$AX = \lambda X$. λ = eigen value

Matrix X = Eigen vector

4. Eigenvalues

Let A be any n-rowed square matrix, λ a scalar and I the unit matrix of the same order. The matrix $\boldsymbol{A} - \lambda \boldsymbol{I}$ is called the **characteristic matrix.**

The determinant $|\boldsymbol{A} - \lambda \boldsymbol{I}|$ is called the **characteristic polynomial** of A.

The equation obtained by equating to zero this determinant *i.e.* the equation $|\boldsymbol{A} - \lambda \boldsymbol{I}| = 0$ is called the **characteristic equation** of the matrix A.

For example, consider a two-rowed square matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

Then, $A - \lambda I = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix}$ is the characteristic matrix of A.

Further, $|A - \lambda I|$ *i.e.* $\begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix}$ *i.e.* $\lambda^2 - 4\lambda - 5$ is the characteristic polynomial of A.

And the equation $|A - \lambda I| = 0$ *i.e.* $\lambda^2 - 4\lambda - 5 = 0$ is the characteristic equation of A.

If we consider an n-th order square matrix A, then its characteristic equation will be

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots\cdots & a_{2n} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ a_{n1} & a_{n2} & \cdots\cdots & a_{nn} - \lambda \end{vmatrix} = 0 \qquad \text{.....................(1)}$$

On expanding this determinant, we get an equation of the form,

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \ldots + a_2\lambda^2 + a_1\lambda + a_0 = 0$$

This is the characteristic equation of A.

The roots of this equation are called the **characteristic roots** or **latent roots** or **characteristic values** or the **eigenvalues** or **proper values** of the matrix A. (Eigen is a German word meaning proper.)

For example if $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ then as seen before the characteristic equation of A is $\lambda^2 - 4\lambda^2 - 5 = 0$.

By solving it we get $(\lambda - 5)(\lambda + 1) = 0$ $\therefore$ $\lambda = -1, 5$.

$\therefore$ $-1, 5$ are the eigenvalues of the matrix A.

5. Eigenvectors

Suppose λ_1 is a root of $|A - \lambda I| = 0$. Then, $|A - \lambda_1 I| = 0$. Suppose further we find a non-zero column matrix X such that

$$[A - \lambda_1 I]\, X = 0. \qquad \text{.....................(1)}$$

This system of equations has non-trivial solutions.

The vector X is called the **eigenvector** or **latent vector** corresponding to the root λ_1.

For example, if $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$, we have seen that $\lambda = 5$ is a root. Corresponding to this root, suppose we have a column matrix X such that

$f(A)$ = function of matrix

Thm: If λ & X are eigen value & eigen vector of A then $f(\lambda)$ & X are eigen value & eigen vector of $f(A)$

If λ & X are eigen value & eigen vector of A then λ^n & X are eigen value & eigen vectors of A^n

Theorem

$$[A-\lambda_1 I]X = 0 \text{ i.e. } [A-5I]X = 0$$

If λ & X are eigen value & eigen vector of A then $\frac{1}{\lambda}$ & X are eigen value & eigen vector of A^{-1} (Provide $|A| \neq 0$)

i.e., $$\left\{\begin{bmatrix}1 & 2\\4 & 3\end{bmatrix} - 5\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right\}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\therefore \begin{bmatrix}1-5 & 2\\4 & 3-5\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \text{ i.e. } \begin{bmatrix}-4 & 2\\4 & -2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

By $R_2 + R_1$ $$\begin{bmatrix}-4 & 2\\0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$\therefore\ -4x_1 + 2x_2 = 0 \quad \therefore\ x_2 = 2x_1.$

Putting $x_1 = t$, we get $x_2 = 2t$, we get, the vector $\begin{bmatrix}t\\2t\end{bmatrix} = t\begin{bmatrix}1\\2\end{bmatrix} \quad \therefore\ X = \begin{bmatrix}1\\2\end{bmatrix}$

or any non-zero multiple k of this vector is the *eigen-vector* corresponding to $\lambda = 5$.

Note .... 1) If $|A| \neq 0$ the $\lambda \neq 0$ // 2) If $|A| = 0$ then atleast one $\lambda = 0$

From (1) we get, $\boxed{AX = \lambda_1 X.}$

Example 1 : If one eigenvalue of a matrix A is $a + ib$ then another eigenvalue must be $a - ib$.

(M.U. 2001)

Sol. : Since eigenvalues are the roots of (its characteristic) equation, the complex roots occur in pairs. Hence, if $a + ib$ is one eigenvalue then another eigenvalue must $a - ib$.

Example 2 : Find the sum and the product of the eigenvalues of A where

$$A = \begin{bmatrix}a_1 & a_2 & a_3\\b_1 & b_2 & b_3\\c_1 & c_2 & c_3\end{bmatrix}$$ **(M.U. 2004)**

Sol. : The characteristic equation is

Sum of diagonal elements

$$\begin{bmatrix}a_1-\lambda & a_2 & a_3\\b_1 & b_2-\lambda & b_3\\c_1 & c_2 & c_3-\lambda\end{bmatrix} = 0$$

minor of a_1, b_2, c_3 Trick

Determinant of A

If we expand the determinant on the l.h.s. then we get,

$$\lambda^3 - (a_1 + b_2 + c_3)\lambda^2 + (a_1b_2 + b_2c_3 + c_3a_1 - a_2b_1 - a_3c_1 - b_3c_2)\lambda - |A| = 0$$

By the theory of roots of an equation, (If $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots.. + a_n = 0$ is the given equation then (i) sum of the roots $= -a_1/a_0$ and (ii) product of the roots $= (-1)^n \dfrac{a_n}{a_0}$)

Sum of the eigenvalues $= a_1 + b_2 + c_3 =$ Sum of the diagonal elements

Product of the eigenvalues $= |A|$.

Note ....

The result can be generalised for a square matrix of order n.

$\boxed{\text{Formula}}$ — sum of EV = sum of dia

Product of EV = $|A|$

Singular matrix : $|A| = 0$ $\quad |A - \lambda I| = 0$

and not $|A| = 0$

Example 3 : If $A = \begin{bmatrix} x & 4x \\ 2 & y \end{bmatrix}$ has eigenvalues 5 and -1, find the values of x and y.

always

(M.U. 2014)

Sol. : By Ex. 2 above, we have

Trick

$$\left\{\begin{array}{l} \text{Sum of the eigenvalues = Sum of the diagonal elements} \\ \text{And Product of the eigenvalues} = |A|. \end{array}\right.$$

$\therefore \quad x + y = 5 + (-1) = 4 \quad$ and $\quad xy - 8x = 5(-1) = -5$

But $\quad y = 4 - x$,

$\therefore \quad x(4 - x) - 8x = -5 \qquad \therefore \quad -x^2 - 4x + 5 = 0$

$\therefore \quad x^2 + 4x - 5 = 0 \qquad \therefore \quad (x + 5)(x - 1) = 0 \qquad \therefore \quad x = -5$ or 1.

When $x = -5$, $y = 9$. When $x = 1$, $y = 3$.

Remember

Example 4 : If A is a non-singular square matrix of order n having eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, prove that (i) $\lambda_1 + \lambda_2 + \ldots + \lambda_n = (a_{11} + a_{22} + \ldots + a_{nn})$ and (ii) $|A| = \lambda_1, \lambda_2, \ldots \lambda_n$. (M.U. 2000)

Sol. : As above.

Imp

Example 5 : If a matrix A is singular then at least one of the eigenvalues is zero and vice versa. (M.U. 2001)

$\hookrightarrow |A| = 0 \rightarrow$ squishes it by one dimension

Sol. : As proved above if $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigenvalues then $|A| = \lambda_1 \lambda_2 \ldots \lambda_n$

But $|A| = 0$ $\therefore$ At least one of $\lambda_1, \lambda_2, \ldots \lambda_n$ must be zero and conversely. [See Ex. 1, page 5-14. One eigenvalue is zero and $|A| = 0$.]

Example 6 : Find the sum and the product of the eigenvalues of the matrix

$(A - \lambda I)\vec{v} = \vec{0}$

if $\lambda = 0 \qquad A\vec{v} = \vec{0}$

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

7:20 Eigenvect E/4

3blue1brown

as A is singular

Sol. : As proved above,

Sum of the eigenvalues = Sum of the diagonal element

$= 8 + 7 + 3 = 18$

Product of the eigenvalues $= |A|$

$= 8(21 - 16) + 6(-18 + 8) + 2(24 - 14)$

$= 40 - 60 + 20 = 0$

Note ....

Note that one of the eigenvalues of A is 0. See Ex. 1, page 5-14. This verifies Ex. 5 above and $|A| = 0$.

Remarks ....

1. The actual eigenvalues are 0, 3, 15.
2. Ex. 6 varifies the result of Ex. 5.

Example 7 : If $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of the matrix $\begin{bmatrix} -2 & -9 & 5 \\ -5 & -10 & 7 \\ -9 & -21 & 14 \end{bmatrix}$

then find $\lambda_1 + \lambda_2 + \lambda_3$ and $\lambda_1 \lambda_2 \lambda_3$. **(M.U. 2000)**

Sol. : As proved above,

Sum of the eigenvalues $= \lambda_1 + \lambda_2 + \lambda_3$

$=$ Sum of the diagonal elements

$= -2 - 10 + 14 = 2$

Product of the eigenvalues $= \lambda_1 \lambda_2 \lambda_3$

$= |A|$

$= -2(-140 + 144) + 9(-70 + 63) + 5(105 - 90)$

$= -8 - 63 + 75 = 4$

Example 8 : If $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$,

where a, b, c are positive integers, then prove that (i) $a + b + c$ is an eigenvalue of A and (ii) if A is non-singular, one of the eigenvalues is negative.

(See Ex. 24, page 5-39)

Sol. : The characteristic equation is

$$\begin{vmatrix} a-\lambda & b & c \\ b & c-\lambda & a \\ c & a & b-\lambda \end{vmatrix} = 0$$

NOTE ✓

By $C_1 + C_2 + C_3$

$$\begin{vmatrix} a+b+c-\lambda & b & c \\ a+b+c-\lambda & c-\lambda & a \\ a+b+c-\lambda & a & b-\lambda \end{vmatrix} = 0 \quad \therefore \ (a+b+c-\lambda)\begin{vmatrix} 1 & b & c \\ 1 & c-\lambda & a \\ 1 & a & b-\lambda \end{vmatrix} = 0$$

$\therefore \ a + b + c - \lambda = 0$ $\qquad \therefore \ \lambda = a + b + c.$

Further if $\lambda_1, \lambda_2, \lambda_3$ are eigenvalues of A, then

$\lambda_1 + \lambda_2 + \lambda_3 = a + b + c$ [By Ex. 2]

But one value, say $\lambda_1 = a + b + c$

$\therefore \ \lambda_2 + \lambda_3 = 0$ .... (1)

Since A is non-singular

$\lambda_1 \lambda_2 \lambda_3 = |A| \neq 0$ .... (2)

$\therefore$ From (1) and (2) it is clear that one of λ_2 and λ_3 is negative.

Example 9 : Show that the following matrices have the same characteristic equation ★

$$\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}, \begin{bmatrix} b & c & a \\ c & a & b \\ a & b & c \end{bmatrix}, \begin{bmatrix} c & a & b \\ a & b & c \\ b & c & a \end{bmatrix}$$ **(M.U. 2010)**

i j k j k i k i j

Sol. : The second matrix can be obtained from the first by interchanging R_1 and R_2 then R_2 and R_3.

The third matrix canbe obtained from the first by interchaning R_1 and R_3 and then R_2 and R_3.

Since the interchange of two rows (or two columns) changes the sign of the determinant, the resulting determinant in the above two cases remains the same as there are two changes.

Hence, the resulting equation remains the same (and the roots of the equation *i.e.* eigenvalues are the same).

Note ....

The eigenvalues of the above three matrices are the same.

Example 10 : Two of the eigenvalues of a 3 × 3 matrix are – 1, 2. If the determinant of the matrix is 4, find its third eigenvalue. **(M.U. 2000)**

Sol. : If the third eigenvalue is x then their product is equal to 4.

$$\therefore (-1)(2)(x) = 4 \quad \therefore \quad x = -2$$

Hence, the third eigenvalue is – 2.

Example 11 : If A and B are two square matrices of the same order then AB and BA have the same characteristic roots.

Sol. : We accept this result without giving proof but we shall verify it by an example.

Example 12 : If A and B are as given below, find the eigenvalues of AB and BA.

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 \\ 5 & 6 \end{bmatrix}$$

Sol. : $AB = \begin{bmatrix} 3 & 0 \\ 26 & 24 \end{bmatrix}, \quad BA = \begin{bmatrix} 3 & 0 \\ 17 & 24 \end{bmatrix}$

The characteristic equations are

$$\begin{vmatrix} 3-\lambda & 0 \\ 26 & 24-\lambda \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} 3-\lambda & 0 \\ 17 & 24-\lambda \end{vmatrix} = 0$$

$$\therefore (3-\lambda)(24-\lambda) = 0 \quad \text{and} \quad (3-\lambda)(24-\lambda) = 0.$$

The eigenvalues of both AB and BA are 3, 24.

6. Procedure to Find Eigenvalues and Eigenvectors of a Given Matrix A of A in λ

1. First write the characteristic equation $|A - \lambda I| = 0$. : characteristic eq
2. Solve the above equation and find the roots, say, $\lambda_1, \lambda_2, \lambda_3, \ldots$
3. For $\lambda = \lambda_1$, consider the matrix equality, $[A - \lambda_1 I]\bar{X} = \bar{0}$. ⇐ to find eigen vector
4. Apply **elementary row transformations** and **reduce the above matrix** $[A - \lambda_1 I]$ to **echelon form.**
5. Now write the equations. If there are n unknowns and the rank of the matrix is r, there will be $(n - r)$ independent parameters.
6. Sometimes you will require to use, Cramer's rule.
7. Solve the equations and get the eigenvectors $X_1 = \ldots, X_2 = \ldots$

$\boxed{|A| = \lambda_1 \lambda_2 \lambda_3}$ (Determinant = product of eigen value)

The following examples will make the procedure clear.

asymmetric

Type I : A is a non-symmetric matrix and eigenvalues are distinct

Example 1 : Find the eigenvalues and eigenvectors of the following matrix. [Verify that the eigenvectors are linearly independent.]

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$ **(M.U. 2004, 05)**

Sol. : The characteristic equation is

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ 1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$\lambda^3 - (6)\lambda^2 + (3+3+5) - 6$

$\therefore\ (2-\lambda)[(2-\lambda)^2 - 1] + 1[1(2-\lambda) + 1] + 1[-1-(2-\lambda)] = 0$

$\therefore\ \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$ $\quad\therefore\ \lambda^3 - \lambda^2 - 5\lambda^2 + 5\lambda + 6\lambda - 6 = 0$

$\therefore\ (\lambda - 1)(\lambda^2 - 5\lambda + 6) = 0$ $\quad\therefore\ (\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$

$\therefore\ \lambda = 1, 2, 3.$

(i) For $\lambda = 1$, $[A - \lambda_1 I]X = O$ gives

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We may use matrix method to obtain the roots of the above equation

By $\begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix}$ $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore\ x_1 - x_2 + x_3 = 0$ $\quad\therefore\ 2x_2 - 2x_3 = 0$ *i.e.* $x_2 - x_3 = 0$

We note that the rank of the matrix is 2 and the number of variables is 3. Hence, there is 3 − 2 = 1 linearly independent solution.

Putting $x_3 = t$, we get $x_2 = t$ and $x_1 = x_2 - x_3 = t - t = 0$.

$$\therefore\ X = \begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Hence, corresponding to $\lambda = 1$, the eigenvector is $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

[The eigenvector can be denoted as a column vector $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ or as the transpose of the row vector $[0, 1, 1]'$.]

Formulas to find cubic eqn in matrix

$\lambda^3 - [\text{sum of Diagonal}]\lambda^2 + [\text{sum of minors} \ldots]\lambda - |A| = 0$

(ii) For $\lambda = 2$, $[A - \lambda_2 I]\, X = O$ gives

$$\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \; -x_2 + x_3 = 0, \quad x_1 - x_3 = 0, \quad x_1 - x_2 = 0$

Putting $x_2 = t$, we get $x_1 = t$, $x_3 = t$.

$\therefore \; X = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Hence, corresponding to $\lambda = 2$, the eigenvector is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(iii) For $\lambda = 3$, $[\, A - \lambda_3 I\,] = O$ gives

$$\begin{bmatrix} -1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } \begin{matrix} R_2 + R_1 \\ R_2 + R_3 \end{matrix} \begin{bmatrix} -1 & -1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \; -x_1 - x_2 + x_3 = 0 \quad \therefore \; -2x_2 = 0 \quad \therefore \; x_2 = 0.$

Putting $x_3 = t$, we get $x_1 = t$.

$\therefore \; X = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Hence, corresponding to $\lambda = 3$, the eigenvector is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

[**Definition : Vectors X_1, X_2, X_3 are said to be independent if $k_1X_1 + k_2X_2 + k_3X_3 = 0$ implies that $k_1 = 0$, $k_2 = 0$, $k_3 = 0$ for all values of k_1, k_2, k_3.**

Here, $k_1X_1 + k_2X_2 + k_3X_3 = 0$ gives

$k_1\,[\,0, 1, 1\,] + k_2\,[\,1, 1, 1\,] + k_3\,[\,1, 0, 1\,] = 0.$

$\therefore \; 0\,k_1 + k_2 + k_3 = 0$ .... (i)

$k_1 + k_2 + 0\,k_3 = 0$ .... (ii)

and $k_1 + 0\,k_2 + k_3 = 0$ .... (iii)

From (i), $k_2 + k_3 = 0$ and from (ii) $k_1 + k_2 = 0$ and from (iii) $k_1 + k_3 = 0$.

Subtracting (iii) from (i), we get, $k_2 - k_1 = 0$.

Adding this to (ii), we get,

$k_2 = 0 \quad \therefore \; k_1 = 0 \quad \therefore \; k_3 = 0.$

$\therefore$ The vectors are linearly independent.]

