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VECTOR SPACES

Let V be a non-empty be a non-empty set of elements called vectors and k, m be the scalars. If the set V satisfies the following axions then V is called as vector space.

$(x,y) \in R^2 \Rightarrow$ 2-dimensional

$(x,y,z) \in R^3 \Rightarrow$ 3-dimensional

$(x_1, x_2, \ldots, x_n) \Rightarrow$ n-dimensional

The axions for vector spaces are :-

(i) Closure Axion :-

~~C_1 :~~ Let $X, Y, Z \in V$ and k & l be the scalars.

$C_1 : (X+Y) \in V$

$C_2 : kX \in V$

(ii) Addition Axion :-

A_1 : Commutative

$(X+Y = Y+X) \in V$

A_2 :

$[(X+Y)+Z = X+(Y+Z)] \in V$

A_3 : Existence of Identity

There exists a $0 \in V$ such that

$X+0 \in V$

Therefore, $0 \in V$ is an additive identity

A_4 : Existence of Inverse

There exists $-X \in V$ such that

$X+(-X) = 0$

(iii) Scalar Multiplication :-

M_1: $k\cdot(X+Y) = kX + kY \in V$

M_2: $(k+l)X = kX + lX \in V$

M_3: $(kl)X = k(lX) \in V$

M_4: Existence of Identity (Multiplicative Identity)

There exists $1 \in V$ such that

$1\cdot X = X$

Q.1) Let V be a set of positive real nos. with addition and scalar multiplication defin as $X+Y = XY$ and $cX = X^c$ where $X, Y \in V$ and c be a scalar.

Soln) To prove that V is a vector space, we prove the 10 axions

let $X, Y, Z \in V$ be the vectors, k & l be the scalars and $X, Y, Z \in R$

(i) Closure Axion :

C_1: $X+Y = XY = YX = Y+X$ $\{\because XY \in R\}$

C_2: $l\cdot X = X^l \in V$ $\{\because X^l \in R\}$

(ii) Addition Axion:

A_1: $X+Y = XY = YX = Y+X \in V$

A_2: $(X+Y)+Z = (XY)+Z = (XYZ) = X(YZ) = X+(Y+Z) \in V$

A_3: There exists $1 \in V$ such that

$X+1 \in V$

A_4: There exists $\frac{1}{X} \in V$ such that

$$X + \frac{1}{X} = X\cdot\frac{1}{X} = 1$$

(iii) Scalar Multiplication:

M1: $k(X+Y) = k \cdot (XY)$
$= (XY)^k$
$= X^k \cdot Y^k$
$= X^k + Y^k$
$= kX + kY \in V$

M2: $(k+l)X = X^{(k+l)}$
$= X^k \cdot X^l$
$= X^k + X^l$
$= kX + lX \in V$

M3: $(kl)X = X^{(kl)}$
$= (X^l)^k$
$= k(X^l)$
$= k(lX)$

M4: There exists $1 \in V$, such that
$1 \cdot X = X^1 = X \in V$
$\therefore 1$ is multiplicative identity of V

Hence V is a vector space.

Q.2) Examine whether the set of all real nos. (X, Y) with operation $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $k(x_1, y_1) = (k^2 x_1, k^2 y_1)$ are vector spaces or not.

Soln) To prove V is a vector space we prove the following 10 axioms.
Let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in V$ ~~x, y, z ∈ R~~ & $\in R$ and k and l be the scalars.

(i) Closure Axiom:

$C_1: (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \in V \; \{\because (x_1 + x_2, y_1 + y_2) \in R\}$

$C_2: k(x_1, y_1) = (k^2 x_1, k^2 y_1) \in V \; \{\because (k^2 x_1, k^2 y_1) \in R\}$

(ii) Addition Axiom:

$A_1: X+Y = (x_1, y_1) + (x_2, y_2)$
$= (x_1+x_2, y_1+y_2)$
$= (x_2+x_1, y_2+y_1)$
$= (x_2, y_2) + (x_1, y_1) \in V \quad \{\because (x_1, y_1) \,\&\, (x_2, y_2) \in R\}$

$A_2: (X+Y)+Z = [(x_1, y_1) + (x_2, y_2)] + (x_3, y_3)$
$= (x_1+x_2, y_1+y_2) + (x_3, y_3)$
$= (x_1+x_2+x_3, y_1+y_2+y_3)$
$= (x_1, y_1) + (x_2+x_3, y_2+y_3)$
$= (x_1, y_1) + [(x_2, y_2) + (x_3, y_3)] \in V \ldots \{(x_1, y_1), (x_2, y_2) \,\&\, (x_3, y_3) \in R\}$

A_3: There exists $(0,0) \in V$ such that
$(x_1, y_1) + (0,0) = (x_1+0, y_1+0) = (x_1, y_1) \in V$
There $(0,0)$ is an additive identity in V.

A_4: There exists $(-x_1, -y_1) \in V$ such that
$(x_1, y_1) + (-x_1, -y_1) = (x_1-x_1, y_1-y_1) = (0,0) \in V$
There exists $(-x_1, y_1)$ as inverse in V.

(iii) Scalar Multiplication:

$M_1: k[(x_1, y_1) + (x_2, y_2)] = k(x_1+x_2, y_1+y_2)$
$= (k^2(x_1+x_2), k^2(y_1+y_2))$
$= (k^2x_1 + k^2x_2, k^2y_1 + k^2y_2)$
$= (k^2x_1, k^2y_1) + (k^2x_2, k^2y_2)$
$= k(x_1, y_1) + k(x_2, y_2) \in V$

$M_2: (k+l)(x_1, y_1) = [(k+l)^2 x_1, (k+l)^2 y_1]$
$= [(k^2+2kl+l^2)x_1, (k^2+2kl+l^2)y_1]$
$\neq k(x_1, y_1) + l(x_{11}, y_1)$

Hence M_2 fails to satisfy the vector space axioms.
$\therefore$ V is not a vector space.

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Q.3) Examine whether the matrices of 2×2 defined as $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ with usual addition and scalar multiples is a vector space.

Soln) let $V = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} / a, b \in R \right\}$

let $X = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, $Y = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$ and $Z = \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix} \in V$ and k & l $\in R$ be the scalars.

(i) Closure Axiom:

$C_1: X+Y = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} a+c & 0 \\ 0 & b+d \end{bmatrix} \in V \ldots \{X, Y \in R\}$

$C_2: kX = k\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} ka & 0 \\ 0 & kb \end{bmatrix} \in V \quad \{\because ka, kb \in R\}$

(ii) Addition Axiom:

$A_1: X+Y = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} a+c & 0 \\ 0 & b+d \end{bmatrix} = \begin{bmatrix} c+a & 0 \\ 0 & d+b \end{bmatrix}$

$= \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = Y + X$

$A_2: (X+Y) + Z = \left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}\right) + \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix} = \begin{bmatrix} a+c & 0 \\ 0 & b+d \end{bmatrix} + \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}$

$= \begin{bmatrix} a+c+e & 0 \\ 0 & b+d+f \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} c+e & 0 \\ 0 & d+f \end{bmatrix}$

$= \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \left(\begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} + \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}\right) = X + (Y+Z)$

A3: There exists $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in V$, such that

$$\cancel{X+} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a+0 & 0 \\ 0 & b+0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in V$$

Therefore, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is an additive identity in V.

