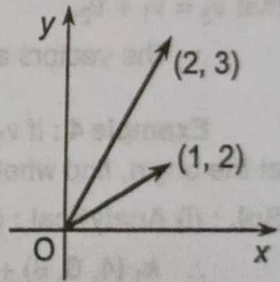


Fig. 3.11

(ii) Geometrical : If we place the vectors with initial points at the origin, we see that there is no line containing both the vectors. (Fig. 3.11)

Hence, the vectors are linearly independent.

Note ....

In examples (1) and (2) we have two vectors in R^2. Referring to the above theorem we see that where $r = n$, the vectors may or may not be independent as above. The theorem does not say any thing when $r = n$.

Example 3 : Determine whether vectors v_1 (1, 3), v_2 (2, 1), v_3 (3, 4) are linearly independent when placed with the initial points at the origin.

Sol. : (i) By the above theorem

Since there are three vectors in R^2 *i.e.* $r > n$, the set is linearly dependent.

(ii) Analytical : Consider the equation

$$k_1\, v_1 + k_2\, v_2 + k_3\, v_3 = 0 \qquad \text{.......... (1)}$$

$$\therefore \quad k_1\,(1, 3) + k_2\,(2, 1) + k_3\,(3, 4) = (0, 0)$$

$$\therefore \quad \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By $R_2 - 3R_1$ $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -5 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

By $-\frac{1}{5}R_2$ $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore\ k_1 + 2k_2 + 3k_3 = 0$

$k_2 + k_3 = 0$

Let $k_3 = t$ $\therefore\ k_2 = -k_3 = -t$

and $k_1 = -2k_2 - 3k_3 = 2t - 3t = -t.$

Putting these values in (1), we get

$-t\,v_1 - t\,v_2 + t\,v_3 = 0$

$\therefore\ v_3 = v_1 + v_2$

$\therefore\ (3, 4) = (1, 3) + (2, 1)$

$\therefore$ The vectors are dependent and $v_3 = v_1 + v_2$.

(iii) Geometrical : In the neighbouring Fig. 3.12, we have shown the three vectors. By the parallelogram of vectors we see that $v_3 = v_1 + v_2$.

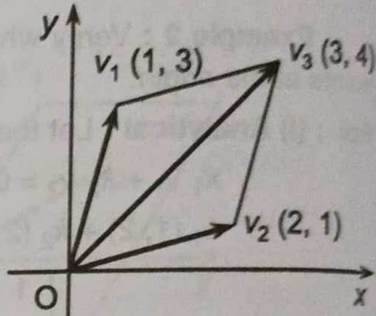


Fig. 3.12

$\therefore$ The vectors are linearly dependent.

Example 4 : If $v_1 = (4, 6, 8)$, $v_2 = (2, 3, 4)$, $v_3 = (-2, -3, -4)$ are three vectors with initial points at the origin, find whether they lie in the same line.

Sol. : (i) Analytical : Consider $k_1 v_1 + k_2 v_2 + k_3 v_3 = 0$ .... (1)

$\therefore\ k_1 (4, 6, 8) + k_2 (2, 3, 4) + k_3 (-2, -3, -4) = (0, 0, 0)$

$\therefore\ (4k_1 + 2k_2 - 2k_3) + (6k_1 + 3k_2 - 3k_3) + (8k_1 + 4k_2 - 4k_3) = (0, 0, 0)$

$\therefore$ $\begin{bmatrix} 4 & 2 & -2 \\ 6 & 3 & -3 \\ 8 & 4 & -4 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

By $\begin{matrix} (1/2)R_1 \\ (1/2)R_3 \end{matrix}$ $\begin{bmatrix} 2 & 1 & -1 \\ 6 & 3 & -3 \\ 4 & 2 & -2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

By $\begin{matrix} R_2 - 3R_1 \\ R_3 - 2R_2 \end{matrix}$ $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore\ 2k_1 + k_2 - k_3 = 0$

Hence, vectors are **not** independent.

If we put $k_1 = t_1$, $k_2 = t_2$, we get $k_3 = 2t_1 + t_2$.

Putting these values in (1), we get

$t_1 v_1 + t_2 v_2 - (2t_1 + t_2) v_3 = 0$

If we put $t_1 = 1$, $t_2 = -1$,

$$v_1 - v_2 + v_3 = 0 \quad i.e. \quad v_1 = v_2 - v_3$$

Even with a little close observation, we find the same relation

$$v_1 = v_2 - v_3.$$

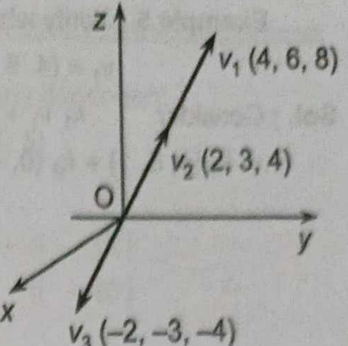


Fig. 3.13

(ii) Geometrical : If we place the vectors with initial points at the origin, we see that the vectors lie on the same line. Each can be expressed in terms of the others. (Fig. 3.13)

$\therefore$ They are **not** independent.

$\therefore$ They lie in the same line.

Example 5 : If $v_1 = (2, -2, 0)$, $v_2 = (4, 3, 0)$ and $v_3 = (6, 1, 3)$ are three vectors with initial points at the origin, find whether the vectors lie in the same plane.

Sol. : (i) Analytical : Consider

$$k_1 v_1 + k_2 v_2 + k_3 v_3 = 0 \qquad \text{.......... (1)}$$

$$\therefore \quad k_1 (2, -2, 0) + k_2 (4, 3, 0) + k_3 (6, 1, 3) = (0, 0, 0)$$

$$\therefore \quad \begin{bmatrix} 2 & 4 & 6 \\ -2 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \quad R_2 + R_1 \begin{bmatrix} 2 & 4 & 6 \\ 0 & 7 & 7 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad 2k_1 + 4k_2 + 6k_3 = 0, \quad 7k_2 + 7k_3 = 0, \quad 3k_3 = 0$$

$$\therefore \quad k_3 = 0, \ k_2 = 0 \ \therefore \ k_1 = 0.$$

$\therefore$ The vectors are linearly independent.

$\therefore$ They do not lie in the same plane.

(ii) Geometrical : We see that the z-ordinates of v_1, v_2 are zero. The points lie on the xy-plane. The z-coordinate of v_3 is 4. Hence, the point is not in the plane of v_1 and v_2. v_3 lies above the xy-plane. (Fig. 3.14)

$\therefore$ The vectors are linearly independent.

[Since the vectors v_1 and v_2 have z-coordinate zero, they lie in the xy-plane whose equation is $z = 0$.

But this equation is not satisfied by the point (6, 1, 3). Hence, v_3 does not lie in the plane of v_1, v_2.

$\therefore$ v_1, v_2, v_3 do not lie in the same plane.]

