Thermalisation of Two-Dimensional Ultracold Bose Gases on Optical Lattices

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Abstract

Systems of ultra-cold gases in light-induced periodic potentials (optical lattices) are of great experimental and theoretical interest, and have been since the first experimental realisation of Bose-Einstein condensation in 1995. This is largely due to the parallels between the behaviour of BECs in an optical lattice and condensed matter systems, where electrons can be modelled as moving on a lattice generated by the periodic array of atom cores [2]. This project investigates the time evolution of a system of bosons prepared in a far-from-equilibrium state on a one-dimensional or two-dimensional optical lattice. The main focus is on determining how the tunneling energies and the strength of the interparticle interactions influence whether or not the system exhibits relaxation to a thermal state. For systems in which revival to the initial state occurs regularly (those which don't thermalise), a method for calculating the revival period is developed. For the systems which exhibit thermalisation, we attempt to characterise the long term states using [ETH, entropy of entanglement, whatever I actually end up doing].

I think I should include some "flavour" here with vaguely interesting general statements about thermodynamics

1 Experimental advantages of working with bosons on optical lattices

Bosons on optical lattices are an excellent arena in which to explore various model Hamiltonians for condensed matter systems and quantum optics, due to the fine degree of experimental control attainable in generating these systems. The shape, depth and spacing of the potential wells can be easily varied in optical lattices of BECs [6]. The strength of the interparticle interaction can also be easily tuned. Moreover, there are always impurities present in the solids we find in nature, which can have significant impacts on the properties of the solid which cannot necessarily be accounted for by a perturbative approach which assumes these effects to be small. Optical lattices are highly uniform, so when dealing with BECs on optical lattices this problem of impurities does not arise. Another benefit of working with bosons on optical lattices is that we can easily control the state of each of the particles on the lattice. By contrast, it would be much more difficult to do things like arranging all of the electrons onto a single atom in an atomic lattice .

reference this at some point from the book I meant to get from David

2 First quantised representation

This paper will deal exclusively with bosons on optical lattices, and not investigate cases involving fermions. There are a number of different contributions to the Hamiltonian for bosons on optical lattices, and these contributions can be seen to have analogues in condensed matter systems. The individual bosons will have kinetic energy, and the potential created by the lattice also contributes to how the system evolves in time, so it must feature in the Hamiltonian. The bosons may be interacting (we will assume this only occurs if they occupy the same lattice sites), and there may also be an external potential imposed that can vary in strength from site to site. From this reasoning, we can construct a first quantised Hamiltonian for a 1D system of interacting bosons on an optical lattice, in the presence of an external potential V_{ext}

$$\hat{H} = \sum_i -\frac{\hbar^2}{2m}\partial_{x_i}^2 + \sum_i V_{\text{lattice}}(R_i) + V_{\text{ext}}(x) + \frac{U}{2}\sum_{i,j}\delta(x_i - x_j). \qquad (1)$$

Similar Hamiltonians are frequently used to describe systems of electrons on atomic lattices.

3 Many-particle wavefunctions and identical particles [7, 4, 1]

The Hamiltonian in the previous section would act on a bosonic many-particle wavefunction. A system of identical bosons must be symmetric under particle exchange, and this has implications for how we must write our many-particle wavefunctions. Let us first consider the case of two indistinguishable bosons. We have not yet determined the eigenstates of our Hamiltonian, but if we suppose

Should spread these references throughout the following sections?

we have the set of normalised single-particle wavefunctions $|\lambda\rangle$, and we have one boson in state $|\lambda_1\rangle$ and another in state $|\lambda_2\rangle$, then we can write the two-particle wavefunction as

$$\Psi(x_1, x_2) = \frac{1}{\sqrt{2}}\big(\langle x_1|\lambda_1\rangle \langle x_2|\lambda_2\rangle + \langle x_1|\lambda_2\rangle \langle x_2|\lambda_1\rangle \big), \tag{2}$$

or in Dirac bra-ket notation, the two-body states would be represented as

$$|\lambda_1, \lambda_2\rangle = \frac{1}{\sqrt{2}}\big(|\lambda_1\rangle \otimes |\lambda_2\rangle + |\lambda_2\rangle \otimes |\lambda_1\rangle \big) \tag{3}$$

The number of permutations that one must account for grows extremely quickly as particle number increases. A properly symmetrised and normalised N-body state can be represented as

$$|\lambda_1, \lambda_2, ..., \lambda_N\rangle = \frac{1}{\sqrt{N! \prod_{\lambda=0}^{\infty} n_\lambda}} \sum_{\mathcal{P}} \lambda_{\mathcal{P}1} \otimes \lambda_{\mathcal{P}2} \otimes ... \otimes \lambda_{\mathcal{P}N} \tag{4}$$

where n_λ is the number of particles in state λ, and the summation runs over all $N!$ permutations $\mathcal{P}$ of the set of quantum numbers $\{\lambda_1, \ldots, \lambda_N\}$.