A4: There exists $\begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix} \in V$, such that

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix} = \begin{bmatrix} a-a & 0 \\ 0 & b-b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in V$$

(iii) Scalar Multiplication:

M1: $k(X+Y) = k\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}\right) = k\begin{bmatrix} a+c & 0 \\ 0 & b+d \end{bmatrix}$

$$= \begin{bmatrix} k(a+c) & 0 \\ 0 & k(b+d) \end{bmatrix} = \begin{bmatrix} ka+kc & 0 \\ 0 & kb+kd \end{bmatrix}$$

$$= \begin{bmatrix} ka & 0 \\ 0 & kb \end{bmatrix} + \begin{bmatrix} kc & 0 \\ 0 & kd \end{bmatrix} = k\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + k\begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} = kX + kY \in V$$

M2: $(k+l)X = (k+l)\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} (k+l)a & 0 \\ 0 & (k+l)b \end{bmatrix}$

$$= \begin{bmatrix} ka+la & 0 \\ 0 & kb+lb \end{bmatrix} = \begin{bmatrix} ka & 0 \\ 0 & kb \end{bmatrix} + \begin{bmatrix} la & 0 \\ 0 & lb \end{bmatrix}$$

$$= k\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + l\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = kX + lX$$

M3: $(kl)X = kl\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} kla & 0 \\ 0 & klb \end{bmatrix} = k\begin{bmatrix} la & 0 \\ 0 & lb \end{bmatrix}$

$$= k(lX)$$

M4: There exists $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ such that

$$1 \cdot X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = X \in V$$

$\therefore$ V is a vector space.

Q. 4) Examine whether set of matrices of 2×2 defined $\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$ with usual addition and scalar multiplication of vector spaces.

Sol^n)

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Q.5) Check whether the set of all pairs of real nos., of the form $(1, x)$ with the operation $(1, x_1) + (1, x_2) = (1, x_1 + x_2)$ & $k(1, x_1) = (1, kx_1)$

* Subspaces

A subset W of vector space V is called subspace of V if W itself is a vector space under the addition & scalar multiplication defined in V.

Theorem:

If W is a non-empty subset of a vector space V then W is called as subspace of V if

(i) $\forall\, u, v \in V$ such that $u + v \in W$

(ii) $\forall$ ~~u, v ∈~~ $u \in V$, k be a scalar such that $k \cdot u \in W$

Q.6) If $V = R^3$ is the vector space of order triplets of real numbers with usual addition and scalar multiplication. Determine which of the following subspaces of V.

(i) $W = \{(a,b,c) \mid b = a + c,\ a, b, c \in R\}$

(ii) $W = \{(a,0,0) \mid a \in R\}$

(iii) $W = \{(x,y,z) \mid x = 1 \ \&\ y = 1\}$

(iv) $W = \{(x,y,z) \mid x + y + z = 3\}$

(v) $W = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle|\ |A| = 0 \right\}$

(vi) $W = \{(x,y,z) \mid x^2 - y^2 = 0\}$

(vii) $W = \{(x,y) \mid x \geq 0,\ y \geq 0\}$ in R^2

Soln) (i) $W = \{(a,b,c) \mid b = a + c,\ a, b, c \in R\}$ is a subset in V i.e. $W \subseteq V (= R^3)$

Let $X = (a_1, b_1, c_1),\ Y = (a_2, b_2, c_2) \in W$ such that $b_1 = a_1 + c_1$, $b_2 = a_2 + c_2$

① $X + Y = (a_1, b_1, c_1) + (a_2, b_2, c_2)$

$= (a_1 + a_2,\ b_1 + b_2,\ c_1 + c_2)$ such that $b_1 + b_2 = (a_1 + c_1) + (a_2 + c_2)$

$= (a_1 + a_2) + (c_1 + c_2)$

$\therefore X + Y \in W$

(2) If k be a scalar then,

$kX = k(a_1, b_1, c_1) = (ka_1, kb_1, kc_1)$ such that $kb_1 = k(a_1 + c_1)$
$= ka_1 + kc_1$

$\therefore kX \in W$

Therefore, W is closed under addition & scalar multiplication.

Hence, W is subspace of $V(=R^3)$

(ii) $W = \{(a, 0, 0) \mid a \in R\}$

Let $X = (a_1, 0, 0)$, $Y = (b_1, 0, 0) \in W$ such that $a_1, b_1 \in R$.

(1) $X + Y = (a_1, 0, 0) + (b_1, 0, 0) = (a_1 + b_1, 0, 0)$ such that $a_1 + b_1 \in R$

$\therefore X + Y \in W$

(2) Let k be a scalar then,

$kX = k(a_1, 0, 0) = (ka_1, 0, 0)$ such that $ka_1 \in R$

$\therefore kX \in W$

Therefore, W is closed under add^n & scalar multiplicat^n.

Hence, W is subspace of $V(=R^3)$.

(iii) $W = \{(x, y, z) \mid x = 1 \ \& \ y = 1\}$

Let $X = (x_1, y_1, z_1)$, $Y = (x_2, y_2, z_2) \in W$ such that $x_1 = x_2 = y_1 = y_2 = 1$

(1) $X + Y = (x_1, y_1, z_1) + (x_2, y_2, z_2)$
$= (x_1 + x_2, y_1 + y_2, z_1 + z_2) \notin W$ such that $x_1 + x_2 = y_1 + y_2 = 1 + 1 = 2$

$\therefore W \not\subseteq V(=R^3)$

(iv) $W = \{(x, y, z) \mid x + y + z = 3\}$

Let $X = (x_1, y_1, z_1)$, $Y = (x_2, y_2, z_2) \in W$ such that $x_1 + y_1 + z_1 = x_2 + y_2 + z_2 = 3$

(1) $X + Y = (x_1, y_1, z_1) + (x_2, y_2, z_2)$
$= (x_1 + x_2, y_1 + y_2, z_1 + z_2) \notin W$ such that $x_1 + x_2 + y_1 + y_2 + z_1 + z_2 = 6$

$\therefore W \not\subseteq V(=R^3)$

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(v) $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \, |A| = 0 \right\}$ You need to find the case where it will fail.