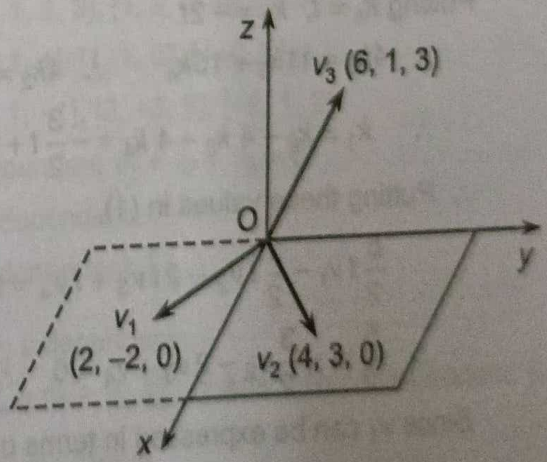


Fig. 3.14

Note ....

In examples 3 and 4 we have three vectors in R^3. Referring to the above theorem we see that when $r = n$, the vectors may be linearly independent or dependent.

Example 5 : Verify whether the following vectors are linearly independent.

$$v_1 = (4, 5, 1),\ v_2 = (0, -1, -1),\ v_3 = (3, 9, 4),\ v_4 = (-4, 4, 4).$$

Sol. : Consider $\quad k_1 v_1 + k_2 v_2 + k_3 v_3 + k_4 v_4 = 0 \quad$ .... (1)

$\therefore\ k_1 (4, 5, 1) + k_2 (0, -1, -1) + k_3 (3, 9, 4) + k_4 (-4, 4, 4) = (0, 0, 0)$

$$\therefore \quad \begin{bmatrix} 4 & 0 & 3 & -4 \\ 5 & -1 & 9 & 4 \\ 1 & -1 & 4 & 4 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \quad R_{31} \begin{bmatrix} 1 & -1 & 4 & 4 \\ 5 & -1 & 9 & 4 \\ 4 & 0 & 3 & -4 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \quad \begin{matrix} R_2 - 5R_1 \\ R_3 - 4R_1 \end{matrix} \begin{bmatrix} 1 & -1 & 4 & 4 \\ 0 & 4 & -11 & -16 \\ 0 & 4 & -13 & -20 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \quad R_3 - R_2 \begin{bmatrix} 1 & -1 & 4 & 4 \\ 0 & 4 & -11 & -16 \\ 0 & 0 & -2 & -4 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad k_1 - k_2 + 4k_3 + 4k_4 = 0$$

$$4k_2 - 11k_3 - 16k_4 = 0$$

$$-2k_3 - 4k_4 = 0$$

$\therefore\ k_3 = -2k_4.$

Putting $k_4 = t,\ k_3 = -2t$

$\therefore\ 4k_2 = 11k_3 + 16k_4 \qquad \therefore\ 4k_2 = -22t + 16t = -6t \qquad \therefore\ k_2 = -(3/2)\,t$

$$\therefore \quad k_1 = k_2 - 4k_3 - 4k_4 = -\frac{3}{2}t + 8t - 4t = \frac{5}{2}t.$$

$\therefore$ Putting these values in (1),

$$\frac{5}{2}tv_1 - \frac{3}{2}tv_2 - 2tv_3 + tv_4 = 0$$

$$\therefore \quad \frac{5}{2}v_1 - \frac{3}{2}v_2 - 2v_3 + v_4 = 0 \quad \therefore \quad \frac{5}{2}v_1 = \frac{3}{2}v_2 + 2v_3 - v_4.$$

Since v_1 can be expressed in terms of v_2, v_3 and v_4 they are linearly dependent.

Note ....

We have here 4 vectors in the space whose dimension is 3, by the above theorem the vectors are linearly dependent.

10. Assume that the vectors v_1, v_2, v_3 in R^3 have their initial points at the origin. Determine whether the three vectors lie in a plane.

(i) $v_1 = (2, -2, 0)$, $v_2 = (6, 1, 4)$, $v_3 = (2, 0, -6)$

(ii) $v_1 = (-6, 7, 2)$, $v_2 = (3, 2, 4)$, $v_3 = (4, -1, 2)$ **[Ans. : (i) No, (ii) Yes]**

11. If the vectors $v_1 = (1, 2, 3)$, $v_2 = (-2, -4, -6)$ and $v_3 = (3, 6, 9)$ have the initial points at the origin, do they lie on the same line. **[Ans. : Yes]**

12. Verify whether the following vectors with initial points at the origin lie on the same line. $(1, 2), (2, 4), (-1, -2), (3, 6)$. **[Ans. : Yes]**

13. Assume that v_1, v_2, v_3 are three vectors in R^3 which have the initial points at the origin. Determine whether the three vectors lie in the same line.

(i) $v_1 = (-1, 2, 3)$, $v_2 = (-3, 6, 0)$, $v_3 = (2, -4, 3)$

(ii) $v_1 = (2, -1, 4)$, $v_2 = (2, 7, -6)$, $v_3 = (4, 2, 3)$

(iii) $v_1 = (2, -1, 0)$, $v_2 = (1, 2, 5)$, $v_3 = (7, -1, 5)$

(iv) $v_1 = (1, 3, -1)$, $v_2 = (5, 7, -3)$, $v_3 = (1, -1, 0)$

[Ans. : (i) Yes, (ii) Yes, (iii) Yes; $3v_1 + v_2 = v_3$, (iv) Yes; $3v_1 + 2v_3 = v_2$]

14. Check whether the following sets of matrices are linearly independent.

(i) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

(ii) $\begin{bmatrix} 3 & -1 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -5 \\ -4 & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & 3 \\ -4 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ -4 & 3 \end{bmatrix}$

[Ans. : (i) Yes, (ii) No; $v_1 = 2v_2 + v_3$, (iii) No; $v_1 + v_2 = v_3$]

14. Linear Independence of Functions of *x*

Linear dependence or independence of a given set of functions in some cases can be determined more conveniently by using the following theorem.

Theorem : If the n functions $f_1, f_2, \ldots, f_n$ have $(n-1)$ continuous derivatives in $(-\infty, \infty)$ and if the determinant, called **Wronskian is not identically zero** in $(-\infty, \infty)$ then these functions are linearly independent in $C^{n-1}(-\infty, \infty)$ where Wronskian denoted by $W(x)$ is defined by

$$W(x) = \begin{vmatrix} f_1 & f_2 & \ldots & f_n \\ f_1' & f_2' & \ldots & f_n' \\ \ldots & \ldots & \ldots & \ldots \\ f_1^{n-1} & f_2^{n-1} & \ldots & f_n^{n-1} \end{vmatrix}$$

Note ....

The converse of the above theorem is **not** true. If the Wronskian is zero, the set may be independent or may not be independent. See Ex. 6.

Jozef Mavia Hoene-Wronski (1776-1853)

He was a Polish-French mathematician and philosopher. He received his early education in Poznan and Warsaw, Poland. He served as an arfillary officer in the Prussian army. He studied philosophy at various German Universities. He became French citizen in 1800 and settled in Paris. His research in analysis lead to some controversial mathematical papers. Much of his mathematical work was found to have errors and imprecision but it contained valuable isolated results and ideas. Some writers think that he exaggerated the importance of his own work.

