TQ-COMPLETION AND THE TAYLOR TOWER OF THE IDENTITY FUNCTOR

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Abstract. The goal of this short paper is to study the convergence of the Taylor tower of the identity functor in the context of operadic algebras in spectra. Specifically, we show that if A is a (-1)-connected $\mathcal{O}$-algebra with 0-connected TQ-homology spectrum $\mathsf{TQ}(A)$, then there is a natural weak equivalence $P_\infty(\mathrm{id})A \simeq A^\wedge_{\mathsf{TQ}}$ between the limit of the Taylor tower of the identity functor evaluated on A and the TQ-completion of A. Since, in this context, the identity functor is only known to be 0-analytic, this result extends knowledge of the Taylor tower of the identity beyond its "radius of convergence."

1. Introduction

Working in the context of symmetric spectra [17, 23] or, more generally, modules over a commutative ring spectrum $\mathcal{R}$, we consider any algebraic structure in the closed symmetric monoidal category $(\mathsf{Mod}_\mathcal{R}, \wedge, \mathcal{R})$ that can be described by a reduced operad $\mathcal{O}$; that is, $\mathcal{O}[0] = *$ and, hence, $\mathcal{O}$-algebras are non-unital. See [10] for another construction of a well-behaved category of spectra. The aim of this short paper is to study convergence properties of the Taylor tower of the identity functor in the category $\mathsf{Alg}_\mathcal{O}$ of $\mathcal{O}$-algebras.

For any homotopy functor F from spaces to spaces or spectra, Goodwillie constructs [12, 13, 14] universal n-excisive approximations P_nF of F, which arrange in a tower as below.

$$\begin{tikzcd}[column sep=small] & & F \arrow[dll] \arrow[d] \arrow[drr] & & & \\ P_1F & & P_2F \arrow[ll] & & P_3F \arrow[ll] & \cdots \arrow[l] \end{tikzcd} \tag{1}$$

For suitably nice functors F and sufficiently connected spaces X, one has $F(X) \simeq \mathrm{holim}_n\, P_nF(X)$. Similar results are obtained in the context of $\mathcal{O}$-algebras in, for instance, [20] and [21].

The Taylor tower of the identity functor in spaces has been widely studied and is known to contain a great deal of non-trivial information and structure; see, for instance, [2], [3], [5], and [18]. It is shown in [13, 4.3] that the identity functor in spaces is 1-analytic and, hence, that the Taylor tower of the identity converges strongly on all 1-connected spaces. In the context of $\mathcal{O}$-algebras, it follows from [6, 1.6] that the identity functor in $\mathsf{Alg}_\mathcal{O}$ is 0-analytic and, hence, its Taylor tower converges strongly on all 0-connected $\mathcal{O}$-algebras. Additional structure possessed by this tower is explored in [8].

The goal of this paper is to study, in $\mathsf{Alg}_\mathcal{O}$, the behavior of the Taylor tower of the identity functor when evaluated on $\mathcal{O}$-algebras which are not assumed to be

0-connected. In particular, our main theorem applies to any (-1)-connected cofibrant $\mathcal{O}$-algebra with 0-connected TQ-homology. To keep this paper appropriately concise, we refer the reader to [22, Section 3] for background on TQ-homology and completion; for a more thorough treatment, see [7], [15], and [16]. The following is our main result.

Theorem 1.1. *Given a (-1)-connected cofibrant $\mathcal{O}$-algebra A, if $\mathsf{TQ}(A)$ is 0-connected, then there is a natural weak equivalence $P_\infty(\mathrm{id})A \simeq A^\wedge_{\mathsf{TQ}}$.*

We also prove, in Section 6, the following sufficient condition under which $\mathsf{TQ}(A)$ is 0-connected.