Let $X = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$, $Y = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \in W$ such that $|X| = |Y| = 0$

① $X + Y = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 7 & 12 \end{bmatrix} \notin W$ such that $|X+Y| = 0$

$\therefore W \not\subseteq V(=R3)$

(vi) $W = \{(x, y, z) \mid x^2 - y^2 = 0\}$

Let $X = (x_1, y_1, z_1)$, $Y = (x_2, y_2, z_2) \in W$ such that $x_1^2 - y_1^2 = x_2^2 - y_2^2 = 0$

① $X + Y = (x_1, y_1, z_1) + (x_2, y_2, z_2)$

$= (x_1 + x_2, y_1 + y_2, z_1 + z_2) \notin W$ such that $(x_1 + x_2)^2 - (y_1 + y_2)^2 = 0$

$\therefore x_1^2 + x_2^2 + 2x_1x_2 - y_1^2 + y_2^2 - 2y_1y_2 = 0$

$\therefore W \subseteq V(=R3)$

(vii) $W = \{(x, y) \mid x \geq 0, y \geq 0\}$

Let $X = (x_1, y_1)$, $Y = (x_2, y_2) \in W$ such that $x_1 \geq 0, y_1 \geq 0, x_2 \geq 0, y_2 \geq 0$

① $X + Y = (x_1, y_1) + (x_2, y_2)$

$= (x_1 + x_2, y_1 + y_2) \in W$ such that $x_1 + x_2 \geq 0, y_1 + y_2 \geq 0$

$\therefore X + Y \in W$

② Let k be a scalar then,

$kX = k(x_1, y_1) = (kx_1, ky_1)$

if $k < 0$ then $= (-kx_1, -ky_1)$ such that $-kx_1 < 0$

$-ky_1 < 0$

$\therefore kX \notin W$

$\therefore W \not\subseteq V(=R3)$

* Linear Combination of Vectors

A vector W is called a linear combination of vectors $V_1, V_2, \ldots, V_n$ if it can expressed in the form $W = k_1V_1 + k_2V_2 + \ldots + k_nV_n$ if all $k_1, k_2, \ldots, k_n$ are not zero (atleast one value of k must be non-zero).

Q.7) If $u=(-2,1,0)$, $v=(1,2,4)$ then show that $w=(-1,8,12)$ is a linear combination of u and v.

Soln) Let k_1 and k_2 be the scalars.

w can be expressed as linear combination of u and v if $w = k_1u + k_2v$ — (1)

$\therefore (-1,8,12) = k_1(-2,1,0) + k_2(1,2,4)$

$\therefore (-1,8,12) = (-2k_1, k_1, 0) + (k_2, 2k_2, 4k_2)$

$\therefore (-1,8,12) = (-2k_1+k_2, k_1+2k_2, 4k_2)$

$$\therefore \begin{bmatrix} -2 & 1 \\ 1 & 2 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \\ 12 \end{bmatrix}$$

By $R_2 \to 2R_2 + R_1$, we get

$$\therefore \begin{bmatrix} -2 & 1 \\ 0 & 5 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} -1 \\ \cancel{8}\,15 \\ 12 \end{bmatrix}$$

By $R_3 \to 5R_3 - 4R_2$, we get

$$\therefore \begin{bmatrix} -2 & 1 \\ 0 & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 15 \\ 0 \end{bmatrix}$$

$\therefore -2k_1 + k_2 = -1$ and $5k_2 = 15 \Rightarrow k_2 = 3$

$\therefore k_1 = 2$

$k_1 = 2, k_2 = 3$

Hence, $(-1,8,12) = 2(-2,1,0) + 3(1,2,4)$ is the linear combination.

Q.8) If $u=(1,2,2)$, $v=(3,4,6)$ then prove that $w=(6,7,4)$ is not a linear combination of u and v.

Soln) Let k_1 and k_2 be the scalars.
Assume w can be expressed as linear combination of u and v if $w = k_1u + k_2v$

$\therefore (6,7,4) = k_1(1,2,2) + k_2(3,4,6)$

$\therefore (6,7,4) = (k_1, 2k_1, 2k_1) + (3k_2, 4k_2, 6k_2)$

$\therefore (6,7,4) = (k_1 + 3k_2, 2k_1 + 4k_2, 2k_1 + 6k_2)$

$$\therefore \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 4 \end{bmatrix}$$

By $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 2R_1$

$$\therefore \begin{bmatrix} 1 & 3 \\ 0 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \\ -8 \end{bmatrix}$$

$\therefore k_1 + 3k_2 = 6$, $-2k_2 = -5$ and $0 = -8$

Since $0 \neq -8$

our assumption was wrong.

Therefore, w cannot be expressed as a linear combination of u and v.

Q.9) A vector $(1,k,5)$ is a linear combination of vector $u=(1,-3,2)$ & $v=(2,-1,1)$. Find k.

Soln) Let $w=(1,k,5)$ and k_1 & k_2 be scalars.
w can be expressed as linear combination of u & v if $w = k_1u + k_2v$ — (1)

$\therefore (1,k,5) = k_1(1,-3,2) + k_2(2,-1,1)$

$\therefore (1,k,5) = (k_1, -3k_1, 2k_1) + (2k_2, -k_2, k_2)$

$\therefore (1,k,5) = (k_1 + 2k_2, -3k_1 - k_2, 2k_1 + k_2)$

$$\therefore \begin{bmatrix} 1 & 2 \\ -3 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 1 \\ k \\ 5 \end{bmatrix}$$

By $R_2 \to R_2 + 3R_1$, $R_3 \to R_3 - 2R_1$

$$\therefore \begin{bmatrix} 1 & 2 \\ 0 & 5 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 1 \\ k+3 \\ 3 \end{bmatrix}$$

By $R_3 \to 5R_3 + 3R_2$,

$$\therefore \begin{bmatrix} 1 & 2 \\ 0 & 5 \\ 0 & \cancel{-3}0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 1 \\ k+3 \\ 3k+24 \end{bmatrix}$$

$\therefore 3k+24=0$

$\therefore k=-8$

* Space spanned by a vector

Let 'V' be a vector space as $S=\{v_1, v_2, \dots, v_n\}$ be spanned set of vectors in V. Then set of all linear combination of the vectors v_1 to v_n is called 'span of a set S' or spans or $L(S)$ or span$\{\cancel{S}, v_1, v_2, \dots, v_n\}$.

Q.9) Check whether the given set of vectors ~~upto~~ span a subspace of R^3 vector space

(i) $(2,1,0), (0,3,-4), (1,-1,2)$

(ii) $(1,2,3), (0,0,1), (0,1,2)$

(iii) $(1,2,6), (3,4,1), (4,3,1), (3,3,1)$

Solⁿ) (i) Let 'S' be the set of 3 vectors $\{(2,1,0), (0,3,-4), (1,-1,2)\}$ i.e.