Example 1 : Use suitable trigonometric identities to determine whether the following sets of functions are linearly independent and find the relation between them.

(i) $8,\ 4\sin^2 x,\ 2\cos^2 x,$ (ii) $\cos 2x,\ \cos^2 x,\ \sin^2 x.$

Sol. : (i) Let $f_1 = 8$, $f_2 = 4\sin^2 x$, $f_3 = 2\cos^2 x$ and consider

$$\begin{aligned} f_1 - 2f_2 - 4f_3 &= 8 - 2(4\sin^2 x) - 4(2\cos^2 x) \\ &= 8 - 8(\sin^2 x + \cos^2 x) \\ &= 8 - 8 = 0 \end{aligned}$$

$\therefore\ f_1 = 2f_2 + 4f_3$

Since f_1 is expressed as a linear combination of f_2 and f_3.

The functions are linearly dependent.

(ii) Let $f_1 = \cos 2x$, $f_2 = \cos^2 x$, $f_3 = \sin^2 x$ and consider

$$\begin{aligned} f_1 - f_2 - f_3 &= \cos 2x - \cos^2 x - \sin^2 x \\ &= \cos^2 x + \sin^2 x - \cos^2 x - \sin^2 x \\ &= 0 \end{aligned}$$

Since $f_1 = f_2 + f_3$, the functions are not linear independent.

Example 2 : Use suitable algebraic or trigonometric relations to show that the following sets of functions are not linearly independent. If not find the relations between them.

(i) $1,\ 1+x,\ 1-x,\ x^2$ (ii) $1,\ \sin^2 x,\ \cos^2 x$

(iii) $1,\ \sin^2 x,\ \cos 2x,$ (iv) $1,\ 1+2x,\ 2-3x,\ x^2$

Sol. : (i) Not independent.

$x^2 + (1-x)(1+x) - 1 = 0$ $\therefore\ x^2 + 1 - x^2 - 1 = 0$

$f_4 + f_3 \cdot f_2 - f_1 = 0$ $\therefore\ f_1 = f_2 \cdot f_3 + f_4$

(ii) Not independent.

$-1\cdot(1) + 1\cdot\sin^2 x + 1\cdot\cos^2 x = 0$

$-1\cdot f_1 + 1\cdot f_2 + 1\cdot f_3 = 0$ $\therefore\ f_1 = f_2 + f_3$

(iii) Not independent

$1 - 2\cdot\sin^2 x - \cos 2x = 0$

$1 f_1 - 2f_2 - f_3 = 0$ $\therefore\ f_1 = 2f_2 + f_3$

(iv) Not independent

$$6x^2 + (2-3x)(1+2x) - [(2-3x) + (1+2x)] + 1$$
$$= 6x^2 + (2 - x - 6x^2) - (3 - x) + 1 = 0$$
$$6f_4 + f_3 \cdot f_2 - (f_3 + f_2) + f_1 = 0.$$

Example 3 : Is the following set of functions linearly independent

$$6 - x^2,\ 1 + x + x^2 ?$$

Sol. : The Wronskian is

$$W(x) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \begin{vmatrix} 6-x^2 & 1+x+x^2 \\ -2x & 1+2x \end{vmatrix}$$
$$= (6-x^2)(1+2x) - (-2x)(1+x+x^2)$$
$$W(x) = 6 + 12x - x^2 - 2x^3 + 2x + 2x^2 + 2x^3$$
$$= 6 + 14x + x^2$$

But $x^2 + 14x + 6$ is **not** identically **zero.**

Hence, the set is **independent.**

Example 4 : Show that $f_1 = x$, $f_2 = \sin x$ are linearly independent in $C'(-\infty, \infty)$.

Sol. : The Wronskian is

$$W(x) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \begin{vmatrix} x & \sin x \\ 1 & \cos x \end{vmatrix} = x\cos x - \sin x$$

But $x \cos x - \sin x$ is **not** equal to **zero** for all x in $(-\infty, \infty)$.

This is so because, for instance

$$\text{if } x = \frac{\pi}{4}, \text{ then } x\cos x - \sin x = \frac{\pi}{4} \cdot \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \neq 0$$

Hence, x, $\sin x$ are independent. Alternatively, we may argue as follows. For linear dependence we should have

$$k_1 x + k_2 \sin x = 0 \quad i.e. \ \sin x = -\frac{k_1}{k_2} x = \lambda x$$

We cannot find x (except zero) for which $\sin x$ is λx. Hence, the set is independent.

Example 5 : Show that $f_1 = 1,\ f_2 = e^x,\ f_3 = e^{2x}$ form a linearly independent set of vectors in $C^2(-\infty, \infty)$.

Sol. : The Wronskian is

$$W(x) = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix} = \begin{vmatrix} 1 & e^x & e^{2x} \\ 0 & e^x & 2e^{2x} \\ 0 & e^x & 4e^{2x} \end{vmatrix}$$
$$= 4e^x \cdot e^{2x} - e^x \cdot 2e^{2x} = 2e^{3x}.$$

This fuhction is **not** equal to **zero** for all x (in fact for any x) in $(-\infty, \infty)$.

Hence, the functions are linearly **independent.**

Example 6 : Check whether $S = \{\sin(x+1), \sin x, \cos x\}$ is linearly independent in $C^2(0, \infty)$.

Sol. : The Wronskian is

$$W(x) = \begin{vmatrix} \sin(x+1) & \sin x & \cos x \\ \cos(x+1) & \cos x & -\sin x \\ -\sin(x+1) & -\sin x & -\cos x \end{vmatrix}$$

By $\quad R_3 + R_1 \begin{vmatrix} \sin(x+1) & \sin x & \cos x \\ \cos(x+1) & \cos x & -\sin x \\ 0 & 0 & 0 \end{vmatrix}$

$\therefore \quad W(x) = 0$ for all x in $(0, \infty)$. $\quad \therefore \; W(x)$ does not help us.

Now, $\sin(x+1) = \sin x \cos 1 + \cos x \sin 1$

$\therefore \quad f_1 = \cos 1 \cdot f_2 + \sin 1 \cdot f_3$

$\therefore$ The functions are **not** linearly independent.

Example 7 : Show that the set $S = \{e^x, x e^x, x^2 e^x\}$ is linearly independent in $C^2(-\infty, \infty)$.

Sol. : The Wronskian is

$$W(x) = \begin{vmatrix} e^x & x e^x & x^2 e^x \\ e^x & (x+1) e^x & (x^2+2x) e^x \\ e^x & (x+2) e^x & (x^2+4x+2) e^x \end{vmatrix} = e^x \begin{vmatrix} 1 & x & x^2 \\ 1 & x+1 & x^2+2x \\ 1 & x+2 & x^2+4x+2 \end{vmatrix}$$

By $R_2 - R_1$ and $R_3 - R_2$

$$W(x) = e^x \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 1 & 2x+2 \end{vmatrix} = e^x(2x+2-2x) = 2e^x$$

But e^x is **not zero** for all x (in fact for any x) in $(-\infty, \infty)$.

Hence, the set is linearly **independent**.