Example 2 : *Find the eigenvalues and eigenvectors of the following matrix.*

$$A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$$

(M.U. 2010, 16)

Sol. : The characteristic equation is

$$\begin{vmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0$$

$\therefore \quad (8-\lambda)[(3+\lambda)(\lambda-1)-8]+8[4-4\lambda+6]-2[-16+9+3\lambda]=0$

$\therefore \quad \lambda^3-6\lambda^2+11\lambda-6=0$

$\therefore \quad \lambda^3-\lambda^2-5\lambda^2+5\lambda+6\lambda-6=0$

$\therefore \quad (\lambda-1)(\lambda^2-5\lambda+6)=0$

$\therefore \quad (\lambda-1)(\lambda-2)(\lambda-3)=0$

$\therefore \quad \lambda=1,\ 2,\ 3.$

(I) For $\lambda=1$, $[A-\lambda_1 I]\,X=O$ gives

$$\begin{bmatrix} 7 & -8 & -2 \\ 4 & -4 & -2 \\ 3 & -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We may obtain the eigenvector by solving simultaneous equations obtained from the above matrix equation.

$\therefore$ From the first two rows, we get

$7x_1-8x_2-2x_3=0\,; \qquad 4x_1-4x_2-2x_3=0$

Solving by Cramer's rule, ⟸ we do this when all eigen values are diff.

$$\frac{x_1}{\begin{vmatrix} -8 & -2 \\ -4 & -2 \end{vmatrix}}=\frac{-x_2}{\begin{vmatrix} 7 & -2 \\ 4 & -2 \end{vmatrix}}=\frac{x_3}{\begin{vmatrix} 7 & -8 \\ 4 & -4 \end{vmatrix}}$$

$\therefore \quad \dfrac{x_1}{8}=\dfrac{x_2}{6}=\dfrac{x_3}{4} \qquad \therefore \quad \dfrac{x_1}{4}=\dfrac{x_2}{3}=\dfrac{x_3}{2}=t$, say

$\therefore \quad x_1=4t,\ x_2=3t,\ x_3=2t$

$\therefore \quad X=\begin{bmatrix} 4t \\ 3t \\ 2t \end{bmatrix}=t\begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$. Hence, corresponding to $\lambda=1$, the eigenvector is $\begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$.

[If k is a non-zero scalar then kX also is an eigenvector. The eigenvector can be denoted as a column matrix or as the transpose of a row matrix.]

(II) For $\lambda=2$, $[A-\lambda_2 I]\,X=O$ gives

$$\begin{bmatrix} 6 & -8 & -2 \\ 4 & -5 & -2 \\ 3 & -4 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From the first two rows, we get,

$6x_1-8x_2-2x_3=0\,; \qquad 4x_1-5x_2-2x_3=0$

Solving by Cramer's rule,

$$\frac{x_1}{\begin{vmatrix} -8 & -2 \\ -5 & -2 \end{vmatrix}}=\frac{-x_2}{\begin{vmatrix} 6 & -2 \\ 4 & -2 \end{vmatrix}}=\frac{x_3}{\begin{vmatrix} 6 & -8 \\ 4 & -5 \end{vmatrix}}$$

$\therefore \quad \dfrac{x_1}{6}=\dfrac{x_2}{4}=\dfrac{x_3}{2} \qquad \therefore \quad \dfrac{x_1}{3}=\dfrac{x_2}{2}=\dfrac{x_3}{1}=t$, say

$\therefore \quad x_1=3t,\ x_2=2t,\ x_3=t.$

$$\therefore \quad X = \begin{bmatrix} 3t \\ 2t \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$ Hence, corresponding to $\lambda = 2$, the eigenvector is $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

(If k is a non-zero scalar then kX also is an eigenvector.)

(iii) For $\lambda = 3$, $[A - \lambda_3 I]\, X = O$ gives

$$\begin{bmatrix} 5 & -8 & -2 \\ 4 & -6 & -2 \\ 3 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From the first two rows, we get,

$$5x_1 - 8x_2 - 2x_3 = 0 \; ; \qquad 4x_1 - 6x_2 - 2x_3 = 0$$

Solving by Cramer's rule

$$\frac{x_1}{\begin{vmatrix} -8 & -2 \\ -6 & -2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 5 & -2 \\ 4 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 5 & -8 \\ 4 & -6 \end{vmatrix}}$$

$$\therefore \quad \frac{x_1}{4} = \frac{x_2}{2} = \frac{x_3}{2} \qquad \therefore \quad \frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{1} = t, \text{ say}$$

$$\therefore \quad x_1 = 2t, \; x_2 = t, \; x_3 = t.$$

$$\therefore \quad X = \begin{bmatrix} 2t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$ Hence, corresponding to $\lambda = 3$, the eigenvector is $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

(If k is a non-zero scalar then kX also is an eigenvector.)

Remark ....

Since there is no 1 in the first column Cramer's rule is more convenient than elementary operations.

Type II : *A* is a non-symmetric matrix and eigenvalues are repeated

Example 1 : Find the eigenvalues and eigenvectors of the following matrix.

See in bk ~~also~~
& up also

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

(M.U. 2001, 09)

Sol. : The characteristic equation is

$$\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

After simplification, we get,

$$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0 \qquad \therefore \quad (\lambda - 1)(\lambda - 1)(\lambda + 5) = 0 \qquad \therefore \quad \lambda = 1, 1, 5$$

Hence, 1, 1, 5 are the eigenvalues.

(i) For $\lambda = 1$, $[A - \lambda_1 I]X = O$ gives

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore\ x_1 + 2x_2 + x_3 = 0.$

We see that the rank of the matrix is 1 and number of variables is 3. Hence, there are $3 - 1 = 2$ linearly independent solutions *i.e.*, there are two parameters. We shall denote these parameters by s and t.

Putting $x_2 = -s$, $x_3 = -t$, we get $x_1 = -2x_2 - x_3 = 2s + t$

$$\therefore\ X = \begin{bmatrix} 2s+t \\ -s+0 \\ 0-t \end{bmatrix} = s\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \therefore\ X_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix},\ X_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

The vectors $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ are linearly independent.

[Let us verify that the vectors X_1 and X_2 are independent.

Consider $K_1X_1 + K_2X_2 = 0$.

$\therefore\ K_1(2, -1, 0) + K_2(1, 0, -1) = 0$

$\therefore\ 2K_1 + K_2 = 0,\ -K_1 = 0,\ -K_2 = 0$

Hence, the vectors are linearly independent.]

Hence, corresponding to $\lambda = 1$, the eigenvectors are $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

(ii) For $\lambda = 5$, $[A - \lambda_2 I]X = O$ gives

$$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } R_{13} \begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ -3 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By } \begin{matrix} R_2 - R_1 \\ R_3 + 3R_1 \end{matrix} \begin{bmatrix} 1 & 2 & -3 \\ 0 & -4 & 4 \\ 0 & 8 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } R_3 + 2R_2 \begin{bmatrix} 1 & 2 & -3 \\ 0 & -4 & 4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore\ x_1 + 2x_2 - 3x_3 = 0$ and $-4x_2 + 4x_3 = 0$.

Putting $x_3 = t$, we get $x_2 = t$ and $x_1 = -2x_2 + 3x_3 = -2t + 3t = t$.

$\therefore\ X = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Hence, corresponding to $\lambda = 5$, the eigenvector is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Note ....

When the matrix A is non-symmetric and the eigenvalues are repeated, the eigenvectors corresponding to **repeated root** may or may not be linearly independent. Verify that the vectors X_1 and X_2 are independent. See Examples 1, 2, 3. In Ex. 1, page 5-23, we see that the eigen vectors corresponding to the repeated root $\lambda = 2$ are **not** independent.

Example 2 : Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$ (M.U. 2016)

Sol. : The characteristic equation is

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

$\therefore\ (-2-\lambda)[(1-\lambda)(-\lambda)-(-2)(-6)]-2[-2\lambda-6]-3[-4+1(1-\lambda)]=0$

$\therefore\ (-2-\lambda)[-\lambda+\lambda^2-12]+2[2\lambda+6]+3[3+\lambda]=0$

$\therefore\ (2+\lambda)[12+\lambda-\lambda^2]+4\lambda+12+9+3\lambda=0$

$\therefore\ 24+2\lambda-2\lambda^2+12\lambda+\lambda^2-\lambda^3+7\lambda+21=0$

$\therefore\ -\lambda^3-\lambda^2+21\lambda+45=0$

$\therefore\ \lambda^3+\lambda^2-21\lambda-45=0$ $\quad\therefore\ \lambda^3-5\lambda^2+6\lambda^2-30\lambda+9\lambda-45=0$

$\therefore\ \lambda^2(\lambda-5)+6(\lambda-5)+9(\lambda-5)=0$ $\quad\therefore\ (\lambda-5)(\lambda^2+6\lambda+9)=0$

$\therefore\ (\lambda-5)(\lambda+3)^2=0$ $\quad\therefore\ \lambda=5,\ -3,\ -3.$

Hence, the eigenvalues are 5, – 3, – 3.

(i) For $\lambda=5$, $[A-\lambda_1 I]X=O$ gives

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } \begin{matrix} R_{13} \\ -R_1 \end{matrix} \begin{bmatrix} 1 & 2 & 5 \\ 2 & -4 & -6 \\ -7 & 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By } \begin{matrix} R_2-2R_1 \\ R_3+7R_1 \end{matrix} \begin{bmatrix} 1 & 2 & 5 \\ 0 & -8 & -16 \\ 0 & 16 & 32 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } \begin{matrix} R_3+2R_2 \\ -\frac{1}{8}R_2 \end{matrix} \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore\ x_1+2x_2+5x_3=0$ and $x_2+2x_3=0.$

Putting $x_3=t$, we get $x_2=-2t$.

Then $x_1-4t+5t=0$ $\quad\therefore\ x_1=-t.$

Changing the sign of t, $x_1=t,\ x_2=2t,\ x_3=-t.$

$\therefore\ X=\begin{bmatrix} t \\ 2t \\ -t \end{bmatrix}=t\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$. Hence, corresponding to $\lambda=5$, the eigenvector is $X=\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$.

(ii) For $\lambda=-3$, $[A-\lambda_2 I]X=O$ gives

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } \begin{matrix} R_2-2R_1 \\ R_3+R_1 \end{matrix} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore\ x_1+2x_2-3x_3=0$

We see that the rank of the matrix is 1 and the number of variables is 3. Hence, there are $3-1=2$ linearly independent solutions.

Putting $x_3 = s$, $x_2 = -t$, we get $x_1 - 2t - 3s = 0$ $\therefore$ $x_1 = 2t + 3s$.

$$\therefore \quad X = \begin{bmatrix} 2t+3s \\ -t+0s \\ 0t+s \end{bmatrix} = t\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \quad \therefore \quad X_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

[The vectors are $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ linearly independent. (Verify it.)]

Example 3 : Find the eigenvalues and eigenvectors of the following matrix.

$$\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$$

(M.U. 1996, 2005, 14, 16)

Sol. : The characteristic equation is

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 2 & 3-\lambda & 2 \\ 3 & 3 & 4-\lambda \end{vmatrix} = 0$$

$$\therefore \quad (2-\lambda)[(3-\lambda)(4-\lambda)-6] - 1[2(4-\lambda)-6] + 1[6-3(3-\lambda)] = 0$$

$$\therefore \quad \lambda^3 - 9\lambda^2 + 15\lambda - 7 = 0 \qquad \therefore \quad \lambda^3 - \lambda^2 - 8\lambda^2 - 8\lambda + 7\lambda - 7 = 0$$

$$\therefore \quad (\lambda-1)(\lambda^2 - 8\lambda + 7) = 0 \qquad \therefore \quad (\lambda-1)(\lambda-1)(\lambda-7) = 0$$

$$\therefore \quad \lambda = 1, 1, 7.$$

(i) For $\lambda = 1$, $[A - \lambda_1 I] X = O$ gives

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{By } \begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad x_1 + x_2 + x_3 = 0$$

We see that the rank of the matrix is 1 and the number of variables is 3. Hence, there are $3 - 1 = 2$ linearly independent solutions.

Putting $x_2 = -s$, $x_3 = -t$, we get $x_1 = -x_2 - x_3 = s + t$.

$$\therefore \quad X = \begin{bmatrix} s+t \\ -s+0 \\ 0-t \end{bmatrix} = s\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \therefore \quad X_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \text{ and } X_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Now, vectors $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ are linearly independent. (Verify it)

Hence, corresponding to $\lambda = 1$, the eigenvectors are $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

Further $k X_1 + k X_2$ is also an eigenvector.

(ii) For $\lambda = 7$, $[A - \lambda_2 I]X = O$ gives

$$\begin{bmatrix} -5 & 1 & 1 \\ 2 & -4 & 2 \\ 3 & 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By} \quad \begin{matrix} R_2 - 2R_1 \\ R_3 + 3R_1 \end{matrix} \begin{bmatrix} -5 & 1 & 1 \\ 12 & -6 & 0 \\ -12 & 6 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \quad \begin{matrix} R_2 + R_3 \\ (1/6)R_2 \end{matrix} \begin{bmatrix} -5 & 1 & 1 \\ 2 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore\ -5x_1 + x_2 + x_3 = 0$ and $2x_1 - x_2 = 0$.

Putting $x_2 = 2t$, we get $2x_1 = x_2 = 2t$ $\therefore$ $x_1 = t$.

$\therefore\ x_3 = 5x_1 - x_2 = 5t - 2t = 3t$

$\therefore\ X = \begin{bmatrix} t \\ 2t \\ 3t \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Hence, corresponding to $\lambda = 7$, the eigenvector is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Type III : A is symmetric and eigenvalues are distinct

Example 1 : Find the eigenvalues and eigenvectors of the following matrix.

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$ **(M.U. 2003)**

Sol. : (a) The characteristic equation is

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

On simplification, we get,

$$\lambda^3 - 18\lambda^2 + 45\lambda = 0 \quad \therefore\ \lambda(\lambda - 3)(\lambda - 15) = 0 \quad \therefore\ \lambda = 0, 3, 15.$$

Hence, 0, 3, 15 are the eigenvalues.

(i) For $\lambda = 0$. $[A - \lambda_1 I]X = O$ gives

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By} \quad R_{31} \begin{bmatrix} 2 & -4 & 3 \\ -6 & 7 & -4 \\ 8 & -6 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \quad \begin{matrix} R_2 + 3R_1 \\ R_3 - 4R_1 \end{matrix} \begin{bmatrix} 2 & -4 & 3 \\ 0 & -5 & 5 \\ 0 & 10 & -10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } R_3 + 2R_1 \begin{bmatrix} 2 & -4 & 3 \\ 0 & -5 & 5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore\ 2x_1 - 4x_2 + 3x_3 = 0$

$-5x_2 + 5x_3 = 0$ *i.e.* $x_2 = x_3$

We note that the rank of the matrix is 2 and the number of variables is 3. Hence, there is 3 − 2 = 1 linearly independent solution.

Putting $x_3 = 2t$, we get $x_2 = 2t$. $\therefore\ 2x_1 = 4x_3 - 3x_3 = 8t - 6t = 2t$ $\therefore\ x_1 = t$.

$$\therefore \quad X = \begin{bmatrix} t \\ 2t \\ 2t \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$ Hence, corresponding to $\lambda = 0$, the eiven vector is $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

(If k is a non-zero scalar, then kX is also an eigenvector.)

(ii) For $\lambda = 3$, $[A - \lambda_2 I]\, X = O$ gives

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_1 + R_2$ $\begin{bmatrix} -1 & -2 & -2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ By $\begin{matrix} R_2 - 6R_1 \\ R_3 + 2R_1 \end{matrix}$ $\begin{bmatrix} -1 & -2 & -2 \\ 0 & 16 & 8 \\ 0 & -8 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

By $\begin{matrix} R_3 + (1/2) R_2 \\ -R_1 \end{matrix}$ $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 16 & 8 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore \quad x_1 + 2x_2 + 2x_3 = 0$ and $16x_2 + 8x_3 = 0$ *i.e.* $2x_2 + x_3 = 0$

Putting $x_2 = t$, we get $x_3 = -2x_2 = -2t$. $\quad \therefore \; x_1 = -2x_2 - 2x_3 = -2t + 4t = 2t$.

$$\therefore \quad X = \begin{bmatrix} 2t \\ t \\ -2t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}.$$ Hence, corresponding to $\lambda = 3$, the eigenvector is $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$.

(If k is a non-zero scalar then kX is also an eigenvector.)

(iii) For $\lambda = 15$, $[A - \lambda_3 I]\, X = O$ gives

$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ By $R_1 - R_2$ $\begin{bmatrix} -1 & 2 & 6 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

By $\begin{matrix} R_2 - 6R_1 \\ R_3 + 2R_1 \end{matrix}$ $\begin{bmatrix} -1 & 2 & 6 \\ 0 & -20 & -40 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore \; -x_1 + 2x_2 + 6x_3 = 0 \quad \therefore \; x_1 - 2x_2 - 6x_3 = 0$

$-20x_2 - 40x_3 = 0 \quad x_2 = -2x_3$

Putting $x_3 = t$, $x_2 = -2t$, $\quad \therefore \; x_1 = 2x_2 + 6x_3 = 2t$.

$$\therefore \quad X = \begin{bmatrix} 2t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$ Hence, corresponding to $\lambda = 15$, the eigenvector is $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$.

(If k is a non-zero scalar then kX is also an eigenvector.)

Note ....

The eigen vectors $X_1 = [\,1, 2, 2\,]'$, $X_2 = [\,2, 1, -2\,]'$ and $X_3 = [\,2, -2, 1\,]$ are orthogonal because $X_1 \cdot X_2 = (2 + 2 - 4) = 0$, $X_1 \cdot X_3 = (2 + 2 - 4) = 0$ and $X_2 \cdot X_3 = (4 - 2 - 2) = 0$.

Example 2 : Find the eigenvalues and eigenvectors of the following matrix.

$$\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

(M.U. 1993, 2005)

Sol. : The characteristic equation is

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\therefore \quad (3-\lambda)[(5-\lambda)(3-\lambda)-1]+1[-1(3-\lambda)+1]+1[1-(5-\lambda)]=0$$

$$\therefore \quad \lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

$$\therefore \quad \lambda^3 - 2\lambda^2 - 9\lambda^2 + 18\lambda + 18\lambda - 36 = 0$$

$$\therefore \quad (\lambda-2)(\lambda^2 - 9\lambda + 18) = 0$$

$$\therefore \quad (\lambda-2)(\lambda-3)(\lambda-6) = 0$$

$$\therefore \quad \lambda = 2, 3, 6$$

(i) For $\lambda = 2$, $[A - \lambda_1 I]X = O$ gives

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By} \begin{array}{c} R_2 + R_1 \\ R_3 - R_1 \end{array} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \quad x_1 - x_2 + x_3 = 0,\ x_2 = 0.$

Putting $x_1 = t$, we get $x_3 = -t$.

$$\therefore \quad X = \begin{bmatrix} t \\ 0 \\ -t \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Hence, corresponding to $\lambda = 2$, the eigenvector is $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

(If k is a non-zero scalar kX also is an eigenvector.)

