modality

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Context

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Modalities, Closure and Reflection

1. Idea

In full linear logic/linear type theory there is assumed a (comonadic) modality denoted "!" and called the *exponential modality*, whose role is, roughly, to give linear types also a non-linear interpretation. This is also called the "of course" modality or the *storage modality*, and sometimes the "bang"-operation.

In classical linear logic (meaning with involutive de Morgan duality), the de Morgan dual of "!" is denoted "?" and called the "why not"-modality.

In categorical semantics of linear type theory the $!$-modality typically appears as a kind of Fock space construction. If one views linear logic as quantum logic (as discussed there), then this means that the $!$- modality produces free second quantization.

2. Categorical semantics

Everyone agrees that $!$ should be a comonad (and $?$ should be a monad, see also at *monads in computer science*), but there are different ways to proceed from there. The goal is to capture the syntactic rules allowing assumptions of the form $!A$ to be duplicated and discarded.

Intuitionistic case

The original definition from Seely, adapted to the intuitionistic case and modernized is:

Definition 2.1. Let C be an symmetric monoidal category with cartesian products. A ***Seely comonad*** on C is a comonad that is a strong monoidal functor from the cartesian monoidal structure to the symmetric monoidal structure, i.e. we have $!(A\times B)\cong !A \otimes !B$ and $!1\cong I$ coherently. (There is also an additiona coherence axiom that should be imposed; see Mellies, section 7.3.)

(Note that in linear logic, the cartesian monoidal structure $\times$ is sometimes denoted by $\&$.) This implies that the Kleisli category of ! (i.e.\ the category of cofree !-coalgebras) is cartesian monoidal. If C is *closed* symmetric monoidal then the Kleisli category of a cartesian closed category, which is a categorical version of the translation of intuitionistic logic into linear logic.

Of course, the above definition depends on the existence of the cartesian product. A different definition that doesn't require the existence of $\times$ was given by Benton Bierman, de Paiva, and Hyland:

Definition 2.2. Let C be a symmetric monoidal category; a ***linear exponentia comonad*** on C is a lax monoidal comonad such that every cofree !-coalgebra naturally carries the structure of a comonoid object in the category of coalgebras (i.e. the cofree-coalgebra functor lifts to the category of comonoids in the category of coalgebras).

It follows automatically that all !-coalgebras are comonoids, and therefore that the category of all !-coalgebras (not just the cofree ones) is cartesian monoidal. Note that for a comonad on a poset, every coalgebra is free; thus the world of pure propositional "logic" doesn't tell us whether to consider the Kleisli category or the Eilenberg-Moore category for the translation.

A more even-handed approach is the following (see Benton (1995) and Mellies (2009)), based on the observation that both Kleisli and Eilenberg-Moore categories are instances of adjunctions.

Definition 2.3. A ***linear-nonlinear adjunction*** is a monoidal adjunction $F \colon M \rightleftarrows L \colon G$ in which L is symmetric monoidal and M is cartesian monoidal. The induced !-modality is the comonad $F G$ on L.

This includes both of the previous definitions where M is taken respectively to be the Kleisli category or the Eilenberg-Moore category of !. Conversely, in any linear nonlinear adjunction the induced comonad $F G$ can be shown to be a linear exponential comonad. Moreover, if ! is a linear exponential comonad on a symmetric monoidal category C with finite products, then the cofree !-coalgebra functor is a right adjoint and hence preserves cartesian products; but the cartesian products o coalgebras are the tensor product in C, so we have $!(A\times B) \cong !A \otimes !B$, the

Seely condition.