Example 8 : Determine whether the given set of vectors

$S = \{1 + x, x + x^2, 1 + x^2\}$ is linearly independent.

Sol. : The Wronskian is

$$W(x) = \begin{vmatrix} 1+x & x+x^2 & 1+x^2 \\ 1 & 1+2x & 2x \\ 0 & 2 & 2 \end{vmatrix}$$

By $C_2 - C_3$, $\quad W(x) = \begin{vmatrix} 1+x & x-1 & 1+x^2 \\ 1 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix}$

Expanding the determinant by the last row

$$2[1(1+x) - 1(x-1)] = 2(1+1) = 4 \neq 0$$

Hence, the set is linearly **independent**.

15. Bases and Dimensions

In plane analytic geometry we have learnt that any point in the plane is associated with a pair of ordered numbers (a, b) called its coordinates. Conversely to every pair of ordered pair (a, b), there corresponds a point in the plane. We describe this by saying that there is **one-to-one correspondence** between the points in the plane and ordered pairs of real numbers. Although we generally use rectangular coordinate system, we may as well use oblique coordinate system *i.e.* system given by two non-parallel lines. Coordinates of a point in oblique coordinate system are obtained by drawing lines through P parallel to the axes (and not perpendicular to the axes). Similarly, in R^3 also we may have rectangular system or oblique coordinate system defined by three non-coplanar coordinate axes.

We shall soon extend the concept of coordinate system from R^2 and R^3 to R^n. For this we shall talk of vectors rather than coordinates. Instead of referring to the point P as $P(a, b)$, we shall denote it as a vector in terms of vectors of unit length in the directions of the axes. We have,

$$OP = av_1 + bv_2$$

where v_1 and v_2 are unit vectors in the positive direction of the axes. Thus, we have expressed a vector OP in R^2 as a **linear combination** of unit vectors v_1, v_2. Similarly, a vector OP in R^3 can be expressed as a linear combination of three vectors v_1, v_2, v_3 in the positive direction of the axes as

$$OP = av_1 + bv_2 + cv_3.$$

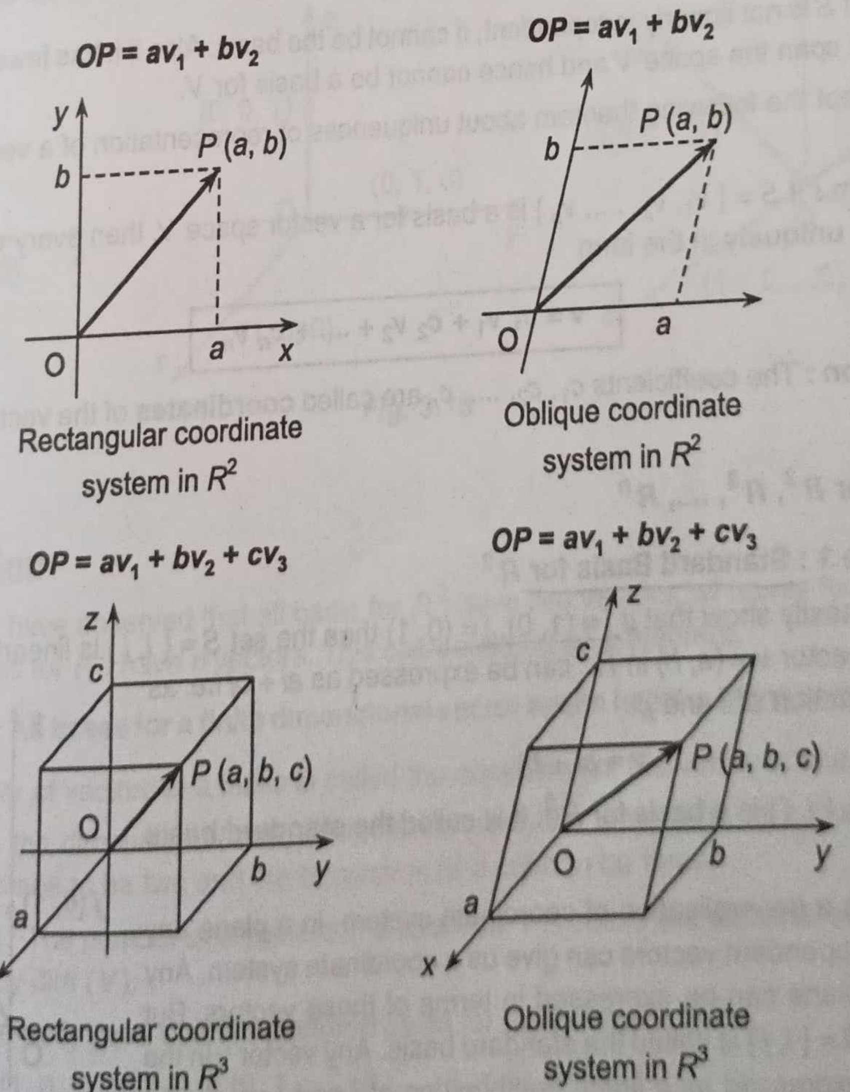


Fig. 3.15

The vectors that specify the coordinate system are called **"basis vectors"**. Any set of vectors can be basis vectors all that we need is that (i) they are linearly independent and (ii) they span the space V.

Although generally basis vectors of length 1 unit are used, it is not always very convenient. For instance, we may use 100 cms on one axes for length and 100 degrees for temperature on the other axis.

We also note that the x and y axes are linearly independent and they span the plane. Also the three coordinate axes x, y, z in R^3 are linearly independent and they span the spare of R^3.

We can now generalise the concept of coordinate system to general vector spaces.

16. Basis

Definition : If V is any vector space and $S = \{ v_1, v_2, ..., v_3 \}$ is a set of vectors in V then S is called a **basis** for V if the following two conditions hold good.

(1) **S is linearly independent,**

(2) **S spans V.**

The definition lays down two conditions for basis.

(i) The set $S = \{ v_1, v_2, ..., v_n \}$ must be linearly independent.

(ii) The set S spans the vector space V.

If the set S is not linearly independent, it cannot be the basis. Also if it has fewer than n vectors then it cannot span the space V and hence cannot be a basis for V.

We accept the following theorem about uniqueness of representation of a vector in V without proof.

Theorem : If $S = \{ v_1, v_2, ..., v_n \}$ is a basis for a vector space V, then every vector v in V can be expressed **uniquely** in the form

$$\boxed{v = c_1 v_1 + c_2 v_2 + ... + c_n v_n}$$

Definition : The coefficients $c_1, c_2, ..., c_n$ are called **coordinates** of the vector v with respect to the basis.

(a) Basis for $R^2, R^3, ..., R^n$

Example 1 : Standard Basis for R^2

We can easily show that if $i = (1, 0)$, $j = (0, 1)$ then the set $S = \{ i, j \}$ is linearly independent in R^2. Also any vector $v = (a, b)$ in R^2 can be expressed as $ai + bj$ *i.e.* as a linear combination of i and j.

$$v = ai + bj$$

Thus, $S = \{ i, j \}$ is a basis for R^2. It is called the **standard basis** for R^2.

