Hochschule Konstanz
Fakultät Elektrotechnik
und Informationstechnik

Automotive Control Systems

Path Planning / Trajectory Generation

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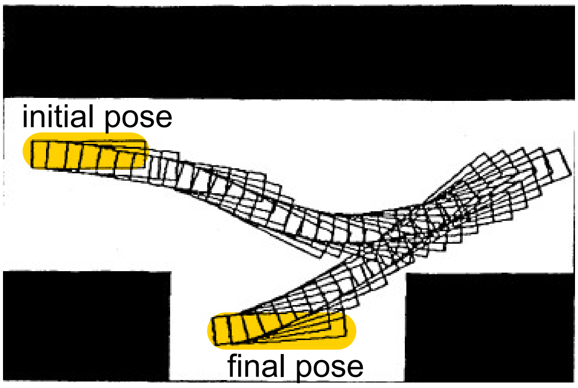
Learning Objectives

For the truck system, being able to plan and implement polynomial reference trajectories that fulfill the desired / given constraints

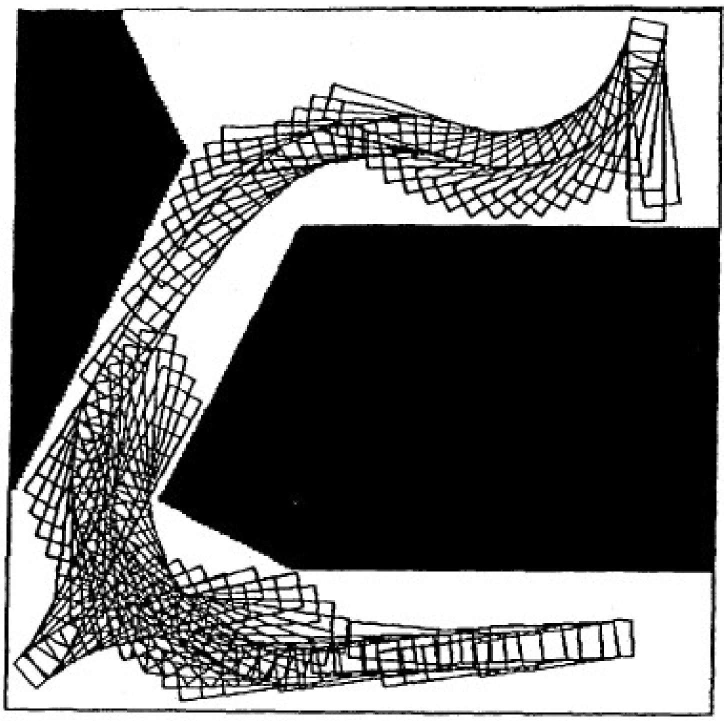
Motion Planning

möglich
Describes the task of determining a <u>feasible</u> way to move a vehicle/robot/ship from an initial pose (Position/Orientation) to a desired final pose.

Examples:



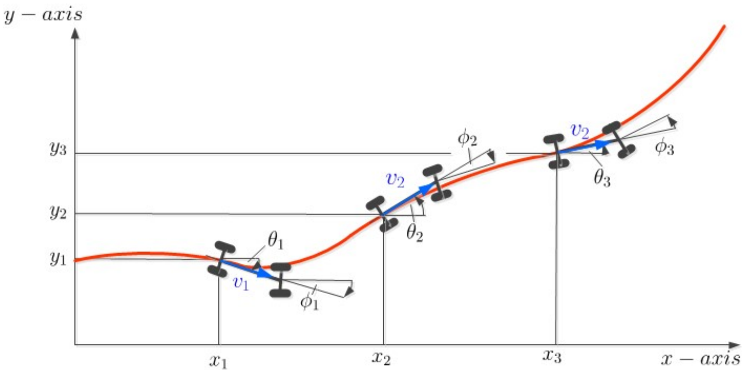
in reality take surrounding obstacles into account

