

FIBRATION THEOREMS FOR TQ-COMPLETION OF STRUCTURED RING SPECTRA

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ABSTRACT. The aim of this short paper is to establish a spectral algebra analog of the Bousfield-Kan “fibration lemma” under appropriate conditions. We work in the context of algebraic structures that can be described as algebras over an operad \mathcal{O} in symmetric spectra. Our main result is that completion with respect to topological Quillen homology (or **TQ**-completion, for short) preserves homotopy fibration sequences provided that the base and total \mathcal{O} -algebras are connected. Our argument essentially boils down to proving that the natural map from the homotopy fiber to its **TQ**-completion tower is a $\text{pro-}\pi_*$ isomorphism. More generally, we also show that similar results remain true if we replace “homotopy fibration sequence” with “homotopy pullback square.”

1. INTRODUCTION

This paper is written in the context of symmetric spectra [16, 20], and more generally, modules over a commutative ring spectrum \mathcal{R} ; see [10] for another approach to a well-behaved category of spectra. We consider any algebraic structure in the closed symmetric monoidal category $(\mathbf{Mod}_{\mathcal{R}}, \wedge, \mathcal{R})$ of \mathcal{R} -modules that can be described as algebras over a reduced operad \mathcal{O} ; that is, $\mathcal{O}[0] = *$ is the trivial \mathcal{R} -module and hence \mathcal{O} -algebras are non-unital (see, for instance, [6, 14]).

Topological Quillen homology, or **TQ**-homology for short, is the precise \mathcal{O} -algebra analog of ordinary homology for spaces and is weakly equivalent to the stabilization of \mathcal{O} -algebras; see, for instance, [1, 2, 14, 17]. The **TQ**-completion of an \mathcal{O} -algebra X , denoted $X_{\mathbf{TQ}}^\wedge$, is supposed to be the “part of the \mathcal{O} -algebra that **TQ**-homology sees” ([6, 13, 15]). To keep this paper appropriately concise, we freely use the notation in [6].

Suppose we start with a fibration sequence $F \rightarrow E \rightarrow B$ of \mathcal{O} -algebras and consider the associated commutative diagram of the form

$$(1) \quad \begin{array}{ccccc} F & \longrightarrow & E & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ F_{\mathbf{TQ}}^\wedge & \longrightarrow & E_{\mathbf{TQ}}^\wedge & \longrightarrow & B_{\mathbf{TQ}}^\wedge \end{array}$$

(*)

in $\mathbf{Alg}_{\mathcal{O}}$. The aim of this short paper is to establish conditions on $E \rightarrow B$ such that the bottom row is also a fibration sequence. In other words, we are interested in establishing a **TQ**-completion analog of the Bousfield-Kan “fibration lemma” [4, II.2.2], under appropriate additional conditions on $E \rightarrow B$. If we are in the special situation where E, B are **TQ**-complete (i.e., their coaugmentation maps in (1) are weak equivalences), then this amounts to verifying that (*) is a weak equivalence. The following theorem is our main result.

Theorem 1.1 (TQ-completion fibration theorem). *Let $E \rightarrow B$ be a fibration of \mathcal{O} -algebras with fiber F . If E, B are connected, then the TQ-completion map $F \simeq F_{\text{TQ}}^\wedge$ is a weak equivalence; furthermore, the natural map from F to its TQ-completion tower is a pro- π_* isomorphism.*

This idea generalizes. Suppose we instead start with a fibration $p: X \rightarrow Y$ that fits into a left-hand pullback square of the form

$$(2) \quad \begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow p \\ B & \longrightarrow & Y \end{array} \quad \begin{array}{ccc} A_{\text{TQ}}^\wedge & \longrightarrow & X_{\text{TQ}}^\wedge \\ \downarrow & & \downarrow \\ B_{\text{TQ}}^\wedge & \longrightarrow & Y_{\text{TQ}}^\wedge \end{array}$$

in $\text{Alg}_{\mathcal{O}}$. We would like to establish conditions on the pullback data $B \rightarrow Y \leftarrow X$ such that the right-hand square of the indicated form is also a homotopy pullback diagram. Similar to above, if B, X, Y are TQ-complete, then this amounts to verifying that the TQ-completion map $A \simeq A_{\text{TQ}}^\wedge$ is a weak equivalence. The following theorem is a generalization of our main result.