This formalism has a number of shortcomings. The most important of these for this project is that it is extremely cumbersome for practical computation because of the large number of entities that need to be represented. To avoid this, we shall adopt the second quantised formalism, which is much better suited to dealing concisely with large numbers of indistinguishable particles.

4 Second Quantisation

4.1 The Occupation Number Representation

The formalism that we have hitherto discussed explicitly represents a significant amount of redundant information, in the sense that it deals separately with the scenarios "particle 1 in state λ_1 and particle 2 in state λ_2" and "particle 2 in state λ_1 and particle 1 in state λ_2". Taking into account the indistinguishability of the particles, it is clear that these two scenarios are identical. A more efficient approach consists of describing the number of particles in a particular state λ_i, i.e., using the occupation number representation. When doing this, a general state can be written as a linear superposition

$$|\Psi\rangle = \sum_{n_1, n_2, \ldots} c_{n_1, n_2, \ldots} |n_1, n_2, ...\rangle \,. \tag{5}$$

In the scenario which this project will be working with, the occupation numbers refer to the number of bosons on a particular site of the lattice. Having established this, we can define creation and annihilation operators that create and annihilate particles from number eigenstates, that is

$$\hat{a}_j^\dagger \, |n_1, n_2, \ldots, n_j, \ldots\rangle = \sqrt{n_j + 1} \, |n_1, n_2, \ldots, n_j + 1, \ldots\rangle \tag{6}$$

and

$$\hat{a}_j \left|n_1, n_2, \ldots, n_j, \ldots\right\rangle = \sqrt{n_j} \left|n_1, n_2, \ldots, n_j - 1, \ldots\right\rangle . \tag{7}$$

These operators have very important commutation relations

$$[\hat{a}_j^\dagger, \hat{a}_k^\dagger] = 0, \qquad [\hat{a}_j, \hat{a}_k] = 0, \qquad [\hat{a}_j, \hat{a}_k^\dagger] = \delta_{jk}. \tag{8}$$

We can also define the number operator $\hat{n}_j = \hat{a}_j^\dagger \hat{a}_j$ that has the property that

$$\hat{n}_j \left|n_1, n_2, \ldots, n_j, \ldots\right\rangle = n_j \left|n_1, n_2, \ldots, n_j, \ldots\right\rangle \tag{9}$$

The occupation number eigenstates form the basis of the N-particle Hilbert space that we are working in (the Fock space $\mathcal{F}^N$), and it is useful to observe that any occupation number eigenstate can be created from the empty (or "vacuum") state $|0\rangle$ by repeated action of creation operators

$$\left|n_1, n_2, \ldots\right\rangle = \prod_i \frac{1}{\sqrt{n_i!}} (\hat{a}_i^\dagger)^{n_i} \left|0\right\rangle \tag{10}$$

The N-particle Hilbert space is built up from N single-particle Hilbert spaces $\mathcal{H}$:.

$$\mathcal{H}^N = \mathcal{H} \otimes \mathcal{H} \otimes \cdots \otimes \mathcal{H}. \tag{11}$$

The Fock space $\mathcal{F}^N$ is a subspace of $\mathcal{H}^N$ subject to the symmetry constraints.

is this true? Needs to account for different values of N?

We have established the space and states that we are operating with, we now need to rewrite the Hamiltonian in such a form that it can operate on number eigenstates, i.e., we need to "second quantise" it.

4.2 Second Quantisation of the Hamiltonian

We can second quantise the Hamiltonian initially used to describe our system through the boson field operators, $\hat{\psi}^\dagger(x)$ and $\hat{\psi}(x)$, which create and destroy particles at particular spatial locations (we will define these more explicitly later). The first two terms in (1) are single-particle operators for the kinetic and potential energy contributions, and can be transformed to

$$\int \hat{\psi}^\dagger(x) \left(\sum_i -\frac{\hbar^2}{2m} \partial_{x_i}^2 + \sum_j V_{\text{lattice}}(R_j) \right) \hat{\psi}(x) dx. \tag{12}$$