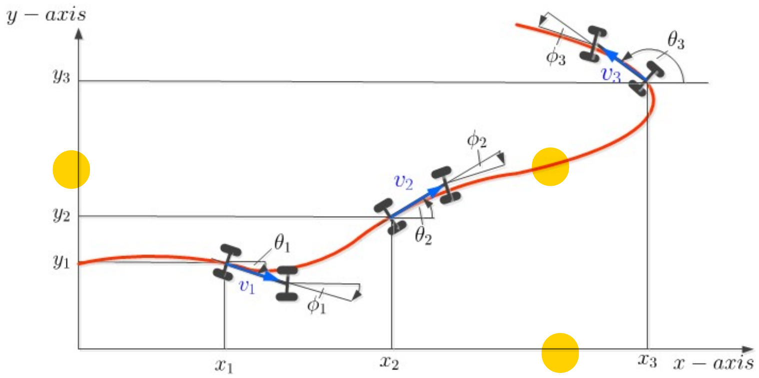


Source: Nonholonomic Motion Planning - Standford

Parametrization of a Path

The path parameter must be monotonically in(de)creasing along the path





1.) $y = f(x)$

- x is the independent variable (the parameter)
- x is also a path coordinate
- simple but restrictive

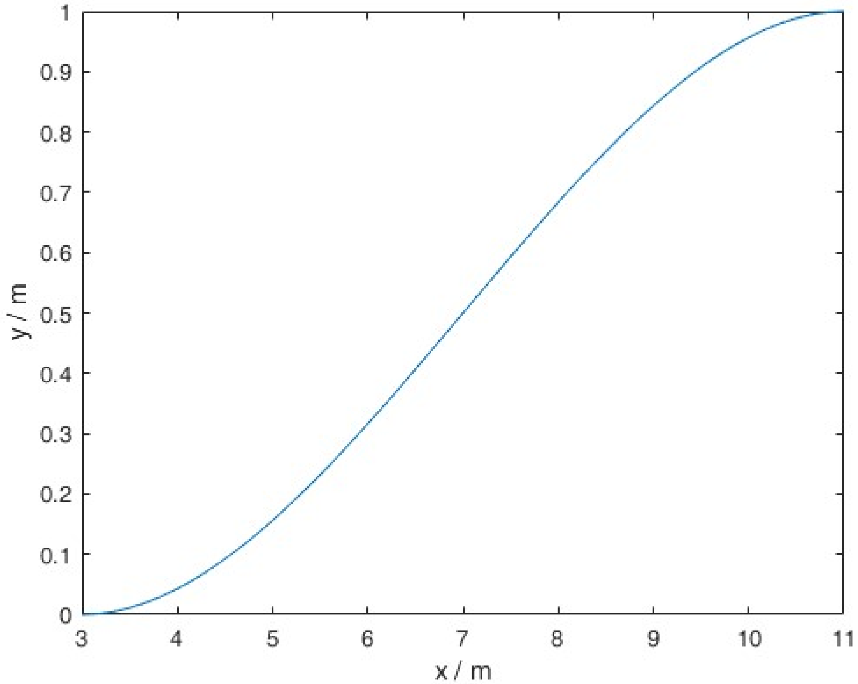
2.) $\begin{pmatrix} x(s) \\ y(s) \end{pmatrix}$

- s is the independent variable (the parameter)
- flexible

Example case 1

A path is often, but not necessarily, described by polynomials. (e.g. Bezier Splines)

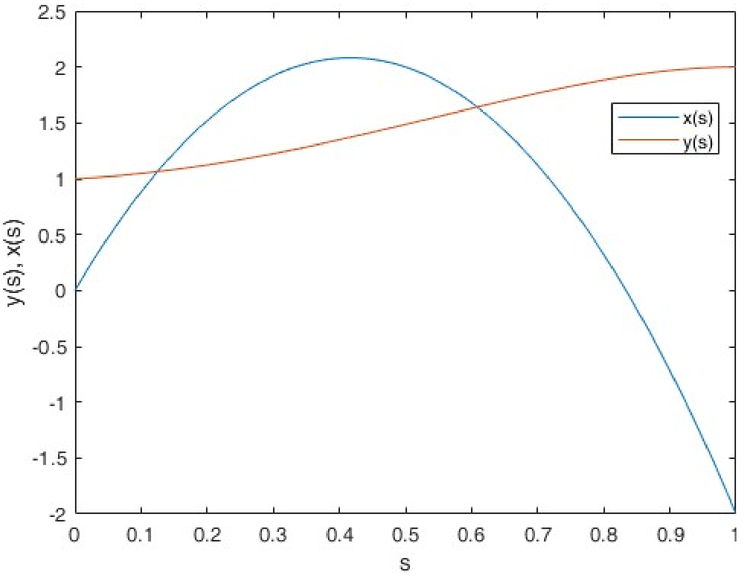
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ f(x) \end{pmatrix} \qquad y = \frac{3}{64}(x-3)^2 - \frac{2}{512}(x-3)^3 \quad x \in [3, \dots, 11]$$