Theorem 1.2. *Given a (-1)-connected cofibrant $\mathcal{O}$-algebra A, if for some $k \geq 2$, the map*

$$\pi_0\big(\mathcal{O}[k] \wedge_{\Sigma_k} A^{\wedge k}\big) \to \pi_0(A) \tag{2}$$

induced by the $\mathcal{O}$-algebra structure of A is a surjection, then $\mathsf{TQ}(A)$ is 0-connected.

Remark 1.3. It follows from the identification (see [16, 1.14] and [19, 2.7]) of the layers of the Taylor tower of the identity

$$D_n(\mathrm{id}) \simeq \mathcal{O}[n] \wedge^{\mathsf{L}}_{\Sigma_n} \mathsf{TQ}^{\wedge^{\mathsf{L}} n} \tag{3}$$

in $\mathsf{Alg}_{\mathcal{O}}$ that, under the assumptions of Theorem 1.1, the Goodwillie spectral sequence associated to the identity functor converges strongly in the sense of [7, 1.8]. Furthermore, by considering the associated $\lim^1$ short exact sequence, one can show that the conditions of Theorem 1.1 imply that $P_\infty(\mathrm{id})A \simeq A^\wedge_{\mathsf{TQ}}$ is 0-connected. Hence, if A is as in Theorem 1.1 and $\pi_0(A) \neq 0$, then $A \not\simeq P_\infty(\mathrm{id})$; in other words, it is not the case that $P_\infty(\mathrm{id})A \simeq A^\wedge_{\mathsf{TQ}}$ simply because both are weakly equivalent to A.

Remark 1.4. The condition that $\mathsf{TQ}(A)$ is 0-connected can be thought of as analogous to assuming, for a connected space X, that $\pi_1(\widetilde{\mathbb{Z}}X) \cong \tilde{H}_1(X;\mathbb{Z}) = 0$ or, equivalently, that $\pi_1(X)$ is a perfect group.

The following connectivity assumptions on $\mathcal{O}$ and $\mathcal{R}$ guarantee that the results of [6] used throughout this paper apply, while the cofibrancy condition on $\mathcal{O}$ implies [16, 5.49] that the forgetful functor $\mathsf{Alg}_J \to \mathsf{Alg}_{\mathcal{O}}$ preserves cofibrant objects, enabling the construction [16, 3.16] of an iterable point-set model of TQ-homology used to define TQ-completion.

Assumption 1.5. *In this paper, $\mathcal{O}$ will denote a reduced operad in the category $(\mathsf{Mod}_{\mathcal{R}}, \wedge, \mathcal{R})$ of $\mathcal{R}$-modules (see, e.g.,* [17], [23], *or* [24]*). We assume that $\mathcal{R}$ and each $\mathcal{O}[n]$ is (-1)-connected and, furthermore, that $\mathcal{O}$ satisfies the following cofibrancy condition, which appears also in* [7, 2.1]*. Consider the unit map $I \to \mathcal{O}$; we assume, for each $r \geq 0$, that $I[r] \to \mathcal{O}[r]$ is a flat stable cofibration (see* [16, 7.7]*) between flat stable cofibrant objects in $\mathsf{Mod}_{\mathcal{R}}$. Unless stated otherwise, we consider $\mathsf{Alg}_{\mathcal{O}}$ with the positive flat stable model structure* [16, 7.14].

Relationship to previous work. It is known [13, 4.3] that the identity functor on spaces converges strongly on all 1-connected spaces. Arone-Kankaanrinta [1, Appendix A] are able to determine the behavior of the Taylor tower more generally, showing that for any 0-connected space X, there is a weak equivalence $P_\infty(\mathrm{id})X \simeq X^\wedge_{\Omega^\infty\Sigma^\infty}$. In the setting of $\mathcal{O}$-algebras, it follows from [6, 1.6] that the identity functor converges strongly on all 0-connected $\mathcal{O}$-algebras. Similarly to Arone-Kankaanrinta, this paper studies the behavior of the Taylor tower of the identity outside of its "radius of convergence," but in the context of $\mathcal{O}$-algebras. Given that $\mathsf{TQ} \simeq \Omega^\infty\Sigma^\infty$, the main result of this paper is a partial analogue to the above result of [1].

