fourier transforms

x-space $\rightarrow$ k-space

for some function $f(x)$, its fourier transform is:

$$\tilde{F}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

to go from the fourier transform $F(k)$, back to $f(x)$:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{F}(k)\, e^{ikx}\, dk$$

discretizing the fourier transform

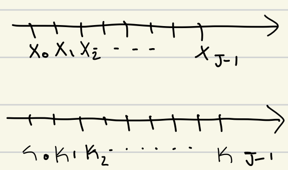
note: generally we use "k" for both the (possibly non-integer) wave # and for the (integer) index because usually $L = 2\pi$ so κ is usually equal to integer k $(\kappa = \frac{2\pi}{L} k)$
i use kappa "κ" for the wave # for clarity

* for 331 HW07 #3 $L = 2\pi$, so $\kappa = k$ & this subtlety doesn't matter

integrals become sums, infinitesimals become discrete changes

$$\int \rightarrow \sum \qquad dx \rightarrow \Delta x \qquad dk \rightarrow \Delta k$$

set up x & κ grids w/ J support points



$x \rightarrow x_j = j\,\Delta x$ where $\Delta x = L/J$ and $j \in [0, 1, 2, \ldots, J-1]$

$\kappa \rightarrow \kappa_k = k\,\Delta\kappa$ where $\Delta\kappa = 2\pi/L$ and $k \in [0, 1, 2, \ldots J-1]$

$f(x) \rightarrow f_j = f(x_j)$

$F(k) \rightarrow F_k = F(k_k)$

exponential term

$$ikx = i k_k x_j$$
$$= i\left(\frac{2\pi k}{L}\right)\left(j \frac{L}{J}\right)$$
$$= i 2\pi kj/J$$

$$F_k = \frac{1}{\sqrt{2\pi}} \sum_{j=0}^{n-1} f_j e^{-i2\pi kj/J} \Delta x$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{j=0}^{n-1} f_j e^{-i2\pi kj/J} \left(\frac{L}{J}\right)$$

$$f_j = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{n-1} F_k e^{i2\pi kj/J} \Delta k$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{n-1} F_k e^{i2\pi kj/J} \left(\frac{2\pi}{L}\right)$$

L and $1/L$

and

$\frac{1}{\sqrt{2\pi}}$ factors are commonly left off since the cancel w/ the 2π from Δk when doing a foreword then backward transform

$$\boxed{F_k = \frac{1}{J} \sum_{j=0}^{n-1} f_j e^{-i2\pi kj/J}}$$ (forward)

$$\boxed{f_j = \sum_{k=0}^{n-1} F_k e^{i2\pi kj/J}}$$ (backward)