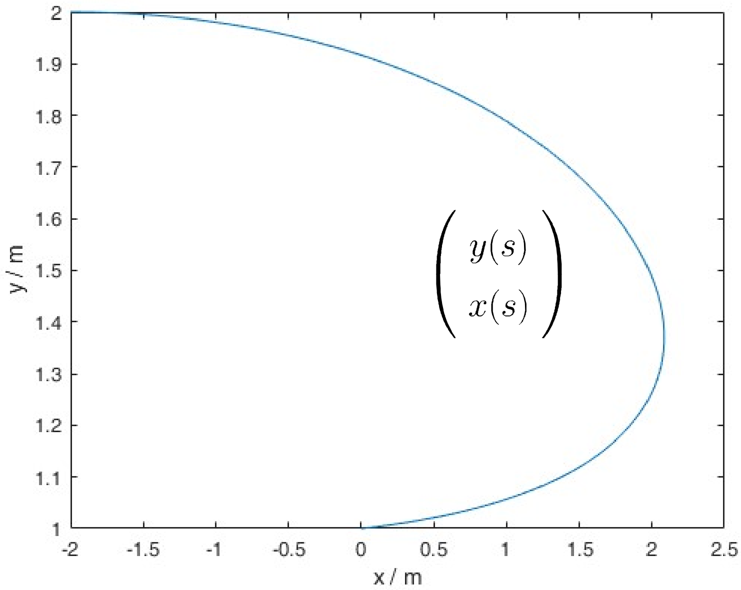
Example case 2

$$
\begin{aligned}
x(s) &= -12s^2 + 10s; \\
y(s) &= -0.8429s^4 + 1.52s^2 + 0.3143s + 1
\end{aligned}
\qquad s \in [0, \ldots, 1]
$$

Graphs of the two polynomials

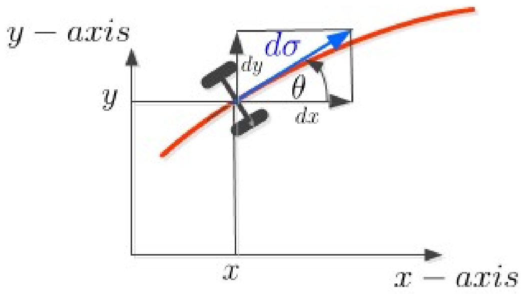


Resulting path



Geometric features of a Path y = f(x)

1.) Coordinates $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ f(x) \end{pmatrix}$



2.) Arc length

$$d\sigma = \sqrt{dx^2 + dy^2} = \sqrt{1 + \frac{dy^2}{dx^2}}dx = \sqrt{1 + \left(\frac{df}{dx}\right)^2}dx$$

3.) Orientation (slope)

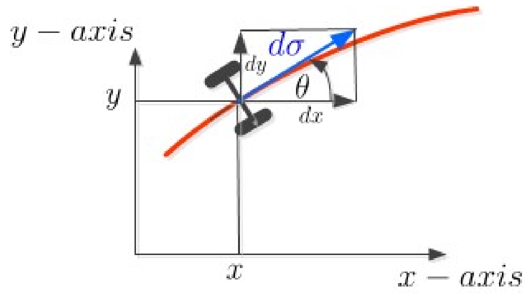
$$\tan(\theta) = \frac{dy}{dx} = \frac{df(x)}{dx} \Rightarrow \theta = \arctan\left(\frac{df(x)}{dx}\right)$$

4.) Change in orientation subject to x

dtheta / dx $\cancel{\kappa} = \frac{1}{1+\left(\frac{df(x)}{dx}\right)^2}\left(\frac{d^2 f(x)}{dx^2}\right)$