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2. Outline of the main argument

We will now outline the proof of Theorem 1.1. The basic idea, motivated by [1], is that it it is easier to analyze the Taylor towers of iterates $(UQ)^k$ of the stabilization functor $UQ \simeq \Omega^\infty\Sigma^\infty$ than it is to analyze the Taylor tower of the identity directly. In fact, since the Taylor tower construction plays nicely with finite homotopy limits, it is equally advantageous to consider finite homotopy limits of functors of the form $(UQ)^k$. To make this more precise, consider the canonical resolution

$$\mathrm{id} \longrightarrow UQ \rightrightarrows (UQ)^2 \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} (UQ)^3 \cdots \tag{4}$$

of the identity functor by TQ-homology (see, for instance [7, 3.10]) and let $\mathsf{TQ}_n := \mathrm{holim}_{\Delta^{\leq n}}(UQ)^{\bullet+1}$. This gives rise to a map of towers of functors

$$\{\mathrm{id}\} \to \{\mathsf{TQ}_n\} \tag{5}$$

in $\mathsf{Alg}_\mathcal{O}$, where the tower on the left is taken to be constant. The strategy now is to analyze the Taylor tower of the identity by analyzing the Taylor towers of each TQ_n. That is, let A be as in Theorem 1.1, and consider the diagram of the form

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
P_2(\mathsf{TQ}_0)A \arrow[d] & P_2(\mathsf{TQ}_1)A \arrow[l] \arrow[d] & P_2(\mathsf{TQ}_2)A \arrow[l] \arrow[d] & \cdots \arrow[l] \\
P_1(\mathsf{TQ}_0)A & P_1(\mathsf{TQ}_1)A \arrow[l] & P_1(\mathsf{TQ}_2)A \arrow[l] & \cdots \arrow[l] \\
\mathsf{TQ}_0 A \arrow[u, dotted, "(\#)"] & \mathsf{TQ}_1 A \arrow[l] \arrow[u, dotted, "(\#)"] & \mathsf{TQ}_2 A \arrow[l] \arrow[u, dotted, "(\#)"] & \cdots \arrow[l]
\end{tikzcd} \tag{6}$$

in $\mathsf{Alg}_\mathcal{O}$ obtained by forming Taylor towers above each functor TQ_n and let $P_*(\mathsf{TQ}_*)A$ denote the bi-tower in (6) formed by the solid arrows above the bottom row. By taking the homotopy limit of $P_*(\mathsf{TQ}_*)A$ in two different ways, we will obtain the desired weak equivalence $P_\infty(\mathrm{id})A \simeq A^\wedge_{\mathsf{TQ}}$.

On one hand, it follows from Proposition 5.7 that each of the maps (#) induce a weak equivalence after applying holim. Hence, taking homotopy limits of each vertical tower in $P_*(\mathsf{TQ}_*)A$ yields a diagram of the form

$$\mathsf{TQ}_0(A) \longleftarrow \mathsf{TQ}_1(A) \longleftarrow \mathsf{TQ}_2(A) \longleftarrow \cdots \tag{7}$$

the homotopy limit of which is $A^{\wedge}_{\mathsf{TQ}}$. Since taking homotopy limits of $P_*(\mathsf{TQ}_*)A$ "vertically" and then "horizontally" calculates holim $P_*(\mathsf{TQ}_*)A$, this establishes the following weak equivalence.

$$\operatorname{holim} P_*(\mathsf{TQ}_*)A \simeq A^{\wedge}_{\mathsf{TQ}} \tag{8}$$

On the other hand, Proposition 4.4 gives a natural weak equivalence

$$\operatorname{holim} P_*(\mathsf{TQ}_*)A \simeq P_\infty(\mathrm{id})A \tag{9}$$

and, together, (8) and (9) prove Theorem 1.1.

