Sturm - Liouville : ODEs w/ eigenvalues

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$$y''(x) + (\lambda\pi)^2 \rho(x) y(x) = 0$$

in the code : $y \to u$, $\lambda \to \nu$

$$u''(x) + (\nu\pi)^2 \rho(x) u(x) = 0$$

this is one 2nd order ODE,

we can break it up into two 1st order ODEs

$$u''(x) = -(\nu\pi)^2 \rho(x) u(x)$$

let's define $q(x) \equiv u'(x)$

$$u' = q$$

$$q' = -(\nu\pi)^2 \rho(x) u(x)$$

we will also include an ODE for the eigenvalue ν since we are also solving for it our coupled ODES:

$$u' = q$$

$$q' = -(\nu\pi)^2 \varphi(x)\, u(x)$$

$$\nu' = 0$$

in the code:

$$\vec{y} = \begin{bmatrix} u \\ q \\ \nu \end{bmatrix} \begin{matrix} y[0] \\ y[1] \\ y[2] \end{matrix} \qquad \frac{d\vec{y}}{dx} = \begin{bmatrix} u' \\ q' \\ \nu' \end{bmatrix} = \begin{bmatrix} \text{RHS} \end{bmatrix}$$

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although this is a BVP, we iterate over an IVP driver in our root finder (we use ode_ivp to solve the ODE & check the boundary condition root $u(x=x_N, \nu)=0$ & iterate until this is satisfied)

let's think initial conditions...

in part A, we defined $u'(0) \equiv a$

then we determined $a = V\pi$ in order to satisfy our condition of a solution w/ an amplitude of 1.

as we change V, our initial slope changes

$$\vec{y}_0 = \begin{bmatrix} u_0 \\ u'_0 \\ V \end{bmatrix} = \begin{bmatrix} 0 \\ V\pi \\ V \end{bmatrix}$$

note V is an array of length N (w/ the same value at each index) so use V[0] in code here

note:

any time you want to run ode_ivp w/ a new V value, you need to call function <u>load_string</u> to reset the initial conditions why this is an eigenvalue problem

$$\frac{d^2}{dx^2} y(x) = -(\lambda\pi)^2 \, y(x)$$

$$A\vec{x} = \lambda\vec{x}$$