(ii) For $\lambda = 3$, $[A - \lambda_2 I]X = O$ gives

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } R_2 + R_1 \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \quad R_3 + R_2 \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \therefore \ -x_2 + x_3 = 0,\ -x_1 + x_2 = 0.$$

Putting $x_2 = t$, we get $x_1 = t$, $x_3 = t$.

$$\therefore \quad X = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Hence, corresponding to $\lambda = 3$, the eigenvector is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(If k is a non-zero scalar kX is also an eigenvector.)

(iii) For $\lambda = 6$, $[A - \lambda_3 I]\, X = O$ gives

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By} \quad \begin{matrix} R_2 + R_1 \\ R_3 + 3R_1 \end{matrix} \begin{bmatrix} -3 & -1 & 1 \\ -4 & -2 & 0 \\ -8 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \quad \begin{matrix} R_3 - (1/2) R_2 \\ (1/2) R_2 \end{matrix} \begin{bmatrix} -3 & -1 & 1 \\ -2 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \; -3x_1 - x_2 + x_3 = 0; \qquad -2x_1 - x_2 = 0$

Putting $x_2 = -2t$, we get $2x_1 = -x_2 = 2t \quad \therefore \; x_1 = t$

$\therefore \; x_3 = 3x_1 + x_2 = 3t - 2t = t.$

$\therefore \; X = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. Hence, corresponding to $\lambda = 6$, the eigenvector is $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

(If k is a non-zero scalar kX also is an eigenvector.)

(Verify that the vectors are orthogonal.)

Example 3 : Find the eigenvalues and eigenvectors of the matrix

$\boxed{?}$

$$A = \begin{bmatrix} -2 & 5 & 4 \\ 5 & 7 & 5 \\ 4 & 5 & -2 \end{bmatrix} \qquad \textbf{(M.U. 2014)}$$

Sol. : The characteristic equation is

$$\begin{vmatrix} -2-\lambda & 5 & 4 \\ 5 & 7-\lambda & 5 \\ 4 & 5 & -2-\lambda \end{vmatrix} = 0$$

On simplification, we get

$(\lambda + 3)(\lambda - 12)(\lambda + 6) = 0 \qquad \therefore \; \lambda = -3, 12, -6$

$\therefore \; -3, 12, -6$ are the eigenvalues.

(i) For $\lambda = -3$, $[A - \lambda_1 I]\, X = O$ gives

$$\begin{bmatrix} 1 & 5 & 4 \\ 5 & 10 & 5 \\ 4 & 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By} \quad R_2 - R_3 \begin{bmatrix} 1 & 5 & 4 \\ 1 & 5 & 4 \\ 4 & 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \quad R_2 - R_1 \begin{bmatrix} 1 & 5 & 4 \\ 0 & 0 & 0 \\ 4 & 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By} \quad R_{32} \begin{bmatrix} 1 & 5 & 4 \\ 4 & 5 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \quad R_2 - R_1 \begin{bmatrix} 1 & 5 & 4 \\ 3 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \; x_1 + 5x_2 + 4x_3 = 0; \qquad 3x_1 - 3x_3 = 0$

Putting $x_3 = t,\ x_1 = t$ and $5x_2 + 5t = 0 \quad \therefore\ x_2 = -t$.

$$\therefore \quad X = \begin{bmatrix} t \\ -t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$ Hence, corresponding to $\lambda = -3$, the eigenvector is $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

(ii) For $\lambda = 12$, $[A - \lambda_2 I]X = O$ gives

$$\begin{bmatrix} -14 & 5 & 4 \\ 5 & -5 & 5 \\ 4 & 5 & -14 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore\ -14x_1 + 5x_2 + 4x_3 = 0;\quad 5x_1 - 5x_2 + 5x_3 = 0;\quad 4x_1 + 5x_2 - 14x_3 = 0$

Solving the first two equations by Crammer's rule, we get

$$\frac{x_1}{45} = \frac{-x_2}{-90} = \frac{x_3}{45} \qquad \therefore\ \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1} = t$$

$\therefore\ x_1 = t,\ x_2 = 2t,\ x_3 = t.$

$$\therefore \quad X = \begin{bmatrix} t \\ 2t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$ Hence, corresponding to $\lambda = 12$, the eigenvector is $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

(iii) For $\lambda = -6$, $[A - \lambda_3 I]X = O$ gives

$$\begin{bmatrix} 4 & 5 & 4 \\ 5 & 13 & 5 \\ 4 & 5 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore\ 4x_1 + 5x_2 + 4x_3 = 0;\quad 5x_1 + 13x_2 + 5x_3 = 0;\quad 4x_1 + 5x_2 + 4x_3 = 0$

Solving the first two equations by Cramer's rule, we get

$$\frac{x_1}{-27} = \frac{-x_2}{0} = \frac{x_3}{27} \qquad \therefore\ \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{-1} = t$$

$$\therefore \quad X = \begin{bmatrix} t \\ 0t \\ -t \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$ Hence, corresponding to $\lambda = -6$, the eigenvector is $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

(Verify that the vectors are orthogonal.)

Type IV : *A* is symmetric and eigenvalues are repeated

Example 1 : Find the eigenvalues and eigenvectors of the following matrix.

$$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$ **(M.U. 1993, 96, 2003)**

Sol. : The characteristic equation is

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

On simplification, we get, $(2-\lambda)(\lambda-2)(\lambda-8)=0$

Hence, 2, 2, 8 are the eigenvalues. $\therefore \lambda = 2, 2, 8.$

For $\lambda = 2$, $[A-\lambda_1 I]X = O$ gives

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } R_{12} \begin{bmatrix} -2 & 1 & -1 \\ 4 & -2 & 2 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $\begin{matrix} R_2 + 2R_1 \\ R_3 + R_1 \end{matrix}$ $$\begin{bmatrix} -2 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \; -2x_1 + x_2 - x_3 = 0 \quad i.e. \quad 2x_1 - x_2 + x_3 = 0$

We again see that the rank of the matrix is 1 and number of variables is 3. Hence, there are 3 − 1 = 2 linearly independent solutions.

Putting $x_2 = 2s$ and $x_3 = -2t$, we get

$2x_1 = x_2 - x_3 = 2s + 2t \qquad \therefore \; x_1 = s + t$

$$\therefore \; X = \begin{bmatrix} s+t \\ 2s+0 \\ 0-2t \end{bmatrix} = s\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

The vectors $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ are linearly independent. (Verifity it.)

Hence, corresponding to $\lambda = 2$, the eigenvectors are $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$.

Further $k_1X + k_2X_2$ is also an eigenvector.

(ii) For $\lambda = 8$, $[A-\lambda_2 I]X = O$ gives,

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } \begin{matrix} R_2 - R_1 \\ R_3 + R_1 \end{matrix} \begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_3 - R_2$ $$\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \; -2x_1 - 2x_2 + 2x_3 = 0 \quad i.e., \quad x_1 + x_2 - x_3 = 0$

and $\; -3x_2 - 3x_3 = 0 \quad i.e., \quad x_2 = -x_3$

Putting $x_3 = t$, we get $x_2 = -t \qquad \therefore \; x_1 = -x_2 + x_3 = t + t = 2t.$

$$\therefore \; X = \begin{bmatrix} 2t \\ -t \\ t \end{bmatrix} = t\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}.$$ Hence, corresponding to $\lambda = 8$, the eigenvectors is $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.

(If k is a non-zero scalar then kX also is an eigenvector.)

(ii) For $\lambda = 5$, $[A - \lambda_2 I]X = O$ gives

$$\begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The first two rows give

$$-4x_1 + 2x_2 + 2x_3 = 0 ; \qquad 2x_1 - 4x_2 + 2x_3 = 0$$

i.e. $\quad -2x_1 + x_2 + x_3 = 0 ; \qquad x_1 - 2x_2 + x_3 = 0$

By Crammer's rule

$$\frac{x_1}{1+2} = \frac{-x_2}{-2-1} = \frac{x_3}{4-1}$$

$$\therefore \frac{x_1}{3} = \frac{x_2}{3} = \frac{x_3}{3} \quad \therefore \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1} = t, \text{ say} \qquad \therefore x_1 = t,\ x_2 = t,\ x_3 = t.$$

$\therefore X = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Hence, corresponding to $\lambda = 5$, the eigenvectors is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Remark ....

Verify that the vectors corresponding to **distinct** eigenvalues of a symmetric matrix are orthogonal *i.e.* $(X_1\ X_3) = 0$ and $(X_2\ X_3) = 0$. (See Theorem 1, page 5-35)

Type V : *A* has repeated Eigenvalues but same eigenvectors

Example 1 : Find the eigenvalues and eigenvectors of the following matrix.

$$A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$$ **(M.U. 2015, 18)**

Sol. : The characteristic equation is

$$\begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -5 & -2-\lambda \end{vmatrix} = 0$$

On simplification, we get

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0 \qquad \therefore (\lambda - 1)(\lambda - 2)(\lambda - 2) = 0 \qquad \therefore \lambda = 1, 2, 2.$$

(I) For $\lambda = 1$, $[A - \lambda_1 I]X = O$ gives

$$\begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ -1 & -5 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } \frac{1}{3}R_1 \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \\ -1 & -5 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By } R_2 - R_1 \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 0 \\ -1 & -5 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } \begin{matrix} R_2 - R_1 \\ R_3 + R_1 \\ R_{2,3} \end{matrix} \begin{bmatrix} 1 & 2 & 2 \\ 0 & -3 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore x_1 + 2x_2 + 2x_3 = 0 \text{ and } -3x_2 - x_3 = 0.$$

Putting $x_3 = -3t$, we get, $3x_2 = 3t$ $\quad \therefore \; x_2 = t$.

Then $x_1 = -2x_2 - 2x_3 = -2t + 6t = 4t$.

$$\therefore \; X = \begin{bmatrix} 4t \\ t \\ -3t \end{bmatrix} = t \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix}$$

Since, the rank is 2 and number of variables is 3 there is only $3 - 2 = 1$. linearly independent solution.

Hence, corresponding to $\lambda = 1$, the eigenvector is $\begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix}$.

ii) For $\lambda = 2$, $[A - \lambda_2 I]X = O$ gives

$$\begin{bmatrix} 2 & 6 & 6 \\ 1 & 1 & 2 \\ -1 & -5 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } R_{1,2} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 6 & 6 \\ -1 & -5 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By } R_3 - R_1 + R_2 \begin{bmatrix} 1 & 1 & 2 \\ 2 & 6 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } R_2 - 3R_1 \begin{bmatrix} 1 & 1 & 2 \\ -1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore x_1 + x_2 + 2x_3 = 0$ and $-x_1 + 3x_2 = 0$.

Putting $x_1 = 3t$, we get,

$3x_2 = 3t$ $\quad \therefore \; x_2 = t$.

Then from $x_1 + x_2 + 2x_3 = 0$, we get,

$3t + t + 2x_3 = 0$ $\quad \therefore \; x_3 = -2t$.

$$\therefore \; X = \begin{bmatrix} 3t \\ t \\ -2t \end{bmatrix} = t \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} \quad \therefore \; X_3 = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$$

Since, the number variables is 3 and the rank of the matrix is 2, there is $3 - 2 = 1$ independent solution.

Hence, corresponding to $\lambda = 2$, the eigenvector is $\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$.

Note ....

Corresponding to the repeated root 2, we get only one eigenvector. The other vector is dependent on this vector. It may be taken as any multiple of this vector. For $\lambda = 2$, the eigen vectors **are not independent.**

Example 2 : Find the eigenvalues and eigenvectors of the following matrix.

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Sol. : The characteristic equation is

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

On simplification, we get

$$(2-\lambda)^3 = 0 \qquad \therefore \lambda = 2, 2, 2.$$

For $\lambda = 2$ $[A - \lambda_1 I]X = O$ gives

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore\ 0x_1 + x_2 + 0x_3 = 0\ ;\qquad 0x_1 + 0x_2 + x_3 = 0$

By Cramer's rule

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{0} = t, \text{ say.} \quad \therefore\ x_1 = t,\ x_2 = 0t \text{ and } x_3 = 0t. \quad \therefore\ X = \begin{bmatrix} t \\ 0t \\ 0t \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

We note for the last time that the rank of the matrix is 2 and the number of variables is 3. Hence, there is $3 - 2 = 1$ linearly independent solution.

Hence, corresponding to $\lambda = 2$, the eigenvector is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

7. Certain Relations Between Eigenvalues and Eigenvectors

We state below certain theorems without proof.

Theorem 1 : ***λ is an eigenvalue of the matrix A if and only if there exists a non-zero vector X such that $AX = \lambda X$.***

Theorem 2 : ***If X is an eigenvector of a matrix A corresponding to an eigenvalue λ then kX (k is a non-zero scalar) is also an eigenvector of A corresponding to the same eigenvalue λ.***

Theorem 3 : **(Uniqueness of Eigenvector) :** ***If X is an eigenvector of a matrix A, then X cannot correspond to more than one eigenvalues of A.***

Theorem 4 : **(Linear Independence of Eigenvectors) :** ***Eigenvectors corresponding to distinct eigenvalues of a matrix are linearly independent.***

(Eigenvectors corresponding to a repeated eigenvalue may or may not be linearly independent. See note page 5-11.)

Theorem 5 : ***The eigenvectors of a real symmetric matrix whose eigenvalues are distinct are orthogonal.***

(For proof, see page 5-35) (See Ex. 1, page 5-14.)

8. Certain Results about the Nature of Eigenvalues

Theorem 1 : *Eigenvalues of a Hermitian matrix are real.*

(M.U. 1996, 2001, 03, 04, 06, 14)

Proof : Let A be a Hermitian matrix with λ as an eigenvalue and X as a corresponding eigenvector. Then,

$$AX = \lambda X.$$

Premultiplying by X^{θ} we get

$$X^{\theta} A X = X^{\theta} \lambda X = \lambda X^{\theta} X \qquad \text{.................... (1)}$$

Taking complex conjugate transpose of both sides

$$(X^{\theta} A X)^{\theta} = (\lambda X^{\theta} X)^{\theta}$$

$$X^{\theta} A^{\theta} (X^{\theta})^{\theta} = \bar{\lambda} X^{\theta} (X^{\theta})^{\theta}$$

Since, A is Hermitian $A^{\theta} = A$ and also $(X^{\theta})^{\theta} = X$

$$\therefore \quad X^{\theta} A X = \bar{\lambda} X^{\theta} X \qquad \text{.................... (2)}$$

From (i) and (ii), we get

$$\lambda X^{\theta} X = \bar{\lambda} X^{\theta} X$$

$$\therefore \quad (\lambda - \bar{\lambda}) X^{\theta} X = 0$$

Since, X is not a zero vector $X^{\theta} X \neq 0$.

$$\therefore \quad \lambda - \bar{\lambda} = 0 \quad \therefore \quad \lambda = \bar{\lambda}$$

Hence, λ is real.

Example 1 : Verify that the eigenvalues of the following Hermitian matrix are real.

$$A = \begin{bmatrix} 2 & 1+i \\ 1-i & 1 \end{bmatrix}.$$

Sol. : The characteristic equation of A is

$$\begin{vmatrix} 2-\lambda & 1+i \\ 1-i & 1-\lambda \end{vmatrix} = 0$$

$$\therefore \quad (2-\lambda)(1-\lambda) - (1+1) = 0 \qquad \therefore \quad 2 - 3\lambda + \lambda^2 - 2 = 0$$

$$\therefore \quad \lambda^2 - 3\lambda = 0 \qquad \therefore \quad \lambda(\lambda - 3) = 0 \qquad \therefore \quad \lambda = 0, 3.$$

Thus, the eigenvalues of the given Hermitian matrix are real.

Corollary 1 : The determinant of a Hermitian matrix is real.

Proof : The determinant of a matrix is equal to the product of its eigenvalues. Since all eigenvalues of a Hermitian matrix are real, their product is real.

Example 2 : Verify corollary 1 for $A = \begin{bmatrix} 1 & 2i \\ -2i & -1 \end{bmatrix}$. *i.e.*, verify that the determinant of Hermitian matrix is real for A given above.

Sol. : The determinant of A is $\Delta = 2 - (1+1) = 0$

$\therefore$ Thus, Δ is real.

Example 3 : Verify corollery 1 for $A = \begin{bmatrix} 4 & 2i \\ -2i & 5 \end{bmatrix}$.

Sol. : Left to you.

Corollary 2 : ***Eigenvalues of a real symmetric matrix are all real.***

(M.U. 1997, 99, 2003)

If the elements of Hermitian matrix are all real, it becomes a symmetric matrix. Thus, a real symmetric matrix is Hermitian and hence, its all eigenvalues are real.

Example 4 : Verify that the eigenvalues of a real symmetric matrix are real for $A = \begin{bmatrix} 12 & 6 \\ 6 & 3 \end{bmatrix}$.

Sol. : The matrix A is symmetric and the characteristic equation is

$$\begin{vmatrix} 12-\lambda & 6 \\ 6 & 3-\lambda \end{vmatrix} = 0$$

$\therefore \ (12-\lambda)(3-\lambda) - 36 = 0 \qquad \therefore \ 36 - 15\lambda + \lambda^2 - 36 = 0$

$\therefore \ \lambda^2 - 15\lambda = 0 \qquad \therefore \ \lambda(\lambda - 15) = 0 \qquad \therefore \ \lambda = 0, 15.$

The eigenvalues of A are real.

(See Ex. 1, Type - III, page 5-14)

Corollary 3 : ***Eigenvalues of a Skew-Hermitian matrix are either purely imaginary or zero.***

(M.U. 2003, 04)

Suppose A is a Skew-Hermitian matrix. Then iA is Hermitian. Let λ be an eigenvalue and X be the corresponding eigenvector

$\therefore \ AX = \lambda X$ [By Theorem 1]

$\therefore \ (iA)X = (i\lambda)X$

$\therefore \ i\lambda$ is an eigenvalue of iA which is Hermitian. Hence, $i\lambda$ is real. Therefore, λ must be either purely imaginary or zero.