Geometric features of a general Path

1.) Coordinates $\begin{pmatrix} x(s) \\ y(s) \end{pmatrix}$



2.) Arc length

dy

$$d\sigma = \sqrt{dx^2 + dy^2} = \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dx}{ds}\right)^2} ds = \sqrt{y'(s)^2 + x'(s)^2} ds$$

' means derivative subject to s

3.) Orientation (slope)

$$\tan(\theta) = \frac{dy}{dx} = \frac{\frac{dy}{ds}}{\frac{dx}{ds}} = \frac{y'(s)}{x'(s)} \Rightarrow \theta = \arctan\left(\frac{y'(s)}{x'(s)}\right)$$

4.) Change in orientation subject to s

$$\theta' = \frac{1}{1+\left(\frac{y'(s)}{x'(s)}\right)^2} \left(\frac{y''(s)}{x'(s)} - \frac{y'(s)x''(s)}{x'(s)^2}\right) = \frac{y''(s)x'(s) - y'(s)x''(s)}{x'(s)^2 + y'(s)^2}$$

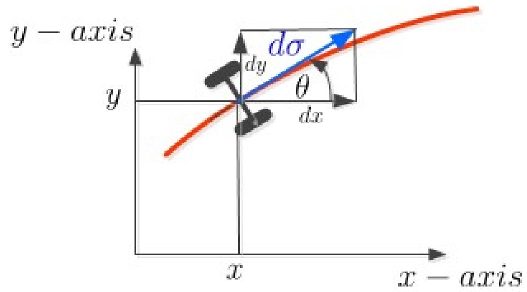
dtheta / dx

Quotient Rule

Note, if $s = x$, the results of case 1 and 2 coincide

Natural parametrization

The length from the starting point to an arbitrary point $(x(s_0), y(s_0))$ on the curve is given by



$$l(s_0) = \int_0^l d\sigma = \int_0^{s_0} \sqrt{y'(s)^2 + x'^2(s)}ds$$

Since $l(s)$ is monotonically increasing, l can be used to parameterize the curve. This is called the "natural parametrization" of the curve. We will see that it is sometimes convenient to use this parametrization.

y(s)(dy/ds)*(ds/dsigma)

The derivatives subject to the arc length are obtained using the chain rule $\frac{dy}{d\sigma} = \frac{dy}{ds}\frac{ds}{d\sigma}$

Orientation

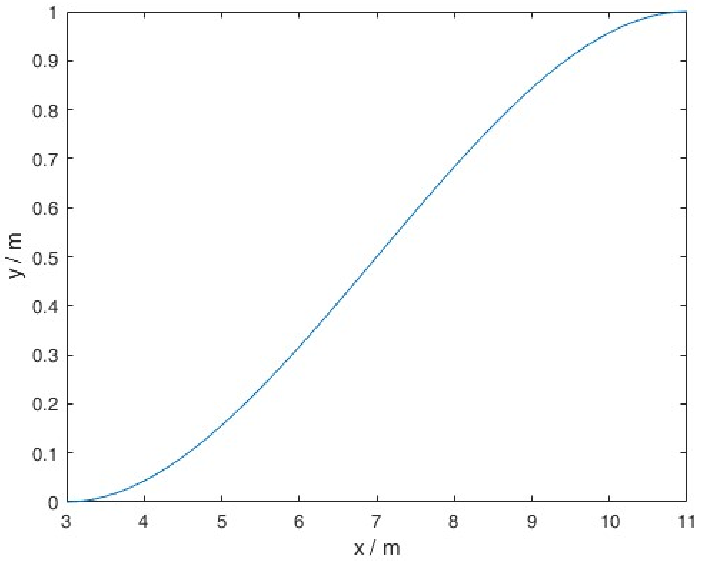
$$\tan(\theta) = \frac{dy}{dx} = \frac{\frac{dy}{d\sigma}}{\frac{dx}{d\sigma}} \Rightarrow \theta = \arctan\left(\frac{y'(s)}{x'(s)}\right) = \arctan\left(\frac{\frac{dy}{d\sigma}}{\frac{dx}{d\sigma}}\right)$$