3. Background on functor calculus in $\mathsf{Alg}_{\mathcal{O}}$

Because the proofs given in Sections 4, 5, and 7 work so directly with the construction of the Taylor tower, we review, in this section, the necessary background of this construction. It is essentially a recapitulation of [20], with minor changes, and the expert can safely skip this section.

Remark 3.1. We will use Alg to denote one of $\mathsf{Alg}_{\mathcal{O}}$, $\mathsf{Alg}_{\tau_1\mathcal{O}}$, or Alg_J when making statements that apply to all three of these categories.

Many of the functors we consider in later sections are not on-the-nose homotopy functors, but do become so after precomposition with a functorial cofibrant replacement. Hence, we have the following.

Definition 3.2. Call a functor $F\colon \mathsf{Alg} \to \mathsf{Alg}$ *left homotopical* if it preserves weak equivalences between cofibrant objects.

Definition 3.3. Given a set W, let $\mathcal{P}(W)$ denote the power set of W and $\mathcal{P}_0(W)$ the set of all nonempty subsets of W. Note that $\mathcal{P}(W)$ and $\mathcal{P}_0(W)$ are naturally partially ordered by inclusion and, hence, are categories.

Remark 3.4. If W is finite, then the simplicial nerve of $\mathcal{P}(W)$ has finitely many nondegenerate simplices, i.e., $\mathcal{P}(W)$ is "very small" in the language of [9].

Definition 3.5. Let $W = \{1, 2, \dots, n\}$. Given an algebra A in Alg, let $\mathcal{X}^n(A)$ denote the $\mathcal{P}(W)$-shaped diagram in Alg obtained via left Kan extension of the $P_{\leq 1}(W)$-diagram

(10)
$$\begin{tikzcd}[column sep=small] & & & A \arrow[dlll] \arrow[dll] \arrow[dr] \arrow[drr] & & \\ C(A) & C(A) & \cdots & & C(A) & C(A) \end{tikzcd}$$

where $C(A)$ denotes the cone on A. Equivalently, $\mathcal{X}^n(A)$ is the n-cube in Alg obtained by taking iterated pushouts of the maps in (10) and, hence, is a pushout cube in the language of [13].

Remark 3.6. If A is cofibrant, then each of the maps in (10) is a cofibration; hence, the n-cube $\mathcal{X}^n(A)$ is strongly cocartesian.

We now define the n^{th} Taylor approximation of a functor $F\colon \mathsf{Alg} \to \mathsf{Alg}$ as in [20], except that we define $\bar{T}_n$ and $\bar{P}_n$ for all such functors, rather than only homotopy functors that take values in cofibrant objects. The reason for this is that we would like to apply these constructions to, e.g., left homotopical functors, which can be made into homotopy functors with values in cofibrant objects via appropriate replacements.

Definition 3.7. Given a functor $F\colon \mathsf{Alg} \to \mathsf{Alg}$, define $\bar{T}_n F$ objectwise by functorial factorization

$$F(A) \to \bar{T}_n F(A) \xrightarrow{\sim} \operatorname*{holim}_{\mathcal{P}_0(W)} \mathcal{X}^{n+1}(A) \tag{11}$$

as a cofibration followed by an acyclic fibration.

The $\bar{T}_n F$ construction of Definition 3.7 takes the place of Goodwillie's [14] $T_n F$. Accordingly, $\bar{P}_n F$ is defined by the analogous sequential homotopy colimit.

Definition 3.8. Given a functor $F\colon \mathsf{Alg} \to \mathsf{Alg}$, define $\bar{P}_n(F)A$ by the following:

$$\bar{P}_n(F)A := \operatorname{hocolim}\Big(\bar{T}_n(F)A \to \bar{T}_n\big(\bar{T}_n(F)\big)A \to (\bar{T}_n^3 F)A \to \cdots\Big) \tag{12}$$

Remark 3.9. It is worth noting that, even if F is a homotopy functor, the constructions $\bar{T}_n F$ and $\bar{P}_n F$ are *not* homotopical. This is because, in general, the construction $\mathcal{X}^n$ of Definition 3.5 is only homotopical on cofibrant inputs. However, to make these constructions homotopical, we only need to add in appropriate cofibrant replacement.