Theorem 1.2 (TQ-completion homotopy pullback theorem). *Consider any pullback square of the form*

$$(3) \quad \begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow p \\ B & \longrightarrow & Y \end{array}$$

in $\text{Alg}_{\mathcal{O}}$, where p is a fibration. If B, X, Y are connected, then the TQ-completion map $A \simeq A_{\text{TQ}}^\wedge$ is a weak equivalence; furthermore, the natural map from A to its TQ-completion tower is a pro- π_ isomorphism.*

Remark 1.3. It is probably worth pointing out that our strategy of attack works with \mathcal{O} -algebras replaced by pointed simplicial sets. In more detail: Consider any pullback diagram of the form (3) in pointed simplicial sets, where p is a fibration. If B, X, Y are 1-connected, then the Bousfield-Kan \mathbb{Z} -completion map $A \simeq A_{\mathbb{Z}}^\wedge$ is a weak equivalence; furthermore, the natural map from A to its \mathbb{Z} -completion tower is a pro- π_* isomorphism. This provides a new proof of the result in Bousfield-Kan (see, for instance [4, III.5.3]) that such A are \mathbb{Z} -complete.

Assumption 1.4. *Throughout this paper, \mathcal{O} denotes a reduced operad in the closed symmetric monoidal category $(\text{Mod}_{\mathcal{R}}, \wedge, \mathcal{R})$ of \mathcal{R} -modules (see, for instance, [16, 20, 21]). For homotopical reasons, we will assume that \mathcal{O}, \mathcal{R} are (-1) -connected, and that \mathcal{O} satisfies the following cofibrancy condition: Consider the unit map $I \rightarrow \mathcal{O}$; we assume that $I[r] \rightarrow \mathcal{O}[r]$ is a flat stable cofibration ([14, 7.7]) between flat stable cofibrant objects in $\text{Mod}_{\mathcal{R}}$ for each $r \geq 0$. This is exactly the cofibrancy condition appearing in [6, 2.1]. Unless stated otherwise, we work in the positive flat stable model structure [14] on $\text{Alg}_{\mathcal{O}}$.*

Acknowledgments. The author wishes to thank John E. Harper for his support, advice, and constructive criticism, Yu Zhang for many helpful conversations, and Jake Blomquist, Duncan Clark, and Sarah Klanderma for useful discussions. The author was supported in part by National Science Foundation grant DMS-1547357 and the Simons Foundation: Collaboration Grants for Mathematicians #638247.

2. OUTLINE OF THE MAIN ARGUMENT

We will now outline the proof of Theorem 1.1. It suffices to consider the case of a fibration $E \rightarrow B$ in $\mathbf{Alg}_{\mathcal{O}}$ between cofibrant objects (otherwise, cofibrantly replace). The first step is (i) to build the associated cosimplicial resolutions of E, B with respect to TQ-homology by iterating the TQ-Hurewicz map $\mathrm{id} \rightarrow UQ$ (see [6, 3.10]) and (ii) to construct the coaugmented cosimplicial diagram $F \rightarrow \tilde{F}$ that is built by taking (functorial) homotopy fibers vertically, followed by objectwise (functorial) cofibrant replacements. In this way, we obtain a commutative diagram of the form

$$(4) \quad \begin{array}{ccccccc} F & \longrightarrow & \tilde{F}^0 & \rightrightarrows & \tilde{F}^1 & \rightrightarrows & \tilde{F}^2 \dots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ E & \longrightarrow & (UQ)E & \rightrightarrows & (UQ)^2E & \rightrightarrows & (UQ)^3E \dots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ B & \longrightarrow & (UQ)B & \rightrightarrows & (UQ)^2B & \rightrightarrows & (UQ)^3B \dots \end{array}$$

in $\mathbf{Alg}_{\mathcal{O}}$, where the vertical columns are homotopy fibration sequences. Replacing if needed, we may also assume that \tilde{F} is a Reedy fibrant cosimplicial \mathcal{O} -algebra.

Remark 2.1. For ease of notational purposes, we usually suppress the codegeneracy maps in Δ -shaped diagrams appearing throughout this paper.

Applying holim_{Δ} (see [6, Section 8]) to the maps of Δ -shaped diagrams in (4), where we regard the left-hand vertical column as maps of constant Δ -shaped diagrams, gives a commutative diagram in $\mathbf{Alg}_{\mathcal{O}}$ of the form

$$\begin{array}{ccccc} F & \longrightarrow & E & \longrightarrow & B \\ \downarrow & & \downarrow \simeq & & \downarrow \simeq \\ \mathrm{holim}_{\Delta} \tilde{F} & \longrightarrow & E_{\mathrm{TQ}}^{\wedge} & \longrightarrow & B_{\mathrm{TQ}}^{\wedge} \end{array}$$

where each row is a homotopy fibration sequence. The indicated maps are weak equivalences by [6, 1.6] since E, B are assumed to be 0-connected. It follows that the left-hand map $F \rightarrow \mathrm{holim}_{\Delta} \tilde{F}$ is a weak equivalence as well.