Classical case

For “classical” linear logic, we want C to be not just (closed) symmetric monoidal bu $\ast$-autonomous. If an $\ast$-autonomous category has a linear exponential comonad one can derive a ? from the ! by de Morgan duality, $?A = (!(A^*))^*$. The resultin relationship between ! and ? was axiomatized in a way not requiring the de Morga duality by Blute, Cockett & Seely (1996):

Definition 2.4. Let C be a linearly distributive category with tensor product $\otimes$ and cotensor product $\parr$. A (!,?)-modality on C consists of:

1. a $\otimes$-monoidal comonad ! and a $\parr$-comonoidal monad ?
2. ? is a !-strong monad, and ! is a ?-strong comonad
3. all free !-coalgebras are naturally commutative $\otimes$-comonoids, and all free ?-algebras are naturally commutative $\parr$-monoids.

Here a functor F is strong with respect to a lax monoidal functor G if there is a natural transformation $F A \otimes G B \to F(A\otimes G B)$ satisfying some natura axioms, and we similarly require compatibility of the monad and comonad structure transformations. BCS96 showed that if C is in fact $\ast$-autonomous, it follows fron the above definition that $?A = \big(!(A^*)\big)^*$ as expected.

3. Examples

Relation to Chu construction

Theorem 3.1. *Suppose $F \colon M \rightleftarrows C : G$ is a linear-nonlinear adjunction where C is closed symmetric monoidal with finite limits and colimits, and $\bot \in$ is an object. Then there is an induced linear-nonlinear adjunctior $M \rightleftarrows Chu(C,\bot)$ where $Chu(C,\bot)$ is the Chu construction $Chu(C,\bot)$ which is $\ast$-autonomous with finite limits and colimits. Hence $Chu(C,\bot)$ admits a !-modality.*

Proof. The embedding of C in $Chu(C,\bot)$ as $A \mapsto (A, [A,\bot], ev)$ is coreflective the coreflection of (B^+, B^-, e_B) is $(B^+, [B^+,\bot], ev)$. Moreover, this subcategory is closed under the tensor product of $Chu(C,\bot)$, i.e. the embedding $C\hookrightarrow Chu(C,\bot)$ is strong monoidal, hence the adjunction is a monoida adjunction. Therefore, the composite adjunctior $M \rightleftarrows C \rightleftarrows Chu(C,\bot)$ is again a linear-nonlinear-adjunction. ▮

Since a Chu construction is $\ast$-autonomous, this !- modality implies a dual ?- modality.

Corollary 3.2. *If C is a cartesian closed category with finite limits and colimits anc*

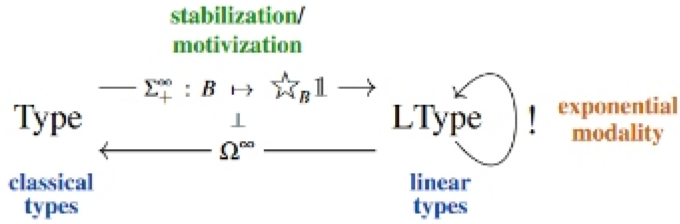
$\bot$ in C is an object, then there is a linear-nonlinear adjunction $C \rightleftarrows Chu(C,\bot)$, and hence $Chu(C,\bot)$ admits a !-modality.

Proof. Apply the previous theorem to the identity adjunction $C \rightleftarrows C$. ▮

Note that the !- modality obtained from the corollary is idempotent, while tha obtained from the theorem is idempotent if and only if the original one was. Othe ways of constructing !- modalities, such as by cofree coalgebras, may produce examples that are not idempotent.

Realization in linear homotopy type theory

In dependent linear homotopy type theory the “linear-nonlinear adjunction” is naturally identified (Ponto & Shulman (2012), Ex. 4.2, see also Schreiber (2014) Sec. 4.2) with the stabilization adjunction between homotopy types and stable homotopy types (Riley (2022), Prop. 2.1.31), whose left adjoint (forming suspension spectra Σ^∞_+) is equivalently given (cf. Riley (2022), Rem. 2.4.13) by sending $B \,\colon\, Type$ to the linear dependent sum $\star_B \mathbb{1} \;\coloneqq\; (p_B)_! (p_B)^\ast \mathbb{1}$ over the monoidal unit in the context B. In terms of quantum modal logic this is forming the “linear randomization” of the given classical homotopy type (its “motive”):