Corollary 4 : ***The eigenvalues of a real skew-symmetric matrix are purely imaginary or zero.***

(M.U. 2002)

If the elements of a Skew-Hermitian matrix are all real, it becomes a skew-symmetric matrix. Thus, a real skew-symmetric matrix is Skew-Harmitian and hence, its all eigenvalues are either purely imaginary or zero.

Theorem 2 : ***The Eigenvalues of a unitary matrix are of unit modulus. (have absolute value one).***

(M.U. 2001, 03, 15)

Proof : Suppose A is a unitary matrix. Then, $A^\theta A = I$

Let λ be an eigenvalue of A and X be the corresponding eigenvector.

Then, $AX = \lambda X$ .... (1)

Taking complex conjugate transposes of both sides of (1), we get

$(AX)^\theta = (\lambda X)^\theta \qquad \therefore \ X^\theta A^\theta = \bar{\lambda} X^\theta$ .... (2)

From (1) and (2) by multiplication, we get,

$(X^{\theta}A^{\theta})(AX) = (\bar{\lambda}X^{\theta})(\lambda X)$ $\quad \therefore \; X^{\theta}(A^{\theta}A)X = \lambda\bar{\lambda}X^{\theta}X$

$\therefore \; X^{\theta}IX = \lambda\bar{\lambda}X^{\theta}X$ $\quad \therefore \; X^{\theta}X = \lambda\bar{\lambda}X^{\theta}X$

$\therefore \; X^{\theta}X(\lambda\bar{\lambda}-1) = 0$

Since, X is not a zero vector,

$X^{\theta}X \neq 0 \quad \therefore \; \lambda\bar{\lambda}-1=0 \quad \therefore \; \lambda\bar{\lambda}=1 \quad \therefore \; |\lambda| = 1.$

Example 5 : Verify that the eigenvalues of a unitary matrix are of unit modulus for

$$A = \frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}.$$

Sol. : We leave it to you to verify that A is a unitary matrix.

Now, consider $\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}$.

Its characteristic equation is

$$\begin{vmatrix} -4-\lambda & -2-4i \\ 2-4i & -4-\lambda \end{vmatrix} = 0$$

$\therefore \; (4+\lambda)^2 + (4+16) = 0 \quad \therefore \; \lambda^2 + 8\lambda + 36 = 0$

$$\therefore \; \lambda = \frac{-8 \pm \sqrt{64-144}}{2} = \frac{-8 \pm 4\sqrt{5}\,i}{2} = -4 \pm 2\sqrt{5}\,i$$

$$\therefore \; \text{Eigenvalues of } A \text{ are } \frac{-4 \pm 2\sqrt{5}\,i}{6} \; i.e. \; -\frac{2}{3} \pm \frac{\sqrt{5}\,i}{3}.$$

$$\therefore \; |\lambda_1| = \left| -\frac{2}{3} + \frac{\sqrt{5}}{3} \right| = \sqrt{\frac{4}{9} + \frac{5}{9}} = \sqrt{\frac{9}{9}} = 1$$

Similarly, $|\lambda_2| = 1$.

Example 6 : Prove that the eigenvalues of $\begin{bmatrix} \frac{(1+i)}{2} & \frac{-(1-i)}{2} \\ \frac{(1+i)}{2} & \frac{(1-i)}{2} \end{bmatrix}$ are of unit modulus.

(M.U. 1999, 2006)

Sol. : The characteristic equation is

$$\begin{bmatrix} \frac{(1+i)}{2} - x & \frac{-(1-i)}{2} \\ \frac{(1+i)}{2} & \frac{(1-i)}{2} - x \end{bmatrix} = 0$$

$$\therefore \; \left[\left(\frac{1+i}{2}\right) - x\right]\left[\frac{(1-i)}{2} - x\right] + \frac{(1+i)}{2}\frac{(1-i)}{2} = 0$$

$$\frac{(1+i)}{2}\frac{(1-i)}{2} - \frac{(1+i)}{2}x - \frac{(1-i)}{2}x + x^2 + \left(\frac{1+i}{2}\right)\left(\frac{1-i}{2}\right) = 0$$

$$\therefore \quad \frac{(1-i^2)}{4} - \frac{1}{2}x - \frac{i}{2}x - \frac{1}{2}x + \frac{i}{2}x + x^2 + \frac{1-i^2}{4} = 0$$

$$\therefore \quad \frac{1}{2} - x + x^2 + \frac{1}{2} = 0 \qquad \therefore \quad x^2 - x + 1 = 0$$

The roots of this equation are $x = \dfrac{1 \pm \sqrt{1-4}}{2} = \dfrac{1 \pm i\sqrt{3}}{2}$.

Hence, the eigenvalues are $\dfrac{1+i\sqrt{3}}{2}$ and $\dfrac{1-i\sqrt{3}}{2}$.

$$\therefore \quad |\lambda_1| = \left|\frac{1}{2} + \frac{i\sqrt{3}}{2}\right| = \left|\frac{1}{4} + \frac{3}{4}\right| = \sqrt{1} = 1. \text{ Similarly, } |\lambda_2| = 1.$$

Hence, eigenvalues are of unit modulus.

Corollary : *Eigenvalues of an orthogonal matrix are of unit modulus.* **(M.U. 2003)**

If the elements of a unitary matrix A are all real, then A becomes an orthogonal matrix. Thus, an orthogonal matrix is unitary and hence, its eigenvalues are of unit modulus. (See also Ex. 12, page 5-31)

Example 7 : Verify that the eigenvalues of an orthogonal matrix are of unit modulus for

$$A = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}.$$

Solution : It can be verified that A is orthogonal.

The characteristic equation is

$$\begin{vmatrix} (\sqrt{3}/2) - \lambda & 1/2 \\ -1/2 & (\sqrt{3/2}) - \lambda \end{vmatrix} = 0$$

$$\therefore \quad \left(\frac{\sqrt{3}}{2} - \lambda\right)^2 + \frac{1}{4} = 0 \qquad \therefore \quad \frac{3}{4} - \sqrt{3} \cdot \lambda + \lambda^2 + \frac{1}{4} = 0$$

$$\therefore \quad \lambda^2 - \sqrt{3}\lambda + 1 = 0 \qquad \therefore \quad \lambda = \frac{\sqrt{3} \pm \sqrt{3-4}}{2} = \frac{\sqrt{3} \pm i}{2}$$

$$\therefore \quad \lambda_1 = \frac{\sqrt{3}}{2} + \frac{i}{2}, \quad \lambda_2 = \frac{\sqrt{3}}{2} - \frac{i}{2}.$$

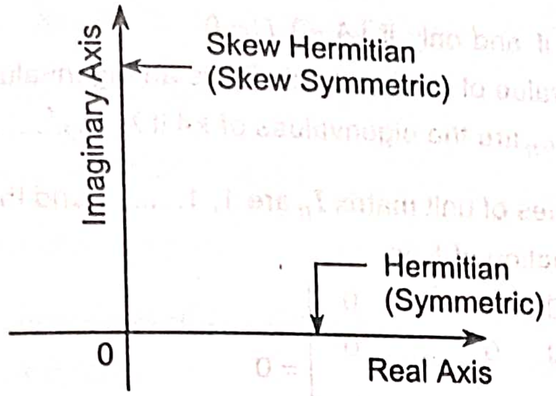
Now, $|\lambda_1| = \sqrt{\dfrac{3}{4} + \dfrac{1}{4}} = 1.$ Also $|\lambda_2| = 1$.

Example 8 : Verify that the matrix $A = \begin{bmatrix} 1/\sqrt{3} & \sqrt{2/3} \\ \sqrt{2/3} & -1/\sqrt{3} \end{bmatrix}$ is orthogonal and that the eigenvalues of A are of unit modulus.

Sol. : Left to you.

Note ....

The following diagram may help you to remember the relationship between different types of matrices and their eigenvalues.



9. Certain Relations between Matrix A and its Eigenvalues

Example 1 : Show that the matrices A and A' have the same eigenvalues. **(M.U. 2003)**

Sol. : We see that $(A-\lambda I)' = A' - \lambda I'$

$\therefore \quad (A-\lambda I)' = A' - \lambda I$

$\therefore \quad |(A-\lambda I)'| = |A'-\lambda I|$

$|A-\lambda I| = |A'-\lambda I| \quad [\because |B'| = |B|]$

$\therefore \quad |A-\lambda I| = 0$ if and only if $|A'-\lambda I| = 0$

This means the roots of the two equations are the same. Hence, eigenvalues of A and A' are the same.

Aliter : The result is true because the value of a determinant is not changed if its rows and columns are interchanged, the diagonal elements remaining the same.

Example 2 : Find the eigenvalues of A' where A is given by Ex. 1, page 5-7.

Sol. : The eigenvalues of A and A' are the same. Hence, the eigenvalues of A' are 1, 2, 3.

Example 3 : If λ is an eigenvalue of A then $\bar{\lambda}$ is an eigenvalue of A^{θ}.

Sol. : We have,

$$|A^{\theta} - \bar{\lambda} I| = |(A-\lambda I)^{\theta}| = \overline{|A-\lambda I|}$$

(Because for any square matrix A, $|A^{\theta}| = |(\bar{A}')| = |\bar{A}'| = \overline{|A|}$)

$\therefore \quad |A^{\theta} - \bar{\lambda} I| = 0$ if and only if $\overline{|A-\lambda I|} = 0$

$\therefore \quad |A^{\theta} - \bar{\lambda} I| = 0$ if and only if $|A-\lambda I| = 0$

(Because for a complex number z, $\bar{z}$ is zero if and only if $z = 0$)

$\therefore \quad \bar{\lambda}$ is an eigenvalue of A^{θ} if λ is an eigenvalue of A.

Example 4 : If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigenvalues of A then show that $k\lambda_1, k\lambda_2, \ldots, k\lambda_n$ are the eigenvalue kA.

Sol. : (i) If $k = 0$ then $kA = O$. Since, each eigenvalue of O (zero matrix) is 0, the eigenvalues of kA will be $0\lambda_1, 0\lambda_2, \ldots 0\lambda_n$.

(ii) If $k \neq 0$, we have

$$|kA - \lambda k I| = |k(A-\lambda I)| = k^n |A-\lambda I|$$

$\therefore$ $|kA - \lambda k I| = 0$ if and only if $|A - \lambda I| = 0$

$\therefore$ $k\lambda$ is an eigenvalue of kA if and only if λ is an eigenvalue of A.

Thus, $k\lambda_1, k\lambda_2, \ldots, k\lambda_n$ are the eigenvalues of kA if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A.

Example 5 : Eigenvalues of unit matrix I_n are 1, 1, ..., 1 and those of kI_n are $k, k, \ldots, k$.

Sol. : The characteristic equation of I_n is

$$\begin{vmatrix} 1-\lambda & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1-\lambda & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

Expanding it, we get,

$(1-\lambda)(1-\lambda) \ldots (1-\lambda) = 0 \quad \therefore \ l = 1, 1, \ldots, 1.$

Eigenvalues of a unit matrix are 1, 1, ..., 1.

And by the above example 3, the eigenvalues of kI_n are $k \cdot 1, k \cdot 1, \ldots k \cdot 1$, *i.e.*, $k, k, \ldots, k$.

Example 6 : Find the eigenvalues of $4A$ where A is given by Ex. 2, page 5-8.

Sol. : The eigenvalues of A are 1, 2, 3. Hence, eigenvalues of $4A$ are 4×1, 4×2, 4×3 *i.e.*, 4, 8, 12.

Example 7 : For the following matrix verify the above result of Ex. 4 for $k = 3$.

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Sol. : In Ex. 1, page 5-7, we have obtained the eigenvalues of the above matrix as 1, 2, 3.

Now, consider $B = 3A = \begin{bmatrix} 6 & -3 & 3 \\ 3 & 6 & -3 \\ 3 & -3 & 6 \end{bmatrix}$

The characteristic equation of B is

$$\begin{vmatrix} 6-\lambda & -3 & 3 \\ 3 & 6-\lambda & -3 \\ 3 & -3 & 6-\lambda \end{vmatrix} = 0$$

$\therefore$ $(6-\lambda)[(6-\lambda)(6-\lambda) - 9] + 3[3(6-\lambda) + 9] + 3[-9 - 3(6-\lambda)] = 0$

$\therefore$ $\lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0 \quad \therefore \ \lambda^3 - 3\lambda^2 - 15\lambda^2 + 45\lambda + 54\lambda - 162 = 0$

$\therefore$ $(\lambda - 3)(\lambda^2 - 15\lambda + 54) = 0 \quad \therefore \ (\lambda - 3)(\lambda - 6)(\lambda - 9) = 0$

$\therefore$ $\lambda = 3, 6, 9$

Thus, the eigenvalues of B are 3 times the eigenvalues of A.

Example 8 : If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigenvalues of A then show that $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots \frac{1}{\lambda_n}$ are the eigenvalues of A^{-1}.

(M.U. 1999, 2000, 01, 16, 18)

Sol. : If λ is an eigenvalue of A and X is the corresponding eigenvector then,

$$A X=\lambda X \qquad \therefore \quad X=A^{-1}(\lambda X)=\lambda\left(A^{-1} X\right)$$

$$\therefore \quad \frac{1}{\lambda} X=A^{-1} X \qquad \therefore \quad \frac{1}{\lambda} \text{ is an eigen value of } A^{-1}.$$

Hence, if $\lambda_1, \lambda_2, \ldots . \lambda_n$ are the eigenvalues of A then $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots . \frac{1}{\lambda_n}$ are the eigenvalues of A^{-1}.

(And X is the eigenvector of A^{-1} corresponding to $1/\lambda$.)

Example 9 : Find the eigenvalues of A^{-1} where A is given by Ex. 2, page 5-16.

Sol. : The eigenvalues of A are 2, 3, 6.

Hence, eigenvalues of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$ *i.e.*, $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$.

Example 10 : If $\lambda_1, \lambda_2, \ldots . \lambda_n$ are the eigenvalues of A then show that $\lambda_1^2, \lambda_2^2, \ldots ., \lambda_n^2$ are the eigenvalues of A^2. **(M.U. 1998)**

Sol. : If λ is an eigen-value of A and X is the corresponding eigenvector then,

$$A X=\lambda X \qquad \therefore \quad A(A X)=A \lambda X$$

$$\therefore \quad A^2 X=\lambda(A X)=\lambda(\lambda X) \qquad [\because \ A X=\lambda X]$$

$$\therefore \quad A^2 X=\lambda^2 X \qquad \therefore \quad \lambda^2 \text{ is an eigenvalue of } A^2.$$

$\therefore$ If $\lambda_1, \lambda_2, \ldots . \lambda_n$ are the eigenvalues of A then $\lambda_1^2, \lambda_2^2, \ldots .., \lambda_n^2$ are the eigenvalues of A^2.

Similarly, we can prove that $\lambda_1^k, \lambda_2^k, \ldots .., \lambda_n^k$ are the eigenvalues of A^k where k is a positive integer. **(M.U. 2006)**

Example 11 : Find the eigenvalues of A^3 where A is given by Ex. 3, page 5-21.

Sol. : The eigenvalues of A are $-1,-1,5$. Hence, the eigenvalues of A^3 are $(-1)^3,(-1)^3,(5)^3$ *i.e.*, $-1,-1,125$.

Example 12 : The eigenvalues of an orthogonal matrix are +1 or −1.

Sol. : Let A be an orthogonal matrix and λ be an eigenvalue.

$$\therefore \quad A A'=A' A=I \qquad [\text{ By definition }]$$

If X is the eighen vector corresponding to λ then

$$A X=\lambda X \qquad \ldots\ldots\ldots\ldots (1)$$

$$\therefore \quad (A X)'=(\lambda X)' \qquad \therefore \quad X' A'=\lambda X' \qquad \ldots\ldots\ldots\ldots (2)$$

Multiply (2) by (1),

$$\therefore \quad \left(X' A'\right)(A X)=(\lambda X)'(\lambda X) \qquad \therefore \quad X'\left(A' A\right) X=\lambda^2 X' X$$

$$\therefore \quad X' I X=\lambda^2 X' X \qquad \therefore \quad X' X=\lambda^2 X' X$$

$$\therefore \quad X' X\left(\lambda^2-1\right)=0.$$

$$\text{But} \quad X' X \neq 0 \qquad \therefore \quad \lambda^2-1=0 \qquad \therefore \quad \lambda= \pm 1.$$

Note ....

Ex. 26 (xvi) on page 5-40 is an orthogonal matrix. Its eigenvalues are ±1.

Example 13 : Show that the eigenvalues of a triangular matrix are just the diagonal elements of the matrix.

Sol. : Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$ be a triangular matrix of order n.

It is characteristic equation is

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ 0 & a_{22}-\lambda & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

After expanding the determinant, we get

$$(a_{11}-\lambda)(a_{22}-\lambda)\ \ldots\ (a_{nn}-\lambda) = 0 \qquad \therefore\ \lambda = a_{11}, a_{22}, \ldots, a_{nn}.$$

Hence, the eigenvalues are the diagonal elements.

Example 14 : If λ is an eigenvalue of a non-singular matrix A, prove that $\dfrac{|A|}{\lambda}$ is an eigenvalue of adj. A. **(M.U. 1998, 2003, 05)**

Proof : By definition A adj. $A = |A|\, I$

Premultiplying by A^{-1},

$$(A^{-1}A)\text{ adj. } A = A^{-1}|A| \qquad \therefore\ \text{adj. } A = A^{-1}|A| = |A|A^{-1}$$

Post multiplying by X where X is an eigenvector corresponding to the eigenvalue λ.

$$\text{adj. } AX = |A|A^{-1}X = |A|\frac{1}{\lambda}X \qquad \left[\because A^{-1}X = \frac{1}{\lambda}X \text{ by Example 5}\right]$$

$$\therefore\ \text{adj. } AX = \frac{|A|}{\lambda}X \qquad \therefore\ \frac{|A|}{\lambda} \text{ is an eigenvalue of adj. } A.$$

Cor. 1 : If $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of a 3×3 matrix A then find Eigenvalues of adj. A. **(M.U. 2000)**

Sol. : As proved above the eigenvalues of adj. A are $\dfrac{|A|}{\lambda_1}, \dfrac{|A|}{\lambda_2}, \dfrac{|A|}{\lambda_3}$.

Example 15 : Find the eigenvalues of adj. A where A is given by Ex. 3, page 5-17.

Sol. : The eigenvalues of A are $-3, 12, -6$. Hence, eigenvalues of adj. A are $\dfrac{|A|}{\lambda_1}, \dfrac{|A|}{\lambda_2}, \dfrac{|A|}{\lambda_3}$.