Bloch's theorem (in 1D) tells us that the eigenstates of a particle in a periodic potential have the form

$$\phi_q(x) = e^{iqx} u_q(x), \tag{13}$$

where q is the quasi-momentum and $u_q(x)$ is periodic with the same period as the lattice, d. We will be considering scenarios in which the strength of the lattice potential is such that the bosons are not completely localised, but such that the overlap between the wavefunctions of particles on particular lattice sites have effectively zero overlap with non-nearest neighbours. Under these conditions, the wavefunctions can be described by the localised Wannier functions

$$\psi(R; r) = \frac{1}{d} \int dq \, e^{iRq} \phi_q(x) \tag{14}$$

which are superpositions of Bloch functions. We can use this to rewrite the field operators as

$$\hat{\psi} = \sum_j \hat{a}_j(t)\psi(R_j - x). \tag{15}$$

With this in mind, we can write

$$\int \sum_j \hat{a}_j^\dagger \psi^*(R_j - x)\left(\sum_i -\frac{\hbar^2}{2m}\partial_{x_i}^2 + \sum_i V_{\text{lattice}}(R_i)\right)\sum_l \hat{a}_l \psi(R_l - x)dx = \sum_{j,l} J_{j,l}\hat{a}_j^\dagger \hat{a}_l. \tag{16}$$

Here

$$J_{j,l} = \int \psi^*(R_j - x)\left(\sum_i -\frac{\hbar^2}{2m}\partial_{x_i}^2 + \sum_i V_{\text{lattice}}(R_i))\right)\psi(R_l - x)dx \tag{17}$$

characterises the strength of the hopping between sites j and l, which (intuitively) depends on the combination of the lattice depth and the kinetic energy. We take this hopping strength to be negligible for sites which are not nearest-neighbours. Of the permitted hopping interactions, we assume that each has identical strength. This assumption allows us to write

$$\int \hat{\psi}^\dagger(x)\left(\sum_i -\frac{\hbar^2}{2m}\partial_{x_i}^2 + \sum_j V_{\text{lattice}}(R_j)\right)\hat{\psi}(x)dx = J\sum_i(\hat{a}_i^\dagger \hat{a}_{i+1} + c.c.), \tag{18}$$

where *c.c.* denotes the complex conjugate. We can go through a similar process for the contributions of the external potential and on-site interactions between bosons. The external potential gives an on-site energy that can vary at different locations in the lattice

$$\int \hat{\psi}^\dagger(x)V_{\text{ext}}(x)\hat{\psi}(x)dx = \sum_i \epsilon_i \hat{a}_i^\dagger \hat{a}_i. \tag{19}$$

Second quantising the term for on-site interaction between bosons (which is a two-particle operator) requires integration between two sets of boson field operators

$$\int \hat{\psi}^\dagger(x)\hat{\psi}^\dagger(x)\frac{U}{2}\sum_{i,j}\delta(x_i - x_j)\hat{\psi}(x)\hat{\psi}(x)dx = \frac{U}{2}\sum_i \hat{a}_i^\dagger \hat{a}_i^\dagger \hat{a}_i \hat{a}_i \tag{20}$$

bad math here?? Revisit and check

Putting these together, we arrive at the Bose-Hubbard model described in the next section.

5 The Bose-Hubbard model

The Hamiltonian for a weakly interacting BEC in an 1-dimensional optical lattice and subject to harmonic trapping potential is given by

$$\hat{H_{1D}} = J\sum_i(\hat{a}_i^\dagger\hat{a}_{i+1} + c.c.) + \frac{U}{2}\sum_i \hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_i + \sum_i \epsilon_i\hat{a}_j^\dagger\hat{a}_j, \tag{21}$$

The ϵ_i's refer to on-site energies at each lattice site (influenced by the harmonic trap), and the middle term gives an interaction energy when there is more than one particle on a particular site.

Should I have a diagram here or elsewhere where I actually draw in the laser beam and the wavefunction overlap?

This project will look at scenarios where there is no external harmonic potential which produces different on-site energies for different sites. Under these conditions, the Hamiltonian in one dimension reduces to

$$\hat{H_{1D}} = J\sum_i(\hat{a}_i^\dagger\hat{a}_{i+1} + c.c.\ ) + \frac{U}{2}\sum_i \hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_i. \tag{22}$$

The atoms in the lattice are still subject to the lattice potential. We can see this from the definition of J as the "hopping integral" in terms of the Hamiltonian of the system in the wavefunction representation.

