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Vector Spaces

Let V be a non Empty set of elements called as vectors and K, m be the scalars. If the set V satisfies following axioms then it is called as Vector space.

I) Axioms

a) Closure property

Let $X, Y, Z \in V$, K & l be scalars

$C_1 : X + Y \in V$

$C_2 : KX \in V$

b) Addition axioms

A_1: Commutative $X + Y = Y + X \in V$

A_2: Associative $(X+Y)+Z = X+(Y+Z)$

A_3: Existence of identity

$\exists\, 0 \in V$ such that $X + 0 = X$

$0 \in V$ is called as additive identity

A_4: Existence of inverse

$\exists\, -X \in V$ such that $X + (-X) = 0$

c) Scalar Multiplication

$M_1 : K(X+Y) = KX + KY \in V$

$M_2 : (K+l)X = KX + lX \in V$

$M_3 : (Kl)X = K(lX) \in V$

M_4: Existence of multiplicative identity

(There must exist) $\exists\, 1 \in V$ such that $1.X = X$

Q Let V be a set of positive real no. with Addition and scalar multiplication define as $X + Y = XY$ and $cX = X^c$ where $X, Y \in V$ and c be a scalar.

Ans To prove V is a Vector space we prove the following 10 axioms

$X + Y = XY$ and $cX = X^c$ ——— (i) where $X, Y, \in V$ and c be a scalar classmate

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Let $X, Y, Z \in V$ be vectors

K & l be the scalars

1) Closure axiom

$C_1 : X + Y = XY \in V$ $\quad \therefore xy \in R$

$C_2 : l . X = X^l \in V$ $\quad$ as $X^l \in$ real (R)

2) Addition axiom

A_1 : Commutative

$X + Y = XY = YX = Y + X \in V$

A_2 : Associative

$(X + Y) + Z = (XY) + Z = X(YZ) = X + (YZ) = X + (Y + Z)$

A_3 : Existence of identity

$\exists \; 1 \in V$ such that $\quad X + Y = XY$

$X + 1 = 1 . X = X$

A_4 : Existence of inverse

$X + \frac{1}{X} = X . \frac{1}{X} = 1$

$\therefore$ $1/X$ is the additive inverse in V

3) Scalar Multiplication

$M_1 : K(X + Y) = K(XY) = (XY)^K = X^K Y^K = X^K + Y^K$

$= KX + KY$

$\therefore K(X + Y) = KX + KY$

$M_2 : (K + l) X = X^{(K+L)} = X^K . X^L = X^K + X^L = KX + LX$

$M_3 : (Kl) X = X^{Kl} = (X^l)^K = K(X^l) = K(lX)$

M_4 : Multiplicative identity

$\exists \; 1 \in V$ such that

$1 . X = X^1 = X$

1 is multiplicative identity.

$\therefore$ V is a vector space.

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Q.2] Examine wheather the set of all real numbers (x, y) with operatio

$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ — (i)

$K(x_1, y_1) = (K^2 x_1, K^2 y_1)$

Ans. $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in V$ and

K and l be the scalars

1) Closure axiom

C_1 : $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \in V$

C_2 : $K(x_1, y_1) = (K^2 x_1, K^2 y_1) \in V$

2) Addition axiom

A_1 : Commutative

$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) = (x_2 + x_1, y_2 + y_1)$

$= (x_2, y_2) + (x_1, y_1)$

A_2 : Associative

$[(x_1, y_1) + (x_2, y_2)] + (x_3, y_3) = (x_1 + x_2, y_1 + y_2) + (x_3, y_3)$

$= (x_1 + x_2 + x_3, y_1 + y_2 + y_3)$

$(x_1, y_1) + [(x_2 + x_3, y_2 + y_3)] \Leftarrow [x_1 + (x_2 + x_3), y_1 + (y_2 + y_3)]$

$= (x_1, y_1) + [(x_2, y_2) + (x_3, y_3)]$

A_3 : Existence of identity

$\exists (0, 0) \in V$ such that

$(x_1, y_1) + (0, 0) = (x_1 + 0, y_1 + 0) = (x_1, y_1)$

A_4 : Existence of inverse

$\exists (-x_1, -y_1) \in V$ such that

$(x_1, y_1) + (-x_1, -y_1) = (0, 0)$

3) Scalar Multiplication Axiom

M_1 : $K[(x_1, y_1) + (x_2, y_2)] = K[(x_1 + x_2, y_1 + y_2)]$

$= [K^2(x_1 + x_2), K^2(y_1 + y_2)]$

$= [K^2 x_1 + K^2 x_2, K^2 y_1 + K^2 y_2]$

$= (K^2 x_1, K^2 y_1) + (K^2 x_2, K^2 y_2)$

$= K(x_1, y_1) + K(x_2, y_2)$

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$M_2 : (k+l)(x_1, y_1) = [(k+l)^2 x_1, (k+l)^2 y_1]$

$= [(k^2+2kl+l^2)x_1, (k^2+2kl+l^2)y_1]$

$\neq k(x_1, y_1) + l(x_1, y_1)$

$\therefore M_2$ fail

$M_3 : (kl)(x_1, y_1) = [(kl)^2 x_1, (kl)^2 y_1]$

$= (k^2 l^2 x_1, k^2 l^2 y_1)$

$= [k^2(l^2 x_1), k^2(l^2 y_1)] = k(l^2 x_1, l^2 y_1)$

$= k[l(x_1, y_1)]$

$M_4 : \exists\ 1 \in V$

$1.(x_1, y_1) = (1^2 x_1, 1^2 y_1) = (x_1, y_1)$

$\therefore$ V is not a vector space.