$S=\{v_1, v_2, v_3\}$

let $w=(w_1, w_2, w_3) \in R^3$

w can be expressed as linear combination of v_1, v_2, v_3 if

$w = k_1 v_1 + k_2 v_2 + k_3 v_3$ — (1)

$\therefore (w_1, w_2, w_3) = k_1(2,1,0) + k_2(0,3,-4) + k_3(1,-1,2)$

$\therefore (2k_1, k_1, 0) + (0, 3k_2, -4k_2) + (k_3, -k_3, 2k_3) = (w_1, w_2, w_3)$

$\therefore (2k_1 + k_3, k_1 + 3k_2 - k_3, -4k_2 + 2k_3) = (w_1, w_2, w_3)$

NOTE:- $AX=B$ or $B=AX$, if A^{-1} exist, $|A| \neq 0$

$\therefore AX = B$

$$\therefore A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 3 & -1 \\ 0 & -4 & 2 \end{bmatrix}$$

$|A| = 2(6-4) + 1(-4) = 0$

$\therefore A^{-1}$ does not exist

Therefore, set $S = \{v_1, v_2, v_3\}$ does not span R^3 vector space.

(ii) Let 's' be the set of 3 vectors $S = \{v_1, v_2, v_3\} = \{(1,2,3), (0,0,1), (0,1,2)\}$

Let $w = (w_1, w_2, w_3) \in R^3$

w can be expressed as linear combination of v_1, v_2, v_3

$w = k_1 v_1 + k_2 v_2 + k_3 v_3$ — (1)

$\therefore (w_1, w_2, w_3) = k_1(1,2,3) + k_2(0,0,1) + k_3(0,1,2)$

$\therefore (w_1, w_2, w_3) = (k_1, 2k_1, 3k_1) + (0, 0, k_2) + (0, k_3, 2k_3)$

$\therefore (w_1, w_2, w_3) = (k_1, 2k_1 + k_3, 3k_1 + k_2 + 2k_3)$

$\therefore AX = B$

$$\therefore A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

$|A| = -1 \neq 0$

$\therefore A^{-1}$ exists and system of equations is consistent.

Therefore, set $S = \{v_1, v_2, v_3\}$ does not span R^3 vector space.

(iii) Let 's' be the set of 4 vectors $S = \{v_1, v_2, v_3, v_4\} = \{(1,2,6), (3,4,1), (4,3,1), (3,3,1)\}$

Let $w = (w_1, w_2, w_3) \in R$

w can be expressed as linear combination of v_1, v_2, v_3, v_4

$w = k_1 v_1 + k_2 v_2 + k_3 v_3 + k_4 v_4$ — (1)

$\therefore (w_1, w_2, w_3, w_4) = (k_1 + 3k_2 + 4k_3 + 3k_4,\ 2k_1 + 4k_2 + 3k_3 + 3k_4,\ 6k_1 + k_2 + k_3 + k_4)$

$\therefore AX = B$

$$\therefore \begin{bmatrix} 1 & 3 & 4 & 3 \\ 2 & 4 & 3 & 3 \\ 6 & 1 & 1 & 1 \end{bmatrix} \begin{array}{c} X \\ \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} \end{array} = \begin{array}{c} X \\ \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \end{array}$$

⇐ It works if you find the Rank of A and get answer

By $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 6R_1$, we get

$$\therefore \begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & -2 & -5 & -3 \\ 0 & -17 & -23 & -17 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

& By $R_3 \to 2R_3 - 17R_2$, we get

$$\therefore \begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & -2 & -4 & -3 \\ 0 & 0 & 39 & 17 \end{bmatrix} \begin{bmatrix} '' \\ \\ \\ \end{bmatrix} = \begin{bmatrix} '' \\ \\ \end{bmatrix}$$

Rank of $A(r) = 3$, $n = 4$

$\therefore n > r \Rightarrow$ The vectors are dependent.

Therefore, the set S do not span R^3 vector space.

* **Basis & Dimension**

Basis :- If 'v' is any vector space and $S = \{v_1, v_2, \dots, v_n\}$ in V, then 's' is called basis for 'v', if it satisfy following two cond^ns,

1.) S is linearly independent

2.) S span V

Dimension :- The no. of vector in basis for V called dimension of the vector 'v' and is denoted by dimension of v(dim(v)).

Q.10) Find basis and dimension of a vector space spanned by vectors $V_1=(3,1,1)$, $V_2=(2,0,-1)$, $V_3=(4,2,1)$.

Sol^n) Let k_1, k_2, k_3 be scalars of the matrix equation of vectors,

$k_1V_1 + k_2V_2 + k_3V_3 = 0$ — (1)

$\therefore k_1(3,1,1) + k_2(2,0,-1) + k_3(4,2,1) = 0$

$\therefore (3k_1 + 2k_2 + 4k_3,\ k_1 + 2k_3,\ k_1 - k_2 + k_3) = (0,0,0)$

$\therefore 3k_1 + 2k_2 + 4k_3 = 0$

$k_1 + 0k_2 + 2k_3 = 0$

$k_1 - k_2 + k_3 = 0$

$$\therefore \begin{bmatrix} 3 & 2 & 4 \\ 1 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2 \to 3R_2 - R_1$, $R_3 \to 3R_3 - R_1$, we get

$$\therefore \begin{bmatrix} 3 & 2 & 4 \\ 0 & -2 & 2 \\ 0 & -5 & -1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_3 \to 2R_3 - 5R_2$, we get

$$\therefore \begin{bmatrix} 3 & 2 & 4 \\ 0 & -2 & 2 \\ 0 & 0 & -12 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore 3k_1 + 2k_2 + 4k_3 = 0 \Rightarrow k_1 = 0$

$-2k_2 + 2k_3 = 0 \Rightarrow k_2 = 0$

$-12k_3 = 0 \Rightarrow k_3 = 0$

Therefore, vectors V_1, V_2 and V_3 are linearly independent.
The set V_1, V_2, V_3 is linearly independent and hence the set spans the given vector space.
This implies,
Basis $= \{(3,1,1), (2,0,-1), (4,2,1)\}$
$\therefore$ Dimension$(V) = 3$

** Pivot elements = 0 then independent vector //

Q.11) Find basis & dimension of vector space spanned by vectors (1, 2, 3, 4, 5), (-2, 1, -3, -5, -4), $V_3 = (-1, 8, 3, 2, 7)$.

Soln) Let $V_1 = (1, 2, 3, 4, 5)$, $V_2 = (-2, 1, -3, -5, -4)$, $V_3 = (-1, 8, 3, 2, 7)$

Let k_1, k_2, k_3 be the scalars.