This project aims to explore the dynamics of both this one dimensional system and the two dimensional version where we couple multiple chains of lattice sites together. The system Hamiltonian in two dimensions is

$$\hat{H_{2D}} = (J\sum_{i,j} \hat{a}_{i,j+1}^\dagger\hat{a}_{i,j} + J'\hat{a}_{i,j}^\dagger\hat{a}_{i+1,j}) + c.c. + \frac{U}{2}\sum_i \hat{a}_i^\dagger\hat{a}_i^\dagger\hat{a}_i\hat{a}_i, \tag{23}$$

where the i index denotes which chain is being referred to and the j index denotes how far along the chain a site is. J' is another hopping parameter; it characterises the overlap between adjacent Wannier states on different chains.

factcheck: should this be Bloch states? or neither?

get the captioning sorted out

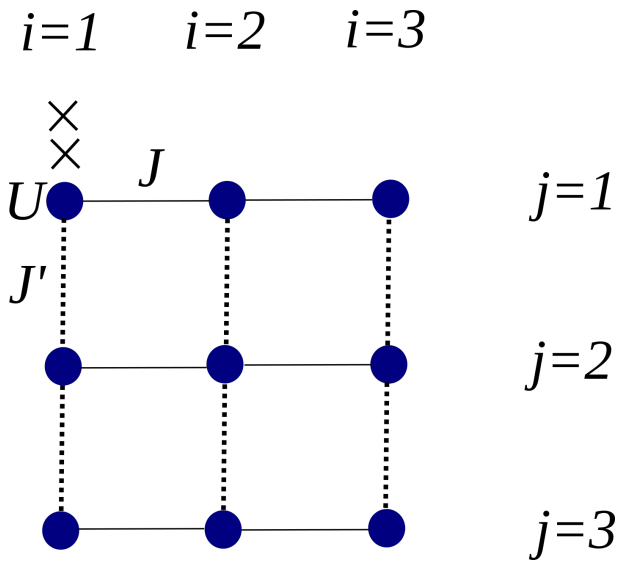


Figure 1: The blue circles here represent lattice sites and the crosses individual bosons. Hopping between any two neighbouring sites on the same chain is characterised by J, whereas hopping between chains is characterised by J'. A three by three lattice is shown here, but the system can be extended to any arbitrary size.

We are interested in the time evolution of these systems as it relates to their thermal behaviour.

6 Thermalisation

6.1 Overview

To say that a system thermalises means that it relaxes to states where the values of macroscopic quantities are stationary, universal with respect to widely differing initial conditions, and predictable using statistical mechanics[8]. We observe thermalisation in a wide variety of classical systems, are there are strong theoretical reasons for anticipating thermalisation here. However, many of these reasons don't apply when considering quantum systems. The question of which, if any, quantum systems exhibit thermalisation and how the thermal states can be characterised is of considerable theoretical and experimental interest.

6.2 Classical systems

To see why thermalisation is common and even expected in many classical systems, we need to understand the property of ergodicity and the impact of chaotic dynamics on it. If you start an isolated system in a particular configuration corresponding to a particular point in phase space, it can move through that phase space along the constant-energy manifold. A system is ergodic if the long-term time average of any single phase space trajectory is equivalent to an ensemble

average. Classical systems exhibit chaotic dynamics, which are strongly non-linear and allow the system to quickly and essentially uniformly explore the constant-energy manifold irrespective of the initial conditions. This promotes ergodicity, which is implicit in the fundamental assumption of statistical mechanics - all accessible microstates are equiprobable. This is what leads us to expect our classical systems to thermalise.

6.3 Quantum systems

Quantum systems do not exhibit dynamic chaos [Rigol's line is "dynamical chaos itself cannot occur in an isolated quantum system, in which the time evolution is linear and the spectrum is discrete" How does that argument work?], so it is not obvious how one would justify applying the assumption of ergodicity in these systems. Because of this, it is unclear if or when we should expect quantum systems to thermalise, or what statistical ensemble we could use to characterise the relaxed states.

7 Integrable systems

There are a number of classical isolated systems that do not display thermalisation. The main difference between these systems and those which approach thermal equilibrium is the extent to which they are constrained relative to the number of degrees of freedom that they posess. This notion is formalised in the property of integrability. A system is said to be integrable if it has as many independent integrals of motion (which are conserved) as it has degrees of freedom. An integral of motion for a Hamiltonian is a smooth function I defined on an open subset of the phase space such that $\dot{I} = 0$ on solutions. So $I(x(t)) =$ constant, where $x(t)$ is the solution of the equations of motion for a particular initial condition. If $x_1(t)$ and $x_2(t)$ are solutions for different initial conditions, then in most cases $I(x_1(t))I(x_2(t))$. Integrability is a useful property, though it is quite rare.

$\neq$ *Theoretically, we can find exact solutions for the equations of motion for a system if it is integrable, otherwise understood and neatly defined. The situation in quantum mechanics is much more challenging. One reason for th*

With respect to the systems that we are dealing with, it has been shown that a one−dimensional lattice chain of non-interacting bosons is an integrable system[9].