But $|A| = 216$. Hence, the eigenvalues of adj. A are $\dfrac{216}{-3}, \dfrac{216}{12}, \dfrac{216}{-6}$ *i.e.*, $-72, 18, -36$.

Cor. 2 : If $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of a 3×3 matrix then prove that the eigenvalues of adj. A are $\lambda_1\lambda_2, \lambda_2\lambda_3$ and $\lambda_3\lambda_1$. **(M.U. 1997, 2003)**

Sol. : As proved above the eigenvalues of adj. A are $\frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2}, \frac{|A|}{\lambda_3}$. But $|A| = \lambda_1\lambda_2\lambda_3$. Hence, the eigenvalues are $\lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_3\lambda_1$.

Example 16 : If λ is an eigenvalue of a matrix A with corresponding eigenvector X, prove that λ^n is an eigenvalue of A^n with corresponding eigenvector X. **(M.U. 2015, 17)**

Hence, find the characteristic roots of A^4 where $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. **(M.U. 1996)**

Sol. : Since λ is an eigenvalue of A if X is the corresponding eigenvector.

$$AX = \lambda X.$$

Premultiply by A.

$$AAX = A\lambda X = \lambda AX \qquad \therefore \quad A^2X = \lambda AX = \lambda(\lambda X) = \lambda^2 X$$

Similarly, $A^3X = \lambda^3 X$.

Continuing in this way $A^n X = \lambda^n X$.

$\therefore$ λ^n is an eigenvalue of A^n and the corresponding eigenvector is X. Now, the characteristic equation of the given matrix A is

$$\begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} = 0$$

$$\therefore \quad (1-\lambda)^2 - 1 = 0 \qquad \therefore \quad 1 - 2\lambda + \lambda^2 - 1 = 0$$

$$\therefore \quad \lambda^2 - 2\lambda = 0 \qquad \therefore \quad \lambda(\lambda - 2) = 0 \qquad \therefore \quad \lambda = 0, \ \lambda = 2.$$

$\therefore$ The characteristic roots of A are 0, 2.

By the above theorem the characteristic roots of A^4 are λ^4 *i.e.* $0^4, 2^4$ *i.e.* 0 and 16.

Example 17 : If A is a real symmetric matrix then the eigenvalues of $A\bar{A}$ are positive. **(M.U. 2005)**

Sol. : Since A is symmetric its eigenvalues are real, positive or negative and also $\bar{A} = A$.

$$\therefore \quad A\bar{A} = AA = A^2$$

If $\lambda_1, \lambda_2, ...$ are the eigenvalues of A, then $\lambda_1^2, \lambda_2^2, ...$ are the eigenvalues of A^2 *i.e.* of $A\bar{A}$.

$\therefore$ The eigenvalues of $A\bar{A}$ are positive.

Example 18 : If A is a square matrix of order n where n is an odd positive integer, defined over a field of real numbers then show that A has at least one real eigenvalue. **(M.U. 2003)**

Sol. : We know that the complex roots of an algebraic polynomial appear in pairs such as $a + ib$ and $a - ib$. Since n is odd, there is atleast one root which is not complex and hence is real.

Example 19 : If λ is an eigenvalue of the matrix A then $\lambda \pm k$ is an eigenvalue of $A \pm kI$.

Sol. : Let X be the eigenvector corresponding to the eigenvalue λ. Then

$$AX = \lambda X$$

$$\therefore \quad (A + kI)X = AX + kIX = \lambda X + kX = (\lambda + k)X$$

Hence, $\lambda + k$ is the characteristic root of $(A + kI)$.

Similarly, we can prove that $\lambda - k$ is an eigenvalue of $A - kI$.

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + a_{n-2} \lambda^{n-2} + \ldots + a_2 \lambda^2 + a_1 \lambda + a_0 = 0$$

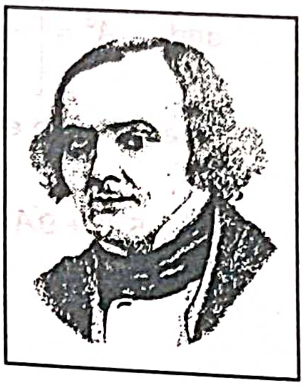
...Hamilton Theorem states that this equation is satisfied by the matrix A itself.

i.e. $a_n A^n + a_{n-1} A^{n-1} + a_{n-2} A^{n-2} + \ldots + a_2 A^2 + a_1 A + a_0 I = 0$

Remark ....

Caylay-Hamilton theorem can be used to find A^{-1}, A^{-2} and A^4 etc. (See Ex. 1, 2).

...Cayley (1821 - 1895)



...great British mathematician. He had shown his talent at the ...17 when he was recognised by his teachers as "above the first". He had published his first paper in mathematics at the age of 20 and in the next five years he published 25 papers, when he was at Cambridge. In 1846 he left Cambridge to study law. He worked as a lawyer for the next 14 years but in the same period he published more than 200 papers. But in 1863 he left law and again joined the faculty at Cambridge University. He pursued his mathematical interest till his death. Cayley knew French, German, Italian, Greek and Latin besides English.

William Rown Hamilton (1805 - 1865)

Hamilton was a child prodigy. He red English at the age of 3, Greek and Hebrew at the age of 5, German, Frech, Italian and Spanish at the age of 12. He had also some command on Syriac, Persian, Arabic, Sanskrit and Hindustani. At the age of 17 he had mastered calculus and astronomy on his own. At the age of 22 he was appointed professor at Trinity College, Dublin. He was knighted at the age of 30. In 1833, he first developed the concept of complex numbers as ordered pairs. In 1843 he discovered quaternions and for the next 22 years he developed them further and wrote two monumental books. He had significant contributions to abstract algebra, dynamics, and optics.

Example 1 : Verify Cayley-Hamilton Theorem for the matrix A and hence, find A^{-1}, A^{-2} and A^4 where

$$A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$ **(M.U. 2005, 06, 10, 15, 16, 17, 18)**

Prove that $A^{-1} = A^2 - 5A + 9I$. **(M.U. 2017)**

Sol. : The characteristic equation is

$$\begin{vmatrix} 1-\lambda & 2 & -2 \\ -1 & 3-\lambda & 0 \\ 0 & -2 & 1-\lambda \end{vmatrix} = 0$$

$\therefore \quad (1-\lambda)[(3-\lambda)(1-\lambda) - 0] - 2[-1(1-\lambda) - 0] - 2[2-0] = 0$

$\therefore \quad (1-\lambda)[3 - 4\lambda + \lambda^2] + 2(1-\lambda) - 4 = 0$

$\therefore \quad 3 - 4\lambda + \lambda^2 - 3\lambda + 4\lambda^2 - \lambda^3 + 2 - 2\lambda - 4 = 0 \qquad \therefore \lambda^3 - 5\lambda^2 + 9\lambda - 1 = 0.$

Cayley-Hamilton theorem states that this equation is satisfied by the matrix A, ... (1)

i.e. $\quad A^3 - 5A^2 + 9A - I = 0$

$$\text{Now,} \quad A^2 = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix}$$

$$\text{and} \quad A^3 = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix}$$

It can now be easily seen that

$$A^3 - 5A^2 + 9A - I = \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix} - 5\begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} + 9\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(a) Now, multiply the above equation by A^{-1}.

$$\therefore \quad A^2 - 5A + 9I - A^{-1} = 0$$

$$\therefore \quad A^{-1} = A^2 - 5A + 9I \qquad \text{..................... (2)}$$

$$= \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

(b) To find A^{-2} multiply (2) by A^{-1}.

$$\therefore \quad A^{-1} \cdot A^{-1} = A^{-1} \cdot A^2 - 5 \cdot A^{-1} \cdot A + 9A^{-1} I$$

$$\therefore \quad A^{-2} = A - 5I + 9A^{-1}$$

$$= \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 9\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 23 & 20 & 52 \\ 8 & 7 & 18 \\ 18 & -16 & 41 \end{bmatrix}$$

(c) To find A^4 multiply (1) by

$$\therefore \quad A^4 - 5A^3 + 9A^2 - A = 0$$

$$\therefore \quad A^4 = 5A^3 - 9A^2 + A$$

$$\therefore \quad A^4 = 5\begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix} - 9\begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -55 & 104 & 24 \\ -20 & -15 & 32 \\ 32 & -42 & 13 \end{bmatrix}$$

(You should verify that $AA^{-1} = 1$)

Example 2 : Find the characteristic equation of the matrix A where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$$

Show that the matrix A satisfies the characteristic equation and hence, find (a) A^{-1}, (b) A^{-2}, (c) A^4.

(M.U. 1998, 2003, 18)

Sol. : The characteristic equation is

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 4 \\ 3 & 1 & -1-\lambda \end{vmatrix} = 0$$

$$\therefore \quad (1-\lambda)[(1+\lambda)(1+\lambda)-4]-2[-2(1+\lambda)-12]+3[2+3(1+\lambda)] = 0$$

$$\therefore \quad (1-\lambda)(-3+2\lambda+\lambda^2)+2(14+2\lambda)+3(5+3\lambda) = 0$$

$$\therefore \quad -3+2\lambda+\lambda^2+3\lambda-2\lambda^2-\lambda^3+28+4\lambda+15+19\lambda = 0$$

$$\therefore \quad \lambda^3+\lambda^2-18\lambda-40 = 0$$

Cayley-Hamilton Theorem states that this equation is satisfied by A *i.e.*

$$A^3 + A^2 - 18A - 40I = 0 \qquad \text{....................} (1)$$

Now, $$A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{bmatrix}$$

It can be seen that $A^3 + A^2 - 18A - 40I$

$$= \begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{bmatrix} + \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} - \begin{bmatrix} 18 & 36 & 54 \\ 36 & -18 & 72 \\ 54 & 18 & -18 \end{bmatrix} - \begin{bmatrix} 40 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 40 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, the theorem is verified.

(a) Now, multiplying (1) by A^{-1}, we get $A^2 + A - 18I - 40A^{-1} = 0.$

$$\therefore \quad 40A^{-1} = A^2 + A - 18I \qquad \text{....................} (2)$$

$$\therefore \quad 40A^{-1} = \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} - 18\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 5 & 11 \\ 14 & -10 & 2 \\ 5 & 5 & -5 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{40}\begin{bmatrix} -3 & 5 & 11 \\ 14 & -10 & 2 \\ 5 & 5 & -5 \end{bmatrix}$$

(b) To find A^{-2}, multiply (2) by A^{-1} again.

$$\therefore \quad 40A^{-1} \cdot A^{-1} = A^{-1} \cdot A^2 + A^{-1} \cdot A - 18A^{-1}$$

$$= A + I - 18A^{-1}$$

$$\therefore \quad 40A^{-2} = A + I - 18A^{-1}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{18}{40}\begin{bmatrix} -3 & 5 & 11 \\ 14 & -10 & 2 \\ 5 & 5 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 3 \\ 2 & 0 & 4 \\ 3 & 1 & 0 \end{bmatrix} - \frac{9}{20}\begin{bmatrix} -3 & 5 & 1 \\ 14 & -10 & 2 \\ 5 & 5 & -5 \end{bmatrix} = \frac{1}{20}\begin{bmatrix} 67 & -5 & 51 \\ -86 & 90 & 62 \\ 15 & -25 & 45 \end{bmatrix}$$

$$\therefore \quad A^{-2} = \frac{1}{800}\begin{bmatrix} 67 & -5 & 51 \\ -86 & 90 & 62 \\ 15 & -25 & 45 \end{bmatrix}$$

(c) Further, multiplying (1) by A, we get,

$$A^4 + A^3 - 18A^2 - 40A = 0$$

$$\therefore \quad A^4 = 40A + 18A^2 - A^3$$

$$\therefore \quad A^4 = \begin{bmatrix} 40 & 80 & 120 \\ 80 & -40 & 160 \\ 120 & 10 & -40 \end{bmatrix} + \begin{bmatrix} 252 & 54 & 144 \\ 216 & 162 & -36 \\ 36 & 72 & 252 \end{bmatrix} - \begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{bmatrix} = \begin{bmatrix} 248 & 101 & 218 \\ 272 & 109 & 50 \\ 104 & 98 & 204 \end{bmatrix}$$

Example 3 : Find the characteristic equation of the matrix $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and verify that it is satisfied by A and hence, obtain A^{-1}. **(M.U. 1995, 98, 2005, 16)**

Sol. : The characteristic equation is

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\therefore \quad (2-\lambda)[(2-\lambda)^2 - 1] + 1[-1(2-\lambda) + 1] + 1[1 - (2-\lambda)] = 0$$

$$\therefore \quad \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

Cayley-Hamilton Theorem states that this equation is satisfied by A,

i.e. $\quad A^3 - 6A^2 + 9A - 4I = 0$ .... (1)

Now $\quad A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$

$$A^3 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

It can be seen that $A^3 - 6A^2 + 9A - 4I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Now multiply (1) by A^{-1}. $\quad \therefore \quad A^2 - 6A + 9I - 4A^{-1} = 0$

$\therefore \quad 4A^{-1} = (A^2 - 6A + 9I)$

$$4A^{-1} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Example 4 : Show that the matrix $A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$ satisfies Cayley-Hamilton Theorem and hence, find A^{-1} if it exists. **(M.U. 2000, 03)**

Sol. : The characteristic equation is

$$\begin{vmatrix} 0-\lambda & c & -b \\ -c & 0-\lambda & a \\ b & -a & 0-\lambda \end{vmatrix} = 0$$

$$\therefore \quad -\lambda(\lambda^2 + a^2) - c(c\lambda - ab) - b(ac + b\lambda) = 0$$

$$\therefore \quad -\lambda^3 - \lambda a^2 - c^2\lambda + abc - abc - b^2\lambda = 0$$

$$\therefore \quad \lambda^3 + (a^2 + b^2 + c^2)\lambda = 0 \qquad \text{.....................(1)}$$

Now $$A^2 = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} = \begin{bmatrix} -c^2-b^2 & ab & ac \\ ab & -c^2-a^2 & bc \\ ac & bc & -b^2-a^2 \end{bmatrix}$$

$$\therefore \quad A^3 = \begin{bmatrix} -c^2-b^2 & ab & ac \\ ab & -c^2-a^2 & bc \\ ac & bc & -b^2-a^2 \end{bmatrix}\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -c^3-cb^2-ca^2 & b^3+bc^2+ba^2 \\ c^3+ca^2+cb^2 & 0 & -ab^2-ac^2-a^3 \\ -bc^2-b^3-a^2b & ac^2+ab^2+a^3 & 0 \end{bmatrix}$$

$$\therefore \quad A^3 = -(a^2+b^2+c^2)\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} = -(a^2+b^2+c^2)A$$

$\therefore \quad A^3 + (a^2+b^2+c^2)A = 0$. $\quad \therefore$ A satisfies the equation (1).

Hence, A satisfies Cayley-Hamilton Theorem.

Now the determinant of the matrix A.

i.e. $$|A| = \begin{vmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{vmatrix} = 0(0+a^2) - c(0-ab) - b(ac-0) = abc - abc = 0.$$

Since the matrix A is singular A^{-1} does not exist.

Example 5 : Verify Cayley-Hamilton theorem and hence, find the matrix represented by $A^6 - 6A^5 + 9A^4 + 4A^3 - 12A^2 + 2A - I$, where A is $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ **(M.U. 2016)**

Sol. : The characteristic equation of A is

$$\begin{vmatrix} 3-\lambda & 10 & 5 \\ -2 & -3-\lambda & -4 \\ 3 & 5 & 7-\lambda \end{vmatrix} = 0$$

$$\therefore\ (3-\lambda)[-(3+\lambda)(7-\lambda)+20] - 10[-2(7-\lambda)+12] + 5[-10-3(-3-\lambda)] = 0$$

$$\therefore\ (3-\lambda)(-1-4\lambda+\lambda^2) - 10(-14+2\lambda+12) + 5(-10+9+3\lambda) = 0$$

$$\therefore\ -3-12\lambda+3\lambda^2+\lambda+4\lambda^2-\lambda^3+20-20\lambda-5+15\lambda = 0$$

$$\therefore\ -\lambda^3+7\lambda^2-16\lambda+12 = 0$$

$$\therefore\ \lambda^3-7\lambda^2+16\lambda-12 = 0$$

Cayley-Hamilton theorem states that this equation is satisfied by the matrix A.

i.e., $A^3 - 7A^2 + 16A - 12I = O$

Now, $$A^2 = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 4 & 25 & 10 \\ -2 & -31 & -26 \\ 20 & 50 & 44 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 4 & 25 & 10 \\ -2 & -31 & -26 \\ 20 & 50 & 44 \end{bmatrix}\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix} = \begin{bmatrix} -8 & 15 & -10 \\ -52 & -157 & -118 \\ 92 & 270 & 208 \end{bmatrix}$$

$$7A^2 = \begin{bmatrix} 28 & 175 & 70 \\ -84 & -217 & -182 \\ 140 & 350 & 308 \end{bmatrix}$$

$$16A = \begin{bmatrix} 48 & 160 & 80 \\ -32 & -48 & -64 \\ 48 & 80 & 112 \end{bmatrix}$$

$$12I = \begin{bmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{bmatrix}$$

It can be easily seen that $A^3 - 7A^2 + 16A - 12I = O$. .... (1)

Thus, the Cayley-Hamilton theorem is verified.

Now, by actual division, we see that

$$A^6 - 6A^5 + 9A^4 + 4A^3 - 12A^2 + 2A - I = (A^3 - 7A^2 + 16A - 12I)(A^3 + A^2) + 2A - I$$

$$= O + 2A - I \qquad [\because\ A^3 - 7A^2 + 16A - 12I = O \text{ by (1)}]$$

$$= 2\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 20 & 10 \\ -4 & -7 & -8 \\ 6 & 10 & 13 \end{bmatrix}$$

Example 6 : Find the characteristic equation of the matrix A given below and hence, find the matrix represented by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$, where

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$ **(M.U. 2000, 03)**

Sol. : The characteristic equation is

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$\therefore\ (2-\lambda)[(1-\lambda)(2-\lambda)-0]-1[0-0]+1[0-1(1-\lambda)]=0$

$\therefore\ (4-4\lambda+\lambda^2)(1-\lambda)-(1-\lambda)=0$

$\therefore\ 4-4\lambda+\lambda^2-4\lambda+4\lambda^2-\lambda^3-1+\lambda=0$

$\therefore\ \lambda^3-5\lambda^2+7\lambda-3=0.$

This equation is satisfied by A.