Consider the matrix equation,

$k_1 V_1 + k_2 V_2 + k_3 V_3 = 0$

~~∴ k1(-2, 1, -3, -5, -4) + k2(~~

$\therefore k_1(1, 2, 3, 4, 5) + k_2(-2, 1, -3, -5, -4) + k_3(-1, 8, 3, 2, 7) = 0$

$\therefore (k_1 - 2k_2 - k_3, 2k_1 + k_2 + 8k_3, 3k_1 - 3k_2 + 3k_3, 4k_1 - 5k_2 + 2k_3, 5k_1 - 4k_2 + 7k_3) = 0$

$$\therefore \begin{bmatrix} 1 & -2 & -1 \\ 2 & 1 & 8 \\ 3 & -3 & 3 \\ 4 & -5 & 2 \\ 5 & -4 & 7 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$, $R_4 \to R_4 - 4R_1$, $R_5 \to R_5 - 5R_1$

$$\therefore \begin{bmatrix} 1 & -2 & -1 \\ 0 & 5 & 10 \\ 0 & 3 & 6 \\ 0 & 3 & 6 \\ 0 & 6 & 12 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_3 \to (1/3)R_3 - (1/5)R_2$, $R_4 \to (1/3)R_4 - (1/5)R_2$, $R_5 \to (1/6)R_5 - (1/5)R_2$

Pivot elements

$$\therefore \begin{bmatrix} 1 & -2 & -1 \\ 0 & 5 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Here, V_1 and V_2 are linearly independent

$\therefore$ Sets $S = \{V_1, V_2\}$ span some subspace of vectors space R^5

Basis of some subspace of $R^5 = \{(1, 2, 3, 4, 5), (-2, 1, -3, -5, -4)\}$

Dimension(subspace) = 2

Q.11) Show that $s=\{1-t-t^2, -2+3t+t^3, 1+t+5t^3\}$ is linearly independent in P_3 & hence find basis and dimension.

soln) $P_1 = 1-t+t^2 \approx (1,-1,1,0)$

$P_2 = -2+3t+2t^3 \approx (-2,3,0,2)$

$P_3 = 1+t+5t^3 \approx (1,1,0,5)$

$P(n) = a+bt+ct^2+dt^3 \approx (a,b,c,d)$

Consider,

$k_1P_1 + k_2P_2 + k_3P_3 = 0$

$\therefore k_1(1,-1,1,0) + k_2(-2,3,0,2) + k_3(1,1,0,5) = 0$

$\therefore (k_1 - 2k_2 + k_3, -k_3 + 3k_2 + k_3, k_1, 2k_2 + 5k_3) = 0$

$$\therefore \begin{bmatrix} 1 & -2 & 1 \\ -1 & 3 & 1 \\ 1 & 0 & 0 \\ 0 & 2 & 5 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2 \to R_2 + R_1$, $R_3 \to R_3 - R_1$

$$\therefore \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 0 & -2 & 1 \\ 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_3 \to R_3 + 2R_2$, $R_4 \to R_4 - R_2$

$$\therefore \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_4 \to 5R_4 - 3R_3$

$$\therefore \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence, P_3 is linearly independent i.e. $S = \{P_1, P_2, P_3\}$ span P_3
Basis = $\{(1,-1,-1,0), (-2,3,0,1), (1,1,0,5)\}$
Dimension $(V) = 3$

* Inner Product Spaces

1) Norm of a Vector :- $u = (u_1, u_2, u_3)$
The length of vector u or magnitude of vector u is called norm of the vector u and is defined and denoted by
$\|u\| = \sqrt{u_1^2 + u_2^2 + u_3^2} = \sqrt{u \cdot u}$

2.) Dot product or inner product :-
If u and v are any non-zero vectors in R^2 or R^3 space, θ be the angle b/w u and v then dot product b/w u and v is defined and denoted by
$\langle u, v \rangle = u \cdot v = \|u\| \cdot \|v\| \cdot \cos\theta$

3.) Euclidean Inner Product :-
If $u = (u_1, u_2, \ldots, u_n)$, $v = (v_1, v_2, \ldots, v_n)$ in R^n space then euclidean inner product for u and v is defined as,
$u \cdot v = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$

4) Cauchy Schwart'z Inequality :-
If u and v in R^n space the cauchy schwart'z inequality states that
$|u \cdot v| \leq \|u\| \cdot \|v\|$

Q.13) Verify cauchy schwart'z Inequality for the vector $u = (2,3,1)$ & $v = (3,0,4)$. Also find the angle b/w u and v.

Sol^n) By cauchy schwart'z Inequality,
$|u \cdot v| \leq \|u\| \cdot \|v\| \quad \text{— (1)}$

Now,

$||u|| = \sqrt{2^2+3^2+1^2} = \sqrt{14}$ — (2)

$||v|| = \sqrt{3^2+0^2+4^2} = 5$ — (3)

and $u \cdot v = (2,3,1) \cdot (3,0,4)$

$= 6+0+4$

$= 10$

$\therefore |u \cdot v| = 10$ — (4)

Put (2), (3) & (4) in (1), we get

$10 \leq \sqrt{14} \cdot 5$

$\therefore 10 \leq 18.7$

Hence proved

$$\cos\theta = \frac{u \cdot v}{||u|| \cdot ||v||} = \frac{10}{18.7}$$

$\therefore \cos\theta = 57.67°$

** NOTE:- If $f(n)$ and $g(n)$ are any two functions then the dot product of f & g is defined as,

$$\langle f, g \rangle = f \cdot g = \int_a^b f \cdot g \cdot dx$$

Q.14) Verify cauchy schwart'z Inequality for f & g where $f(x) = \cos x$ & $g(x) = \sin x$ in $[-\pi, \pi]$.

soln) By cauchy schwart'z Inequality,

$|u \cdot v| \leq ||u|| \cdot ||v||$ — (1)

Now,

$$\langle f, g \rangle = f \cdot g = \int_{-\pi}^{\pi} f(x) \cdot g(x) \cdot dx$$

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$$\langle f,g\rangle = \int_{-\pi}^{\pi} \sin x \cdot \cos x \cdot dx$$

$$= \int_{-\pi}^{\pi} \frac{\sin 2x}{2} \cdot dx$$

$$= -\frac{1}{2}\left[\frac{\cos 2x}{2}\right]_{-\pi}^{\pi}$$

$$= 0$$

$$\|f\| = \sqrt{\langle f,f\rangle} = \left(\int_{-\pi}^{\pi} f \cdot f \cdot dx\right)^{1/2}$$

$$= \left(\int_{-\pi}^{\pi} \cos^2 x \cdot dx\right)^{1/2}$$

$$= \left[\frac{1}{2}\left(x + \frac{\sin 2x}{2}\right)_{-\pi}^{\pi}\right]^{1/2}$$

$$= \left[\frac{1}{2}(\pi + 0 - (-\pi + 0))\right]$$

$$= \left(\frac{2\pi}{2}\right)^{1/2}$$

$$= \sqrt{\pi}$$

$$\|g\| = \sqrt{\langle g,g\rangle} = \left(\int_{-\pi}^{\pi} g \cdot g \cdot dx\right)^{1/2}$$

$$= \left(\int_{-\pi}^{\pi} \sin^2 x \cdot dx\right)^{1/2}$$

$$= \left(\int_{-\pi}^{\pi} (1 - \cos^2 x) \cdot dx\right)^{1/2}$$

$$= \left[\left(x - \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right)\right)_{-\pi}^{\pi}\right]^{1/2}$$

$$= \left[2\pi - \frac{1}{2}(2\pi)\right]^{1/2} = \sqrt{\pi}$$

$\& \langle f,g\rangle \leq \|f\|\cdot\|g\|$

$\therefore\ 0 \leq \sqrt{\pi}\cdot\sqrt{\pi}$

$\therefore\ 0 \leq \pi$

Hence proved.

