fourier transforms

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F2021

a fourier transform

takes a function from "x-space" to "k-space"

it takes a wave and "picks out" the wavelength / frequency

the function $f(x)$ could be a single wave (like $\sin(x)$ or $\cos(x)$)

or a linear combination of waves (like $f(x) = \sin(3x) + \sin(x)$)

fourier transform: $f(x) \rightarrow \tilde{F}(k)$

inverse fourier transform: $\tilde{F}(k) \rightarrow f(x)$

definition:

fourier transform of $f(x)$

$$\boxed{\tilde{F}(k) = \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx}$$

k is called the wavenumber, it is related to the wavelength by,

$$\boxed{k = \frac{2\pi}{\lambda}}$$

definition:

inverse fourier transform

$$\boxed{f(x) = \frac{1}{L} \int_{-\infty}^{\infty} \tilde{F}(k)\, e^{ikx}\, dk}$$

notation issue:

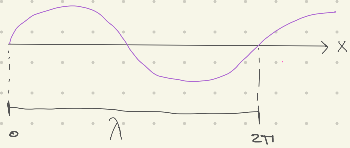
$$F[f](k) = \text{the fourier transform of } f(x)$$

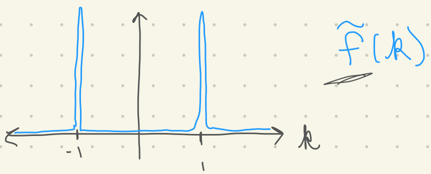
$$= \tilde{F}(k) \text{ in my notation}$$

homework notation

simple example

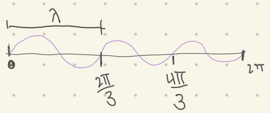
$f(x) = \sin(x) \rightarrow \lambda = 2\pi \rightarrow k = 1$

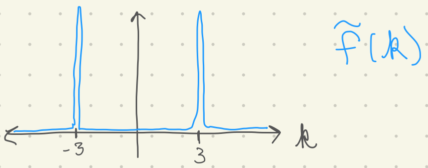




another example

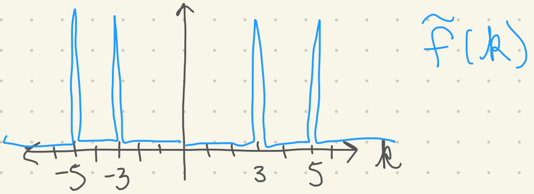
$f(x) = \sin(3x) \rightarrow \lambda = \frac{2\pi}{3} \rightarrow k = 3$





linear combo example

$f(x) = \cos(3x) + \cos(5x)$



convolution

$$\boxed{[f * g](s) = \int_{-\infty}^{\infty} f(x)\, g(s-x)\, dx}$$

show that

$$F(f)\,F(g) = F(f * g)$$

show that the product of the fourier transforms of arbitrary functions $f(x)$ and $g(x)$ is equal to the fourier transform of their convolution

show:

$$\hat{f}(k)\,\tilde{g}(k) = \widetilde{[f*g]}(k)$$

$$\widetilde{[f*g]}(k) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x)\, g(s-x)\, dx \right) e^{-iks}\, ds$$

$f*g$ is a function of s

the fourier transf. goes from $s \to k$

(i.e.) use $\underline{s}$ in place of $\underline{x}$ in fourier transf. def.

$$\widetilde{[f*g]}(k) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x)\, g(s-x)\, dx \right) e^{-iks}\, ds$$

$$= \int_{-\infty}^{\infty} f(x) \int_{-\infty}^{\infty} g(s-x)\, e^{-iks}\, ds\, dx$$

$$= \int_{-\infty}^{\infty} f(x) \left(\int_{-\infty}^{\infty} g(s-x)\, e^{-iks}\, ds \right) dx$$

pull out a factor of e^{-ikx}

$$= \int_{-\infty}^{\infty} f(x)\, e^{-ikx} \underbrace{\int_{-\infty}^{\infty} g(s-x)\, e^{-ik(s-x)}\, ds}\, dx$$

let $u \equiv s-x \rightarrow du = ds$

(note x is "constant" in ds integral)

$$= \int_{-\infty}^{\infty} f(x)\, e^{-ikx} \underbrace{\int_{-\infty}^{\infty} g(u)\, e^{-iku}\, du}_{=\tilde{g}(k)}\, dx$$

$$= \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, \tilde{g}(k)\, dx$$

$$= \tilde{g}(k) \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

$$= \tilde{g}(k)\, \tilde{f}(k) \quad \checkmark$$

fourier transforms

x-space → k-space

for some function $f(x)$, it's fourier transform is:

$$\tilde{F}(k) = \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

to go from the fourier transform $F(k)$, back to $f(x)$:

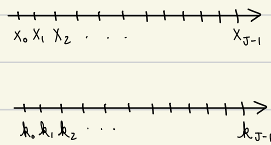
$$f(x) = \frac{1}{L} \int_{-\infty}^{\infty} \tilde{F}(k)\, e^{ikx}\, dk$$

discretizing the fourier transform

integrals become sums, infinitesimals become discrete changes

$$\int \rightarrow \sum \qquad dx \rightarrow \Delta x \qquad dk \rightarrow \Delta k$$

set up x & k grids w/ J support points



$x \rightarrow x_j = j\,\Delta x$ where $\Delta x = L/J$ and $j \in [0,1,2,\ldots,J-1]$

$k \rightarrow k_n = n\,\Delta k$ where $\Delta k = 2\pi/L$ and $n \in [0,1,2,\ldots J-1]$

$K_k = k\,\Delta k$

$f(x) \rightarrow f_j = f(x_j)$

$F(k) \rightarrow F_n = F(k_n)$

exponential term

$$ikx = ik_n x_j$$
$$= i\left(\frac{2\pi n}{L}\right)\left(j\frac{L}{J}\right)$$
$$= i2\pi nj/J$$

$$\tilde{F}(k) = \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

$$F_n = \sum_{j=0}^{J-1} f_j\, e^{-i2\pi nj/J}\, \boxed{\Delta x}$$
$$= \sum_{j=0}^{J-1} f_j\, e^{-i2\pi nj/J}\, \boxed{\left(\frac{L}{J}\right)}$$

these (in green) are usually left off of this derivation. you can leave them off, then 8c will show you that you need the 1/J factor

$$f_j = \sum_{n=0}^{J-1} F_n\, e^{i2\pi nj/J}\, \Delta k$$

$$= \frac{1}{2\pi} \sum_{n=0}^{J-1} F_n\, e^{i2\pi nj/J}\, \boxed{\left(\frac{2\pi}{L}\right)}$$

note: the factors of L and 1/L for F_n and f_j, respectively, are commonly dropped, as they cancel each other when transforming forward and then back again

leave this factor off for part c

$$\boxed{\begin{aligned} F_n &= \frac{1}{J} \sum_{j=0}^{J-1} f_j\, e^{-i2\pi nj/J} \quad \text{(forward)} \\ f_j &= \sum_{n=0}^{J-1} F_n\, e^{i2\pi nj/J} \quad \text{(backward)} \end{aligned}}$$

$e^{ix} = \cos x + i\sin x$

$e^{-ix} = \cos x - i\sin x$

$$f(x) = \sin(2\pi m x)$$
$$= \frac{1}{2i}\left(e^{2\pi i m x} - e^{-2\pi i m x}\right)$$

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$
$$= \frac{1}{2i}\int_{-\infty}^{\infty}\left(e^{2\pi i m x} - e^{-2\pi i m x}\right) e^{-ikx}\, dx$$
$$= \frac{1}{2i}\int_{-\infty}^{\infty}\left(e^{(2\pi m - k)ix} - e^{-(2\pi m + k)ix}\right) dx$$
$$= \frac{1}{2i}\int_{-\infty}^{\infty}\left(e^{2\pi(m-n)ix} - e^{-2\pi(m+n)ix}\right) dx$$

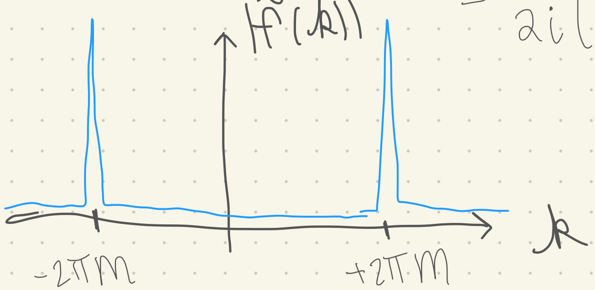
$k = \frac{2\pi n}{L}$

for $\sin(2\pi m x)$

$\to L = 1$

$$= \frac{1}{2i}\left(\delta(m-n) - \delta(m+n)\right)$$

$$= \frac{1}{2i}\left[\delta\left(m - \frac{k}{2\pi}\right) - \delta\left(m + \frac{k}{2\pi}\right)\right]$$



$k = 2\pi n$

8f] poissons equation ...

$$\nabla^2 \Phi(x) = 4\pi G \rho(x)$$

... discretized

$$\boxed{\frac{\Phi_{j-1} - 2\Phi_j + \Phi_{j+1}}{(\Delta x)^2} = 4\pi G \rho_j}$$

show
$$\hat{\Phi}_k = \frac{4\pi G \hat{\rho}_k (\Delta x)^2}{2(\cos\frac{2\pi k}{J} - 1)}$$

by plugging in the eqs (inverse transforms) for ρ_j and Φ_j into

$$\rho_j = \frac{1}{J}\sum_{k=0}^{J-1} \hat{\rho}_k e^{-2\pi i j k/J} \qquad \Phi_j = \frac{1}{J}\sum_{k=0}^{J-1} \hat{\Phi}_k e^{-2\pi i j k/J}$$

8f notation

	continuous	discrete
function of x	$f(x)$	f_j
fourier transform of (function of k)	$\hat{f}(k)$	$\hat{f}_k$

where f_j is f at x_j
and $\hat{f}_k$ is $\hat{f}$ at k_k