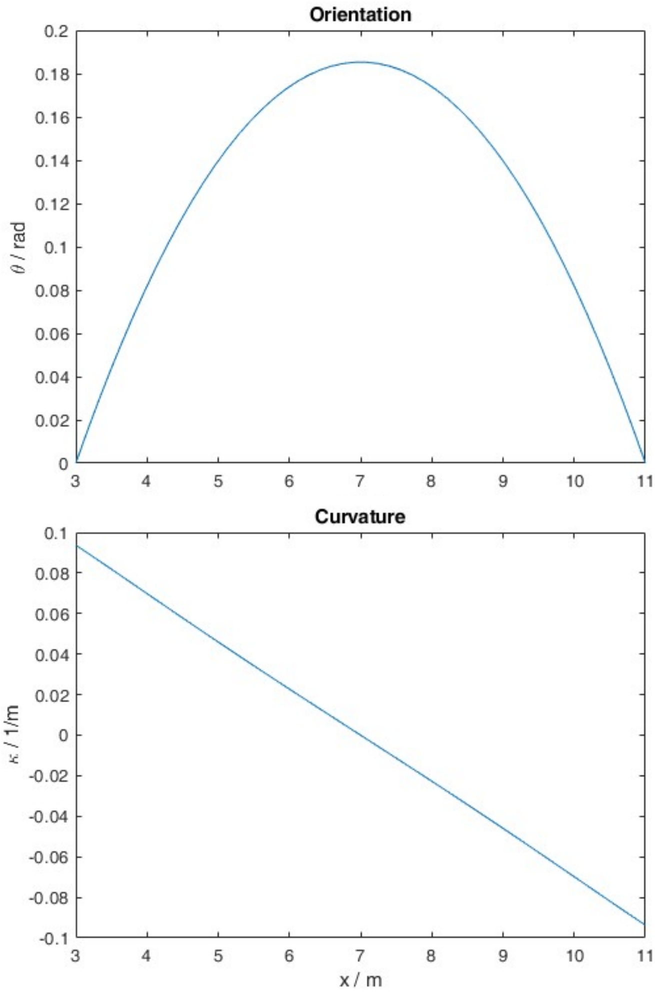
Curvature (Change in Orientation subject to the arc length)

$$\kappa = \frac{d\theta}{d\sigma} = \theta'\frac{ds}{d\sigma} = \theta'\frac{1}{\sqrt{x'(s)^2+y'(s)^2}} = \frac{y''(s)x'(s)-y'(s)x''(s)}{(x'(s)^2+y'(s)^2)^{3/2}}$$

Example case 1

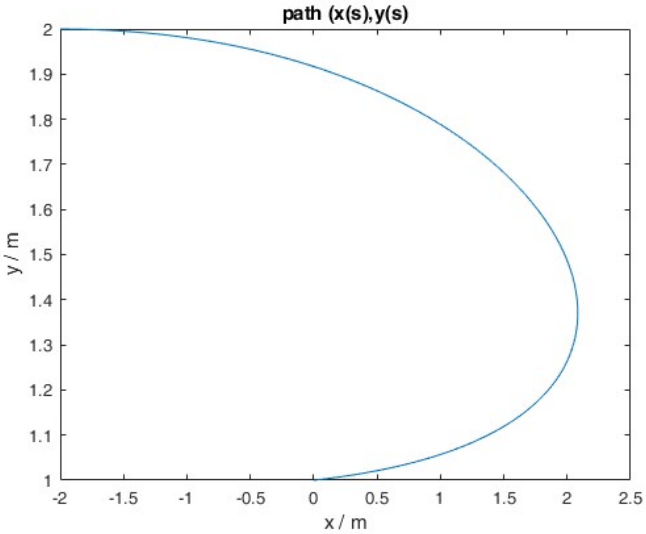
$$y = \frac{2}{512}(x-3)^3 + \frac{3}{64}(x-3)^2 - \qquad x \in [3, \ldots, 11]$$

