Linear

Least squares fitting

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linear regression

y_i are our data points at x_i
with uncertainties σ_i

we have a model $y(x; a_1, a_2, \dots a_m)$

where the "a"s are parameters
that we would like to solve for
their values that give the best
fit for our data.

(a) maximum likelihood estimator

$$P \propto \prod_{i=0}^{n-1} \exp\left(-\frac{1}{2}\left(\frac{y_i - y(x_i; a, b)}{\sigma_i}\right)^2\right)$$

show $P \uparrow$ for $\chi^2 \downarrow$

$$\chi^2 \equiv \sum_{i=0}^{n-1} \left(\frac{y_i - y(x_i; a, b)}{\sigma_i}\right)^2$$

note: $\exp(A)\exp(B) = \exp(A+B)$

(b) minimizing χ^2 $\quad y(x_i; a, b) = a + bx_i$

$$\chi^2 \equiv \sum_{i=0}^{n-1} \left(\frac{y_i - y(x_i; a, b)}{\sigma_i} \right)^2$$

we want to minimize χ^2 with respect to our parameters a & b

for a:

$$\frac{\partial \chi^2}{\partial a} = \frac{\partial}{\partial a} \left(\sum_i \left(\frac{y_i - a - bx_i}{\sigma_i} \right)^2 \right)$$

$$= 2 \left(\sum_i \left(\frac{y_i - a - bx_i}{\sigma_i} \right) \left(-\frac{1}{\sigma_i} \right) \right)$$

$$= -2 \left[\sum_i \frac{y_i}{\sigma_i^2} - a \sum_i \frac{1}{\sigma_i^2} - b \sum_i \frac{x_i}{\sigma_i^2} \right]$$

$$0 = -2 (S_y - aS - bS_x)$$ ✓

(c) parameter uncertainties

for a

$$\sigma_a^2 \equiv \sum_{i=0}^{n-1} \sigma_i^2 \left(\frac{\partial a}{\partial y_i}\right)^2 \overset{\text{show}}{=} \frac{S_{xx}}{\Omega}$$

from (b): $a = \dfrac{S_{xx} S_y - S_{xy} S_x}{\Omega}$

$$\frac{\partial a}{\partial y_k} = \frac{\partial}{\partial y_k}\left(\frac{S_{xx} S_y - S_{xy} S_x}{\Omega}\right)$$

$$= \frac{1}{\Omega}\left(S_{xx} \frac{\partial S_y}{\partial y_k} - S_x \frac{\partial S_{xy}}{\partial y_k}\right)$$

$$= \frac{1}{\Omega}\left[S_{xx} \frac{\partial}{\partial y_k}\left(\sum_i \frac{y_i}{\sigma_i^2}\right) - S_x \frac{\partial}{\partial y_k}\left(\sum_i \frac{x_i y_i}{\sigma_i^2}\right)\right]$$

$$= \frac{1}{\Omega}\left[S_{xx}\left(\frac{1}{\sigma_k^2}\right) - S_x\left(\frac{x_k}{\sigma_k^2}\right)\right]$$

$$\frac{\partial a}{\partial y_k} = \frac{S_{xx} - S_x x_k}{\sigma_k^2 \, \Omega}$$

$$\sigma_a^2 \equiv \sum_i \sigma_i^2 \left(\frac{\partial a}{\partial y_i}\right)^2$$

$$= \sum_i \sigma_i^2 \left(\frac{S_{xx} - S_x X_i}{\sigma_i^2 \Omega}\right)^2$$

$$= \frac{1}{\Omega^2} \sum_i \left(\frac{S_{xx}^2 - 2 S_x S_{xx} X_i + S_x^2 X_i^2}{\sigma_i^2}\right)$$

$$= \frac{1}{\Omega^2}\left[S_{xx}^2 \sum_i \frac{1}{\sigma_i^2} - 2 S_x S_{xx} \sum_i \frac{X_i}{\sigma_i^2} + S_x^2 \sum_i \frac{X_i^2}{\sigma_i^2}\right]$$

$$= \frac{1}{\Omega^2}\left[S_{xx}^2 S - 2 S_x^2 S_{xx} + S_x^2 S_{xx}\right]$$

$$= \frac{S_{xx}}{\Omega^2}\left(S_{xx} S - S_x^2\right)$$

$$= \frac{S_{xx}}{\Omega}$$