The next step is to get the TQ-completion of F into the picture; the basic idea is to prove that $F \rightarrow \mathrm{holim}_{\Delta} \tilde{F}$ is weakly equivalent to the natural coaugmentation $F \rightarrow F_{\mathrm{TQ}}^{\wedge}$. Our strategy of attack is to objectwise resolve, with respect to TQ-homology, the upper horizontal diagram $F \rightarrow \tilde{F}$

$$\begin{array}{ccccccc}
(5) & \begin{array}{c} \vdots \\ (UQ)^3 F \end{array} & \xrightarrow{(\#)} & \begin{array}{c} \vdots \\ (UQ)^3 \tilde{F}^0 \end{array} & \rightrightarrows & \begin{array}{c} \vdots \\ (UQ)^3 \tilde{F}^1 \end{array} & \rightrightarrows & \begin{array}{c} \vdots \\ (UQ)^3 \tilde{F}^2 \dots \end{array} \\
& \uparrow \uparrow \uparrow & & \uparrow \uparrow \uparrow & & \uparrow \uparrow \uparrow & & \uparrow \uparrow \uparrow \\
& \begin{array}{c} (UQ)^2 F \end{array} & \xrightarrow{(\#)} & \begin{array}{c} (UQ)^2 \tilde{F}^0 \end{array} & \rightrightarrows & \begin{array}{c} (UQ)^2 \tilde{F}^1 \end{array} & \rightrightarrows & \begin{array}{c} (UQ)^2 \tilde{F}^2 \dots \end{array} \\
& \uparrow \uparrow & & \uparrow \uparrow & & \uparrow \uparrow & & \uparrow \uparrow \\
& \begin{array}{c} (UQ) F \end{array} & \xrightarrow{(\#)} & \begin{array}{c} (UQ) \tilde{F}^0 \end{array} & \rightrightarrows & \begin{array}{c} (UQ) \tilde{F}^1 \end{array} & \rightrightarrows & \begin{array}{c} (UQ) \tilde{F}^2 \dots \end{array} \\
& \uparrow (*)' & & \uparrow (**) & & \uparrow (**) & & \uparrow (**) \\
& F & \xrightarrow{(\#)} & \tilde{F}^0 & \rightrightarrows & \tilde{F}^1 & \rightrightarrows & \tilde{F}^2 \dots
\end{array}$$

in (4), and show that the maps $(\#)$ and $(**)$ induce weak equivalences after applying holim_Δ (Propositions 3.8 and 4.5). Once this has been accomplished, we obtain a commutative diagram of the form

$$\begin{array}{ccc}
(6) & \text{holim}_\Delta (UQ)^{\bullet+1} F & \xrightarrow{\simeq} \text{holim}_{\Delta \times \Delta} (UQ)^{\bullet+1} \tilde{F} \\
& \uparrow (*)' & \uparrow \simeq \\
& F & \xrightarrow{\simeq} \text{holim}_\Delta \tilde{F}
\end{array}$$

and conclude that the natural coagumentation map $F \simeq F_{\text{TQ}}^\wedge = \text{holim}_\Delta (UQ)^{\bullet+1} F$ is a weak equivalence. This proves the first part of Theorem 1.1.

The second part of Theorem 1.1 requires additional work. In order to precisely formulate this stronger result, we introduce the following two definitions.

Definition 2.2. A map of towers of \mathcal{O} -algebras $\{X_s\}_s \rightarrow \{Y_s\}_s$ is a *pro- π_* isomorphism* if the induced map

$$\{\pi_n X_s\}_s \rightarrow \{\pi_n Y_s\}_s$$

of (abelian) groups towers is a pro-isomorphism for each $n \in \mathbb{Z}$. (Throughout this paper, we assume all homotopy groups are derived [19, 20].)

Remark 2.3. Given a pro- π_* isomorphism as above, it follows from the associated \lim^1 short exact sequence that the induced map $\text{holim}_s X_s \simeq \text{holim}_s Y_s$ is a weak equivalence; see, for instance, [6, Section 8].

Definition 2.4. We define Tot to be the right derived functor of Tot in $(\text{Alg}_\mathcal{O})^\Delta$ equipped with the Reedy model structure. In other words, given a cosimplicial \mathcal{O} -algebra X , we define $\text{Tot}(X)$ to be $\text{Tot}(RX)$ where RX is the (functorial) Reedy fibrant replacement of X in $\text{Alg}_\mathcal{O}^\Delta$.

