ph 332: ODE examples

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Rider
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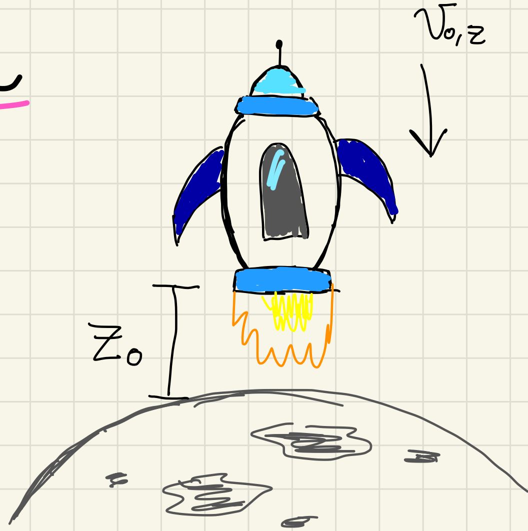
lunar lander



given in problem:

$M_{tot} = M_{ship} + M_{fuel}$

$v_{0,z} = -5 \text{ m/s}$

$z_0 = 500 \text{ m}$

$F = T_{max} k - M_{tot} g$

$k \in [0, 1]$

$v_{nozz} = \text{const.}$

$$M_{tot} \ddot{z} = T_{max} k - M_{tot} g$$

$$\ddot{z} = \frac{T_{max} k}{M_{tot}} - g$$

(2nd order ODE)

we can break this up into 2 1st order ODEs

(1) $$\dot{z} = v_z$$

(2) $$\dot{v}_z = \frac{T_{max} k}{M_{tot}} - g$$

M_{tot} also changes in time, we need an ODE for that.

from lecture:

$$(3) \qquad \dot{M}_{fuel} = -\frac{T_{max}\,k}{v_{nozz}}$$

last, k could depend on time (but doesn't now) so,

$$(4) \qquad \dot{k} = 0$$

we can change this later if we want k to be time-dependent

the code already has

$$y = \begin{bmatrix} z \\ v_z \\ M_{fuel} \\ k \end{bmatrix}$$

you need to write the corresponding derivative vector

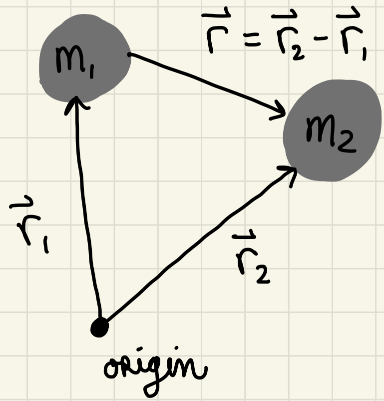
$$\frac{dy}{dx} = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}$$

our "x" is t for this problem

kepler

force of gravity on m_1 from m_2:

$$\vec{F}_g = m_1 \ddot{\vec{r}}_1 = \frac{G m_1 m_2}{r^2} \hat{r}$$



where r is the distance between them

$$\vec{r} = \vec{r}_2 - \vec{r}_1$$

$$r = |\vec{r}| = |\vec{r}_2 - \vec{r}_1|$$

$\hat{r}$ is the unit vector along $\vec{r}$

so, $$\hat{r} = \frac{\vec{r}}{|\vec{r}|}$$

two 2nd order ODEs for m_1

$$\ddot{\vec{r}}_1 = \frac{G m_2}{|\vec{r}_2 - \vec{r}_1|^3} (\vec{r}_2 - \vec{r}_1)$$

we need to break this up into cartesian (x & y) components

general r vector in cartesian:

$$\vec{r} = x\hat{x} + y\hat{y}$$

so, what we need is

$$\vec{r}_2 - \vec{r}_1 = (x_2 - x_1)\hat{x} + (y_2 - y_1)\hat{y}$$

also, let's define

$$u_i = \dot{x}_i \quad \& \quad v_i = \dot{y}_i$$

to break the 2nd order ODEs into 1st order

ODEs for m_1 in the presence of only m_2:

$$\begin{cases} \dot{x}_1 = vx_1 \\ \dot{y}_1 = vy_1 \\ \dot{vx}_1 = +\dfrac{Gm_2}{|\vec{r}_2 - \vec{r}_1|^3}(x_2 - x_1) \\ \dot{vy}_1 = +\dfrac{Gm_2}{|\vec{r}_2 - \vec{r}_1|^3}(y_2 - y_1) \end{cases}$$

note: the sign here depends on the order of this difference

Remember, forces are additive

$$\vec{F}_{tot,i} = \sum_j \vec{F}_j$$

so we can generalize this.

$$\begin{cases} \dot{vx}_i = G \sum_j m_j \dfrac{(x_j - x_i)}{|\vec{r}_j - \vec{r}_i|^3} \\ \dot{vy}_i = G \sum_j m_j \dfrac{(y_j - y_i)}{|\vec{r}_j - \vec{r}_i|^3} \end{cases}$$

* where $i \neq j$ (a body doesn't experience a force from itself!)

& note that:

$$|\vec{r}_2 - \vec{r}_1| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

i would make a variable for this in your loop to use in the $\dot{u}$ & $\dot{v}$ eqs. it will be cleaner than writing the full expression in the denominator

$$\begin{cases} \dot{x}_i = u_i \\ \dot{y}_i = v_i \end{cases}$$

these ODEs are unchanged when we add more bodies

4 first order ODEs per body

optional

a trick to speed things up

now, according to Newton's 3rd law, the force from m_1 on m_2 is equal & opposite to the force from m_2 on m_1

so, we only need one force calculation per pair!

$$\dot{u}_{21} = -\dot{u}_{12}$$

$$\dot{v}_{21} = -\dot{v}_{12}$$

this will save us computing time!