[Maybe fall back on "All 1D systems with conserved energy are integrable" for U!=0, referencing page 6 of the handout Danny gave me. I don't really know where to go for 2D at this point.]

8 Experimental studies

8.1 A quantum Newton's cradle

This study investigated the thermalisation of an out-of-equilibrium 1D Bose gas, which is a nearly-integrable system. Kinoshita et al. [5] started with several thousand arrays of one-dimensional Bose gases, each containing from

get Kinoshita's definition of ergodicity

Do I need to elaborate on this further?

often? usually? sometimes?

Need to explicitly link ergodicity and thermalisation, otherwise this is just free-floating and its relevance is not obvious. What is this link?

usually?

see square brackets

half? according to handout this seems to be the case

reference handout Danny gave me, I can't see anywhere where its from

see square brackets

should expand what I currently have have, Kinoshita and

40 to 250 ^{87}Rb atoms, which they trapped by combining a blue-detuned two dimensional optical lattice, which provides tight transverse confinement, with a red-detuned crossed dipole trap which provides weak axial trapping. In order to create the non-equilibrium momentum distributions, they then pulsed on a 3.2 THz detuned 1D lattice, which depletes the zero momentum state and puts the atoms in a superposition momentum state of $+ - 2\hbar k$. The two parts of the wavefunction then oscillate out of phase with each other, colliding with each other twice every full cycle and either reflecting off each other elastically or transmitting straight through like a "ghostly" Newton's cradle (see diagram below, taken from the paper)

check I understand why this one is blue-detuned and the other red-detuned

what is a crossed dipole trap?

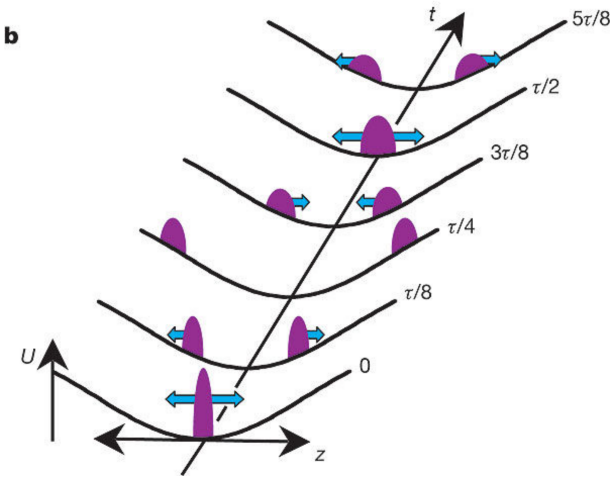


Figure 2: *
Sketches at various times of two out of equilibrium clouds of atoms in a 1D anharmonic trap.At time $t = 0$ the atoms are put into a momentum superposition with $2\hbar k$ to the right and $2\hbar k$ to the left. The two parts of the wavefunction oscillate out of phase with each other with a period τ. Each atom collides with the opposite momentum group twice every full cycle, for instance, at $t = 0$ and $t = \frac{\tau}{2}$

The anharmonicity of the trap caused the atoms to gradually dephase. However,the momentum distribution after dephasing was not gaussian (as one would expect for a thermalised state), and this momentum did not noticeably tend toward this equilibrium distribution, even after thousands of collisions.

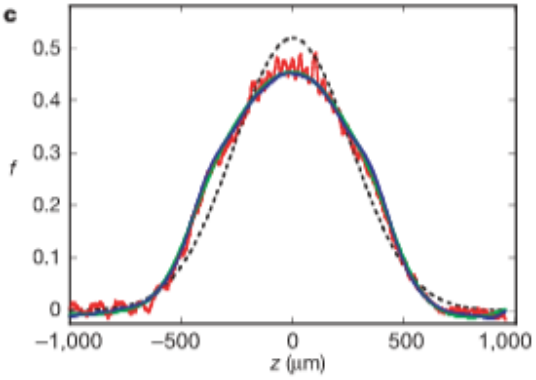


Figure 3: *
The blue and green curves in the above figure are projections from models that take into account loss and heating. The red line is the actual distribution observed. The dashed line is a gaussian with the same number of atoms and r.m.s. width as the actual distribution.

To the extent that the actual distribution conforms to the projected distribution rather than to a gaussian, the atoms have not thermalized.

These observations extended from the Tonks–Girardeau regime, which has very strong repulsive interactions between bosons so only pairwise collisions can occur, to the intermediate coupling regime, where there can be three- (or more) body collisions.

