MATH 387 Midterm Part I. Theory

Niklas Brake, 260602499

March 16, 2018

(a) Let

$$\tilde{D}_h = \frac{f(h) + \delta f(h) - f(-h) - \delta f(-h)}{2h}$$

Then,

$$\tilde{D}_h = \frac{f(h) - f(-h)}{2h} + \frac{\delta f(h) - \delta f(-h)}{2h}$$

By Taylor's Theorem and the Mean Value Theorem, $\exists \xi_1 \in (0, h)$ and $\xi_2 \in (-h, 0)$ such that

$$\begin{aligned} f(h) &= f(0) + hf'(0) + \frac{h^2}{2}f''(0) + \frac{h^3}{6}f'''(\xi_1) \\ f(-h) &= f(0) - hf'(0) + \frac{(-h)^2}{2}f''(0) + \frac{(-h)^3}{6}f'''(\xi_2) \end{aligned}$$

Thus,

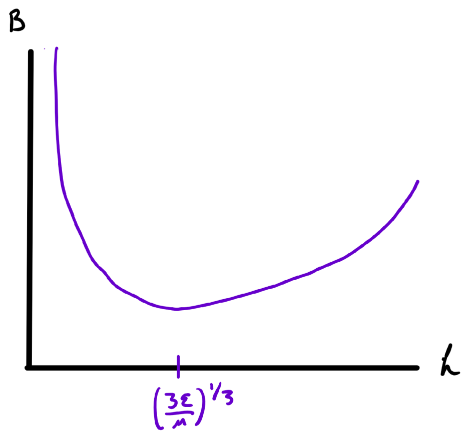
$$f(h) - f(-h) = 2hf'(0) + \frac{h^3}{6}\left(f'''(\xi_1) + f'''(\xi_2)\right)$$

Now since f''' is continuous, by the Intermediate Value Theorem, $\exists \xi \in (-h, h)$ such that $f'''(\xi) = \frac{f'''(\xi_1)+f'''(\xi_2)}{2}$. Therefore,

$$\begin{aligned} f(h) - f(-h) &= 2hf'(0) + \frac{h^3}{3}f'''(\xi) \\ \Rightarrow \tilde{D}_h &= f'(0) + \frac{h^2}{6}f'''(\xi) + \frac{\delta f(h) - \delta f(-h)}{2h} \\ \Leftrightarrow \tilde{D}_h - f'(0) &= \frac{h^2}{6}f'''(\xi) + \frac{\delta f(h) - \delta f(-h)}{2h} \end{aligned}$$

(b) Since $h \in (-a, a)$, $|\delta f(h)| < \varepsilon$. Further, since f''' is continuous on a closed interval, it is bounded by some constant $M < \infty$.

$$
\begin{aligned}
\left|\tilde{D}_h - f'(0)\right| &= \left|\frac{h^2}{6} f'''(\xi) + \frac{\delta f(h) - \delta f(-h)}{2h}\right| \\
&\leq \left|\frac{h^2}{6} f'''(\xi)\right| + \left|\frac{\delta f(h) - \delta f(-h)}{2h}\right| \\
&\leq \frac{h^2}{6} \max_{x \in [-a,a]} |f'''(x)| + 2\left|\frac{\delta f(h)}{2h}\right| \\
&\leq \frac{h^2}{6} M + \frac{\varepsilon}{h}
\end{aligned}
$$



$B(h) = \frac{\varepsilon}{h} + \frac{Mh^2}{6}$

For h large $B(h) \approx \frac{Mh^2}{6}$

$B'(h) = \frac{-\varepsilon}{h^2} + \frac{M}{3} h$

$= 0 \iff h^3 = \frac{3\varepsilon}{M}$

Since $B \to \infty$ as $h \to 0$, if we take h to be too small, we cannot bound our error. Thus, we should take h such that we can bound our error as small as possible. This occurs when $B'(h) = 0$, which gives us $h = \left(\frac{3\varepsilon}{M}\right)^{1/3}$