H01

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5) a) 1. $y_k = \sum_{i=0}^{k} \frac{f^{(i)}(x_0)}{i!} (x-x_0)^i$; $y = f(x) = \sqrt{x}$

für $k = 0, 1, 2, \dots, n$

$a_0 = y_0 = \sqrt{x_0}$

$y_1 = y_0 + a_1 = \underbrace{\sqrt{x_0}}_{y_0} + \underbrace{\left(\frac{3}{2} - 1\right)\left(\frac{x}{x_0} - 1\right) \cdot \underbrace{\sqrt{x_0}}_{a_0}}_{a_1}$

$y_2 = y_1 + a_2 = \underbrace{\sqrt{x_0} + \left(\frac{3}{2} - 1\right)\left(\frac{x}{x_0} - 1\right) \cdot \sqrt{x_0}}_{y_1} + \underbrace{\left(\frac{3}{4} - 1\right)\left(\frac{x}{x_0} - 1\right) \underbrace{\left(\frac{3}{2} - 1\right)\left(\frac{x}{x_0} - 1\right) \cdot \sqrt{x_0}}_{a_1}}_{a_2}$

$a_k = \left(\frac{3}{2k} - 1\right)\left(\frac{x}{x_0} - 1\right) a_{k-1}$

$y_k = y_{k-1} + a_k$

$\Rightarrow f^{(i)}(x)$ für $f(x) = \sqrt{x}$

$f'(x) = \frac{1}{2\sqrt{x}}$

$f''(x) = -\frac{1}{4x^{\frac{3}{2}}}$

$f'''(x) = \frac{3}{8x^{\frac{5}{2}}}$

$\Rightarrow f^{(i)}(x) = (-1)^{i-1} \cdot \frac{(2i-1)!!}{2^i} \cdot x^{-\frac{2i-1}{2}}$

$\Rightarrow a_i = \frac{f^{(i)}(x_0)}{i!} (x-x_0)^i$

$= \frac{(-1)^{i-1} \cdot \frac{(2i-1)!!}{2^i} \cdot x_0^{-\frac{2i-1}{2}}}{i!} (x-x_0)^i$

$\Rightarrow \frac{a_i}{a_{i-1}} = \frac{\frac{(-1)^{i-1} \cdot \frac{(2i-1)!!}{2^i} \cdot x_0^{-\frac{2i-1}{2}}}{i!} (x-x_0)^i}{\frac{(-1)^{i-2} \cdot \frac{(2i-3)!!}{2^{i-1}} \cdot x_0^{-\frac{2i-3}{2}}}{(i-1)!} (x-x_0)^{i-1}}$

$= \frac{(-1)^{\cancel{i-1}} \cdot \frac{\cancel{(2i-1)!!}}{\cancel{2^i}} \cdot x_0^{-\frac{2i-1}{2}} \cdot (x-x_0)^{\cancel{i}} \cdot \cancel{(i-1)!}}{\cancel{(-1)^{i-2}} \cdot \frac{\cancel{(2i-3)!!}}{\cancel{2^{i-1}}} \cdot \cancel{x_0}^{-\frac{2i-3}{2}} \cdot \cancel{(x-x_0)^{i-1}} \cdot i!}$

$= \frac{(-1) \cdot \frac{2i-3}{2} \cdot x_0^{-1} \cdot (x-x_0)}{i}$ $\quad 2!!$

$= \frac{(-2i+3)(x-x_0)}{2i\,x_0}$ $\quad \left(\frac{(2i-1)!!}{(2i-3)!!} = 2i-3 \text{ ; double factorial of odd numbers} \right)$

$= \left(\frac{3-2i}{2i}\right)\left(\frac{x-x_0}{x_0}\right)$

$= \left(\frac{3}{2i} - 1\right)\left(\frac{x}{x_0} - 1\right)$

$\Rightarrow \prod_{i=0}^{k} a_i = a_0 \cdot a_1 \cdot \ldots \cdot a_k$

$\Rightarrow a_k = \left(\frac{3}{2i} - 1\right)\left(\frac{x}{x_0} - 1\right) \cdot a_{k-1}$ mit $a_0 = \sqrt{x_0}$