Stated precisely, the second part of Theorem 1.1 asserts that the map of towers

$$\{F\}_s \xrightarrow{(*)'} \{\text{Tot}_s(UQ)^{\bullet+1} F\}_s$$

is a $\text{pro-}\pi_*$ isomorphism. To show that this assertion is true, note that the proofs of Propositions 3.8 and 4.5 imply that the tower maps

$$\{(UQ)^k F\}_s \rightarrow \{\text{Tot}_s(UQ)^k \tilde{F}\}_s \quad \text{and} \quad \{\tilde{F}^n\}_s \rightarrow \{\text{Tot}_s(UQ)^{\bullet+1} \tilde{F}^n\}_s$$

are actually $\text{pro-}\pi_*$ isomorphisms for each $k, n \geq 0$. Now consider the commutative diagram of towers of the form

$$(7) \quad \begin{array}{ccc} \{\text{Tot}_s(UQ)^{\bullet+1}\}_s & \longrightarrow & \{\text{Tot}_s \text{Tot}_s(UQ)^{\bullet+1} \tilde{F}\}_s \\ \uparrow (*)' & & \uparrow \\ \{F\}_s & \longrightarrow & \{\text{Tot}_s \tilde{F}\}_s \end{array}$$

It follows from the tower lemma below that the horizontal and right-hand vertical maps are $\text{pro-}\pi_*$ isomorphisms, and hence the map $(*)'$ is as well.

Proposition 2.5 (Tower lemma for \mathcal{O} -algebras). *Suppose we are given a map from the Reedy fibrant cosimplicial \mathcal{O} -algebra X to a tower of Reedy fibrant cosimplicial \mathcal{O} -algebras $\{Y_s\}_s$*

$$\begin{array}{ccccc} X^0 & \rightrightarrows & X^1 & \rightrightarrows & X^2 \dots \\ \downarrow & & \downarrow & & \downarrow \\ \{Y_s^0\}_s & \rightrightarrows & \{Y_s^1\}_s & \rightrightarrows & \{Y_s^2\}_s \dots \end{array}$$

If $\{X^k\}_s \rightarrow \{Y_s^k\}_s$ induces a $\text{pro-}\pi_*$ isomorphism for each fixed k , then

$$\{\text{Tot}_n X\}_s \rightarrow \{\text{Tot}_n Y_s\}_s$$

induces a $\text{pro-}\pi_*$ isomorphism for each fixed n .

Proof. In the context of spaces, this is proven in [9, 1.4] and the same argument remains valid in our setting. \square

3. ANALYSIS OF THE HORIZONTAL DIRECTION

The purpose of this section is to analyze the maps $(\#)$ and, in particular, to prove Proposition 3.8. The basic idea is to, first, establish uniform cartesian estimates on the canonical coface cubes (see Definition 3.1) associated to the coaugmented cosimplicial \mathcal{O} -algebra $F \rightarrow \tilde{F}$. This is the content of Proposition 3.3 and is accomplished by analyzing the corresponding coface cubes of $E \rightarrow (UQ)^{\bullet+1}E$ and $B \rightarrow (UQ)^{\bullet+1}B$. We next show, in Proposition 3.6, that objectwise application of the TQ-homology spectrum functor preserves this cartesian estimate. Proposition 3.8 then follows inductively.

Our analysis in this section will involve a number of arguments about (and conventions of) cubical diagrams. Good references for cubical diagrams of spaces include [12], [18], and [8, A.8]. In the context of \mathcal{O} -algebras, see [5] and [6].

Definition 3.1. Let \mathcal{E}_{n+1} be the canonical coface $(n+1)$ -cube [6, 2.4] associated to the coaugmented cosimplicial \mathcal{O} -algebra $E \rightarrow (UQ)^{\bullet+1}E$ and define \mathcal{B}_{n+1} similarly. Let $\tilde{\mathcal{F}}_{n+1}$ be the canonical coface $(n+1)$ -cube associated to the coaugmented cosimplicial \mathcal{O} -algebra $F \rightarrow \tilde{F}$.

The following proposition gives the uniform cartesian estimates on \mathcal{E}_{n+1} and \mathcal{B}_{n+1} (by setting $k = 0$) that we will ultimately use to analyze $\tilde{\mathcal{F}}_{n+1}$. It is proven in [3, 7.1]; a special case is dealt with also in [6]. The proposition is a spectral algebra analogue of Dundas's [7, 2.6] higher Hurewicz theorem.

Proposition 3.2 (Higher TQ-Hurewicz theorem). *Let $k \geq 0$ and \mathcal{X} be a W -cube in \mathcal{O} -algebras that is objectwise cofibrant. If \mathcal{X} is $(\mathrm{id} + 1)(k + 1)$ -cartesian, then so is $\mathcal{X} \rightarrow UQ\mathcal{X}$.*

The uniform cartesian estimates given by Proposition 3.2 applied to $\mathcal{E}_{n+1}, \mathcal{B}_{n+1}$ imply a (slightly weaker) uniform cartesian estimate on $\tilde{\mathcal{F}}_{n+1}$.