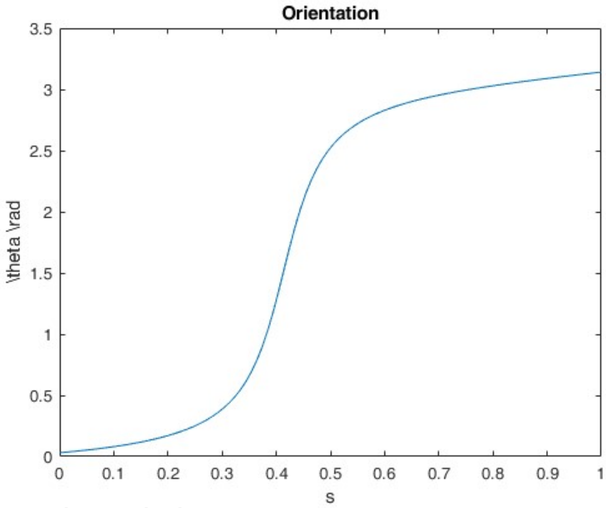




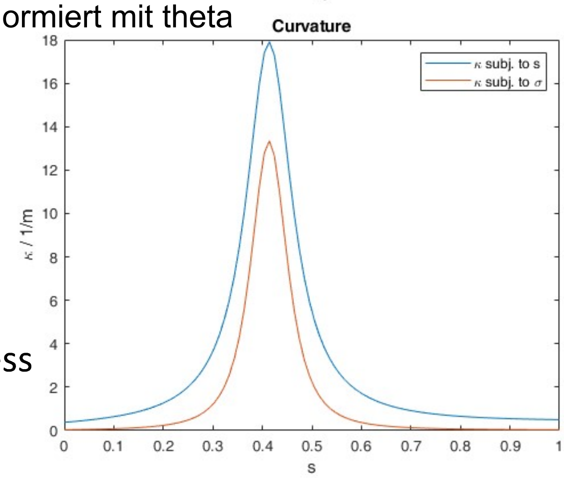
Example case 2

$$x(s) = -12s^2 + 10s; \qquad s \in [0, \ldots, 1]$$
$$y(s) = -0.8429s^4 + 1.52s^2 + 0.3143s + 1$$





normiert mit theta



The polynomial are output of a path planning process

parametrized by sigma

Re-parametrization of the vehicle model

Goal: To remove the time dependency, describe the vehicle state just by the path parameter s

$$\frac{dx}{dt} = v\cos(\theta) \quad \text{recall} \quad v = \frac{d\sigma}{dt}$$

Formally we can enhance and cancel out

$$\frac{dx}{dt}\frac{dt}{ds} = \frac{d\sigma}{dt}\frac{dt}{ds}\cos(\theta) \Rightarrow \frac{dx}{ds} = \frac{d\sigma}{ds}\cos(\theta)$$

Recall – arc length

$$\begin{aligned} d\sigma &= \sqrt{y'(s)^2 + x'(s)^2}ds := \gamma ds \\ &\Rightarrow \frac{d\sigma}{ds} = \gamma \end{aligned}$$

Leads to the system model parametrized by s

$$\begin{aligned} \frac{dx}{ds} = x'(s) &= \gamma\cos(\theta) \\ \frac{dy}{ds} = y'(s) &= \gamma\sin(\theta) \\ \frac{d\theta}{ds} = \theta'(s) &= \frac{\gamma}{l_0}\tan(\varphi) \end{aligned}$$

$$\begin{aligned} \frac{d\varphi}{ds} = \varphi' &= \tilde{u}_2 \\ \frac{d\gamma}{ds} = \gamma' &= \tilde{u}_1 \end{aligned}$$

Simple Motion Planning

Goal: Find a curve, that connects the current (initial) and final (goal) vehicle pose

Initial State $(x_0, y(x_0), \theta(x_0), \varphi(x_0))$ Goal State $(x_1, y(x_1), \theta(x_1), \varphi(x_1))$

The parameters have to be chosen such that the features of the curve *f(x)* match the state components at the initial and goal pose.

$$\tan(\theta) = \frac{dy}{dx} = \frac{df(x)}{dx}$$

=> Therefore the first derivative of *f(x)* is known

$$\theta'(s) = \frac{\gamma}{l_0}\tan(\varphi)$$

Ref. Slide 6

$$= \frac{1}{\sqrt{1+\left(\frac{df(x)}{dx}\right)^2}}\left(\frac{d^2f(x)}{dx^2}\right)$$