4. Modal term calculi

Girard’s original presentation of linear logic involved rules that explicitly assumed the presence of ! on hypotheses or on entire contexts, such as , dereliction? weakening and contraction:

$$\frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B} \qquad \frac{\Gamma \vdash B}{\Gamma, !$$

and “promotion”:

$$\frac{!\Gamma \vdash A}{!\Gamma\vdash !A}$$

If this is translated into a natural deduction style term calculus, the resulting rules are more complicated than those of most type formers. This can be avoided using adjoint type theory with two context zones, one “nonlinear” one where contraction and weakening are permitted (and admissible) and one “linear” one where they are not, with ! as a modality relating the two zones.

Such a “modal” presentation of linear logic was first introduced by Girard in his work on LU? and then developed by a number of other people such as Plotkin

Wadler, Benton, and Barber. See the references for details.

This presentation also generalizes naturally to dependent linear type theory, with the nonlinear type theory being dependent, and the linear types depending on the nonlinear ones but nothing depending on linear types. In this context, the !-modality decomposes into "context extension" and a "dependent sum".

5. Related concepts

- exponential map

symbol	**in logic**
\in	element relation
\,:	typing relation
=	equality
\vdash	entailment / sequent
\top	true / top
\bot	false / bottom
\Rightarrow	implication
\Leftrightarrow	logical equivalence
\not	negation
\neq	negation of equality / apartness
\notin	negation of element relation
\not \not	negation of negation
\exists	existential quantification
\forall	universal quantification
\wedge	logical conjunction
\vee	logical disjunction
symbol	***in type theory (propositions as types)***
\to	function type (implication)
\times	product type (conjunction)
+	sum type (disjunction)
0	empty type (false)
1	unit type (true)

1	unit type (true)
=	identity type (equality)
\simeq	equivalence of types (logical equivalence)
\sum	dependent sum type (existential quantifier)
\prod	dependent product type (universal quantifier)
symbol	***in linear logic***
\multimap	linear implication
\otimes	multiplicative conjunction
\oplus	additive disjunction
\&	additive conjunction
\invamp	multiplicative disjunction
\;!	exponential conjunction

6. References

General

Brief survey in a context of computer science/linear type theory:

- Daniel Mihályi, Valerie Novitzká, Section 2.2 of: *What about Linear Logic in Computer Science?*, Acta Polytechnica Hungarica ***10*** 4 (2013) 147-160 [pdf, pdf]

On the semantics of exponential conjunction as a comonad:

- R. A. G. Seely, *Linear logic, |ast-autonomous categories and cofree coalgebras* in *Categories in Computer Science and Logic*, Contemporary Mathematics ***92*** (1989) [pdf, ps.gz, ISBN:978-0-8218-5100-5]
- Nick Benton, Gavin Bierman, Valeria de Paiva, Martin Hyland, *Linear |lambda Calculus and Categorical Models Revisited*, in *Computer Science Logic. CSL 1992*, Lecture Notes in Computer Science ***702***, Springer (1993) [doi:10.1007/3-540-56992-8_6
- R. F. Blute , J. R. B. Cockett, R. A. G. Seely, *! and ? – Storage as tensorial strength*, Mathematical Structures in Computer Science ***6*** 4 (1996) 313-351 [doi:10.1017/S0960129500001055]
- Paul-André Melliès, *Categorical semantics of linear logic*, 2009. pdf
- Martin Hyland and Andreas Schalk, *Glueing and orthogonality for models of linear logic*, pdf

Construction of such comonads based on cofree comonoids can be found in (among

Construction of such comonads based on coffee comonoids can be found in (among
other places):

- Mellies and Tabareau and Tasson, *An explicit formula for the free exponentia modality of linear logic*. Mathematical Structures in Computer Science, 28(7) 1253-1286. doi:10.1017/S0960129516000426
- Sergey Slavnov, *On Banach spaces of sequences and free linear logic exponential modality*, Math. Struct. Comp. Sci. 29 (2019) 215-242 arXiv:1509.03853