Proposition 3.3. *Let $n \geq -1$. The coface $(n + 1)$ -cube $\tilde{\mathcal{F}}_{n+1}$ associated to $F \rightarrow \tilde{F}$ is id -cartesian.*

Proof. It follows from 3.2 that both \mathcal{E}_{n+1} and \mathcal{B}_{n+1} are $(n + 2)$ -cartesian and so [5, 3.8] the cube $\mathcal{E}_{n+1} \rightarrow \mathcal{B}_{n+1}$ is $(n + 1)$ -cartesian. This means that the iterated homotopy fiber [6, 2.6] of $\mathcal{E}_{n+1} \rightarrow \mathcal{B}_{n+1}$ is n -connected. Since this is weakly equivalent to the iterated homotopy fiber of $\tilde{\mathcal{F}}_{n+1}$, we conclude that $\tilde{\mathcal{F}}_{n+1}$ is $(n + 1)$ -cartesian. Repeating this argument on all subcubes completes the proof. \square

The following two short lemmas are used in the proof of Proposition 3.6, which states that levelwise application of the TQ-homology spectrum functor preserves this cartesian estimate on $\tilde{\mathcal{F}}_{n+1}$.

Lemma 3.4. *Let $k \in \mathbb{Z}$ and let \mathcal{Y} be a W -cube in \mathbf{Alg}_J . If \mathcal{Y} is k -cartesian, then so is $U\mathcal{Y}$.*

Proof. This is because U is a right Quillen functor and preserves connectivity of all maps, since this connectivity is calculated in the underlying category $\mathbf{Mod}_{\mathcal{R}}$. \square

Lemma 3.5. *Let $k \geq -1$ and let \mathcal{X} be an objectwise cofibrant W -cube of \mathcal{O} -algebras. If \mathcal{X} is k -cocartesian, then so is $Q\mathcal{X}$.*

Proof. If $|W| = 0$ or 1 , note that an \mathcal{O} -algebra (resp. a map between \mathcal{O} -algebras) is k -cartesian if and only if it is k -connected. The result now follows from [14, 1.9(b)] and the observation that if X is cofibrant, then $\mathrm{TQ}(X) \simeq UQ(X)$. To show, more generally, that $Q\mathcal{X}$ is k -cocartesian, let \mathcal{X}_\circ be the punctured W -cube obtained by removing the terminal vertex \mathcal{X}_W from \mathcal{X} . By assumption, $\mathrm{hocolim} \mathcal{X}_\circ \rightarrow \mathcal{X}_W$ is a k -connected map of cofibrant objects, so $\mathrm{hocolim} Q\mathcal{X}_\circ \simeq Q \mathrm{hocolim} \mathcal{X}_\circ \rightarrow Q\mathcal{X}_W$ is also k -connected, by the first part of the proof. \square

Proposition 3.6. *Let \mathcal{X} be a W -cube in $\mathbf{Alg}_{\mathcal{O}}$. If \mathcal{X} is objectwise cofibrant and is (id) -cartesian, then so is $UQ\mathcal{X}$.*

Remark 3.7. Before we give the proof of Proposition 3.6 in full generality, here is the argument assuming that \mathcal{X} is a 2-cube, i.e., that $W = \{1, 2\}$. In this case, \mathcal{X} is the commutative diagram

$$\begin{array}{ccc} \mathcal{X}_\emptyset & \longrightarrow & \mathcal{X}_{\{1\}} \\ \downarrow & & \downarrow \\ \mathcal{X}_{\{2\}} & \longrightarrow & \mathcal{X}_{\{1,2\}} \end{array}$$

in \mathbf{Alg}_0 , where each object is (-1) -connected (i.e., 0-cartesian as a 0-cube), each map is 1-connected (i.e., 1-cartesian as a 1-cube), and the entire square is 2-cartesian.

That the objects and maps of $UQ\mathcal{X}$ are appropriately connected follows as in the proof of Lemma 3.5. Let us now show that $UQ\mathcal{X}$ is 2-cartesian. The dual Blakers-Massey theorem of Ching-Harper [5, 1.9] implies that \mathcal{X} is k -cocartesian, where

$$k = \min \{k_{12} + 1, k_1 + k_2 + 2\} = \min \{2 + 1, 1 + 1 + 2\} = 3$$

By Lemma 3.5, this means that $Q\mathcal{X}$ is also 3-cocartesian. It is now important to observe that $Q\mathcal{X}$ is a diagram in the *stable* category \mathbf{Alg}_J , so the fact that it is 3-cocartesian implies it is 2-cartesian; see [5, 3.10]. Hence, by Lemma 3.4, $UQ\mathcal{X}$ is also 2-cartesian.

To see that $UQ\mathcal{X}$ is objectwise cofibrant, recall that Q is a left Quillen functor and that [14, 5.49] the functor U preserves cofibrant objects.

