ECE 66100 Computer Vision **Name:** *Nikita Ravi (ravi30)*
Homework 1
08/30/2022

---

1. What are all the points in the representational space $\mathbb{R}^3$ that are the homogeneous coordinates of the origin in the physical space $\mathbb{R}^2$?

   Answer: The origin represented in the $\mathbb{R}^2$ space can be expressed as

$$k \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

   where $k \in \mathbb{R}$ and $k > 0$ in the $\mathbb{R}^3$ space.

2. Are all points at infinity in the physical plane $\mathbb{R}^2$ the same? Justify your answer

   Answer: The intersection of two parallel lines

$$l_1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{ and } l_2 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

   is $\begin{bmatrix} b & -a & 0 \end{bmatrix}^T$. The third coordinate is equal to zero because the intersection point is at infinity in $\mathbb{R}^2$ along a specific direction dependent on the pair (a, b). However, not all the points at infinity in the $\mathbb{R}^2$ space will be the same because two ideal points $\begin{bmatrix} \mathrm{x}_1 & \mathrm{y}_1 & 0 \end{bmatrix}^T$ and $\begin{bmatrix} \mathrm{x}_2 & \mathrm{y}_2 & 0 \end{bmatrix}^T$ for example, will approach infinity in different directions if $x_1 \neq x_2$ and $y_1 \neq y_2$ and slopes $\frac{y_1}{x_1} \neq \frac{y_2}{x_2}$

3. Argue that the matrix rank of a degenerate conic can never exceed 2

   Answer: The matrix of a degenerate conic is given by

$$C = lm^T + ml^T$$

   And the rank of the above equation will always definitely be two because the rank of an outer product (e.g. lm^T or ml^T) will be equal to one due to the fact that every column is a constant times the first column. As a result, the sum of two outer product matrices such as that of C will always have a rank of two.

4. A line in $\mathbb{R}^2$ is defined by two points. That raises the question – how many points define a conic in $\mathbb{R}^2$? Justify your answer.

   Answer: The implicit form of a conic in the physical plane $\mathbb{R}^2$ is $ax^2 + bxy + cy^2 + dx + ey + f = 0$. To rephrase this in the form of matrices:

$$\begin{bmatrix} \mathrm{x}^2 & \mathrm{xy} & \mathrm{y}^2 & \mathrm{x} & \mathrm{y} & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix}$$

Since there are six unknowns (i.e. $(a, b, c, d, e, f)^T$), we require five systems of linear equations to solve for these unknowns. These equations represent points so there must be five points.

5. (a) Derive in just 3 steps the intersection of two lines l_1 and l_2 with l_1 passing through the points (0,0) and (1,2), and with l_2 passing through the points (3,4) and (5,6).

Answer: The first step is to compute the homogeneous vector of l_1 given that we know the two points it passes through as shown below

$$l_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

The second step is to compute the homogeneous vector of l_2 given that we know the two points it passes through as shown below

$$l_2 = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \times \begin{bmatrix} 5 \\ 6 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix}$$

Finally the third step is to find the intersection of lines l_1 and l_2 using the cross product of the two vectors as shown below

$$l_1 \times l_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ -2 \end{bmatrix}$$

Therefore lines l_1 and l_2 intersect at coordinate $(\frac{-2}{-2}, \frac{-4}{-2}) = (1, 2)$

(b) How many steps would take you if the second line passed through (7, −8) and (−7, 8)?

Answer: The first step is to calculate the homogeneous vector for l_2 given the new points as shown below

$$l_2 = \begin{bmatrix} 7 \\ -8 \\ 1 \end{bmatrix} \times \begin{bmatrix} -7 \\ 8 \\ 1 \end{bmatrix} = \begin{bmatrix} -16 \\ -14 \\ 0 \end{bmatrix}$$

The second and final step is to calculate the intersection of the new line l_2 with the already calculated l_1 as shown below

$$l_1 \times l_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} -16 \\ -14 \\ -0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 44 \end{bmatrix}$$

The point of intersection is at the origin. Therefore, there are **two** steps.