Definition 3.10. Define $P_n(F)$ as $P_n(F) := \bar{P}_n(F) \circ L$, where L is the functorial cofibrant replacement functor in Alg.

The following proposition says that if F is a left homotopical functor, then, up to weak equivalence, there is no difference between working with $P_n(F)$ and $\bar{P}_n(F)$.

Lemma 3.11. *If F is left homotopical, then the natural map $P_n(F)A \xrightarrow{\sim} \bar{P}_n(F)A$ is a weak equivalence on all cofibrant objects A.*

Proof. It follows by construction that if F is left homotopical, the functors $\bar{T}_n^k(F)$ are also left homotopical, for $k \geq 1$. Hence, $\operatorname{hocolim}_i \bar{T}_n^k(F) = \bar{P}_n(F)$ is as well. So, if $A^c \xrightarrow{\sim} A$ is the functorial cofibrant replacement of A, we have that

$$P_n(F)A = \bar{P}_n(F)A^c \xrightarrow{\sim} \bar{P}_n(F)A \tag{13}$$

is a weak equivalence. □

The following two propositions follow from the construction of $\mathcal{X}^n(A)$ and the fact [6, 3.8(a)] that homotopy pushouts of k-connected maps are k-connected. They mirror the fact that, in spaces, joining with a nonempty set increases connectivity of maps and preserves strongly cocartesian cubes (see, for instance, the proof of [13, 1.4]).

Proposition 3.12. *Given a k-connected map $A \to B$ between cofibrant objects in Alg and any nonempty subset $U \subseteq \{1, 2, \ldots, n\}$, the induced map $\mathcal{X}^n_U(A) \to \mathcal{X}^n_U(B)$ is $(k+1)$-connected.*

Proposition 3.13. *If $\mathcal{Y}$ is an objectwise cofibrant, strongly cocartesian cube in Alg, then for any fixed $U \subseteq \{1, 2, \ldots, n\}$, the cube $\mathcal{X}^n_U(\mathcal{Y})$ is strongly cocartesian.*

4. Analysis of the diagonal of $P_*(\mathsf{TQ}_*)A$

The purpose of this section is to prove Proposition 4.4. After observing that the analogue of [14, 1.6] holds in our setting, the key technical maneuver is Proposition 4.3, which is a consequence of [4, 7.1].

The following definition is essentially [14, 1.2], but restricted to cofibrant objects. Since we make frequent use of this definition, we have included it for the sake of completeness.

Definition 4.1. A map $F \to G$ between functors $\mathsf{Alg} \to \mathsf{Alg}$ is said to satisfy $O'_n(c,\kappa)$ if for every cofibrant $A \in \mathsf{Alg}$ such that $A \to *$ is k-connected, with $k \geq \kappa$, the map $F(A) \to G(A)$ is $(-c+(n+1)k)$-connected. We say that F and G *agree to order n on cofibrant objects* if this holds for some constants c and κ.

Proposition 4.2. *If a map $F \to G$ between functors $\mathsf{Alg} \to \mathsf{Alg}$ satisfies $O'_n(c,\kappa)$, then*

(i) $\bar{T}^i_n(F) \to \bar{T}^i_n(G)$ *satisfies* $O'_n(c-i,\kappa-i)$

(ii) $\bar{P}_n(F)A \xrightarrow{\sim} \bar{P}_n(G)A$ *is a weak equivalence for (-1)-connected, cofibrant A*

Proof. Using Propositions 3.12 and 3.13, this is proven as in [14, 1.6]. □

We are now in a position to prove the key technical result of this section, which states that the identity functor agrees to order n on cofibrant objects with the functor TQ_{n-1}. To keep this section appropriately brief, we refer the reader to [22, Section 7] for the relevant background on cubical diagrams used below.