=> Therefore the second derivative of *f(x)* is known

recall $\sqrt{1+\left(\frac{df(x)}{dx}\right)^2} = \gamma$

$$\frac{d^2f(x)}{dx^2} = \frac{\gamma^3}{l_0}\tan(\varphi)$$

Simple Motion Planning

Minimum Requirements: Path needs to be compliant with

$$(x_0, y(x_0), \theta(x_0), \varphi(x_0)) \quad \text{and} \quad (x_1, y(x_1), \theta(x_1), \varphi(x_1))$$

Therefore – six conditions needs to be fulfilled (6 DOF in the polynomial):

$$y(x_0) = y_0$$

$$\left.\frac{dy}{dx}\right|_{x_0} = \tan(\theta_0)$$

$$\left.\frac{d^2y}{dx^2}\right|_{x_0} = \gamma(x_0)^3 \frac{1}{l_0} tan(\varphi_0)$$

$$y(x_1) = y_1$$

$$\left.\frac{dy}{dx}\right|_{x_1} = \tan(\theta_1)$$

$$\left.\frac{d^2y}{dx^2}\right|_{x_1} = \gamma(x_1)^3 \frac{1}{l_0} tan(\varphi_1)$$

Prototype:

$$y(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$$

Simple Motion Planning

$$y(x) = a\tilde{x}^5 + b\tilde{x}^4 + c\tilde{x}^3 + d\tilde{x}^2 + e\tilde{x} + f$$

Using $\tilde{x} = x - x_0$ $\quad \tilde{x}_0 = 0 \quad \tilde{x}_1 = x_1 - x_0$

Coefficients, immediately available:

$$y(0) = y_0 = f, \quad y'(0) = \tan(\theta_0) = e, \quad y''(0) = \gamma(x_0)^3 \frac{1}{l_0} tan(\varphi_0) = 2d$$

dy/dx $\qquad$ d²x/dx²

Further relations:

$$y(\tilde{x}_1) = y_1 = a\tilde{x}_1^5 + b\tilde{x}_1^4 + c\tilde{x}_1^3 + d\tilde{x}_1^2 + e\tilde{x}_1 + f$$

$$y'(\tilde{x}_1) = \tan(\theta_1) = 5a\tilde{x}_1^4 + 4b\tilde{x}_1^3 + 3c\tilde{x}_1^2 + 2d\tilde{x}_1 + e$$

$$y''(\tilde{x}_1) = \gamma(x_1)^3 \frac{1}{l_0} tan(\varphi_1) = 20a\tilde{x}_1^3 + 12b\tilde{x}_1^2 + 6c\tilde{x}_1 + 2d$$

The three unknown parameters can be obtained by solving the system of three linear equations

$'$ represents the derivative subject to $\tilde{x}$

Transformation to a Trajectory (time parameterization)

Suppose, the path should be traveled within a time span of T

Suppose path is defined as $y(x)$ and $x(0) = x_0$ and $x(T) = x_1$

Defining a scaling parameter $s(\tau) = 3\tau^2 - 2\tau^3 \quad s \in [0,1]$ $\tau(t) = \frac{t}{T}$

Possible time parametrization of x

$$x(t) = x_0 + (x_1 - x_0)s(\tau(t))$$

References for x(t)

$$\dot{x} = \frac{dx}{ds}\frac{ds}{d\tau}\frac{d\tau}{dt} \qquad \dot{x}(t) = \frac{1}{T}(x_1 - x_0)(6\tau - 6\tau^2)$$

$$\ddot{x} = \frac{d\dot{x}}{d\tau}\frac{d\tau}{dt} \qquad \ddot{x}(t) = \frac{1}{T^2}(x_1 - x_0)(6 - 12\tau)$$

Later needed for state feedback tracking control

Transformation to a Trajectory (time parameterization)

Reference state components are now available using

$$y(x(t)) = y_{ref}(t)$$

$$\arctan(y'(x(t)) = \theta_{ref}(t)$$

$$\arctan\left(l_0 \frac{y''(x(t))}{\gamma(x(t))^3}\right) = \varphi_{ref}(t)$$

$$\gamma(x(t)) = v_{ref}(t)$$