6. Let l_1 be the line passing through points (4, 0) and (2, 8) and l_2 be the line passing through points (0, 2) and (4, 14). Find the intersection between these two lines. Comment on your answer.

   Answer: $l_1 = \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} -2 \\ 8 \\ 1 \end{bmatrix} = \begin{bmatrix} -8 \\ 2 \\ -32 \end{bmatrix}$

   $l_2 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \times \begin{bmatrix} 4 \\ 14 \\ 1 \end{bmatrix} = \begin{bmatrix} -16 \\ 4 \\ 8 \end{bmatrix}$

   $l_1 \times l_2 = \begin{bmatrix} -8 \\ 2 \\ -32 \end{bmatrix} \times \begin{bmatrix} -16 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 144 \\ 576 \\ 0 \end{bmatrix}$

   l_1 and l_2 are parallel to each other since the intersection is an ideal point

7. Find the intersection of two lines whose equations are given by x = 1 and y = .

   Answer: A line in $\mathbb{R}^2$ is a set of points $(x, y) \in \mathbb{R}^2$ that obey $ax + by + c = 0$. Given that $x = 1$ and $y = -1$, let $a = 1$ and $b = 1$.

   To find the homogeneous coordinates of the line whose equation is $x = 1$, we need to figure out what the third coordinate c will be. This is done by substituting $x = 1$ and $a = 1$ and $y = 0$ into the linear equation as shown: $(1)(1) + c = 0$, so $c = -1$. Similarily, to find the homogeneous coordinates of the line whose equation is $y = -1$, the third coordinate c is calculated to be: $c = -(1)(-1) = 1$.

   Therefore $l_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $l_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

   The intersection of these two lines is as follows:

   $l_1 \times l_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

   So the point of intersection is $(1, -1)$

8. As you know, when a point p is on a conic C, the tangent to the conic at that point is given by l = Cp. That raises the question as to what Cp would correspond to when p was outside the conic. As you'll see later in class, when p is outside the conic, Cp is the line that joins the two points of contact if you draw tangents to C from the point p. This line is referred to as the polar line. Now let our conic C be an ellipse that is centered at the coordinates (2, 3), with a = 1/2 and b = 1, where a and b, respectively, are the lengths of semi-minor

and semi-major axes. For simplicity, assume that the minor axis is parallel to x-axis and the major axis is parallel to y-axis. Let p be the origin of the $\mathbb{R}^2$ physical plane. Find the intersections points of the polar line with x- and y-axes.

Answer: The equation of an ellipse whose major axis is parallel to the y-axis and whose minor axis is parallel to the x-axis with its center at $2, 3$ is $\frac{(x-2)^2}{1^2} + \frac{(y-3)^2}{\frac{1}{2}^2} = 1$. Which can be further simplified to $(x-2)^2 + 4(y-3)^2 = 1$. Expanding this equation will result it to be $x^2 - 4x - 24y + 4y^2 + 39 = 0$.

Let a point have coordinates of $\begin{bmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \\ \mathrm{x}_3 \end{bmatrix}$ so $x = \frac{x_1}{x_3}$ and $y = \frac{x_2}{x_3}$. Substituting this x and y expressions into the expanded form of the ellipse equation gives $x_1^2 - 4x_1x_3 + 24x_2x_3 + 4x_2^2 + 39x_3^2 = 0$. Therefore, the homogeneous representation of a conic is

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 4 & 12 \\ -2 & 12 & 39 \end{bmatrix}$$

The homogeneous coordinates of the polar line is therefore calculated to be

$$l_{polar} = Cp = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 4 & 12 \\ -2 & 12 & 39 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 12 \\ 39 \end{bmatrix}$$

(a) Intersection with the x-axis

Answer: $l_{polar} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -39 \\ 0 \\ -2 \end{bmatrix}$

(b) Intersection with the y-axis

Answer: $l_{polar} \times \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 39 \\ -12 \end{bmatrix}$