→ Properties of Eucledian Inner Product

Let $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ be any two vectors in R^n space, and k be any scalar,

(i) $u\cdot v = v\cdot u$

(ii) $(u+v)\cdot w = u\cdot w + v\cdot w$

(iii) $k u\cdot v = k(u\cdot v)$

(iv) $v\cdot v \geq 0$, and $v\cdot v = 0$ if and only if $v = 0$

→ Properties of Norm in R^n space

If u and v belongs to R^n space and k be any scalar then,

(i) $\|u\| \geq 0$

(ii) $\|u\| = 0$ if and only if $u = 0$

(iii) $\|k u\| = |k|\cdot\|u\|$

(iv) $\|u+v\| \leq \|u\| + \|v\|$

→ Relation b/w Norm & Inner Product

If u and v are any two vectors then,

$$\boxed{u\cdot v = \frac{1}{4}\|u+v\|^2 - \frac{1}{4}\|u-v\|^2}$$

Proof:

$\|u+v\|^2 = \langle u+v, u+v \rangle$

$= (u+v)\cdot(u+v)$

$= u\cdot u + u\cdot v + v\cdot u + v\cdot v$

$= \|u\|^2 + 2(u\cdot v) + \|v\|^2$ — (1)

$\|u-v\|^2 = \langle u-v, u-v \rangle$

$= (u-v)\cdot(u-v)$

$= u\cdot u - u\cdot v - v\cdot u + v\cdot v$

$= \|u\|^2 - 2(u\cdot v) + \|v\|^2$ — (2)

Consider RHS,

$= \frac{1}{4}\|u+v\|^2 - \frac{1}{4}\|u-v\|^2$

$= \frac{1}{4}(\|u\|^2 + 2(u\cdot v) + \|v\|^2) - \frac{1}{4}(\|u\|^2 - 2(u\cdot v) + \|v\|^2)$

$= u\cdot v$

$= L.H.S$

Hence proved.

→ Orthogonally

Two vectors u and v are said to be orthogonal if $\langle u, v \rangle = u\cdot v = 0$.

→ Orthonormal

Two vectors u and v are said to be orthonormal if dot product $u\cdot v = 0$ and $\|u\| = 1$ & $\|v\| = 1$.

Q.15) Which of the following sets of vectors are orthogonal with euclidean inner product.

(i) $\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

(ii) $\left(\frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$

(iii) $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1\right), \left(0, \frac{-1}{\sqrt{2}}, \frac{1}{2}\right), \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right)$

Soln) (i) $u = (1/\sqrt{3}, -1/\sqrt{3})$, $v = (1/\sqrt{3}, 1/\sqrt{3})$

$u \cdot v = \frac{1}{3} - \frac{1}{3} = 0$

(ii) $u = (1/\sqrt{3}, 0, 1/\sqrt{3})$, $v = (1/\sqrt{2}, 1/\sqrt{2}, -1/\sqrt{2})$

$u \cdot v = \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{6}} = 0$

(iii) $u = (1/\sqrt{2}, 1/\sqrt{2}, 1)$, $v = (0, -1/\sqrt{2}, 1/2)$, $w = (1/\sqrt{2}, -1/\sqrt{2}, 0)$

$u \cdot v = -1/2 + 1/2 = 0$

$v \cdot w = +1/2 \neq 0$

$u \cdot w = 1/2 - 1/2 = 0$

Therefore, u and v are orthogonal, u and w are orthogonal, v and w are not orthogonal.

∴ The set u, v, w are not orthogonal.

Q.16) For which k are following vectors orthogonal $u = (k, k, 4)$ & $v = (k, -7, 3)$.

Soln) It is given, u & v are orthogonal.

$u \cdot v = 0$

$\therefore (k, k, 4) \cdot (k, -7, 3) = 0$

$\therefore k^2 - 7k + 12 = 0$

$\therefore k = 3, 4$

Q.17) Find two vectors orthogonal to $u=(a,0,b)$.

Soln) Let $v=(x,y,z)$ be a vector orthogonal to u

$\therefore u\cdot v = 0$

$\therefore (a,0,b)\cdot(x,y,z)=0$

$\therefore ax+bz=0$

If $x=-b$ then $z=a$

and if $x=b$ then $z=-a$

Therefore,

Two orthogonal vectors are $(-b,0,a)$ & $(b,0,-a)$

Q.18) Find a vector orthogonal to both $u=(-6,4,2)$ & $v=(3,1,5)$.

Soln) Let $w=(x,y,z)$ be the vector orthogonal to u and v.

$\therefore u\cdot w = 0$ & $v\cdot w=0$

$(-6,4,2)\cdot(x,y,z)=0$ $\qquad$ $(3,1,5)\cdot(x,y,z)=0$

$-6x+4y+2z=0 \text{ --- } (1)$ $\qquad$ $3x+y+5z=0 \text{ --- } (2)$

By Cramer's Rule on (1) & (2), we get

$$\frac{x}{\begin{vmatrix}4 & 2\\ 1 & 5\end{vmatrix}} = \frac{y}{\begin{vmatrix}2 & -6\\ 5 & 3\end{vmatrix}} = \frac{z}{\begin{vmatrix}-6 & 4\\ 3 & 1\end{vmatrix}}$$

$$\therefore \frac{x}{18} = \frac{y}{36} = \frac{z}{-18}$$

$\therefore x=1,\ y=2,\ z=-1$

Therefore, vector orthogonal to both u and v is $(1,2,-1)$.

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Q.19) Find unit vector in R^3, orthogonal to both $u=(1,0,1)$ & $v=(0,1,1)$.

Sol^n) Let $w=(x,y,z)$ be a vector orthogonal to u & v.

~~$\therefore u \cdot v = 0$~~

$\therefore u \cdot w = 0$ & $v \cdot w = 0$

$(1,0,1)\cdot(x,y,z)=0$ $\quad$ $(0,1,1)\cdot(x,y,z)=0$

$x+0y+z=0 \text{ --- } (1)$ $\quad$ $0x+y+z=0 \text{ --- } (2)$

By Cramer's Rule on (1) & (2), we get

$$\frac{x}{\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}}$$

$$\therefore \frac{x}{-1} = \frac{y}{-1} = \frac{z}{1}$$

Therefore, $w=(-1,-1,1)$

$$\hat{w} = \text{unit vector } w = \frac{w}{\|w\|}$$

$$= \left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

Q.20) $S=\{(0,1,0),(-4/5,0,3/5),(3/5,0,4/5)\}$

(i) Prove that S is orthonormal basis in R^3.