Q.3] Examine wheather the set of matrices of order 2x2 defined as $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ with usual addition and scalar multiplication is a vector space.

Ans. Let $V = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ such that $a, b \in R$

Let $X = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, Y = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}, Z = \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix} \in V$

K and L be scalar

i) Closure axiom

$C_1 : X + Y = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$

$= \begin{bmatrix} a+c & 0 \\ 0 & b+d \end{bmatrix} \in V$

$C_2 : KX = K\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} Ka & 0 \\ 0 & Kb \end{bmatrix} \in V$ $\quad Ka, Kb \in R$

ii) Addition axiom

A_1: Commutative

$X + Y = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} a+c & 0 \\ 0 & b+d \end{bmatrix} = \begin{bmatrix} c+a & 0 \\ 0 & d+b \end{bmatrix}$

$= \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = Y + X$

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A_2 : Associative

$$(X+Y)+Z = \left\{\begin{bmatrix} a & 0 \\ 0 & b\end{bmatrix} + \begin{bmatrix} c & 0 \\ 0 & d\end{bmatrix}\right\} + \begin{bmatrix} e & 0 \\ 0 & f\end{bmatrix}$$

$$= \begin{bmatrix} a+c & 0 \\ 0 & b+d\end{bmatrix} + \begin{bmatrix} e & 0 \\ 0 & f\end{bmatrix} = \begin{bmatrix} a+c+e & 0 \\ 0 & b+d+f\end{bmatrix} = \begin{bmatrix} a+(c+e) & 0 \\ 0 & b+(d+f)\end{bmatrix}$$

$$= \begin{bmatrix} a & 0 \\ 0 & b\end{bmatrix} + \begin{bmatrix} c+e & 0 \\ 0 & d+f\end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b\end{bmatrix} + \left\{\begin{bmatrix} c & 0 \\ 0 & d\end{bmatrix} + \begin{bmatrix} e & 0 \\ 0 & f\end{bmatrix}\right\}$$

$$= X + (Y+Z)$$

A_3 : Existence of identity

$\exists \begin{bmatrix} 0 & 0 \\ 0 & 0\end{bmatrix} \in V$ such that

$$\begin{bmatrix} a & 0 \\ 0 & b\end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0\end{bmatrix} = \begin{bmatrix} a+0 & 0 \\ 0 & b+0\end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b\end{bmatrix}$$

$\begin{bmatrix} 0 & 0 \\ 0 & 0\end{bmatrix}$ additive iden. $\in V$

A_4 :

$\exists \begin{bmatrix} -a & 0 \\ 0 & -b\end{bmatrix} \in V$ such that

$$\begin{bmatrix} a & 0 \\ 0 & b\end{bmatrix} + \begin{bmatrix} -a & 0 \\ 0 & -b\end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0\end{bmatrix}$$

iii) Scalar Multiplication Axiom

$$M_1 : K(X+Y) = K\left\{\begin{bmatrix} a & 0 \\ 0 & b\end{bmatrix} + \begin{bmatrix} c & 0 \\ 0 & d\end{bmatrix}\right\} = K\begin{bmatrix} a+c & 0 \\ 0 & b+d\end{bmatrix}$$

$$= \begin{bmatrix} K(a+c) & 0 \\ 0 & K(b+d)\end{bmatrix} = \begin{bmatrix} Ka+Kc & 0 \\ 0 & Kb+Kd\end{bmatrix}$$

$$= \begin{bmatrix} Ka & 0 \\ 0 & Kb\end{bmatrix} + \begin{bmatrix} Kc & 0 \\ 0 & Kd\end{bmatrix}$$

$$= KX + KY$$

M_2 : $(k+l)X$

$$= (k+l)\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} (k+l)a & 0 \\ 0 & (k+l)b \end{bmatrix} = \begin{bmatrix} ka+la & 0 \\ 0 & kb+lb \end{bmatrix}$$

$$= \begin{bmatrix} ka & 0 \\ 0 & kb \end{bmatrix} + \begin{bmatrix} la & 0 \\ 0 & lb \end{bmatrix} = kX + lX$$

M_3 : $(kl)X = (kl)\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} kla & 0 \\ 0 & klb \end{bmatrix} = \begin{bmatrix} k(la) & 0 \\ 0 & k(lb) \end{bmatrix}$

$$= k\begin{bmatrix} la & 0 \\ 0 & lb \end{bmatrix} = k\left\{ l\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \right\}$$

$$= k(lX)$$

M_4 : $\exists \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in V$ such that

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = X$$

$\therefore \quad 1 . X = X$

$\therefore$ V is a vector space.

* Subspaces.

A subset W of a vector space V is called subspace of V if W is itself a vector space under the addition and scalar multiplication define on V

Theorem :-

If W is non empty subset of a vector space V than W is called as subspace of V if

i) For every $(\forall)$ $u, v \in V$ such that $u + v \in W$

ii) $(\forall)$ $u \in V$ and k be a scalar such that $ku \in W$