Week 1

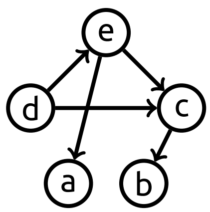
- A graph $G = (V, E)$ consists of the set of vertices V and the set of edges E.
- For an edge $e = \{u, v\}$, we say:
  - e connects u and v;
  - u and v are end points of e;
  - u and e are incident (v and e are incident);
  - u and v are adjacent or neighbors.
- The degree $\deg(v)$ of a vertex v is the number of edges incident to it. A vertex of degree 0 is called isolated.
- In a directed graph, the indegree (outdegree) of a vertex v is the number of edges ending at v (leaving v).
- The degree of a graph is the maximum degree of its vertex. A k-regular graph is a graph where each vertex has degree k.
- The complement of a graph $G = (V, E)$ is a graph $\overline{G} = (V, \overline{E})$ s.t. $(u, v) \in \overline{E}$ if and only if $(u, v) \notin E$.
- A walk in a graph is a sequence of edges, where each edge (except for the 1st one) starts with a vertex where the previous edge ended. The length of a walk is the number of edges in it.
- A path is a walk where all edges are distinct.
- A simple path is a walk where all vertices are distinct.
- A cycle in a graph is a path whose 1st vertex is the same as the last one.
- A simple cycle is a cycle where all vertices except for the 1st one are distinct. (And there 1st vertex is taken twice.)
- A graph is called connected if there is a path between every pair of its vertices.
- A connected component of a graph G is a maximal connected subgraph of G.
- The path graph P_n consists of n vertices $v_1, \ldots, v_n$ and $n-1$ edges $\{v_1, v_2\}, \ldots, \{v_{n-1}, v_n\}$.
- The cycle graph C_n consists of n vertices $v_1, \ldots, v_n$ and n edges $\{v_1, v_2\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$.
- The complete graph (clique) K_n contains n vertices $v_1, \ldots, v_n$ and all $n(n-1)/2$ edges between them.
- Three equivalent definitions of a tree:
  - a connected graph without cycles;
  - a connected graph on n vertices with $n-1$ edges;
  - a graph with a unique simple path between any pair of its vertices.
- A graph G is bipartite if its vertices can be partitioned into two disjoint sets L and R s.t. every edge of G connects a vertex in L with a vertex in R.

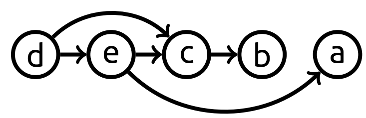
Week 2

- Degree sum formula: for any undirected graph $G(V, E)$, the sum of degrees of all its nodes is twice the number of edges: $\sum_{v \in V} \text{degree}(v) = 2 \cdot |E|$
- Lower bound on the number of connected components: an undirected graph $G(V, E)$ has at least $|V| - |E|$ connected components.
- A directed acyclic graph (DAG) is a directed graph without cycles.

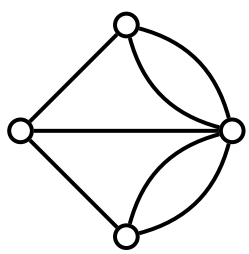
(Strongly connected components only for DAGs)

- A topological ordering of a directed graph is an ordering of its vertices such that, for each edge (u, v), u comes before v. Such an ordering exists, if and only if the graph is acyclic.

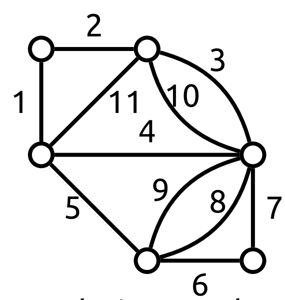




- An Eulerian cycle (or path) visits every edge exactly once.



non-Eulerian graph

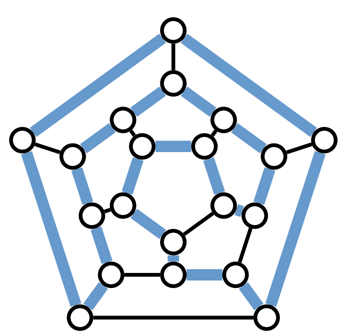


Eulerian graph

Criteria:

  - A connected *undirected* graph contains an Eulerian cycle, if and only if the degree of every node is even.
  - A strongly connected *directed* graph contains an Eulerian cycle, if and only if, for every node, its in-degree is equal to its out-degree.

- A Hamiltonian cycle visits every node exactly once.



Week 3

- A spanning tree of a graph G, is a subgraph of G which is a tree and contains all vertices of G.
- A minimum spanning tree of a weighted graph is a spanning tree of the smallest weight.
- Kruskal's minimum spanning tree algorithm:
  - Start with an empty graph T.
  - Repeat $n-1$ times:
    - Add to T an edge of the smallest weight which doesn't create a cycle in T.
- A graph is bipartite if and only if it has no cycles of odd length.
- A matching in a graph is a set of edges without common vertices.
- A maximal matching is a matching which cannot be extended to a larger matching.
- A maximum matching is a matching of the largest size.
- If $G = (V, E)$ is a graph, and $S \subseteq V$ is its subset of vertices, then the neighborhood $N(S)$ of S is the set of all vertices connected to at least one vertex in S.
- Hall's theorem: In a bipartite graph $G = (L \cup R, E)$, there is a matching which covers all vertices from L if and only if for every subset of vertices $S \subseteq L$, $|S| \leq |N(S)|$.
- A graph is planar if it can be drawn in the plane such that its edges do not meet except at their end points. (All cycle graphs are planar)
- A face of a planar drawing of a graph is a region bounded by the edges of the graph. (There is always one infinitely large outer face.)
- Euler's formula: for a planar drawing of a connected planar graph: $v - e + f = 2$.
- Every planar graph has a vertex of degree ≤ 5.
- In every planar graph on $n \geq 3$ vertices: $e \leq 3v - 6$.
- In every bipartite planar graph on $n \geq 4$ vertices: $e \leq 2v - 4$.