8.2 Relaxation in a Completely Integrable Many-Body Quantum System

Inspired by the "A Quantum Newton's Cradle" study, investigations were made into a completely integrable many-body quantum system. In this experiment, hard-core bosons on a 1D lattice were used. The authors, Rigol et. al, [9] started with a one dimensional bosonic Hamiltonian with no interactions and periodic boundary conditions for a lattice with L lattice sites.

find a good description of what makes bosons "hard core"

$$\hat{H} = -J \sum_{i=1}^{L} (\hat{a_i^\dagger}$$

They then mapped their bosonic system to a free ferminonic one using a Jordan-Wigner transformation.

$$\hat{H} = -J \sum_{i=1}^{L} (\hat{c_i^\dagger}$$

[struggling with the math here. Do I actually need to find an explicit form for all of the Lagrange multipliers for my system so I can plug them into eqn 8 of https://journals.aps.org/prl/pdf/10.1103/PhysRevLett.98.050405 to make predictions?]

see square brackets

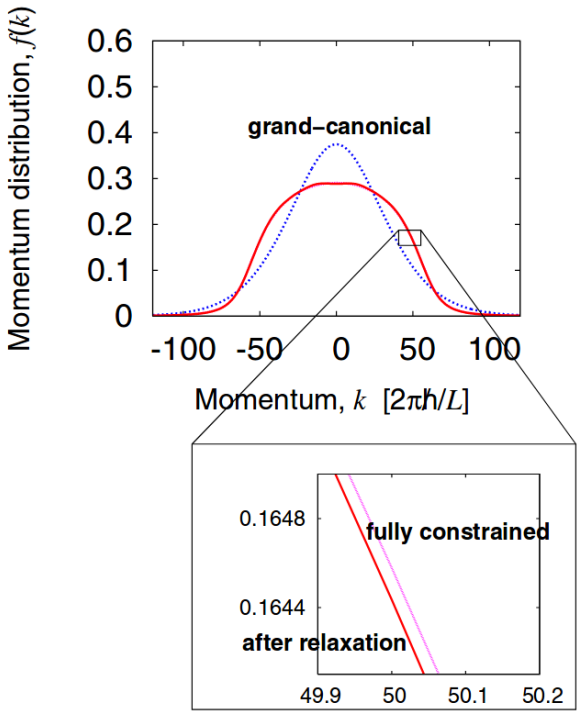


Figure 4: Image taken from [9]. Equilibrium (quasi-)momentum distribution after relaxation in comparison with the predictions of the grand-canonical and of the fully constrained thermodynamical ensembles. The pre- diction of the fully constrained ensemble is virtually indistinct from the results of the dynamical simulation

They further showed that their generalized equilibrium state carries more memory of the initial conditions than the usual thermodynamic one.

the above caption (except the first sentence) is currently copy-pasted from the paper. I'm not sure how to reword those sentences any other way

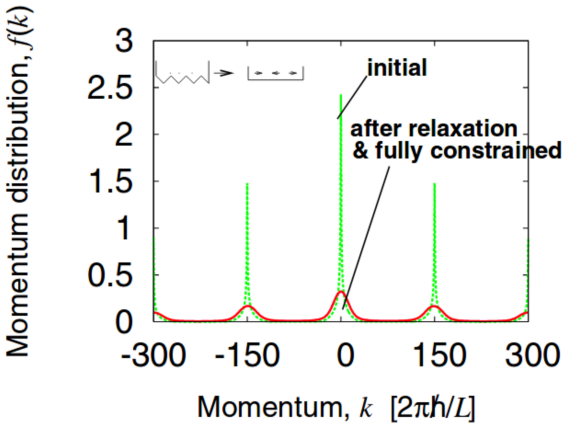


Figure 5: Image taken from [9]. It shows the quasi-momentum distribution after relaxation and compares it to the predictions of the GGE and the grand canonical ensemble

The momentum peaks remain clear and distinct during the whole duration of propagation;$t_{\text{fin}} = 3000\hbar/J$

need to introduce Greiner's entanglement entropy stuff somewhere, probably around here

9 Results

10 Revival

should this section come within results or background?

In a subset of the integrable systems considered, we observed regular instances in which the system would return to the state in which it was initially prepared (it "revives"). To see why we might expect to see revival in some cases, consider the initial state of the system $|\psi(0)\rangle$, which evolves in time according to

$$|\psi(t)\rangle = e^{i\hat{H}t} |\psi(0)\rangle . \tag{26}$$

Now noting that we can use the completeness relation for the energy eigenstates to expand the initial state in the energy basis,

$$\begin{aligned} |\psi(t)\rangle =& e^{i\hat{H}t} \sum_j |E_j\rangle \langle E_j|\psi(0)\rangle \\ =& \sum_j e^{i\hat{H}t} |E_j\rangle \langle E_j|\psi(0)\rangle \\ =& \sum_j e^{iE_j t} |E_j\rangle \langle E_j|\psi(0)\rangle \\ =& \sum_j c_j e^{iE_j t} |E_j\rangle \end{aligned} \tag{27}$$

mention something about absorbing the hbar into the timescale

From here it can be seen that the only time dependence is in the complex exponentials, each of which is 2π-periodic. If we can find some common time

t_r such that $E_j t_r = 2\pi k_j$, where $k_j \in Z$ for all j, we will recover the initial state exactly, i.e., $|\psi(t_r)\rangle = |\psi(0)\rangle$. The next section describes the conditions necessary for the existence of the revival time.