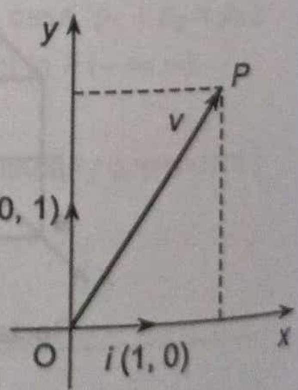


Fig. 3.16

A basis is a generalisation of coordinate system. In a plane any two linearly independent vectors can give us a coordinate system. Any vector in that plane can be expressed in terms of these vectors. But the above set $S = \{ i, j \}$ is called the standard basis. Any vector v in the plane can be expressed as a linear combination of i and j as $ai + bj$. The coefficients (a, b) are called the coordinates of the point P.

Example 2 : Standard Basis for R^3

We can also show that if

$i = (1, 0, 0)$, $j = (0, 1, 0)$, $k = (0, 0, 1)$

then the set $S = \{i, j, k\}$ is linearly independent in R^3. Also any vector $v = (a, b, c)$ in R^3 can be expressed as $ai + bj + ck$ i.e. as a linear combination of i, j and k.

$$v = ai + bj + ck$$

Thus, $S = \{i, j, k\}$ is a basis for R^3. It is called the **standard basis** for R^3.

Any vector v can be expressed as $ai + bj + ck$.

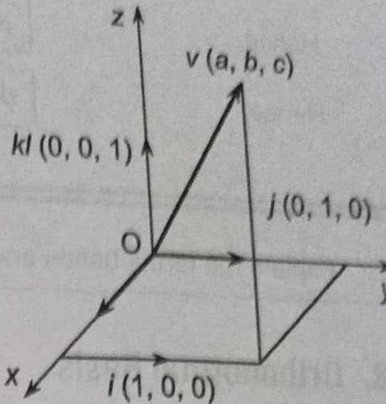


Fig. 3.17

Example 3 : Standard Basis for R^n

We can show that if $e_1 = (1, 0, ..., 0)$, $e_2 = (0, 1, ..., 0)$, ..., $e_n = (0, 0, ..., 1)$ then the set $S = \{e_1, e_2, ..., e_n\}$ is linearly independent. Also any vector $v = (v_1, v_2, ..., v_n)$ can be expressed as a linear combination of $e_1, e_2, ..., e_n$ i.e. as

$$v = v_1 e_1 + v_2 e_2 + ... + v_n e_n$$

Thus, $S = \{e_1, e_2, ..., e_n\}$ is a basis for R^n. It is called the **standard basis** for R^n.

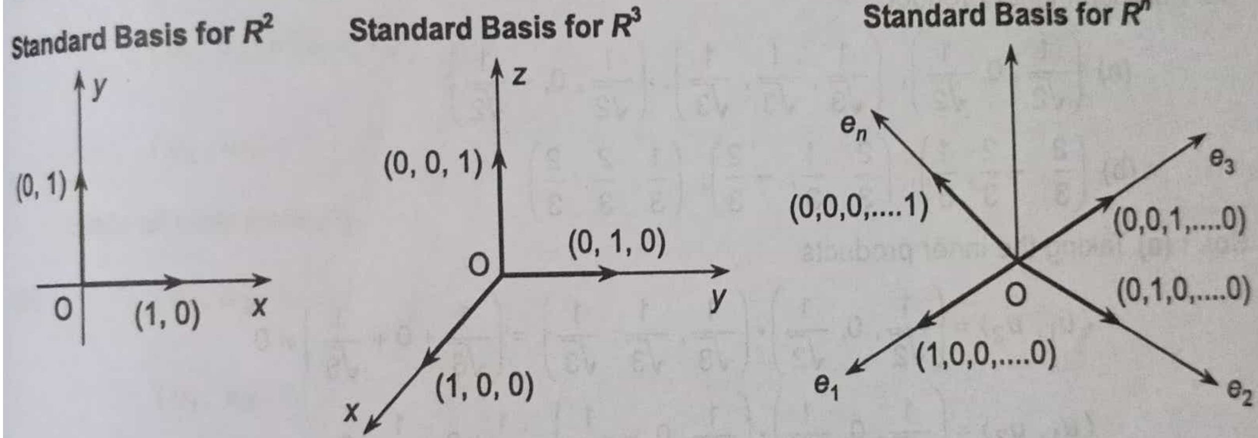


Fig. 3.18

17. Dimension

You might have observed that all basis for R^2 have two vectors, all bases for R^3 have three vectors, all bases for R^n have n vectors. This can be stated as a theorem.

Theorem : All bases for a finite dimensional vector space has the same number of vectors.

The number of vectors in a basis is called the dimension of the vector space.

We define the dimension of zero vector to be zero, the dimension of a line to be one, the dimension of a plane to be two and the dimension of a cube to be three.

Definition : The number of a vectors in a basis for V is called the **dimension** of the vector V and is denoted by dim (V).

We have already seen that the standard basis for vector space R^n is

$$e_1 = (1, 0, ..., 0), \; e_2 = (0, 1, ..., 0), \; ...., \; e_n = (0, 0, ..., 1)$$

Hence, $\boxed{\dim(R^n) = n}$

Hence, $\boxed{\dim(P_n) = n+1}$

EXERCISE - VII

Explain the terms bases and dimensions of a vector space.

18. Orthonormal Basis

In this section we shall learn orthonormal vectors, orthonormal basis. Then we shall learn how to construct orthogonal basis from an ordinary basis of an inner product space by using Gram-Schmidt process.

Definition : A set of vectors in an inner product space is called an **orthogonal set** if all the distinct vectors are orthogonal. If the norm of each vector in the orthogonal set is 1, then it is called **orthonormal set.**

Example 1 : Check whether the following sets of vectors in R^3 are orthogonal with respect to the Euclidean inner product.

(a) $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$

(b) $\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

Sol. : (a) Taking the inner products

$$\langle u_1, u_2\rangle = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\cdot\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \left(\frac{1}{\sqrt{6}} + 0 + \frac{1}{\sqrt{6}}\right) \neq 0$$

$$\langle u_1, u_3\rangle = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\cdot\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right) = \frac{1}{2} + 0 - \frac{1}{2} = 0$$

$$\langle u_2, u_3\rangle = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)\cdot\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{6}} + 0 - \frac{1}{\sqrt{6}} = 0$$

Since $\langle u_1, u_2\rangle \neq 0$, the vectors are not orthogonal.

(b)

$$\langle u_1, u_2\rangle = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)\cdot\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right) = \frac{4}{3} - \frac{2}{3} - \frac{2}{3} = 0$$

$$\langle u_1, u_3\rangle = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)\cdot\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) = \frac{2}{3} - \frac{4}{3} + \frac{2}{3} = 0$$

$$\langle u_2, u_3\rangle = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)\cdot\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) = \frac{2}{3} + \frac{2}{3} - \frac{4}{3} = 0$$

Since all inner products are zero, the vectors are orthogonal.

Example 2 : Check whether the above vectors are orthonormal.

Sol. : (a) Since the vectors are not orthogonal, they are not orthonormal.