Proof of Proposition 3.6. Objectwise cofibrancy is proven in the same way as in Remark 3.7. To show that $UQ\mathcal{X}$ is (id)-cartesian, we induct on n . The cases $|W| = 0, 1, 2$ are handled in Remark 3.7. Suppose now \mathcal{X} is a W -cube with $|W| = n \geq 3$ and that the result holds for all k -cubes with $k < n$. This verifies that $UQ\mathcal{X}$ is (id)-cartesian on all strict subcubes, so we must only further show that $UQ\mathcal{X}$ is itself n -cartesian.

As in Remark 3.7, we first establish a cocartesian estimate on \mathcal{X} , but now use the *higher* dual Blakers-Massey Theorem of Ching-Harper. Adopting the notation of [5, 1.11], observe that each k_V (the cartesianness of a particular $|V|$ -dimensional subcube of \mathcal{X}) is equal to $|V|$ by assumption that \mathcal{X} is (id)-cartesian. It follows that for any partition λ of W , we have

$$|W| + \sum_{V \in \lambda} k_V = n + \sum_{V \in \lambda} |V| = n + n = 2n$$

On the other hand,

$$k_W + |W| - 1 = n + n - 1 = 2n - 1$$

Hence, \mathcal{X} is $(2n - 1)$ -cocartesian. By Lemma 3.5, this means $Q\mathcal{X}$ is also $(2n - 1)$ -cocartesian. Since $Q\mathcal{X}$ is in the stable category \mathbf{Alg}_J , the proof of [5, 3.10] implies that $Q\mathcal{X}$ is $(2n - 1) - n + 1 = n$ -cartesian. Therefore, by Lemma 3.4, $UQ\mathcal{X}$ is also n -cartesian. \square

We are now in a position to prove the main result of this section.

Proposition 3.8. *Let $n \geq -1$ and $k \geq 0$. The coface $(n + 1)$ -cube associated to $(UQ)^k F \rightarrow (UQ)^k \tilde{F}$ is id-cartesian. In particular, the natural map $(UQ)^k F \rightarrow \operatorname{holim}_\Delta (UQ)^k \tilde{F}$ is a weak equivalence.*

Proof. The first part follows inductively from Propositions 3.3 and 3.6. The second part follows by observing that this cartesian estimate implies [6, 8.27] that the natural map $(UQ)^k F \rightarrow \operatorname{holim}_{\Delta \leq n} (UQ)^k \tilde{F}$ is $(n + 1)$ connected, then using the associated \lim^1 short exact sequence. \square

Remark 3.9. In particular, the increasing connectivity proven in Proposition 3.8 implies that, for all $k \geq 0$, the map of towers $\{(UQ)^k F\}_s \rightarrow \{\operatorname{Tot}_s(UQ)^k \tilde{F}\}_s$ is a pro- π_* isomorphism.

4. ANALYSIS OF THE VERTICAL DIRECTION

The purpose of this section is to analyze the maps $(**)$ and, in particular, to prove Proposition 4.5. The basic idea is to first show that, up to homotopy, there is an extra codegeneracy in each coaugmented cosimplicial diagram $\tilde{F}^n \rightarrow (UQ)^{\bullet+1}\tilde{F}^n$. This is accomplished by showing that each \tilde{F}^n is weakly equivalent to an \mathcal{O} -algebra of the form UY and observing that the diagram $UY \rightarrow (UQ)^{\bullet+1}UY$ has an extra codegeneracy on the nose. A short spectral sequence argument then completes the analysis.

Lemma 4.1. *For each $n \geq 0$, there is a fibrant and cofibrant J -algebra G^n with a natural zigzag of weak equivalences $UG^n \simeq \tilde{F}^n$ in $\mathbf{Alg}_{\mathcal{O}}$.*

Proof. We will prove the $n = 0$ case. The proof is essentially the same for $n \geq 1$. By definition, and commuting U past a homotopy limit, we have a natural zigzag of weak equivalences

$$\tilde{F}^0 \simeq \mathrm{hofib}(UQE \rightarrow UQB) \simeq U \mathrm{hofib}(QE \rightarrow QB)$$

and the lemma follows by letting G^0 be the functorial cofibrant replacement of $\mathrm{hofib}(QE \rightarrow QB)$ in \mathbf{Alg}_J . \square

Lemma 4.2. *If Y is in \mathbf{Alg}_J , the diagram $UY \rightarrow (UQ)^{\bullet+1}UY$ has an extra codegeneracy.*

Proof. One obtains an extra codegeneracy by defining $s^n = U(QU)^{n+1}Y \xrightarrow{U(\overrightarrow{QU})^n \epsilon} U(QU)^n Y$ for all $n \geq 0$, where ϵ is the counit associated to the (Q, U) adjunction. \square

Lemma 4.3. *If the coaugmented cosimplicial \mathcal{O} -algebra $X^{-1} \rightarrow X$ has an extra codegeneracy and X^{-1} is fibrant, then the natural map $\{X^{-1}\}_s \rightarrow \{\mathrm{Tot}_s X\}_s$ is a $\mathrm{pro}\text{-}\pi_*$ isomorphism.*

Remark 4.4. In the proof below, we use the spectral sequence associated to a tower of fibrations of \mathcal{O} -algebras. For details of the construction, see [6, 8.31]. It is essentially the same as the homotopy spectral sequence [4, X.6] of Bousfield-Kan; see also [11, VIII.1].