(ii) Write $w=(1,1,1)$ as a linear combination of S and hence find coordinate vector of w w.r.t S i.e. $[w]_S$

Sol^n) Let $v_1=(0,1,0)$, $v_2=(-4/5,0,3/5)$ & $v_3=(3/5,0,4/5)$ be the vectors and k_1,k_2,k_3 be the scalars.

The linear combination is given by,

$k_1v_1+k_2v_2+k_3v_3=0$

$k_1(0,1,0)+k_2(-4/5,0,3/5)+k_3(3/5,0,4/5)=0$

$\therefore(-4/5k_2+3/5k_3, k_1, 3/5k_2+4/5k_3)=0$

$\therefore \frac{-4}{5}k_2 + \frac{3}{5}k_3 = 0$, $k_1 = 0$, $\frac{3}{5}k_2 + \frac{4}{5}k_3 = 0$

On solving, we get $k_2 = 0$, $k_3 = 0$

Hence, vectors v_1, v_2, v_3 are linearly independent.

$\therefore S = \{v_1, v_2, v_3\}$ spans R^3 and it forms basis for R^3

Consider,

$v_1 \cdot v_2 = (0,1,0) \cdot (-4/5, 0, 3/5)$

$= 0$

$v_2 \cdot v_3 = (-4/5, 0, 3/5) \cdot (3/5, 0, 4/5)$

$= 0$

$v_1 \cdot v_3 = (0,1,0) \cdot (-4/5, 0, 3/5)$

$= 0$

Therefore, v_1, v_2, v_3 are orthogonal to each other.

Now,

$||v_1|| = \sqrt{0+1+0} = 1$

$||v_2|| = \sqrt{16/25 + 0 + 9/25} = 1$

$||v_3|| = \sqrt{9/25 + 0 + 16/25} = 1$

Therefore, S is the orthonormal basis of R^3 space.

(ii) If w is the linear combination of v_1, v_2, v_3 where

$w = k_1 v_1 + k_2 v_2 + k_3 v_3$ — (1)

$\therefore (1,1,1) = k_1(0,1,0) + k_2(-4/5, 0, 3/5) + k_3(3/5, 0, 4/5)$

$\therefore (1,1,1) = (-4/5 k_1 + 3/5 k_2, k_1, 3/5 k_2 + 4/5 k_3)$

Hence, we get

$-4/5 k_1 + 3/5 k_2 = 1$ — (2) , $k_1 = 1$, $3/5 k_2 + 4/5 k_3 = 1$ — (3)

On solving (2) & (3) simultaneously, we get

$k_2 = -1/5$, $k_3 = 7/5$

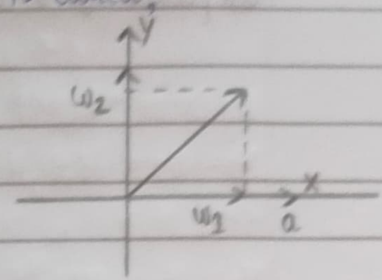
→ **Orthogonal Projection of vector 'u'**

If $u = w_1 + w_2$ be a sum of 2 vectors, one in the direction of given non-zero vector 'a' and other in the direction perpendicular to vector 'a'

The vector component w_1 in the direction of vector 'a' is called,

ω_1 = Orthogonal Projection of u on a = proj_a^u

$$\omega_1 = \frac{(u \cdot a) \cdot a}{\|a\|^2}$$



The vector component w_2 in the direction $\perp^{ar}$ to 'a' is called,

W_2 = Vector component of 'u' orthogonal to 'a' = $u - w_1 = u - \text{proj}_a^u$

$$= u - \frac{(u \cdot a)\, a}{\|a\|^2}$$

→ **Projection Theorem**

If W is a finite dimensional subspace of inner product space V ($W \subseteq V$) then every vector $\forall\, u \in V$ can be uniquely expressed as $u = w_1 + w_2$ where $w_1 \in W$ and $w_2 \in W^{\perp}$ (Perpendicular set)

→ Orthogonal compliment of W

Q.22) Find orthogonal projection of u and v where $u = (3, 1, -7)$ and $v = (1, 0, 5)$. Also find the vector component of u orthogonal to v.

Soln) Projection of 'u' along 'v' = $W_1 = \text{proj}_v^u$

$$= \frac{(u \cdot v)\, v}{\|v\|^2}$$

$$= \frac{((3,1,-7)\cdot(1,0,5))}{(\sqrt{1^2+0^2+5^2})^2}(1,0,5)$$

$$= \frac{-32}{26}(1,0,5)$$

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$\therefore w_1 = \left(\frac{-16}{13}, 0, \frac{-80}{13}\right)$

Component of 'u' orthogonal to 'v' $= w_2 = u - w_1$

$$= (3,1,-7) - \left(\frac{-16}{13}, 0, \frac{-80}{13}\right)$$

$$w_2 = \left(\frac{55}{13}, 1, \frac{-11}{13}\right)$$

* Gram Schmidt Process (To find orthogonal set of vectors)

It gives us a method of finding orthogonal vectors. Let V be a non-zero inner product space in R^3. Let $u = (u_1, u_2, u_3)$ be any basis for V. The following steps leads to an orthonormal basis (v_1, v_2, v_3) for V.

Step 1: $v_1 = u_1$

Step 2: $v_2 = u_2 - \text{proj}_{v_1}^{u_2} = u_2 - \frac{(u_2 \cdot v_1)\, v_1}{\|v_1\|^2}$

Step 3: $v_3 = u_3 - \text{proj}_{v_1}^{u_3} - \text{proj}_{v_2}^{u_3} = u_3 - \frac{(u_3 \cdot v_1)\, v_1}{\|v_1\|^2} - \frac{(u_3 \cdot v_2)\, v_2}{\|v_2\|^2}$

For orthonormal $\Rightarrow$ Divide by norm of the vector.

Thus, we get orthogonal set $V = \{v_1, v_2, v_3\}$

Q-23) Let R^3 have the euclidean inner product. Use Gram schmidt process to transform the basis (u_1, u_2, u_3) into orthonormal basis (v_1, v_2, v_3) where $u_1 = (1,1,1)$, $u_2 = (0,1,1)$, $u_3 = (0,0,1)$.