Now dividing $\lambda^8-5\lambda^7+7\lambda^6-3\lambda^5+\lambda^4-5\lambda^3+8\lambda^2-2\lambda+1$ by $\lambda^3-5\lambda^2+7\lambda-3$, we get the quotient $\lambda^5+\lambda$ and the remainder $\lambda^2+\lambda+1$.

In terms of the matrix A this means

$$A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I = (A^3-5A^2+7A-3I)(A^5+A)+(A^2+A+I)$$

But $(A^3-5A^2+7A-3I)=0$

$\therefore$ L.H.S. $= A^2+A+I$

$$\text{Now, } A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$\therefore\ A^2+A+I = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

$$\therefore\ \text{The given expression} = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

Example 7 : Use Cayley-Hamilton theorem to find $2A^4-5A^3-7A+6I$ where $A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$.

(M.U. 2004, 10, 16)

Sol. : The characteristic equation of A is

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = 0$$

$\therefore\ (1-\lambda)(2-\lambda)-4=0 \qquad \therefore\ 2-3\lambda+\lambda^2-4=0 \qquad \therefore\ \lambda^2-3\lambda-2=0$

By Cayley-Hamilton theorem, this equation is satisfied by A.

$\therefore \quad A^2 - 3A - 2I = 0$

Now, dividing $2\lambda^4 - 5\lambda^3 - 7\lambda + 6$ by $\lambda^2 - 3\lambda - 2$, we get

$$2\lambda^4 - 5\lambda^3 - 7\lambda + 6I = (\lambda^2 - 3\lambda - 2)(2\lambda^2 + \lambda + 7) + 16\lambda + 20$$

In terms of matrix A, this means

$$2A^4 - 5A^3 - 7A + 6 = (A^2 - 3A - 2I)(2A^2 + A + 7I) + 16A + 20I$$

But as seen above $\quad A^2 - 3A - 2I = 0$

$$\therefore \quad 2A^4 - 5A^3 - 7A + 6I = 16A + 20I$$

$$= 16\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} + 20\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 36 & 32 \\ 32 & 52 \end{bmatrix}$$

Example 8 : Apply Cayley-Hamilton theorem to $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ and deduce that $A^8 = 625\,I$.

(M.U. 2004, 14, 15)

Sol. : The characteristic equation is

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0$$

$$\therefore \quad -(1-\lambda)(1+\lambda) - 4 = 0 \qquad \therefore \quad 1 - \lambda^2 + 4 = 0 \qquad \therefore \quad \lambda^2 = 5$$

By Cayley-Hamilton theorem this equation is satisfied by A, *i.e.*, $A^2 = 5\,I$.

Squaring, we get $\quad A^4 = 25\,I \quad$ and again squaring, we get $\quad A^8 = 625\,I$.

IMP

Example 9 : If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, show that for every integer $n \geq 3$, $A^n = A^{n-2} + A^2 - I$.

Hence, find A^{50}. **(M.U. 1997, 98, 2006)**

Sol. : The characteristic equation of A is

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 0-\lambda & 1 \\ 0 & 1 & 0-\lambda \end{vmatrix} = 0$$

$$\therefore \quad (1-\lambda)[\lambda^2 - 1] - 0 + 0 = 0$$

$$\therefore \quad \lambda^2 - 1 - \lambda^3 + \lambda = 0 \;\therefore\; \lambda^3 - \lambda^2 - \lambda + 1 = 0.$$

By Cayley-Hamilton theorem, this equation is satisfied by A.

$$\therefore \quad A^3 - A^2 - A + I = 0 \qquad \text{.................. (1)}$$

$$\therefore \quad A^3 = A + A^2 - I \qquad \text{.................. (2)}$$

We prove the required result by the method of mathematical induction.

Let the result be true for $n = k$ *i.e.* suppose $A^k = A^{k-2} + A^2 - I$ be true.

Now, multiply the equation by A.

$$\therefore \quad A^{k+1} = A^{k-1} + A^3 - A$$

But by (1), $\quad A^3 - A = A^2 - I$

$\therefore \quad A^{k+1} = A^{k-1} + A^2 - I = A^{(k+1)-2} + A^2 - I.$

Hence, the result is true for $n = k + 1$.

But by (2), the result is true for $n = 3$.

Hence, by mathematical induction, it is true for $n = 4, 5, \ldots$ for all $n \geq 3$.

Hence, $\quad A^n = A^{n-2} + A^2 - I \quad \ldots\ldots\ldots\ldots (3)$

To find A^{50}, we put successively, $n = 2, 4, \ldots, 46, 48, 50$ in (3).

(1) $\quad A^2 = I + A^2 - I$

(2) $\quad A^4 = A^2 + A^2 - I$

(3) $\quad A^6 = A^4 + A^2 - I$

....

....

(23) $\quad A^{46} = A^{44} + A^2 - I$

(24) $\quad A^{48} = A^{46} + A^2 - I$

(25) $\quad A^{50} = A^{48} + A^2 - I$

note this step

Adding these results, columnwise. Since there are 25 equalities, we get

good

$$A^{50} = 25A^2 - 24I$$

But $$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\therefore \quad A^{50} = 25 \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} - 24 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}$$

EXERCISE - II

1. Using Cayley-Hamilton Theorem,

(1) for the matrix $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$, prove that $A^{-1} = A^2 - 5A + 9I$. **(M.U. 2006)**

(2) for the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$, prove that $A^{-1} = \frac{1}{40}\left[A^2 + A - 18I\right]$.

(3) for the matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, prove that $A^{-1} = \frac{1}{4}\left[A^2 - 6A + 9I\right]$.

2. Find the characteristic equation of each of the following matrices and obtain the inverse.

1. $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ **(M.U. 2017)**

2. $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ **(M.U. 2004)**

3. $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

We list below the properties which two similar matrices A and B have in common.

Property	Statement
1. Determinant	A, B have the same determinant.
2. Rank	A, B have the same rank.
3. Trace	A, B have the same trace. [Trace means the sum of the diagonal elements of a square matrix.]
4. Characteristic Polynomial	A, B have the same characteristic polynomial.
5. Eigenvalues	A, B have the same eigenvalues.
6. Dimension	Corresponding to an eigenvalue λ, the eigenspaces of A and B have the same dimension.

13. Algebraic and Geometric Multiplicity of an Eigenvalue

Definition (Algebraic Multiplicity) : If an eigenvalue λ_1 of matrix A is repeated t times then t is called the **albebraic multiplicity** of λ_1.

If λ_1 is an eigenvalue occurring once then the algebraic multiplicity of λ_1 is one.

If λ_1 is an eigenvalue occurring twice then the algebraic multiplicity of λ_1 is two, etc.

Definition (Geometric Multiplicity) : If corresponding to an eigenvalue λ_1, there are s linearly independent eigenvectors then s is called the **geometric multiplicity** of λ_1.

If there is only one eigenvector corresponding to an eigenvalue λ_1 then the geometric multiplicity of λ_1 is one.

If there are two linearly independent eigenvectors corresponding to an eigenvalue λ_1 then the geometric multiplicity of λ_1 is two.

Relation between the number of variables n in the matrix, the rank r of the matrix and the number s of independent eigenvectors of λ_1.

We note the relation that $\boxed{n - r = s}$

Example 1 : Determine the algebraic and geometric multiplicity of each eigenvalue of $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$ if the eigenvalues of A are 7, 1, 1 and the eigenvectors are $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ corresponding to $\lambda = 7$ and $\lambda = 1$ in this order.

Sol. : Since the value $\lambda = 7$ is occurs only once, the algebraic multiplicity of $\lambda = 7$ is 1.

Since there is only one eigenvector corresponding to $\lambda = 7$, the geometric multiplicity of $\lambda = 7$ is 1.

Since the value $\lambda = 1$ is repeated twice, the algebraic multiplicity of $\lambda = 1$ is two.

Since there are two linearly independent eigenvectors corresponding to $\lambda = 1$, the geometric multiplicity of $\lambda = 1$ is 2.

Example 2 : Find the eigenvalues and eigenvectors of the matrix A and discuss algebraic and geometric multiplicity of each eigenvalue where

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Sol. : The characteristic equation of A is

$$\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$$

$\therefore\ \lambda^2(\lambda^2 - 1) - 1(-\lambda - 1) + 1(1 + \lambda) = 0$

$\therefore\ -\lambda^3 + \lambda + \lambda + 1 + \lambda + 1 = 0$

$\therefore\ \lambda^3 - 3\lambda - 2 = 0$

$\therefore\ \lambda^3 + \lambda^2 - \lambda^2 - \lambda - 2\lambda - 2\lambda = 0$

$\therefore\ (\lambda^2 - \lambda - 2)(\lambda + 1) = 0$

$\therefore\ (\lambda - 2)(\lambda + 1)(\lambda + 1) = 0$

$\therefore\ \lambda = -1, -1, 2.$

As seen above the algebraic multiplicity of $\lambda = -1$ is 2 and algebraic multiplicity of $\lambda = 2$ is 1.

(i) For $\lambda = -1$, $[A - \lambda_1 I]X = O$ gives

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since there are three variables and rank is 1, there are $3 - 1 = 2$ independent eigenvectors.

Now, we have $x_1 + x_2 + x_3 = 0$.

Putting $x_2 = -s$, $x_3 = -t$, we get $x_1 = -x_2 - x_3 = s + t$

$$\therefore\ X = \begin{bmatrix} s + t \\ -s + 0 \\ 0 - t \end{bmatrix} = s\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$\therefore\ X_1 = [1, -1, 0]'$ and $X_2 = [1, 0, -1]'$.

Since there are two linearly independent eigenvectors corresponding to $\lambda = -1$, the geometric multiplicity of $\lambda = -1$ is 2.

(ii) For $\lambda = 2$, $[A - \lambda_2 I]X = O$ gives

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } R_{2,1} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By } \begin{matrix} R_2 + 2R_1 \\ R_3 - R_1 \end{matrix} \begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By } \begin{matrix} R_3 + R_2 \\ -(1/3)R_2 \end{matrix} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since there are three variables and the rank is two, there is $3 - 2 = 1$, eigenvector.

$\therefore\ x_1 - 2x_2 + x_3 = 0,\ \ x_2 - x_3 = 0$

Putting $x_3 = t$, we get $x_2 = t$, $x_1 = 2x_2 - x_3 = 2t - t = t$.

$$\therefore \quad X = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \therefore \quad X = [\,1,\,1,\,1\,]'.$$

As seen above the geometric multiplicity of $\lambda = 2$ is 1.

Example 3 : Find the algebraic multiplicity and geometric multiplicity of each eigenvalue of the matrix.

$$\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$$

(M.U. 2005)

Sol. : The characteristic equation of A is

$$\begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -5 & -2-\lambda \end{vmatrix} = 0$$

$$\therefore \quad (4-\lambda)[-(3-\lambda)(2+\lambda)+10] - 6[-(2+\lambda)1+2] + 6[-5+(3-\lambda)] = 0$$

$$\therefore \quad \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0 \qquad \therefore \quad (\lambda-2)(\lambda-2)(\lambda-1) = 0$$

$$\therefore \quad \lambda = 1, 2, 2.$$

(i) For $\lambda = 1$, $[A - \lambda_1 I]X = O$ gives

$$\begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ -1 & -5 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \therefore \text{ By } \frac{R_1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \\ -1 & -5 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{matrix} R_2 - R_1 \\ R_3 + R_1 \end{matrix}\begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & -3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad x_1 + 2x_2 + 2x_3 = 0, \qquad 3x_2 + x_3 = 0$$

Putting $x_2 = t$, $x_3 = -3t$, $x_1 = 4t$.

$$\therefore \quad X_1 = \begin{bmatrix} 4t \\ t \\ -3t \end{bmatrix} = t\begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix} \qquad \therefore \text{ Eigenvector is } [\,4,\,1,\,-3\,]'.$$

There are 3 variables and the rank is 2, hence, there is only $3 - 2 = 1$ independent eigenvector.

$\therefore$ For $\lambda = 1$, algebraic multiplicity = 1, geometric multiplicity = 1.

(ii) For $\lambda = 2$, $[A - \lambda_2 I] = O$ gives

$$\begin{bmatrix} 2 & 6 & 6 \\ 1 & 1 & 2 \\ -1 & -5 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{By } \frac{R_1}{2}\begin{bmatrix} 1 & 3 & 3 \\ 1 & 1 & 2 \\ -1 & -5 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By } \begin{matrix} R_2 - R_1 \\ R_3 + R_2 \end{matrix}\begin{bmatrix} 1 & 3 & 3 \\ 0 & -2 & -1 \\ 0 & -4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{By } \begin{matrix} R_3 - 2R_2 \\ -R_2 \end{matrix}\begin{bmatrix} 1 & 3 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore\ x_1 + 3x_2 + 3x_3 = 0, \quad 2x_2 + x_3 = 0$

Putting $x_2 = -t,\ x_3 = 2t,\ x_1 = -3t.$

$$\therefore\ X_2 = \begin{bmatrix} -3t \\ -t \\ 2t \end{bmatrix} = t\begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix} \qquad \therefore \text{ Eigenvector is } [-3, -1, 2]'.$$

There are 3 variables and rank is 2, hence there is only $3 - 2 = 1$ independent eigenvector.

$\therefore$ For $\lambda = 2$, algebraic multiplicity = 2, geometric multiplicity = 1.

Theorem : ***The necessary and sufficient condition of a square matrix to be similar to a nal matrix is that the geometric multiplicity of each of its eigenvalues coincides with gebraic multiplicity.***

We shall accept this theorem without proof. The theorem states that we can diagonalise a square matrix if and only if algebraic multiplicity of each of its eigenvalues is equal to the tric multiplicity. If corresponding to any eigenvalue, if algebraic multiplicity is **not equal** to tric multiplicity then the matrix is **not diagonalisable.**

See Ex. 6, page 5-67 and Ex. 10, page 5-71)

or. Every matrix whose eigenvalues are distinct is similar to a diagonal matrix.

odal Matrix

e shall now define an important matrix *viz.* modal matrix.

heorem : ***A square non singular matrix A whose eigenvalues are all distinct can be alised by a similarity transformation $D = M^{-1}AM$ where M is the matrix whose columns e eigenvectors of A and D is the diagonal matrix whose diagonal elements are the alues of A.***

roof : Let the roots of the characteristic equation $|A - \lambda I| = 0$ be $\lambda_1, \lambda_2, \ldots, \lambda_n$ which are Let the corresponding eigenspaces be $X_1, X_2, X_3, \ldots, X_n$. Let M be the matrix whose columns $X_2, \ldots, X_n$. The matrix M is called the **modal** matrix.

e. $M = [X_1, X_2, \ldots, X_n] \qquad \therefore\ AM = [AX_1, AX_2, \ldots, AX_n]$

ince $X_1, X_2, \ldots, X_n$ are eigenspaces of A we have,

$$AX_1 = \lambda_1 X_1, \quad AX_2 = \lambda_2 X_2, \ldots, AX_n = \lambda_n X_n.$$

$$\therefore\ AM = [\lambda_1 X_1, \lambda_2 X_2, \ldots, \lambda_n X_n] = [X_1, X_2, \ldots, X_n]\begin{bmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \lambda_n \end{bmatrix} = MD$$

$\therefore\ AM = MD$

perating by M^{-1}, $\qquad M^{-1}AM = M^{-1}MD = D.$

hus, if a non-singular square matrix A has all distinct eigenvalues then, $M^{-1}AM = D$.

rom the two theorems we learn that :-

i) If all eigenvalues of A are distinct then A can be diagonalised (See Ex. 1, 2, 3)

ii) If an eigenvalue of A is repeated then A **may be** diagonalisable or A **may not be** diagonalisable.

(iii) If the algebraic multiplicity and geometric multiplicity of a repeated value **are equal** then A is diagonalisable. (See Ex. 5).

(iv) If the algebraic multiplicity and geometric multiplicity of a repeated value **are not equal** then A is not diagonalisable (See Ex. 6 and 10).

15. Diagonalising a Given Matrix

The process of finding a diagonal matrix similar to a given matrix is known as **diagonalising the given matrix**. The procedure is given below.

Procedure to diagonalise a given matrix

(a) Distinct Eigenvalues

1. First find the eigenvalues of the given matrix.
2. If all eigenvalues are **distinct** the matrix is **diagonalisable.** (See Theorem of § 14, page 5-57)
3. The diagonal matrix whose diagonal elements are the eigenvalues obtained above is the required diagonal matrix.
4. Now, find the eigenvectors corresponding to the eigenvalues obtained above.
5. The matrix of eigenvectors obtained above is the diagonalising or transforming matrix.
6. If $\lambda_1, \lambda_2, \lambda_3$ are the (distinct) eigenvalues of A, then the diagonal matrix D is

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

7. If $[u_1, u_2, u_3]'$, $[v_1, v_2, v_3]'$ and $[w_1, w_2, w_3]'$ are the eigenvectors **corresponding** to $\lambda_1, \lambda_2, \lambda_3$ **in this order**, then the diagonalising matrix or the modal matrix M is

$$M = \begin{bmatrix} u_1 & v_1 & w_3 \\ u_2 & v_2 & w_2 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

(b) Repeated Eigenvalues

1. If an eigenvalue λ is repeated 't' times then its **algebraic multiplicity** is 't'.
2. Now, find the eigenvectors corresponding to eignevalue λ. If there are 's' linearly independent eigenvectors then the **geometric multiplicity** of λ is 's'.
3. If the algebraic multiplicity 't' of λ **is equal** to the geometric multiplicity 's', *i.e.*, if $t = s$, then the matrix is **diagonalisable.**
4. The matrix of the eigenvalues are the diagonal matrix and the matrix of the corresponding eigenvectors are the diagonalising or modal matrix as explained above.
5. If the algebraic multiplicity of λ is **not equal** the geometric multiplicity, then the matrix A **is not diagonalisable.**

(c) Symmetric Matrix (Discussed in § 5 and § 12 of the next chapter)

If A is a symmetric matrix, then A can be diagonalised to diagonal form through orthogonal transformations also.

(i) If further all the eigenvalues are distinct, then the corresponding eigenvectors are orthogonal.

(ii) If some eigenvalues are repeated, then the corresponding eigenvectors are obtained by considering the orthogonality of the vectors.

(iii) If we further normalise the above eigenvectors, we get an orthogonal matrix that diagonalises the given symmetric matrix.