Proof. Fix $n \in \mathbb{Z}$ and consider the coaugmented cosimplicial abelian group $\pi_n X^{-1} \rightarrow \pi_n X$. The assumed extra codegeneracy implies that for any $s \geq 0$, we have $\pi^s \pi_n X^{-1} \xrightarrow{\cong} \pi^s \pi_n X$. It follows that there is an induced isomorphism on E^2 pages of the homotopy spectral sequences associated to $\{X^{-1}\}_s$ and $\{\mathrm{Tot}_s X\}_s$. (Here, we are using the fact that, since X^{-1} is fibrant, the constant cosimplicial diagram with value X^{-1} is Reedy fibrant.) The result now follows from [6, 8.36]. \square

Proposition 4.5. *For each $n \geq 0$, the TQ -completion map $\tilde{F}^n \simeq (\tilde{F}^n)_{\mathrm{TQ}}^{\wedge}$ is a weak equivalence.*

Proof. First, note that both \tilde{F}^n and UG^n (as constructed in Lemma 4.1) are cofibrant. By taking further functorial replacements, it follows from Lemma 4.1 that there is a natural zigzag of weak equivalences $UG^n \simeq (UG^n)^c \simeq (\tilde{F}^n)^c \simeq \tilde{F}^n$ in

which each object is cofibrant. This induces a zigzag of towers

$$\begin{array}{ccc} \{UG^n\}_s & \simeq & \{\tilde{F}^n\}_s \\ \downarrow & & \downarrow (**) \\ \{\mathrm{Tot}_s(UQ)^{\bullet+1}UG^n\}_s & \simeq & \{\mathrm{Tot}_s(UQ)^{\bullet+1}\tilde{F}^n\}_s \end{array}$$

and, by Lemmas 4.3 and 4.2, the left-hand vertical map is a $\mathrm{pro}\text{-}\pi_*$ isomorphism. It follows that $(**)$ is a $\mathrm{pro}\text{-}\pi_*$ isomorphism as well. Hence, $\tilde{F}^n \simeq (\tilde{F}^n)_{\mathrm{TQ}}^\wedge$ is a weak equivalence (see Remark 2.3). \square

5. TQ-COMPLETION OF HOMOTOPY PULLBACK SQUARES

In this section, we prove Theorem 1.2. The strategy of proof is essentially the same as that used in the proof of Theorem 1.1. The new arguments given in this section are needed to obtain an analogue of Proposition 3.3; this is the content of Proposition 5.1 below.

As in the proof of Theorem 1.1, we may assume that B, X, Y are cofibrant, and we then build the associated cosimplicial resolutions of B, X, Y with respect to TQ-homology; then take levelwise homotopy pullbacks to obtain a coaugmented cosimplicial diagram $A \rightarrow \tilde{A}$. In other words, we obtain maps of coaugmented cosimplicial \mathcal{O} -algebras of the form

$$(8) \quad \begin{array}{ccc} (A \rightarrow \tilde{A}) & \longrightarrow & (X \rightarrow (UQ)^{\bullet+1}X) \\ \downarrow & & \downarrow \\ (B \rightarrow (UQ)^{\bullet+1}B) & \longrightarrow & (Y \rightarrow (UQ)^{\bullet+1}Y) \end{array}$$

such that on each fixed cosimplicial degree, one has a homotopy pullback diagram. For example, in cosimplicial degrees 0, 1 we have homotopy pullback diagrams of the form

$$(9) \quad \begin{array}{ccc} \tilde{A}^0 & \longrightarrow & (UQ)X \\ \downarrow & & \downarrow \\ (UQ)B & \longrightarrow & (UQ)Y \end{array} \quad \begin{array}{ccc} \tilde{A}^1 & \longrightarrow & (UQ)^2X \\ \downarrow & & \downarrow \\ (UQ)^2B & \longrightarrow & (UQ)^2Y \end{array}$$

in $\mathrm{Alg}_{\mathcal{O}}$, and these are coaugmented by the diagram in Theorem 1.2. For the same reasons as in the proof of Theorem 1.1, we may assume $A \rightarrow \tilde{A}$ is objectwise cofibrant, and that \tilde{A} is a Reedy fibrant cosimplicial \mathcal{O} -algebra.