Proposition 4.3. *The natural transformation* $\mathrm{id} \to \mathsf{TQ}_{n-1}$ *of functors* $\mathsf{Alg}_{\mathcal{O}} \to \mathsf{Alg}_{\mathcal{O}}$ *satisfies $O'_n(0,1)$ for all $n \geq 1$.*

Proof. The proposition essentially follows from [4, 7.1]. For instance, if $A \in \mathsf{Alg}_{\mathcal{O}}$ is cofibrant and k-connected with $k \geq 0$, consider the coface cubes

(14)
$$A \longrightarrow UQA \qquad \begin{tikzcd} A \arrow[r] \arrow[d] & UQA \arrow[d] \\ UQA \arrow[r] & (UQ)^2A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr] \arrow[dr] \arrow[dd] & & UQA \arrow[dr] \arrow[dd] & \\ & UQA \arrow[rr] \arrow[dd] & & (UQ)^2A \arrow[dd] \\ UQA \arrow[rr] \arrow[dr] & & (UQ)^2A \arrow[dr] & \\ & (UQ)^2A \arrow[rr] & & (UQ)^3A \end{tikzcd}$$

arising from the cosimplicial TQ-resolution of A. As a 0-cube, A is $(k+1)$-cartesian; repeated application of [4, 7.1] then shows that the cubes in (14) are, respectively, $2(k+1)$-cartesian, $3(k+1)$-cartesian, and $4(k+1)$-cartesian. Hence, the map $A \to \mathsf{TQ}_0(A)$ is $2(k+1)$-connected; the map $A \to \mathsf{TQ}_1(A)$ is $3(k+1)$-connected; and the map $A \to \mathsf{TQ}_2(A)$ is $4(k+1)$-connected. To finish the proof, continue inductively. □

The main result of this section, below, is now a straightforward consequence of Proposition 4.3.

Proposition 4.4. *Given a (-1)-connected cofibrant $\mathcal{O}$-algebra A, there is a natural weak equivalence of the form* $\operatorname{holim} P_*(\mathsf{TQ}_*)A \simeq P_\infty(\mathrm{id})A$.

Proof. It follows from Propositions 4.2 and Proposition 4.3 that there is a natural weak equivalence between the diagonal of $P_*(\mathsf{TQ}_*)A$ and the tower $\{P_n(\mathrm{id})\}$. Since the inclusion of the diagonal into a bi-tower is left cofinal (i.e., homotopy initial), it follows that we have weak equivalences

$$\operatorname{holim} P_*(\mathsf{TQ}_*)A \simeq \operatorname{holim} \operatorname{diag} P_*(\mathsf{TQ}_*)A \simeq \operatorname*{holim}_n P_n(\mathrm{id})A =: P_\infty(\mathrm{id})A \tag{15}$$

and this completes the proof. □

5. Analysis of the columns of $P_*(\mathsf{TQ}_*)A$

The purpose of this section is to prove Proposition 5.7. The key techincal result used is Proposition 5.3, the proof of which we postpone to Section 7 in order to clarify the overall argument. In the results below, we make use the fact that, by construction, the functors Q, $(UQ)^m U$, and TQ_m are left homotopical for all $m \geq 0$.

As in the previous section, we give the following definition and proposition for the sake of completeness; they are the analogues of [13, 4.1] and [14, 1.4], respectively.

Definition 5.1. Given $F\colon \mathsf{Alg} \to \mathsf{Alg}$, we say F is *cofibrantly stably n-excisive* or satisfies *cofibrant stable n^{th} order excision* if the following is true for some numbers c and κ.

$E'_n(c,\kappa)$: If $\mathcal{X}\colon P(S) \to \mathsf{Alg}$ is an objectwise cofibrant, objectwise (-1)-connected, strongly cocartesian $(n+1)$-cube such that for all $s \in S$ the map $\mathcal{X}_\emptyset \to \mathcal{X}_s$ is k_s-connected and $k_s \geq \kappa$, then the diagram $F(\mathcal{X})$ is $(-c + \sum k_s)$-cartesian.