Example 1 : Find a matrix M which diagonalizes the matrix $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$.

Verify that $M^{-1}AM = D$ where D is the diagonal matrix. **(M.U. 2011)**

Sol. : The characteristic equation of A is

$$\begin{vmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$\therefore\ (4-\lambda)(3-\lambda) - 2 = 0 \qquad \therefore\ \lambda^2 - 7\lambda + 10 = 0$

$\therefore\ (\lambda-5)(\lambda-2) = 0 \qquad \therefore\ \lambda = 2, 5.$

Since the eigenvalues are distinct, the matrix A is diagonalisable.

(i) For $\lambda = 2$, $[A - \lambda_1 I]X = O$ gives

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{By } R_2 - R_1 \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore\ 2x_1 + x_2 = 0$

Putting $x_2 = -2t$, we get $2x_1 = -x_2 = 2t \quad \therefore\ x_1 = t.$

$$\therefore\ X = \begin{bmatrix} t \\ -2t \end{bmatrix} = t\begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \text{or} \quad X_1 = [1, -2]'.$$

(ii) For $\lambda = 5$, $[A_1 - \lambda_2 I]X = O$ gives

$$\begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{By } R_2 + 2R_1 \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore\ -x_1 + x_2 = 0$

Putting $x_2 = t$, we get $x_1 = x_2 = t$.

$$\therefore\ X = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{or} \quad X_2 = [1, 1]'.$$

Thus, the matrix $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$ will be diagonalised to the diagonal matrix $D = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$ by the transforming matrix $M = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$. And $M^{-1}AM = D$.

Verification : We recall that if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$A^{-1} = \frac{1}{|A|}\,\text{adj}\,A = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

$$\therefore\ M^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$$

Now, $M^{-1}AM = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -4 & 5 \end{bmatrix}$

$$= \frac{1}{3}\begin{bmatrix} 6 & 0 \\ 0 & 15 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} = D$$

Example 2 : Show that the matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ is diagonalisable. Find the transforming matrix and the diagonal matrix. **(M.U. 2005, 15)**

Sol. : In Ex. 1, page 5-14, we have obtained the eigenvalues and eigenvectors of the above matrix A. The eigenvalues are 0, 3, 15.

Since all eigenvalues are distinct, the matrix A is diagonalisable.

Now, the eigenvectors are (See page 5-14)

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$\therefore \quad M = [X_1, \ X_2, \ X_3] = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Since, $M^{-1}AM = D$, the given matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ is diagonalised to diagonal matrix

$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$ by transforming matrix $M = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$.

[Let us verify that $M^{-1}AM = D$.

Now, $|M| = 1(1-4) - 2(2+4) + 2(-4-2) = -27$

The cofactors of the first column are

$$\begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} = 1 - 4 = -3, \qquad -\begin{vmatrix} 2 & 2 \\ -2 & 1 \end{vmatrix} = -(2+4) = -6,$$

$$\begin{vmatrix} 2 & 2 \\ 1 & -2 \end{vmatrix} = -4 - 2 = -6$$

The cofactors of the second column are

$$-\begin{vmatrix} 2 & -2 \\ 2 & 1 \end{vmatrix} = -(2+4) = -6, \qquad \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 1 - 4 = -3$$

$$-\begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix} = -(-2-4) = 6$$

The cofactors of the third column are

$$\begin{vmatrix} 2 & 1 \\ 2 & -2 \end{vmatrix} = -4-2=-6, \qquad -\begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix} = -(-2-4)=6,$$

$$\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 1-4=-3$$

$$\therefore \quad M^{-1} = -\frac{1}{27}\begin{bmatrix} -3 & -6 & -6 \\ -6 & -3 & 6 \\ -6 & 6 & -3 \end{bmatrix} = \frac{1}{27}\begin{bmatrix} 3 & 6 & 6 \\ 6 & 3 & -6 \\ 6 & -6 & 3 \end{bmatrix}$$

Now, $M^{-1} A M = D$

$\therefore \quad M(M^{-1} A M) M^{-1} = M D M^{-1}$

$\therefore \qquad A = M D M^{-1}$

Instead of verifying $M^{-1}AM = D$, we shall verify that $MDM^{-1} = A$.

$$\text{Now,} \quad M D M^{-1} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix} \times \frac{1}{27}\begin{bmatrix} 3 & 6 & 6 \\ 6 & 3 & -6 \\ 6 & -6 & -3 \end{bmatrix}$$

$$= \frac{1}{27}\begin{bmatrix} 0 & 6 & 30 \\ 0 & 3 & -30 \\ 0 & -6 & 15 \end{bmatrix}\begin{bmatrix} 3 & 6 & 6 \\ 6 & 3 & -6 \\ 6 & -6 & -3 \end{bmatrix} = \frac{1}{27}\begin{bmatrix} 216 & -162 & 54 \\ -162 & 189 & -108 \\ 54 & -108 & 81 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} = A$$

This shows that M diagonalises A to diagonal matrix D.]

Example 3 : Show that the matrix $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ is diagonalisable. Find the transforming matrix and the diagonal matrix. **(M.U. 1991, 93, 95, 2003, 05, 16)**

Sol. : The characteristic equation of A is

$$\begin{vmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0$$

$\therefore \quad (1-\lambda)(\lambda-2)(\lambda-3) = 0 \qquad \therefore \quad \lambda = 1, 2, 3.$

Since, all eigenvalues are distinct the matrix A is diagonalisable.

(i) For $\lambda = 1$, $[A - \lambda I] X = O$ gives

$$\begin{bmatrix} 7 & -8 & -2 \\ 4 & -4 & -2 \\ 3 & -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \quad 7x_1 - 8x_2 - 2x_3 = 0; \qquad 4x_1 - 4x_2 - 2x_3 = 0$

By Cramer's rule

$$\frac{x_1}{\begin{vmatrix}-8 & -2\\-4 & -2\end{vmatrix}} = \frac{-x_2}{\begin{vmatrix}7 & -2\\4 & -2\end{vmatrix}} = \frac{x_3}{\begin{vmatrix}7 & -8\\4 & -4\end{vmatrix}} \quad \therefore \quad \frac{x_1}{8} = \frac{x_2}{6} = \frac{x_3}{4} \quad \therefore \quad \frac{x_1}{4} = \frac{x_2}{3} = \frac{x_3}{2} = t$$

$$\therefore \quad x_1 = 4t, \ x_2 = 3t, \ x_3 = 2t. \quad \therefore \quad X_1 = \begin{bmatrix}4t\\3t\\2t\end{bmatrix} = t\begin{bmatrix}4\\3\\2\end{bmatrix} \quad \therefore \quad X_1 = \begin{bmatrix}4\\3\\2\end{bmatrix}.$$

$\therefore$ Corresponding to eigenvalue 1, the eigenvector is [4, 3, 2] '.

(ii) For $\lambda = 2$, $[A - \lambda_2 I]X = O$ gives

$$\begin{bmatrix}6 & -8 & -2\\4 & -5 & -2\\3 & -4 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\therefore \quad 6x_1 - 8x_2 - 2x_3 = 0; \qquad 4x_1 - 5x_2 - 2x_3 = 0$$

By Cramer's rule,

$$\frac{x_1}{\begin{vmatrix}-8 & -2\\-5 & -2\end{vmatrix}} = \frac{-x_2}{\begin{vmatrix}6 & -2\\4 & -2\end{vmatrix}} = \frac{x_3}{\begin{vmatrix}6 & -8\\4 & -5\end{vmatrix}}$$

$$\therefore \quad \frac{x_1}{6} = \frac{x_2}{4} = \frac{x_3}{2} \quad \therefore \quad \frac{x_1}{3} = \frac{x_2}{2} = \frac{x_3}{1} = t$$

$$\therefore \quad x_1 = 3t, \ x_2 = 2t, \ x_3 = t. \quad \therefore \quad X_2 = \begin{bmatrix}3t\\2t\\t\end{bmatrix} = t\begin{bmatrix}3\\2\\1\end{bmatrix} \quad \therefore \quad X_2 = \begin{bmatrix}3\\2\\1\end{bmatrix}$$

$\therefore$ Corresponding to eigenvalue, 2 the eigenvector is [3, 2, 1] '.

(iii) For $\lambda = 3$, $[A - \lambda_3 I]X = O$ gives

$$\begin{bmatrix}5 & -8 & -2\\4 & -6 & -2\\3 & -4 & -2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\therefore \quad 5x_1 - 8x_2 - 2x_3 = 0; \qquad 4x_1 - 6x_2 - 2x_3 = 0$$

By Cramer's rule,

$$\frac{x_1}{\begin{vmatrix}-8 & -2\\-6 & -2\end{vmatrix}} = \frac{-x_2}{\begin{vmatrix}5 & -2\\4 & -2\end{vmatrix}} = \frac{x_3}{\begin{vmatrix}5 & -8\\4 & -6\end{vmatrix}}$$

$$\therefore \quad \frac{x_1}{4} = \frac{x_2}{2} = \frac{x_3}{2} \quad \therefore \quad \frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{1} = t$$

$$\therefore \quad x_1 = 2t, \ x_2 = t, \ x_3 = t. \quad \therefore \quad X_3 = \begin{bmatrix}2t\\t\\t\end{bmatrix} = t\begin{bmatrix}2\\1\\1\end{bmatrix} \quad \therefore \quad X_3 = \begin{bmatrix}2\\1\\1\end{bmatrix}$$

$\therefore$ Corresponding to eigenvalue 3, the eigenvector is [2, 1, 1] '.

$$\therefore \quad M = [X_1, X_2, X_3] = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

Since, $M^{-1}AM = D$, the matrix $A = \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ will be diagonalised to the diagonal matrix

$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ by the transforming matrix $M = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$.

Note ....

It should be noted that the columns of matrix M are to be taken **in the order** in which we take the eigenvalues in D. In the above example, if we change the order of eigenvalues in D, then we have to take the columns in M in the new order.

e.g., If $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, then $M = \begin{bmatrix} 3 & 4 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix}$.

Example 4 : Is the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ diagonalisable? If so find the diagonal form and the transforming matrix. **(M.U. 2014, 17)**

Sol. : The characteristic equation of A is

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$\therefore \quad (2-\lambda)[(2-\lambda)(1-\lambda)-0]-1[1(1-\lambda)-0]+1[0-0]=0$

$\therefore \quad (2-\lambda)(2-\lambda)(1-\lambda)-(1-\lambda)=0 \quad \therefore \quad (1-\lambda)[(2-\lambda)(2-\lambda)-1]=0$

$\therefore \quad (1-\lambda)(4-4\lambda+\lambda^2-1)=0 \quad \therefore \quad (1-\lambda)(\lambda^2-4\lambda+3)=0$

$\therefore \quad (1-\lambda)(\lambda-3)(\lambda-1)=0 \quad \therefore \quad \lambda = 1, 1, 3.$

Since the eigenvalues are repeated the matrix A may be or may not be diagonalisable.

[See (ii) at the bottom, page 5-57]

We shall now find algebraic multiplicity and geometric multiplicity of each eigenvalue and apply (iii) and (iv) at the top of page 5-58.

(i) For $\lambda = 3$, $[A - \lambda_1 I]X = O$ gives

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } R_2 + R_1 \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By } \begin{matrix} R_3 + R_2 \\ (1/2)R_2 \end{matrix} \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \quad -x_1 + x_2 + x_3 = 0, \qquad x_3 = 0$

Putting $x_2 = t$, we get $x_1 = t$.

$$\therefore \quad X_1 = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \therefore \text{ Eigenvector is } [\,1,\ 1,\ 0\,].$$

There are three variables and the rank is 2, hence, there is only $3-2=1$ independent solution.

$\therefore$ For $\lambda = 3$, algebraic multiplicity = 1 and the geometric multiplicity = 1.

(ii) For $\lambda = 1$, $[\,A - \lambda_2 I\,]\,X = O$ gives

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $\quad R_2 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \quad x_1 + x_2 + x_3 = 0$

Let $\quad x_2 = -s,\ x_3 = -t \qquad \therefore \ x_1 = s + t$

$$\therefore \quad X_2 = \begin{bmatrix} s+t \\ -s+0 \\ 0-t \end{bmatrix} = \begin{bmatrix} s \\ -s \\ 0 \end{bmatrix} + \begin{bmatrix} t \\ 0 \\ -t \end{bmatrix} = s\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\therefore \quad X_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad X_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

There are three variables and the rank of the matrix is one and hence there are $3-1=2$ independent vectors.

$\therefore$ For $\lambda = 1$, since the eigenvalue is repeated twice, the algebraic multiplicity = 2 and since X_2, X_3 are two independent vectors corresponding to $\lambda = 1$, the geometric multiplicity = 2.

Since the algebraic multiplicity and geometric multiplicity of each eigenvectors are equal, by Theorem of § 14, page 5-57, the matrix is diagonalisable.

The diagonalising matrix is

$$M = \left[X_1,\ X_2,\ X_3\right] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Thus, the given matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ is diagonalised to the diagonal matrix $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

by transforming matrix $M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

... similar to diagonal matrices.

(i) $\begin{bmatrix} \ldots & \ldots & 0 \\ 0 & 2 & 1 \\ \ldots & \ldots & \ldots \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(M.U. 2011)

$\begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ (vi) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (M.U. 2003)

...gonisable ? Justify your answer.

$\begin{bmatrix} \ldots & \ldots & 0 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ (M.U. 2004)

[Ans. : Not diagonisable. The characteristic equation is $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$. The roots 1, 1, 4 are not distinct and geometric multiplicity of $\lambda = 1$ is 1.]

14. Find the symmetric matrix $A_{3 \times 3}$ having the eigenvalues $\lambda_1 = 3$, $\lambda_2 = 6$, $\lambda_3 = 9$ with the ...onding eigenvectors $X_1 = [1, 2, 2]'$, $X_2 = [-2, 2, -1]'$ and X_3.

[Ans. : $\begin{bmatrix} 7 & 0 & -2 \\ 0 & 5 & -2 \\ -2 & -2 & 6 \end{bmatrix}$

15. Find a square symmetric matrix of order 3 whose eigenvalues are 3, –3, 9 with corresponding eigenvectors $[2, 2, -1]'$, $[2, -1, 2]'$ and X_3.

[Ans. : $\begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix}$

16. Functions of A Square Matrix

If A is a non-singular square matrix with distinct eigenvalues, we can find the modal matrix M as explained in § 14. We can also find the diagonal matrix D as seen earlier.

We know how to find A^2 from A by the process of matrix multiplication. From A^2, we can find A^3, and also A^4. In fact we can find any positive integral power A by this process. However, this method becomes obviously tedious if we want a large power A. For obtaining a large power we use the method explained below in (a).

The method is quite general and can also be used to find other functions of A such e^A, 4^A etc. as explained in the examples which follow.

(a) Calculation of the powers of a matrix

If A is a non-singular square matrix with distinct eigenvalues then we can find any power of A *i.e.* A^k (k is a positive integer) by the process explained below.

As seen in § 14 on page 5-57, we have $M^{-1} A M = D$.

Operating by M on the left and by M^{-1} on the right,

$$MM^{-1}AMM^{-1} = MDM^{-1}$$

$$\therefore \quad (MM^{-1})A(MM^{-1}) = MDM^{-1}$$

$$\therefore \quad A = MDM^{-1}$$

$$\therefore \quad A^n = (MDM^{-1})(MDM^{-1})....(MDM^{-1})....(n \text{ times})$$

$$\therefore \quad A^n = MD(M^{-1}M)D(M^{-1}M)....(M^{-1}M)DM^{-1}$$

$$= MD....DM^{-1}$$

$$= MD^nM^{-1}$$

$$= M\begin{bmatrix} \lambda_1^n & 0 & 0 & \dots & 0 \\ 0 & \lambda_2^n & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \lambda^n \end{bmatrix} M^{-1}$$

Example 1 : If $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, find A^{50}. **(M.U. 2000, 04, 15, 17)**

Sol. : The characteristic equation of A is

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\therefore \quad (2-\lambda)^2 - 1 = 0 \qquad \therefore \quad 4 - 4\lambda + \lambda^2 - 1 = 0$$

$$\therefore \quad \lambda^2 - 4\lambda + 3 = 0 \qquad \therefore \quad (\lambda - 3)(\lambda - 1) = 0 \qquad \therefore \quad \lambda = 1, 3.$$

(i) For $\lambda = 1$, $[A - \lambda_1 I]X = O$ gives

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{By } R_2 - R_1 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad x_1 + x_2 = 0$$

Putting $x_2 = -t$, $x_1 = t$. $\quad \therefore \quad X_1 = \begin{bmatrix} t \\ -t \end{bmatrix} = t\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Hence, corresponding to $\lambda = 1$, the eigenvector is $[1, -1]'$.

(ii) For $\lambda = 3$, $[A - \lambda_2 I]X = O$ gives

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{By } R_2 + R_1 \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad -x_1 + x_2 = 0 \text{ and } x_1 = x_2$$

Putting $x_2 = t$, $x_1 = t$. $\quad \therefore \quad X_2 = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Hence, corresponding to $\lambda = 3$, the eigenvector is $[1, 1]'$.

$\therefore$ Modal matrix $M = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

Now, $|M| = 2 \quad \therefore M^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

[If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.]

Now, $D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \quad \therefore D^{50} = \begin{bmatrix} 1^{50} & 0 \\ 0 & 3^{50} \end{bmatrix}$

$$\therefore A^{50} = MD^{50}M^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1^{50} & 0 \\ 0 & 3^{50} \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 3^{50} \\ -1 & 3^{50} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1+3^{50} & -1+3^{50} \\ -1+3^{50} & 1+3^{50} \end{bmatrix}$$

Example 2 : Find e^A and 4^A if $A = \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix}$. **(M.U. 1997, 2001, 03, 05, 06, 15)**

Sol. : The characteristic equation of A is

$$\begin{vmatrix} (3/2-\lambda) & 1/2 \\ 1/2 & (3/2-\lambda) \end{vmatrix} = 0$$

$$\therefore \left(\frac{3}{2}-\lambda\right)^2 - \frac{1}{4} = 0 \quad \therefore \frac{9}{4} - 3\lambda + \lambda^2 - \frac{1}{4} = 0$$

$$\therefore \lambda^2 - 3\lambda + 2 = 0 \quad \therefore (\lambda-1)(\lambda-2) = 0 \quad \therefore \lambda = 1, 2.$$

(i) For $\lambda = 1$, $[A - \lambda_1 I]X = O$ gives

$$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{By } \begin{matrix} 2R_1 \\ 2R_2 \end{matrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By $R_2 - R_1 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \therefore x_1 + x_2 = 0$

Putting $x_2 = -t$, we get $x_1 = t$. $\quad \therefore X_1 = \begin{bmatrix} t \\ -t \end{bmatrix} = t\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Hence, the eigenvector is $[1, -1]'$.