Soln) By using Gram ~~Schmi~~ Schmidt Process,

Assume that v_1 is along u_1

$v_1 = u_1 = (1,1,1)$

$$v_2 = u_2 - \text{proj}_{v_1}^{u_2} = u_2 - \frac{(u_2 \cdot v_1)}{\|v_1\|^2} v_1$$

$$\cancel{= (0,1,1)}$$

$$= (0,1,1) - \frac{((0,1,1)\cdot(1,1,1))}{(\sqrt{1^2+1^2+1^2})} (1,1,1)$$

$$= (0,1,1) - \frac{2}{3}(1,1,1)$$

$$= \left(\frac{-2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$v_3 = u_3 - \text{proj}_{v_1}^{u_3} - \text{proj}_{v_2}^{u_3} = u_3 - \frac{(u_3 \cdot v_1)}{\|v_1\|^2} v_1 - \frac{(u_3 \cdot v_2)}{\|v_2\|^2} v_2$$

$$= (0,0,1) - \frac{((0,0,1)\cdot(1,1,1))}{\sqrt{1^2+1^2+1^2}}(1,1,1) - \frac{((0,0,1)\cdot(-2/3, 1/3, 1/3))}{(\sqrt{4/9+1/9+1/9})^2}(-2/3, 1/3, 1/3)$$

$$= (0,0,1) - (1/3, 1/3, 1/3) - (-1/3, 1/6, 1/6)$$

$$= (0, -1/2, 1/2)$$

$\therefore$ v_1, v_2 & v_3 are orthogonal vectors.

$\|v_1\| = \sqrt{3}$, $\|v_2\| = \sqrt{2/3}$, $\|v_3\| = \sqrt{1/2}$

$\therefore$ Orthonormal basis vectors are

$$q_1 = \frac{v_1}{\|v_1\|} = \frac{(1,1,1)}{\sqrt{3}} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$q_2 = \frac{v_2}{\|v_2\|} = \frac{(-2/3, 1/3, 1/3)}{\sqrt{2/3}} = \left(-\sqrt{\frac{3}{2}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

$$q_3 = \frac{v_3}{\|v_3\|} = \frac{(0, -1/2, 1/2)}{\sqrt{1/2}} = \left(0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

Q.24) Using Gram Schmidt Process for $S = \{(3,0,4), (1,0,-7), (2,9,11)\}$. Construct orthonormal basis of R^3.

Soln) Let $S = \{u_1, u_2, u_3\}$, where

$u_1 = (3,0,4), u_2 = (1,0,+7), u_3 = (2,9,11)$

By Gram Schmidt Process, we assume u_1 is along v_1

$v_1 = u_1 = (3,0,4)$

$$v_2 = u_2 - \text{proj}_{v_1} u_2 = u_2 - \frac{(u_2 \cdot v_1)}{\|v_1\|^2} v_1$$

$$= (1,0,+7) - \frac{((1,0,+7)\cdot(3,0,4))}{(\sqrt{9+0+16})} (3,0,4)$$

$$= (1,0,+7) - \frac{25(3,0,4)}{25}$$

$$= \cancel{(-2,0,-11)}\ (-4,0,3)$$

$$v_3 = u_3 - \text{proj}_{v_1} u_3 - \text{proj}_{v_2} u_3$$

$$= (2,9,11) - \frac{((2,9,11)\cdot(3,0,4))}{(\sqrt{9+0+16})^2}(3,0,4) - \frac{((2,9,11)\cdot(-4,0,3))}{(\sqrt{16+0+9})^2}(-4,0,3)$$

$$= (2,9,11) - \frac{50}{25}(3,0,4) - \frac{25}{25}(-4,0,3)$$

$$= (0,9,0)$$

$\therefore v_1, v_2$ & v_3 are orthogonal vectors

$\|v_1\| = \cancel{\sqrt{}}5, \|v_2\| = 5, \|v_3\| = 9$

The orthonormal basis vectors are

$$q_1 = \frac{v_1}{\|v_1\|} = \frac{(3,0,4)}{5} = \left(\frac{3}{5}, 0, \frac{4}{5}\right)$$

$$q_2 = \frac{V_2}{\|V_2\|} = \cancel{\frac{(1,0,-7)}{5}} = \frac{(-4,0,3)}{5} = \left(\frac{-4}{5}, 0, \frac{3}{5}\right)$$

$$q_3 = \frac{V_3}{\|V_3\|} = \frac{(0,9,0)}{9} = (0,1,0)$$

Q.25) Let vector space P_2 have the inner product defined by $\langle p,q \rangle = \int_{-1}^{1} p(x)\cdot q(x)\cdot dx$
Using Gram schmidt Process to transform the standard basis $S = \{1, x, x^2\}$
into an orthonormal basis.

soln) By Gram Schmidt Process, we assume u_1 is along V_1

$V_1 = u_1 = \cancel{\{}1 — (1)$

$$V_2 = u_2 - \text{proj}_{V_1}^{u_2} = u_2 - \frac{\langle u_2, V_1 \rangle V_1}{\|V_1\|^2}$$

Now,

$$\langle u_2, V_1 \rangle = \int_{-1}^{1} u_2 \cdot V_1 \cdot dx$$

$$= \int_{-1}^{1} x \cdot 1 \cdot dx$$

$$= \left(\frac{x^2}{2}\right)_{-1}^{1} = 0$$

$$\|V_1\|^2 = \cancel{\sqrt{}}\langle V_1, V_1 \rangle$$

$$= \int_{-1}^{1} 1 \cdot dx$$

$$= [x]_{-1}^{1}$$

$$= 2$$

$\therefore V_2 = \cancel{1 - 0 = 1}\ x - 0 = x$

$$V_3 = u_3 - \text{proj}_{V_1}^{u_3} - \text{proj}_{V_2}^{u_3} = u_3 - \cancel{\{}\frac{\langle u_3, V_1 \rangle V_1}{\|V_1\|^2} - \frac{\langle u_3, V_2 \rangle V_2}{\|V_2\|^2}$$

Now,

$$\langle u_3, v_1 \rangle = \int_{-1}^{1} x^2 \cdot 1 \cdot dx$$

$$= \left[\frac{x^3}{3}\right]_{-1}^{1}$$

$$= \frac{2}{3}$$

$$\langle u_3, v_2 \rangle = \int_{-1}^{1} x^2 \cdot x \cdot dx$$

$$= \left[\frac{x^4}{4}\right]_{-1}^{1}$$

$$= 0$$

$$\|v_2\|^2 = \langle v_2, v_2 \rangle = \int_{-1}^{1} 1 \cdot dx = 2$$

$$v_3 = x^2 - \frac{2/3}{2} \cdot (1) = x^2 - \frac{1}{3}$$

$$\|v_3\|^2 = \langle v_3, v_3 \rangle$$

$$= \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 \cdot dx$$

$$= \left[\frac{x^5}{5} - \frac{2x^3}{9} + \frac{x}{9}\right]_{-1}^{1}$$

$$= \frac{8}{45}$$

Therefore, orthonormal vectors are

$$q_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}}, \quad q_2 = \frac{v_2}{\|v_2\|} = \sqrt{\frac{3}{2}}\, x, \quad q_3 = \frac{v_3}{\|v_3\|} = \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right)$$