ph 332: Laplace transform method for BVPs

w/ eigenvalues!

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using the following two equations, let's solve a 2^{nd}-order ODE via Laplace transforms:

(i) $$\mathcal{L}_s[f(t)] = \int_0^\infty f(t)\, e^{-st}\, dt$$

s is some #, it won't be necessary for us to specify it

definition of the Laplace transform of a function $f(t)$

(ii) $$\mathcal{L}[f'(t)] = -f(0) + s\,\mathcal{L}[f(t)]$$

Relationship between the Laplace transform of a function $f(t)$ and the Laplace transform of it's first derivative $f'(t)$ (derived in lecture)

Laplace transforms are useful because they will allow us to solve an ODE by hand without integrating!!!

first, let's derive the Laplace transform of the second derivative of a general function:

- let's define a new function

$$g(t) \equiv f'(t)$$

so that $g'(t) = f''(t)$

- use (ii) for the function $g(t)$

$$\mathcal{L}[g'(t)] = -g(0) + s\,\mathcal{L}[g(t)]$$

- substituting $g(t) = f'(t)$, gives

$$\mathcal{L}[f''(t)] = -f'(0) + s\,\mathcal{L}[f'(t)]$$

- use (ii) for $\mathcal{L}[f'(t)]$ on the RHS

(iii) $$\mathcal{L}[f''(t)] = -f'(0) - sf(0) + s^2\,\mathcal{L}[f(t)]$$

this is the Laplace transform of the second derivative of a general function $f(t)$

example:

solve for $y(x)$ using Laplace transforms given,

$$y''(x) + (\lambda\pi)^2 y(x) = 0$$

where $y(0) = 0$ and $y'(0) = a$

- take the Laplace transform of $y''(x)$

(i) $$\mathcal{L}_s[f(t)] = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$\mathcal{L}_s[y''(x)] = \int_0^\infty y''(x)\, e^{-sx}\, dx$$

$$= \int_0^\infty \left(-(\lambda\pi)^2 y(x)\right) e^{-sx}\, dx$$

$$= -(\lambda\pi)^2 \underbrace{\int_0^\infty y(x)\, e^{-sx}\, dx}_{\text{this is (i) for } y(x)}$$

$$\mathcal{L}_s[y''(x)] = -(\lambda\pi)^2 \mathcal{L}_s[y(x)]$$

- write out (iii) for $y''(x)$

(iii) $\mathcal{L}[f''(t)] = -f'(0) - sf(0) + s^2 \mathcal{L}[f(t)]$

$$\mathcal{L}_s[y''(x)] = -y'(0) - s\,y(0) + s^2 \mathcal{L}[y(x)]$$

- use the initial conditions and set your two expressions for $\mathcal{L}_s[y''(x)]$ equal to each other

$$-(\lambda\pi)^2 \mathcal{L}_s[y(x)] = -a + s^2 \mathcal{L}[y(x)]$$

- isolate $\mathcal{L}_s[y(x)]$ and apply the inverse Laplace transform with an online calculator, such as wolfram alpha

$$\mathcal{L}_s[y(x)] = \frac{a}{s^2 + (\lambda\pi)^2}$$

$$\boxed{y(x) = \frac{a}{\lambda\pi} \sin(\lambda\pi x)}$$