(b) $\|u_1\| = \sqrt{\frac{4}{9}+\frac{4}{9}+\frac{1}{9}} = 1$; $\|u_2\| = \sqrt{\frac{4}{9}+\frac{1}{9}+\frac{4}{9}} = 1$;

$\|u_3\| = \sqrt{\frac{1}{9}+\frac{4}{9}+\frac{4}{9}} = 1.$

Since the vectors u_1, u_2, u_3 are orthogonal and their norms are 1, they are orthonormal.

Example 3 : Which of the following sets are orthonormal with respect to the inner product P_2 defined by

$$\langle p, q\rangle = a_0 b_0 + a_1 b_1 + a_2 b_2 \quad \text{where } p = a_0 + a_1 x + a_2 x^2,\ q = b_0 + b_1 x + b_2 x^2 ?$$

(a) $x,\ \frac{1}{2}+x^2,\ 1-x-\frac{1}{2}x^2$

(b) $\frac{2}{3}-\frac{2}{3}x+\frac{1}{3}x^2,\ \frac{2}{3}+\frac{1}{3}x-\frac{2}{3}x^2,\ \frac{1}{3}+\frac{2}{3}x+\frac{2}{3}x^2$

Sol. : (a) We have

$$\langle u_1, u_2\rangle = (0, 1, 0)\cdot\left(\frac{1}{2}, 0, 1\right) = 0$$

$$\langle u_1, u_3\rangle = (0, 1, 0)\cdot\left(1, -1, -\frac{1}{2}\right) = -1 \neq 0$$

$$\langle u_2, u_3\rangle = \left(\frac{1}{2}, 0, 1\right)\cdot\left(1, -1, -\frac{1}{2}\right) = \frac{1}{2}+0-\frac{1}{2} = 0$$

Since all inner products are not zero, the vectors are not orthogonal and hence not orthonormal.

(b)

$$\langle u_1, u_2\rangle = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)\cdot\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right) = \frac{4}{9}-\frac{2}{9}-\frac{2}{9} = 0$$

$$\langle u_1, u_3\rangle = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)\cdot\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) = \frac{2}{9}-\frac{4}{9}+\frac{2}{9} = 0$$

$$\langle u_2, u_3\rangle = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)\cdot\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) = \frac{2}{9}+\frac{2}{9}-\frac{4}{9} = 0$$

$\therefore$ Vectors are orthogonal.

Further, $\|u_1\| = \sqrt{\frac{4}{9}+\frac{4}{9}+\frac{1}{9}} = 1$; $\|u_2\| = \sqrt{\frac{4}{9}+\frac{1}{9}+\frac{4}{9}} = 1$;

$\|u_3\| = \sqrt{\frac{1}{9}+\frac{4}{9}+\frac{4}{9}} = 1.$

Since the vectors are orthogonal and have norms 1, they are orthonormal.

[Note that this is the same example as Ex. 1 (b).]

Example 4 : Check whether the following vectors are orthogonal with respect to the inner product of two matrices $u = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}$, $v = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$ defined by $\langle u, v\rangle = \text{tr}\,(v'u)$.

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2/3 \\ 2/3 & -2/3 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 2/3 \\ -2/3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2/3 \\ 2/3 & 2/3 \end{bmatrix}$

Sol. : (a) We know that

$$\langle u, v\rangle = a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4$$

$\therefore \quad \langle u, v\rangle = 0+0+0+0 = 0 \qquad \therefore \quad u, v$ are orthogonal.

(b) $\quad \langle u, v\rangle = 0 + \frac{4}{9} - \frac{4}{9} + 0 = 0 \qquad \therefore \quad u, v$ are orthogonal.

Example 5 : Verify that $v_1 = \left(-\frac{3}{5}, \frac{4}{5}, 0\right), \quad v_2 = \left(\frac{4}{5}, \frac{3}{5}, 0\right), \quad v_3 = (0, 0, 1)$ are orthonormal and if $u = (1, -1, 2)$ then express u as a linear combination of v_1, v_2, v_3.

Sol. : We have $\langle v_1, v_2\rangle = -\frac{12}{5} + \frac{12}{5} + 0 = 0;$

$$\langle v_1, v_3\rangle = 0+0+0 = 0; \qquad \langle v_2, v_3\rangle = 0+0+0 = 0.$$

Hence, v_1, v_2, v_3 are orthogonal.

Further, $\quad ||v_1|| = \sqrt{\frac{9}{25} + \frac{16}{25} + 0} = 1; \qquad ||u_2|| = \sqrt{\frac{16}{25} + \frac{9}{25} + 0} = 1;$

$$||u_3|| = \sqrt{0+0+1} = 1.$$

Since, the vectors are orthogonal and of norms one, they are orthonormal.

Now, $\boxed{u = \langle u, v_1\rangle v_1 + \langle u, v_2\rangle v_2 + \langle u, v_3\rangle v_3}$

$$= \left[(1, -1, 2) \cdot \left(-\frac{3}{5}, \frac{4}{5}, 0\right)\right] v_1 + \left[(1, -1, 2) \cdot \left(\frac{4}{5}, \frac{3}{5}, 0\right)\right] v_2 + [(1, -1, 2) \cdot (0, 0, 1)] v_3$$

$$= \left(-\frac{3}{5} - \frac{4}{5} + 0\right) v_1 + \left(\frac{4}{5} - \frac{3}{5} + 0\right) v_2 + (0+0+2) v_3$$

$$= -\frac{7}{5} v_1 + \frac{1}{5} v_2 + 2v_3.$$

(You can verify the equality.)

Example 6 : Find the coordinate vector of u with respect to the orthonormal basis $\{v_1, v_2, v_3\}$ where

$$u = (-1, 0, 2), \quad v_1 = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right), \quad v_2 = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right), \quad v_3 = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

Sol. : We have already checked that the set $\{v_1, v_2, v_3\}$ is an orthonormal set [Ex. 3 (b), page 3-50]

Now, $\boxed{u = \langle u, v_1\rangle v_1 + \langle u, v_2\rangle v_2 + \langle u, v_3\rangle v_3}$

$$\therefore \; u = \left[(-1, 0, 2) \cdot \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)\right] v_1 + \left[(-1, 0, 2) \cdot \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)\right] v_2 + \left[(-1, 0, 2) \cdot \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)\right] v_3$$

$$= 0\, v_1 + \left(-\frac{2}{3} - \frac{4}{3}\right) v_2 + \left(-\frac{1}{3} + \frac{4}{3}\right) v_3$$

$$= 0\, v_1 - 2 v_2 + v_3$$

$$\therefore \; (u)_s = (0, -2, 1).$$

... sets of vectors are orthogonal with respect to the Euclidean

(a) $(0, 1), (2, 0)$

(b) $\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}\right)$

(c) $(1, 0, 0), (0, 1, 0), (0, 0, 1)$

(d) $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0\right)$

[**Ans.** : All]

2. Check whether the following set of polynomials is orthogonal with respect to the inner product in P_2.

$$1, \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}x^2, x^2$$

[**Ans.** : No]

3. Check whether the following sets of matrices are orthogonal with respect to the inner product on M_{22}.

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2/3 \\ 1/3 & -2/3 \end{bmatrix}, \begin{bmatrix} 0 & 2/3 \\ -2/3 & 1/3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

[**Ans.** : Both]

4. Verify that the vectors $v_1 = \left(-\frac{3}{5}, \frac{4}{5}, 0\right)$, $v_2 = \left(\frac{4}{5}, \frac{3}{5}, 0\right)$, $v_3 = (0, 0, 1)$ form an orthonormal basis in R^3 with respect to the Euclidean inner product. Express the vector $(3, -7, 4)$ as a linear combination of v_1, v_2, v_3.