10.1 Exact revival

We observe an exact and regular revival in precisely the cases in which the eigenvalues are mutually rational, i.e. they are either all rational, or are all rational when divided by the same irrational number. For example, the set of hypothetical eigenvalues $\{0, \sqrt{2}, 2\sqrt{2}, \frac{3\sqrt{2}}{10}\}$ is mutually rational, whereas $\{0, \sqrt{2}, \sqrt{3}\}$ is not. In the case of mutually rational eigenvalues, we can write $E_j = \frac{p_j}{q_j}, p_j, q_j \in Z$. The revival time is then given by $t_r = 2\pi Q_{LCM}$, where Q_{LCM} is the lowest common multiple of q_j.

We are yet to find any systems with nonzero interparticle interaction that have mutually rational eigenvalues. In one dimension, the only systems that we have found that meet the criterion of mutually irrational eigenvalues are those with fewer than 4 lattice sites. In two dimensions, .

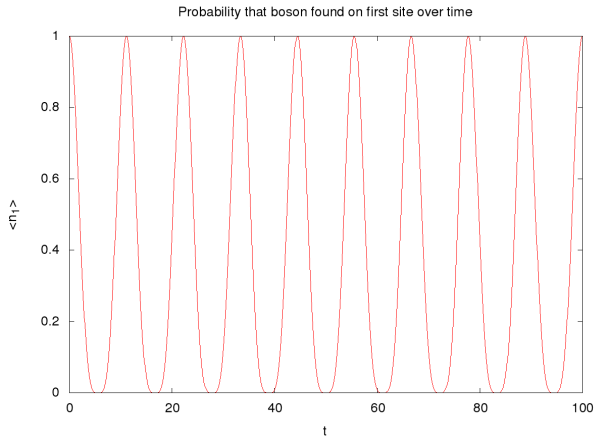
To demonstrate this exact revival, consider a one dimensional lattice with three sites, a single boson and a hopping constant of $J = 0.4$. So the Hamiltonian is

$$\hat{H} = \begin{bmatrix} 0 & -0.4 & 0 \\ -0.4 & 0 & -0.4 \\ 0 & -0.4 & 0 \end{bmatrix} \tag{28}$$

. Diagonalising this matrix yields the eigenvalues $\{-\frac{2\sqrt{2}}{5}, 0, \frac{2\sqrt{2}}{5}\}$. These are mutually rational (If we divide them all by $\sqrt{2}$ they are all rational). Ignoring the $\sqrt{2}$ factor for a moment, our formula for the revival time gives $t_r = 10\pi$, or taking into account the square root of two factor $t_r = 10\sqrt{2}\pi \approx 44.43$. We can see from the graph of the simulation below that this matches up with a return to the initial state.

do a comprehensive batch of simulations to determine what is and isn't ok in 2D

will go through this section with full generality, not hard coded numbers soon



The reason that there appear to be several revivals that our method "misses"

is that states with different phases in the coefficients of the eigenvectors can also produce $\langle n_1 \rangle = 1$, whereas the revival time that is calculated in our method corresponds to exact revival of the initial wavefunction.

10.2 Approximate revival

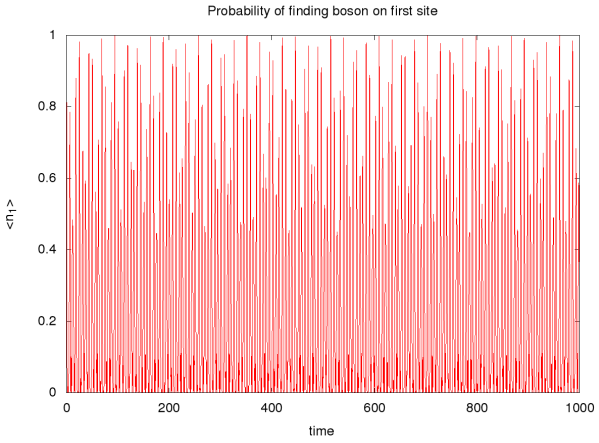
The range of systems for which the spectrum is mutually rational is a small one. All systems which have nonzero interparticle interactions have mutually irrational eigenvalues. One of the most interesting results so far is that even in the one dimensional single particle case, if there are more than three lattice sites, the spectrum will be irrational. We can demonstrate this by considering a single chain of five sites, with one boson and $J = -1$. The Hamiltonian for this system is

is it inaccurate to describle the spectrum (as opposed to the eigenvalues) as mutually irrational?