Proof of Theorem 1.2. Construct a diagram identical to (5). For the same reasons as in Theorem 1.1, the maps $(**)$ induces $\mathrm{pro}\text{-}\pi_*$ isomorphisms after applying Tot_s . The result now follows from Proposition 5.1 below and arguing as in the proof of Theorem 1.1. \square

Proposition 5.1 (cf. Proposition 3.3). *Let $n \geq -1$. The coface $(n+1)$ -cube associated to $A \rightarrow \tilde{A}$ is (id)-cartesian.*

Remark 5.2. As in the case of Proposition 3.3, it may be helpful to first understand a low-dimensional example of Proposition 5.1 before attacking the proof in full generality. Suppose we wish to show the 1-cube $A \rightarrow \tilde{A}^0$ is (id)-cartesian. Consider the corresponding 1-cubes of A, X, Y to obtain a commutative diagram of the form

$$(10) \quad \begin{array}{ccccc} A & \longrightarrow & X & & \\ \searrow & & \downarrow & \searrow & \\ & \tilde{A}^0 & \longrightarrow & UQX & \\ \downarrow & \downarrow & \downarrow & \downarrow & \\ B & \longrightarrow & Y & & \\ \searrow & & \downarrow & \searrow & \\ & UQB & \longrightarrow & UQY & \end{array}$$

in \mathbf{Alg}_O . We will make frequent use of [5, 3.8] in the following analysis.

Since the back and front faces of (10) are both homotopy pullback diagrams (i.e., infinitely cartesian), the entire 3-cube is also infinitely cartesian. The 1-cubes $X \rightarrow UQX$ and $Y \rightarrow UQY$ are both 2-cartesian (i.e., the maps are 2-connected) by Proposition 3.2 and so the right-hand face of (10) is 1-cartesian. Therefore, the left-hand face is 1-cartesian as well. Since $B \rightarrow UQB$ is 2-cartesian (also by Proposition 3.2), we conclude that $A \rightarrow \tilde{A}^0$ is 1-cartesian. One then repeats this argument on all subcubes of $A \rightarrow \tilde{A}^0$, i.e., on the objects A and \tilde{A}^0 .

Proof of Proposition 5.1. Denote by $\tilde{\mathcal{A}}_{n+1}, \mathcal{B}_{n+1}, \mathcal{X}_{n+1}, \mathcal{Y}_{n+1}$ the coface $(n+1)$ cubes associated to the coaugmented cosimplicial diagrams in (8). Let \mathcal{C} be any subcube of $\tilde{\mathcal{A}}_{n+1}$, say of dimension k . Let $\mathcal{C}_B, \mathcal{C}_X, \mathcal{C}_Y$ denote the obvious corresponding subcubes and consider the commutative diagram of subcubes

$$(11) \quad \begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{C}_X \\ \downarrow & & \downarrow \\ \mathcal{C}_B & \longrightarrow & \mathcal{C}_Y \end{array}$$

As in Remark 5.2, the first step is to establish that the cube (11) is infinitely cartesian. This is accomplished by Lemma 5.3 below. Next, Proposition 3.2 implies that \mathcal{C}_X and \mathcal{C}_Y are both $(k+1)$ -cartesian, so the cube $\mathcal{C}_X \rightarrow \mathcal{C}_Y$ is k -cartesian. Therefore, the cube $\mathcal{C} \rightarrow \mathcal{C}_B$ is k -cartesian. Since \mathcal{C}_B is $(k+1)$ -cartesian (also by Proposition 3.2), we conclude that \mathcal{C} is k -cartesian. \square

Lemma 5.3. *For any subcube \mathcal{C} of $\tilde{\mathcal{A}}_{n+1}$, the cube constructed in (11) is infinitely cartesian.*

Proof. The proof is by induction on k . If $k = 0$, then \mathcal{C} is a single object and the lemma follows by construction. If $k \geq 1$, we may write \mathcal{C} as a map of $(k-1)$ -dimensional subcubes $\mathcal{D} \rightarrow \mathcal{E}$ of $\tilde{\mathcal{A}}_{n+1}$. Consider the commutative diagram of

subcubes

$$\begin{array}{ccccc}
 \mathcal{D} & \longrightarrow & \mathcal{D}_X & & \\
 \searrow & & \downarrow & \searrow & \\
 & \mathcal{E} & \longrightarrow & \mathcal{E}_X & \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathcal{D}_B & \longrightarrow & \mathcal{D}_Y & & \\
 \searrow & & \downarrow & \searrow & \\
 & \mathcal{E}_B & \longrightarrow & \mathcal{E}_Y &
 \end{array}$$

and note that this diagram is precisely (11). By induction, the back and front faces (which are themselves both $(k+1)$ -cubes) are both infinitely cartesian, so the whole cube is as well. \square

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