[**Ans.** : $\left(-\frac{37}{5}v_1 - \frac{9}{5}v_2 + 4v_3\right)$

5. Check whether the following vectors are orthonormal and then find the coordinates of w with respect to u_1, u_2.

$u_1 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, $u_2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $w = (3, 7)$ [**Ans.** : $(w)_s = \left(-2\sqrt{2}, 5\sqrt{2}\right)$]

6. If $v_1 = (0, 1, 0)$, $v_2 = \left(-\frac{4}{5}, 0, \frac{3}{5}\right)$, $v_3 = \left(\frac{3}{5}, 0, \frac{4}{5}\right)$, show that the vectors are orthonormal. Express $u = (1, 1, 1)$ as a linear combination of v_1, v_2, v_3 and find the coordinate vectors of u with respect to $S = \{v_1, v_2, v_3\}$.

[**Ans.** : $u = 1 \cdot u_1 - \frac{1}{5}u_2 + \frac{7}{5}u_3$, $(u)_s = \left(1, -\frac{1}{5}, \frac{7}{5}\right)$

19. Projection Theorem

We now state the following theorem known as projection theorem.

Projection Theorem

If W is a finite dimensional subspace of an inner product space V, then every vector u in V can be expressed uniquely as

$$\boxed{u = w_1 + w_2}$$

where w_1 is in W and w_2 is in $W^{\perp}$.

($W^{\perp}$ read as "W perp" denotes the orthogonal complement of W.)

Proof : We shall accept this theorem without proof.

Example 1 : Let $W = \text{span}\left\{(0, 1, 0), \left(-\frac{4}{5}, 0, \frac{3}{5}\right)\right\}$.

Express, $w = (1, 1, 1)$ in the form $w = w_1 + w_2$ where $w_1 \in W$ and $w_2 \in W^{\perp}$.

Sol. : Let $v_1 = (0, 1, 0)$, $v_2 = \left(-\frac{4}{5}, 0, \frac{3}{5}\right)$.

Then projection of w on W

$$\text{proj}_W w = \langle w, v_1\rangle v_1 + \langle w, v_2\rangle v_2$$

$$= [(1, 1, 1) \cdot (0, 1, 0)]\, v_1 + (1, 1, 1) \cdot \left(-\frac{4}{5}, 0, \frac{3}{5}\right) v_2$$

$$\text{proj}_W w = 1 \cdot v_1 - \frac{1}{5} v_2 = 1(0, 1, 0) - \frac{1}{5}\left(-\frac{4}{5}, 0, \frac{3}{5}\right)$$

$$w_1 = \left(\frac{4}{25}, 1, -\frac{3}{25}\right)$$

This is the component of w which is in W.

The component of w orthogonal to W is

$$\text{proj}_{W^{\perp}} w = w - \text{proj}_W w = (1, 1, 1) - \left(\frac{4}{25}, 1, -\frac{3}{25}\right)$$

$$w_2 = \left(\frac{21}{25}, 0, \frac{28}{25}\right)$$

This is the component of w which is in $W^{\perp}$.

Hence, $w = w_1 + w_2$ (w_1 is in W and w_2 is in $W^{\perp}$)

$$= \left(\frac{4}{25}, 1, -\frac{3}{25}\right) + \left(\frac{21}{25}, 0, \frac{28}{25}\right)$$

(Verify that $w_1 + w_2 = (1, 1, 1)$.)

Example 2 : Let $W = \text{span}\left\{(0, 1, 0), \left(-\frac{4}{5}, 0, \frac{3}{5}\right)\right\}$.

Express $w = (1, 2, 3)$ in the form of $w = w_1 + w_2$, where $w_1 \in W$ and $w_2 \in W^{\perp}$.

Sol. : Let $v_1 = (0, 1, 0)$, $v_2 = \left(-\frac{4}{5}, 0, \frac{3}{5}\right)$.

Then the projection of w on W.

$$\text{proj}_W\, w = \langle w, v_1\rangle v_1 + \langle w, v_2\rangle v_2$$

$$= [(1, 2, 3) \cdot (0, 1, 0)]\, v_1 + \left[(1, 2, 3) \cdot \left(-\frac{4}{5}, 0, \frac{3}{5}\right)\right] v_2$$

$$= 2v_1 + 1 \cdot v_2 = 2(0, 1, 0) + 1 \cdot \left(-\frac{4}{5}, 0, \frac{3}{5}\right)$$

$$w_1 = \left(-\frac{4}{5}, 2, \frac{3}{5}\right)$$

This is the component of w in W.

The component of w orthogonal to W is

$$\text{proj}_{W^\perp} w = w - \text{proj}_W w = (1, 2, 3) - \left(-\frac{4}{5}, 2, \frac{3}{5}\right)$$

$$w_2 = \left(\frac{9}{5}, 0, \frac{12}{5}\right)$$

This is the component of w in $W^\perp$.

Hence, $w = w_1 + w_2$ (where $w_1 \in W$ and $w_2 \in W^\perp$)

$$= \left(-\frac{4}{5}, 2, \frac{3}{5}\right) + \left(\frac{9}{5}, 0, \frac{12}{5}\right).$$

Example 3 : Let R^4 have the Euclidean inner product. Express $w = (-1, 2, 6, 0)$ in the form $w = w_1 + w_2$ where $w_1 \in W$ and $w_2 \in W^\perp$ where w is spanned by $v_1 = (-1, 0, 1, 2)$ and $v_2 = (0, 1, 0, 1)$.

Sol. : The project of w on W

$$\text{proj}_W\, w = \langle w, v_1\rangle v_1 + \langle w, v_2\rangle v_2$$

$$\therefore \quad \text{proj}_W\, w = [(-1, 2, 6, 0) \cdot (-1, 0, 1, 2)]\, v_1 + [(-1, 2, 6, 0) \cdot (0, 1, 0, 1)]\, v_2$$

$$= 7v_1 + 2v_2$$

$$= 7(-1, 0, 1, 2) + 2(0, 1, 0, 1)$$

$$\therefore \quad w_1 = (-7, 2, 7, 16)$$

The component of w orthogonal to W is

$$\text{proj}_{W^\perp} w = w - \text{proj}_W w$$

$$= (-1, 2, 6, 0) - (-7, 2, 7, 16)$$

$$\therefore \quad w_2 = (6, 0, -1, -16)$$

$$\therefore \quad w = w_1 + w_2 \quad \text{where } w_1 \in W \text{ and } w_2 \in W^\perp$$

$$= (-7, 2, 7, 16) + (6, 0, -1, -16).$$

20. Gram-Schmidt Process

Gram-Schmidt process gives us a method of finding orthonormal vectors.