$$\hat{H} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix} \tag{29}$$

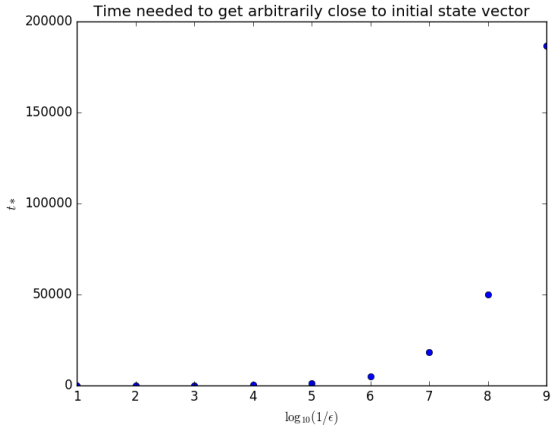
.

the eigenvalues of this system are $\{-\sqrt{3}, 1, 0, 1, \sqrt{3}\}$. These are clearly not mutually rational, so we cannot find an exact revival time for this system, despite the fact that it is fully integrable. Running a simulation of this system where we start the boson off on the first site, we find that the probability of finding the boson on the first site never quite reaches 1 again.



Judging the figure above by eye, it may appear that we have exact revival, but this is not actually the case. If we look at how long it takes for the system to get within epsilon of its initial state (i.e. $|| \, |\psi(t)\rangle - |\psi(0)\rangle \, || < \epsilon$) we get the graph below. Note that t^* denotes the time at which the system got within a particular value of ϵ of the initial state.

Danny- is this just the absolute value of the difference, or the absolute value squared? It looks like the former but I want to confirm



While we see that the system does appear to get arbitrarily close to revival, but it can take arbitrarily long to do so. This result can be explained in terms of the Poincaré recurrence theorem.

11 The Poincaré Recurrence Theorem

In 1890, Henri Poincaré proved the following theorem for classical mechanics: "Any phase-space configuration (q, p) of a system enclosed in a finite volume will be repeated as accurately as one wishes after a finite(be it possibly very long) interval of time'. This theorem was extended to quantum systems in 1956 by P. Bocchieri and A. Loinger[3], where it takes a slightly different form:"Let us consider a system with discrete energy eigenvalues E_n; if $\psi(t_0)$ is its state vector in the Schrödinger picture at the time t_0 and ϵ is any positive number, at least on time T will exist such that the norm $||\psi(T) - \psi(t_0)||$ of the vector $\psi(T) - \psi(t_0)$ is smaller than ϵ."

go through proof of this, ask for help if/when needed.

References

[1] A. Altland and B. Simons. *Condensed Matter Field Theory*. Cambridge books online. Cambridge University Press, 2010.

[2] I. Bloch, J. Dalibard, and S. Nascimbène. Quantum simulations with ultracold quantum gases. *Nature Physics*, 8(4):267–276, Apr 2012.

[3] P. Bocchieri and A. Loinger. Quantum Recurrence Theorem. *Physical Review*, 107(2):337–338, jul 1957.

[4] J. Inkson. *Many-Body Theory of Solids: an Introduction*. Plenum Press, 1984.

[5] T. Kinoshita, T. Wenger, and D. Weiss. A quantum Newton's cradle. *Nature*, 440(7086):900–903, Apr 2006.

[6] O. Morsch and M. Oberthaler. Dynamics of Bose-Einstein condensates in optical lattices. *Reviews of Modern Physics*, 78(1):179–215, Feb 2006.

[7] J. Negele and H. Orland. *Quantum many-particle systems*. Frontiers in physics. Addison-Wesley Pub. Co., 1988.

[8] M. Rigol, V. Dunjko, and M. Olshanii. Thermalization and its mechanism for generic isolated quantum systems. *Nature*, 452(7189):854–858, Apr 2008.

[9] M. Rigol, V. Dunjko, V. Yurovsky, and M. Olshanii. Relaxation in a Completely Integrable Many-Body Quantum System: An Ab Initio Study of the Dynamics of the Highly Excited States of 1D Lattice Hard-Core Bosons. *Physical Review Letters*, 98(5), Feb 2007.