Week 4

- A graph coloring is a coloring of the graph vertices s.t. no pair of adjacent vertices share the same color.
- The chromatic number $\chi(G)$ of a graph G is the smallest number of colors needed to color the graph. (Chromatic number of bipartite graph is 2)
- Four color theorem: every planar graph can be colored in 4 colors.
- Brooks' theorem: A graph G of maximum degree Δ can be colored with Δ colors, unless G is complete (K_n) (χ=n) or a cycle of odd length (C_{2k+1}). (Greedy colouring - Δ+1 colors)
- A clique of a graph is a set of vertices such that every two vertices are connected by an edge. (Start from vertices of highest degree)
- A maximal clique is a clique which is not contained in any other clique.
- A maximum clique is a clique such that there are no cliques with more vertices.
- The clique number $\omega(G)$ of a graph G is the number of vertices in its maximum clique. (Clique of bipartite graph is 2)
- An independent set (IS) of a graph is a set of vertices such that no two vertices are connected by an edge.
- A maximal independent set is an IS which is not contained in any other IS (i.e., cannot be extended to a larger IS).
- A maximum independent set is an IS such that there are no IS's with more vertices.
- The independence number $\alpha(G)$ of a graph G is the number of vertices in its maximum IS.
- $\omega(G) = \alpha(\overline{G})$. (Cliques correspond to IS in the complement graph and vice versa)
- $\chi(G) \cdot \alpha(G) \geq n$.
- Mantel's theorem: A graph on n vertices without triangles has at most $\lfloor n^2/4 \rfloor$ edges.(Bipartite graph)
- Turán's theorem: If a graph G on n vertices contains no K_{r+1}, then it has at most $\left(1 - \frac{1}{r}\right)\frac{n^2}{2}$ edges. (General case)
- For two integers k, ℓ, the Ramsey number $R(k, \ell)$ is the minimum number, s.t. every graph with at least $R(k, \ell)$ vertices must have either a clique of size k or an independent set of size ℓ.
- $R(3,3) = 6$; $R(4,4) = 18$; $43 \leq R(5,5) \leq 48$. R(2,2) = 2; R(2,3) = 3, R(2,n)=R(n,2)= n
- Ramsey's theorem: $R(k, \ell) \leq R(k-1, \ell) + R(k, \ell - 1)$.
- A vertex cover of a graph G is a set of vertices C such that every edge of G is connected to some vertex in C.
- A minimal vertex cover is a vertex cover which does not contain other vertex covers.
- A minimum vertex cover is a vertex cover of the smallest size. (Min. vertex cover of K_n = n-1)
- The size of a minimum vertex cover is denoted by $\beta(G)$.
- $\beta(G) + \alpha(G) = n$.
- König's theorem: in a bipartite graph, the number of edges in a maximum matching equals the number of vertices in a minimum vertex cover.

Week 5

- Network: n vertices $1, 2, \ldots, n$; source and destination vertices(=nodes) are fixed; for each vertices i and j we know (directed) capacities $c[i, j] \geq 0$ and $c[j, i] \geq 0$; "no pipe" is 0; we assume $c[i, i] = 0$ for convenience. (c[i,j] not necessarily equal to c[j,i])
- Flow (in a network): for every two vertices i and j some number $f[i, j]$ is fixed; $f[i, j] \leq c[i, j]$; $f[i, j] = -f[j, i]$; no spill condition: $\sum_j f[i, j] = 0$ for all i except for source and destination.
- Total flow can be computed at the source A as $\sum_j f[A, j]$ or a destination B as $\sum_i f[i, B]$.
- Cut: a set of vertices that contains the source but not the destination.
- Total capacity of a cut C: the sum of all capacities $c[u, v]$ where u is in C and v is outside C.
- Obvious: any total flow is bounded by the total capacity of any cut
- Ford–Fulkerson's theorem: maximal flow equals minimal cut
- Special case: non-zero flow exists $\Leftrightarrow$ there is no zero cut $\Leftrightarrow$ there is a path from source to destination with positive capacities
- Residual network: if a network with capacities $c[i, j]$ is given, and $f[i, j]$ is a flow, then residual network has capacities $c'[i, j] = c[i, j] - f[i, j]$
- Perfect matching: a set of edges that is a one-to-one correspondence between the parts of a bipartite graph (we consider only graphs with parts of equal size).
- Hall's theorem: a perfect matching in a bipartite graph exists if and only if there is no obstacles, where an obstacle is a set of left vertices that has less neighbors that elements.
- Flows in the residual network are exactly the possible increases of the given flow in the given network
- Reduction from tiling problem to perfect matching: each domino tile is an edge the connects left vertex (white cell) and right vertex (black cell).
- Stable matching problem: a bipartite graph with n left vertices ("men") and n right vertices ("women"); each man has an ordered list of women and vice versa ("preferences")
- Stable matching: a perfect matching that has no unstable pairs. Unstable pair: man and woman that prefer each other to their current partner
- Gale–Shapley algorithm: men go along their lists making proposals that are accepted when better than status quo
- Gale–Shapley theorem: the algorithm converges to a stable matching
- Unfair: the algorithm provides for each man a partner that is the most preferable among all possible partners (for all stable matchings).
- Very Unfair: the algorithm provides for each woman a partner that is the least preferable among all possible partners (for all stable matchings).