(ii) For $\lambda = 2$, $[A - \lambda_2 I]X = O$ gives

$$\begin{bmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{By } \begin{matrix} 2R_1 \\ 2R_2 \end{matrix}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By $R_2 + R_1 \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \therefore -x_1 + x_2 = 0 \quad \therefore x_1 = x_2$

Putting $x_2 = t,\ x_1 = t$, we get $\quad \therefore \quad X_2 = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Hence, the eigenvector is [1, 1] '.

$$\therefore \quad M = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \therefore \ |M| = 2 \qquad \therefore \quad M^{-1} = \frac{\text{adj.} M}{|M|} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Now $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

If $f(A) = e^A,\ f(D) = e^D = \begin{bmatrix} e^1 & 0 \\ 0 & e^2 \end{bmatrix}$. If $f(A) = 4^A,\ f(D) = 4^D = \begin{bmatrix} 4^1 & 0 \\ 0 & 4^2 \end{bmatrix}$

$$\therefore \quad e^A = M f(D) M^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} e & 0 \\ 0 & e^2 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e & e^2 \\ -e & e^2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} e + e^2 & -e + e^2 \\ -e + e^2 & e + e^2 \end{bmatrix}$$

Similarly, replacing e by 4 we get,

$$4^A = \frac{1}{2}\begin{bmatrix} 20 & 12 \\ 12 & 20 \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$$

Example 3 : If $A = \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}$ then prove that $3 \tan A = A \tan 3$. **(M.U. 2000, 05, 06)**

Sol. : The characteristic equation of A is

$$\begin{vmatrix} -1-\lambda & 4 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$\therefore \quad -(1+\lambda)(1-\lambda) - 8 = 0 \qquad \therefore \ \lambda^2 - 9 = 0 \qquad \therefore \ \lambda = 3, -3.$$

(i) For $\lambda = 3$, $[A - \lambda_1 I]X = O$ gives

$$\begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{By } R_2 + \frac{1}{2}R_1 \begin{bmatrix} -4 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad -4x_1 + 4x_2 = 0 \qquad \therefore \ x_1 - x_2 = 0$$

Putting $x_2 = t$, we get $x_1 = t$. $\quad \therefore \quad X_1 = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore$ The eigenvector is [1, 1]'.

(i) For $\lambda = -3$, $[A - \lambda_2 I]X = O$ gives

$$\begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{By } R_2 - R_1 \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad 2x_1 + 4x_2 = 0 \ \text{ i.e. } \ x_1 + 2x_2 = 0$$

Putting $x_2 = -t$, we get $x_1 = -2x_2 = 2t$.

$$\therefore \quad X_2 = \begin{bmatrix} 2t \\ -t \end{bmatrix} = t\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$\therefore$ The eigenvector is $[2, -1]'$.

$$\therefore \quad M = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \text{ and } |M| = -3 \qquad \therefore \quad M^{-1} = \frac{\text{adj. } M}{|M|} = -\frac{1}{3}\begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix}$$

Now, $D = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$

$$\because \quad f(A) = \tan A, \qquad f(D) = \begin{bmatrix} \tan 3 & 0 \\ 0 & \tan(-3) \end{bmatrix} = \begin{bmatrix} \tan 3 & 0 \\ 0 & -\tan 3 \end{bmatrix}$$

$$\therefore \quad \tan A = M\, f(D)\, M^{-1}$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \tan 3 & 0 \\ 0 & \tan(-3) \end{bmatrix}\left(-\frac{1}{3}\right)\begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix}$$

$$= -\frac{1}{3}\begin{bmatrix} \tan 3 & -2\tan 3 \\ \tan 3 & -\tan 3 \end{bmatrix}\begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} \tan 3 & -4\tan 3 \\ -2\tan 3 & -\tan 3 \end{bmatrix}$$

$$\therefore \quad 3\tan A = \begin{bmatrix} -\tan 3 & 4\tan 3 \\ 2\tan 3 & \tan 3 \end{bmatrix} = \tan 3\begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix} = \tan 3 \cdot A$$

$$= A\tan 3.$$

Example 4 : If $A = \begin{bmatrix} \alpha & \alpha \\ \alpha & \alpha \end{bmatrix}$, prove that $e^A = e^{\alpha}\begin{bmatrix} \cos h\alpha & \sin h\alpha \\ \sin h\alpha & \cos h\alpha \end{bmatrix}$. **(M.U. 2006)**

Sol. : The characteristic equation of A is

$$\begin{vmatrix} \alpha - \lambda & \alpha \\ \alpha & \alpha - \lambda \end{vmatrix} = 0$$

$$\therefore \quad (\alpha - \lambda)^2 - \alpha^2 = 0 \qquad \therefore \quad \lambda^2 - 2\alpha\lambda = 0$$

$$\therefore \quad \lambda(\lambda - 2\alpha) = 0 \qquad \therefore \quad \lambda = 0, \ \lambda = 2\alpha.$$

(i) For $\lambda = 0$, $[A - \lambda_1 I]\, X = O$ gives

$$\begin{bmatrix} \alpha & \alpha \\ \alpha & \alpha \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{By } R_2 - R_1 \begin{bmatrix} \alpha & \alpha \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad \alpha x_1 + \alpha x_2 = 0 \qquad \therefore \quad x_1 + x_2 = 0$$

Putting $x_2 = -t$, $x_1 = t$, $\qquad \therefore \quad X_1 = \begin{bmatrix} t \\ -t \end{bmatrix} = t\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\therefore$ The eigenvector is $[1, -1]'$.

(ii) For $\lambda = 2\alpha$, $[A - \lambda_2 I]\, X = O$ gives

$$\begin{bmatrix} -\alpha & \alpha \\ \alpha & -\alpha \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{By } R_2 + R_1 \begin{bmatrix} -\alpha & \alpha \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad -\alpha x_1 + \alpha x_2 = 0 \qquad \therefore \quad x_1 - x_2 = 0$$

Putting $x_2 = t$, $x_1 = t$. $\qquad \therefore \quad X_1 = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore$ The eigenvector is [1, 1]'.

$$\therefore\ M=\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\quad \therefore\ |M|=2 \qquad \therefore\ M^{-1}=\frac{\text{adj. } M}{|M|}=\frac{1}{2}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$$

Now, $D=\begin{bmatrix}0 & 0\\ 0 & 2\alpha\end{bmatrix}$

If $f(A)=e^A$, $f(D)=e^D=\begin{bmatrix}e^0 & 0\\ 0 & e^{2\alpha}\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & e^{2\alpha}\end{bmatrix}$

$$\therefore\ e^A=Mf(D)M^{-1}=\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & e^{2\alpha}\end{bmatrix}\frac{1}{2}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\ e^{2\alpha} & e^{2\alpha}\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1+e^{2\alpha} & -1+e^{2\alpha}\\ -1+e^{2\alpha} & 1+e^{2\alpha}\end{bmatrix}$$

Dividing the matrix by e^{α}.

$$e^A=e^{\alpha}\begin{bmatrix}\dfrac{e^{\alpha}+e^{-\alpha}}{2} & \dfrac{e^{\alpha}-e^{-\alpha}}{2}\\[2mm] \dfrac{e^{\alpha}-e^{-\alpha}}{2} & \dfrac{e^{\alpha}+e^{-\alpha}}{2}\end{bmatrix}=e^{\alpha}\begin{bmatrix}\cos h\alpha & \sin h\alpha\\ \sin h\alpha & \cos h\alpha\end{bmatrix}$$

Example 5 : If $A=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$, find e^{At}. Use Another Method only **(M.U. 1999)**

Sol. : The characteristic equation is

$$\begin{vmatrix}0-\lambda & 1\\ -1 & 0-\lambda\end{vmatrix}=0 \qquad \therefore\ \lambda^2+1=0 \qquad \therefore\ \lambda=i,-i.$$

(i) For $\lambda=i$, $[A-\lambda_1 I]X=O$ gives

$$\begin{bmatrix}-i & 1\\ -1 & -i\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$\therefore\ -ix_1+x_2=0$ and $x_1+ix_2=0$

Putting $x_2=it$, we get $x_1=-ix_2=-i^2t=t$.

$$\therefore\ X_1=\begin{bmatrix}t\\ it\end{bmatrix}=t\begin{bmatrix}1\\ i\end{bmatrix} \qquad \therefore\ \text{The eigenvector is } [1, i]'.$$

(ii) For $\lambda=-i$, $[A-\lambda_2 I]X=O$ gives

$$\begin{bmatrix}i & 1\\ -1 & i\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix} \qquad \therefore\ ix_1+x_2=0 \text{ and } -x_1+ix_2=0$$

Putting $x_2=-it$, we get $-x_1=-ix_2=i^2t=-t \qquad \therefore\ x_1=t$

$$\therefore\ X_2=\begin{bmatrix}t\\ -it\end{bmatrix}=t\begin{bmatrix}1\\ -i\end{bmatrix} \qquad \therefore\ \text{The eigenvector is } [1, -i]'.$$

$\therefore$ Modal matrix $M=\begin{bmatrix}1 & 1\\ i & -i\end{bmatrix}$ $\quad\therefore\ |M|=-2i$

$$\therefore\ M^{-1} = -\frac{1}{2i}\begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix} = \frac{1}{2i}\begin{bmatrix} i & 1 \\ i & -1 \end{bmatrix}$$

Now, $D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

$\therefore$ If $f(A) = e^{At}$, $f(D) = e^{D} = \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix}$

$$e^{At} = M\, f(D)\, M^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}\begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix}\frac{1}{2i}\begin{bmatrix} i & 1 \\ i & -1 \end{bmatrix} = \frac{1}{2i}\begin{bmatrix} e^{it} & e^{-it} \\ ie^{it} & -ie^{-it} \end{bmatrix}\begin{bmatrix} i & 1 \\ i & -1 \end{bmatrix}$$

$$= \frac{1}{2!}\begin{bmatrix} i(e^{it}+e^{-it}) & e^{it}-e^{-it} \\ -e^{it}-e^{-it} & i(e^{it}+e^{-it}) \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

Another method

The method used in the above examples is useful **if the matrix A is diagonisable *i.e.*, if the eigenvalues are distinct**. If the diagonal matrix of A cannot be found out, the above method cannot be used. In such cases we use the method stated below.

If the given matrix is of order 3, we assume that the required function $\Phi(A)$ can be written as

$\Phi(A) = \alpha_2 A^2 + \alpha_1 A + \alpha_0 I$ where, $\alpha_0, \alpha_1, \alpha_2$ are constants to be determined.

If the given matrix is of order 2, we assume that the required function

$\Phi(A) = \alpha_1 A + \alpha_0 I$ where, α_0, α_1 are constants to be determined.

We then find the constants $\alpha_0, \alpha_1, \alpha_2$ as illustrated in Ex. 10 and 11.

Remark ....

The method is quite general when the matrix is diagonalisable or not *i.e.* when the eigenvalues are distinct or not distinct as is illustrated in Ex. 11 below.

Example 6 : If $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, prove that $A^{50} - A^{49} = \begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix}$.

Sol. : The characteristic equation of A is

$$\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$\therefore\ (1-\lambda)(3-\lambda) - 8 = 0 \qquad \therefore\ 3 - 4\lambda + \lambda^2 - 8 = 0$

$\therefore\ \lambda^2 - 4\lambda - 5 = 0 \qquad \therefore\ (\lambda - 5)(\lambda + 1) = 0 \qquad \therefore\ \lambda = -1, 5.$

For $\lambda = 5$, $[A - \lambda I]X = O$ gives

$$\begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By $R_2 + (1/2)R_1$, $-(1/4)R_1$ $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore\ x_1 - x_2 = 0.$ Putting $x_2 = t$, we get $x_1 = t$.

$$\therefore \quad X = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \therefore \quad X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(ii) For $\lambda = -1$, $[A - \lambda I]X = O$ gives

$$\begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By $\begin{matrix} R_2 - R_1 \\ (1/2)R_1 \end{matrix}\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore \; x_1 + 2x_2 = 0.$ Putting $x_2 = -t$, we get $x_1 = 2t$.

$$\therefore \quad X = \begin{bmatrix} 2t \\ -t \end{bmatrix} = t\begin{bmatrix} 2 \\ -1 \end{bmatrix} \qquad \therefore \quad X_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Now, $D = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$ and $M = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$

$$\therefore \quad |M| = -3 \qquad \therefore \quad M^{-1} = -\frac{1}{3}\begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$\therefore \quad D^{49} = \begin{bmatrix} 5^{49} & 0 \\ 0 & -1 \end{bmatrix} \qquad \left[\because \text{ If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ then } A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right]$$

$$\therefore \quad A^{49} = MD^{49}M^{-1}$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 5^{49} & 0 \\ 0 & -1 \end{bmatrix}\left(\frac{1}{3}\right)\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 5^{49} & -2 \\ 5^{49} & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 5^{49} - 2 & 2 \cdot 5^{49} + 2 \\ 5^{49} + 1 & 2 \cdot 5^{49} - 1 \end{bmatrix}$$

$$\therefore \quad A^{50} - 5A^{49} = A^{49}[A - 5I]$$

$$= A^{49}\left\{\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\right\} = A^{49}\begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 5^{49} - 2 & 2 \cdot 5^{49} + 2 \\ 5^{49} + 1 & 2 \cdot 5^{49} - 1 \end{bmatrix}\begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 12 & -12 \\ -6 & 6 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -2 & 2 \end{bmatrix}$$

Example 7 : If $A = \begin{bmatrix} 2 & 3 \\ -3 & -4 \end{bmatrix}$, prove that $A^{50} = \begin{bmatrix} -149 & -150 \\ 150 & 151 \end{bmatrix}$. **(M.U. 1999, 2004, 16)**

Sol. : The characteristic equation of A is

$$\begin{vmatrix} 2 - \lambda & 3 \\ -3 & -4 - \lambda \end{vmatrix} = 0$$

$$\therefore \quad (2 - \lambda)(-4 - \lambda) + 9 = 0 \qquad \therefore \quad -8 - 2\lambda + 4\lambda + \lambda^2 + 9 = 0$$

$$\therefore \quad \lambda^2 + 2\lambda + 1 = 0 \qquad \therefore \quad (\lambda + 1)^2 = 0 \qquad \therefore \quad \lambda = -1, -1.$$

The eigenvalues are repeated. Hence, we use the second method.

Let $\quad \Phi(A) = A^{50} = \alpha_1 A + \alpha_0 I$

We assume that the above equality is satisfied by the characteristic roots of A.

$\therefore\ \lambda^{50} = \alpha_1 \lambda + \alpha_0$ .... (1)

Putting $\lambda = -1$, we get $(-1)^{50} = \alpha_1(-1) + \alpha_0$

$\therefore\ 1 = -\alpha_1 + \alpha_0$ .... (2)

Since the characteristic roots are repeated to obtain another equation we differentiate w.r.t. λ.

Now, differentiating (1), we get $\quad 50\lambda^{49} = \alpha_1$

Putting $\lambda = -1$, again, we get $\quad 50(-1)^{49} = \alpha_1 \quad \therefore\ \alpha_1 = -50.$

Putting $\alpha_1 = -50$ in (2), we get $\quad 1 = 50 + \alpha_0 \quad \therefore\ \alpha_0 = -49.$

From $A^{50} = \alpha_1 A + \alpha_0 I$, we get

$$A^{50} = -50\begin{bmatrix} 2 & 3 \\ -3 & -4 \end{bmatrix} - 49\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -149 & 150 \\ 150 & 151 \end{bmatrix}$$

Example 8 : If $A = \begin{bmatrix} \pi & \pi/4 \\ 0 & \pi/2 \end{bmatrix}$, find $\cos A$. **(M.U. 2003, 05, 10)**

Sol. : The characteristic equation is

$$\begin{vmatrix} \pi-\lambda & \pi/4 \\ 0 & (\pi/2)-\lambda \end{vmatrix} = 0 \quad \therefore\ (\pi-\lambda)\left(\frac{\pi}{2}-\lambda\right) = 0 \quad \therefore\ \lambda = \frac{\pi}{2}, \pi.$$

Let $\Phi(A) = \cos A = \alpha_1 A + \alpha_0 I$ .... (1)

Since λ satisfies the above equation, we have

$\cos\lambda = \alpha_1\lambda + \alpha_0$ .... (2)

Putting $\lambda = \pi/2$, we get

$\cos\frac{\pi}{2} = \alpha_1 \cdot \frac{\pi}{2} + \alpha_0 \quad \therefore\ 0 = \alpha_1 \cdot \frac{\pi}{2} + \alpha_0$ .... (3)

$\cos\pi = \alpha_1 \cdot \pi + \alpha_0 \quad \therefore\ -1 = \alpha_1 \cdot \pi + \alpha_0$ .... (4)

From (3) and (4), we get

$$\alpha_1 \cdot \frac{\pi}{2} = -1 \quad \therefore\ \alpha_1 = -\frac{2}{\pi} \quad \therefore\ \alpha_0 = -1+2 = 1.$$

Putting these values in (1), we get

$$\cos A = -\frac{2}{\pi}\begin{bmatrix} \pi & \pi/4 \\ 0 & \pi/2 \end{bmatrix} + 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1/2 \\ 0 & 0 \end{bmatrix}$$

Example 9 : Show that $\cos O_{3\times3} = I_{3\times3}$. **(M.U. 2004)**

Sol. : We have $\quad O_{3\times3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Clearly eigenvalues of $O_{3\times3}$ are 0, 0 and 0.

Let $\quad \cos O_{3\times3} = \alpha_2 O_{3\times3} + \alpha_1 O_{3\times3} + \alpha_0 I_3$ .... (1)

$\therefore\ \lambda$ satisfies this equation. $\quad \therefore\ \cos\lambda = \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0$ .... (2)

Differentiating (2), w.r.t. λ, $\quad -\sin\lambda = 2\alpha_2\lambda + \alpha_1$ .... (3)

Again differentiating w.r.t. λ, $\quad -\cos\lambda = 2\alpha_2$ .... (4)