Proposition 5.2. *If $F\colon \mathsf{Alg} \to \mathsf{Alg}$ satisfies $E'_n(c,\kappa)$, then*

(i) $\bar{T}_n F$ satisfies $E'_n(c-1,\kappa-1)$

(ii) $\bar{t}_n F\colon F \to \bar{T}_n F$ satisfies $O'_n(c,\kappa)$

Proof. Using Propositions 3.12 and 3.13, this is proven in the same way as [14, 1.4] (replacing [13, 1.20] equivalently with repeated application of [6, 3.8]). □

The following proposition is the key ingredient used to prove Proposition 5.7. Since the proof of the following result is somewhat technical, we have deferred it to Section 7.

Proposition 5.3. *For each $m \geq 0$, the functor $(UQ)^m U\colon \mathsf{Alg}_J \to \mathsf{Alg}_{\mathcal{O}}$ satisfies $E'_n(-1,0)$ for all $n \geq 1$.*

We analyze the functors $(UQ)^m U$ rather than $(UQ)^{m+1}$ because the latter make it easier to use the assumption of our main result that $\mathsf{TQ}(A)$ is 0-connected. The following lemma says that, up to weak equivalence, there is no difference in analyzing the Taylor towers of the two functors.

Lemma 5.4. *For any left homotopical functor F and cofibrant $A \in \mathsf{Alg}$, there is a natural commutative diagram*

$$\begin{tikzcd} (FQ)A \arrow[r] \arrow[dr] & \bar{T}^i_n(FQ)A \arrow[d, "\sim"] \\ & \bar{T}^i_n(F)QA \end{tikzcd} \tag{16}$$

for all $n, i \geq 1$.

Proof. Since $\mathcal{X}^{n+1}(A)$ is a pushout cube and Q is a left adjoint, there is a natural isomorphism $Q\mathcal{X}^{n+1}(A) \cong \mathcal{X}^{n+1}(QA)$. This proves the lemma for $i = 1$, and one then continues inductively. □

Remark 5.5. Applying $\mathrm{hocolim}_i$ to (16) shows that Lemma 5.4 remains true if $\bar{T}_n^i$ is replaced by $\bar{P}_n$.

We will now work towards proving Proposition 5.7, assuming Proposition 5.3.

Proposition 5.6. *For all $n, i \geq 1$ and $m \geq 0$, if $A \in \mathsf{Alg}_{\mathcal{O}}$ is cofibrant and QA is 0-connected, then the map*

$$(UQ)^{m+1}A \to \bar{T}_n^i\big((UQ)^{m+1}\big)A \tag{17}$$

is $(n+2)$-connected.

Proof. Proposition 5.3 and the fact that π_* commutes with filtered homotopy colimits implies that $(UQ)^m UQA \to \bar{T}_n^i\big((UQ)^m U\big)QA$ is $(n+2)$-connected; the result follows from Lemma 5.4 with $F = (UQ)^m U$. □

The connectivity estimates provided by Proposition 5.6 now translate to similar estimates for the functors TQ_n which we use to prove the following.

Proposition 5.7. *For any $k \geq 0$, the natural map*

$$\mathsf{TQ}_k A \to \bar{P}_\infty\big(\mathsf{TQ}_k\big)A \tag{18}$$

is a weak equivalence.

Proof. Since $\bar{T}_{n+k}^i$ commutes with homotopy limits, it follows from Proposition 5.6 and appealing to [13, 1.20] or repeated application of [6, 3.8] that for any $i \geq 1$, the map

$$\mathsf{TQ}_k A \to \bar{T}_{n+k}^i\Big(\mathsf{TQ}_k\Big)A \tag{19}$$

is n-connected.