The relation to Fock space is discussed in:

- Richard Blute, Prakash Panangaden, R. A. G. Seely, *Fock Space: A Model o Linear Exponential Types* (1994) (web, pdf)
- Marcelo Fiore, *Differential Structure in Models of Multiplicative Biadditive Intuitionistic Linear Logic*, Lecture Notes in Computer Science Volume 4583 2007, pp 163-177 (pdf)
- Jamie Vicary, *A categorical framework for the quantum harmonic oscillator* International Journal of Theoretical Physics December 2008, Volume 47, Issue 12, pp 3408-3447 (arXiv:0706.0711)
  (in the context of finite quantum mechanics in terms of dagger-compact categories)

The interpretation of $\Omega^\infty \Sigma^\infty_+$ as an exponential in the context o Goodwillie calculus is due to

- Gregory Arone, Marja Kankaanrinta, *The Goodwillie tower of the identity is a logarithm*, 1995 (web)

based on

- Gregory Arone, Mark Mahowald, *The Goodwillie tower of the identity functor and the unstable periodic homotopy of spheres*, 1998 (pdf)

The modal approach to a term calculus for the !-modality can be found in:

- Jean-Yves Girard. *On the unity of logic*. Annals of Pure and Applied Logic 59:201-217, 1993.
- G. Plotkin. *Type theory and recursion*. In Proceedings of the Eigth Symposium of Logic in Computer Science, Montreal , page 374. IEEE Computer Society Press, 1993.
- Nick Benton, *A mixed linear and non-linear logic: Proofs, terms and models*, in *Computer Science Logic. CSL 1994*, Lecture Notes in Computer Science ***93*** [doi:10.1007/BFb0022251, pdf]
- Gavin Bierman, *On Intuitionistic Linear Logic*, Cambridge (1993) [pdf]
- Philip Wadler. *A syntax for linear logic*. In Ninth International Coference on the Mathematical Foundations of Programming Semantics , volume 802 of LNCS , Springer Verlag, April 1993.

Springer Verlag, April 1993

- Andrew Barber, *Dual Intuitionistic Linear Logic*, Technical Report ECS-LFCS-96-347, University of Edinburgh, Edinburgh (1996), web

A quantum programming language based on this linear/non-linear type theory adunction is QWIRE:

- Jennifer Paykin, Robert Rand, Steve Zdancewic, *QWIRE: a core language for quantum circuits*, POPL 2017: Proceedings of the 44th ACM SIGPLAN Symposium on Principles of Programming LanguagesJanuary 2017 Pages 846-858 (doi:10.1145/3009837.3009894)

theoretical background:

- Jennifer Paykin, *Linear/non-Linear Types For Embedded Domain-Specific Languages*, 2018 (upenn:2752)

applied to verified programming after implementation in Coq:

- Robert Rand, Jennifer Paykin, Steve Zdancewic, *QWIRE Practice: Formal Verification of Quantum Circuits in Coq*, EPTCS 266, 2018, pp. 119-132 (arXiv:1803.00699)

and using ambient homotopy type theory:

- Jennifer Paykin, Steve Zdancewic, *A HoTT Quantum Equational Theory*, talk at QPL2019 (arXiv:1904.04371)

In dependent linear type theory

Discussion of the exponential modality via stabilization in dependent linear homotopy type theory:

- Kate Ponto, Mike Shulman, Ex. 4.2 in: *Duality and traces in indexed monoidal categories*, Theory and Applications of Categories ***26*** 23 (2012) [arXiv:1211.1555, tac:26-23, blog]
- Urs Schreiber, Sec. 4.2 of: *Quantization via Linear homotopy types* [arXiv:1402.7041]
- Mitchell Riley, §2.1.2 in: *A Bunched Homotopy Type Theory for Synthetic Stable Homotopy Theory*, PhD Thesis (2022) [doi:10.14418/wes01.3.139]

category: logic