Let for convenience V be a non-zero inner product space in R^3 and let $u = \{u_1, u_2, u_3\}$ be any base for V. The following steps lead to an orthogonal basis $\{v_1, v_2, v_3\}$ for V.

Step 1 : Let $v_1 = u_1$.

Step 2 : As shown in the Fig. 4.3 (a), we obtain a vector v_2, orthogonal to v_1.

$v_2 = u_2 - \text{proj } u_2$ which is given by

$$\Rightarrow \quad \boxed{v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1}$$

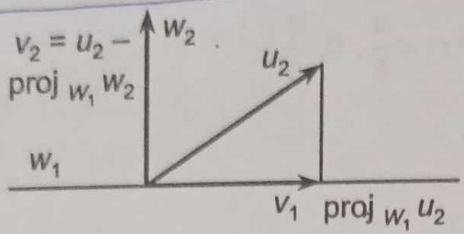


Fig. 3.19 (a)

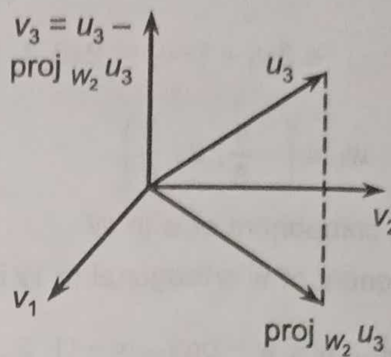


Fig. 3.19 (b)

Step 3 : To find v_3 that is orthogonal to both v_1 and v_2, we find u_3 orthogonal to the space w_2 spanned by v_1 and v_2 which is given by

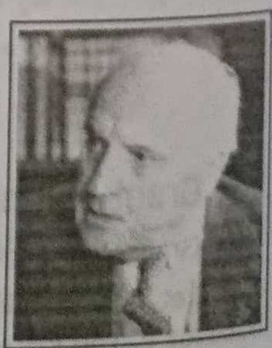
$$\Rightarrow \quad v_3 = v_3 - \text{proj } u_3 = \boxed{u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_2, v_2 \rangle}{\|v_2\|^2} v_2}$$

Thus, we get the orthogonal set $\{v_1, v_2, v_3\}$.

Jörgen Pederson Gram (1850-1916)

Jörgen Pederson Gram was an insurance expert in Denmark. He got his master's degree in mathematics with specialisation in the then new developing subject modern algebra. He then joined the Hafnia Life Insurance Company where he developed mathematical foundations of accident insurance. During this period he got his Ph.D. on "On Series Development Utilising the Least Square Methods". On this thesis Gram-Schmidt process is based. In 1910 he became director of the Danish (Denmark) Insurance Board. Gram eventually developed interest in number theory and won a gold medal from "The Royal Danish Society of Sciences And Letters". He had keen interest in the interplay between applied mathematics and pure mathematics, which led to four treatises on Danish forest management. Unfortunately he died in a an accident while going to attend a meeting of the Royal Danish Society.

Erhardt Schmidt (1876-1959)



Erhardt Schmidt was a reputed German mathematician. He got his Ph.D. degree from Göttingen University, Germany in 1905. He was a student of another giant German mathematician David Hilbert. He then went to teach at Berlin University in 1917 and stayed there for the rest of his life. He made important contributions to various mathematical fields. Schmidt first described 'Gram-Schmidt process' in a paper on integral equations published in 1907. He is credited to have developed the general concept "Hilbert Space" which is fundamental in the study of infinite-dimensional vector spaces.

Example 1 : Construct an orthonormal basis of R^2 by applying Gram-Schmidt orthogonalisation to $S = \{ (3, 1), (4, 2) \}$. **(M.U. 2014)**

Sol. : Let $u_1 = (3, 1)$ and $u_2 = (4, 2)$.

Step 1 : $v_1 = u_1 = (3, 1)$

Step 2 : $v_2 = u_2 - \text{proj. } u_2 = u_2 - \dfrac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1$

Now, $\quad \langle u_2, v_1 \rangle = (4, 2) \cdot (3, 1) = 14$

$\quad \| v_1 \|^2 = 9 + 1 = 10$

$$\therefore \quad v_2 = (4, 2) - \frac{14}{10}(3, 1) = \left(4 - \frac{7}{5} \cdot 3,\ 2 - \frac{7}{5} \cdot 1\right) = \left(-\frac{1}{5}, \frac{3}{5}\right)$$

Hence, $v_1 = (3, 1)$, $v_2 = \left(-\dfrac{1}{5}, \dfrac{3}{5}\right)$ form the orthogonal basis for R^2.

Now, the norms of these vectors are

$$\| v_1 \| = \sqrt{9 + 1} = \sqrt{10}$$

$$\| v_2 \| = \sqrt{\frac{1}{25} + \frac{9}{25}} = \sqrt{\frac{10}{25}} = \sqrt{\frac{2}{5}}$$

Hence, the orthonormal basis are

$$q_1 = \frac{v_1}{\|v_1\|} = \left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)$$

$$q_2 = \frac{v_2}{\|v_2\|} = \left(-\frac{1}{5} \cdot \sqrt{\frac{5}{2}},\ \frac{3}{5} \cdot \sqrt{\frac{5}{2}}\right) = \left(-\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$$

Example 2 : Find an orthonormal basis for the subspaces of R^3 by applying Gram-Schmidt process where $S = \{ (1, 2, 0), (0, 3, 1) \}$. **(M.U. 2014, 15)**

Sol. : Let $u_1 = (1, 2, 0)$ and $u_2 = (0, 3, 1)$.

Step 1 : $v_1 = u_1 = (1, 2, 0)$

Step 2 : $v_2 = u_2 - \text{proj. } u_2 = u_2 - \dfrac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1$

Now, $\langle u_2, v_1 \rangle = (0, 3, 1) \cdot (1, 2, 0) = 0 + 6 + 0 = 6$

and $\| v_1 \|^2 = 1 + 4 + 0 = 5$

$$\therefore \quad v_2 = (0, 3, 1) - \frac{6}{5}(1, 2, 0) = \left(-\frac{6}{5}, \frac{3}{5}, 1\right)$$

Norms of these vectors are

$$\| v_1 \| = \sqrt{5}, \ \| v_2 \| = \sqrt{\frac{36}{25} + \frac{9}{25} + 1} = \sqrt{\frac{70}{25}} = \sqrt{\frac{14}{5}}.$$

Hence, the orthonormal basis are

$$q_1 = \frac{v_1}{\|v_1\|} = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right)$$

$$q_2 = \frac{v_2}{\|v_2\|} = \sqrt{\frac{5}{14}}\left(-\frac{6}{5}, \frac{3}{5}, 1\right) = \left(-\frac{6}{\sqrt{70}}, \frac{3}{\sqrt{70}}, \frac{5}{\sqrt{70}}\right)$$