Since π_* commutes with filtered homotopy colimits, this implies that the map

$$\mathsf{TQ}_k A \to \bar{P}_{n+k}\Big(\mathsf{TQ}_k\Big)A \tag{20}$$

is also n-connected. The result now follows by letting n tend to infinity and considering the associated $\lim^1$ short exact sequence. □

6. A sufficient condition

The purpose of this section is to prove Theorem 1.2. The strategy is to, first, use the fact [16, 4.10] that $\mathsf{TQ}(A)$ can be calculated as the realization of a simplicial bar construction, i.e., $\mathsf{TQ}(A) \simeq |\mathrm{Bar}(\tau_1\mathcal{O}, \mathcal{O}, A)|$. Theorem 1.2 then follows by analyzing the spectral sequence [16, 4.43] associated to this bar construction.

Proof of Theorem 1.2. Let A be a (-1)-connected cofibrant $\mathcal{O}$-algebra and suppose the following.

$(*)$: For some $k \geq 2$, the map $\mathcal{O}[k] \wedge_{\Sigma_k} A^{\wedge k} \to A$ induces a surjection on π_0.

Consider the simplicial $\mathcal{R}$-module $\mathrm{Bar}(\tau_1\mathcal{O}, \mathcal{O}, A)$ and associated spectral sequence, as below.

$$E^2_{p,q} = H_p\Big(\pi_q\Big(\mathrm{Bar}(\tau_1\mathcal{O}, \mathcal{O}, A)\Big)\Big) \implies \pi_{p+q}\big(|\mathrm{Bar}(\tau_1\mathcal{O}, \mathcal{O}, A)|\big) \tag{21}$$

The assumption that A is (-1)-connected implies that $\mathrm{Bar}(\tau_1\mathcal{O},\mathcal{O},A)$ is objectwise (-1)-connected. Hence, the only contribution to $\pi_0(|\mathrm{Bar}(\tau_1\mathcal{O},\mathcal{O},A)|)$ comes from $p=q=0$. Furthermore, for dimensional reasons, we have $E^2_{0,0}=E^\infty_{0,0}$. The result will therefore follow by showing that $E^2_{0,0}=*$.

It follows from (21) and properties of simplicial abelian groups (see, e.g., [11, III.2]) that

$$H_0\big(\pi_0\big(\mathrm{Bar}(\tau_1\mathcal{O},\mathcal{O},A)\big)\big)\cong \pi_0(\tau_1\mathcal{O}\circ(A))/\operatorname{im}(d_0-d_1) \tag{22}$$

We will now show that $(*)$ implies that the map

$$\begin{aligned} \pi_0\big(\tau_1\mathcal{O}\circ\mathcal{O}\circ(A)\big) &\xrightarrow{d_0-d_1} \pi_0(\tau_1\mathcal{O}\circ(A)) \ \cong \\ \pi_0\Big(\mathcal{O}[1]\wedge\coprod_{k=1}^{\infty}\mathcal{O}[k]\wedge_{\Sigma_k}A^{\wedge k}\Big) &\xrightarrow{d_0-d_1} \pi_0\big(\mathcal{O}[1]\wedge A\big)\end{aligned} \tag{23}$$

is surjective. Indeed, the cofibrancy condition of Assumption 1.5 guarantees that $\mathcal{O}[1]$ is flat in the language of [23] and, hence, it suffices by [23, 5.23] to show that

$$\pi_0\Big(\coprod_{k=1}^{\infty}\mathcal{O}[k]\wedge_{\Sigma_k}A^{\wedge k}\Big)\cong\bigoplus_{k=1}^{\infty}\pi_0\big(\mathcal{O}[k]\wedge_{\Sigma_k}A^{\wedge k}\big)\xrightarrow{d_0-d_1}\pi_0(A) \tag{24}$$

is surjective. Since the map d_0 is trivial when $k\geq 2$, the result now follows immediately from $(*)$. ☐

7. Proof of Proposition 5.3

The purpose of this section is to prove Proposition 5.3. The strategy is a double induction, on n and m. More precisely, for fixed $m\geq 0$ and $n\geq 1$, let $P(n,m)$ be the following statement:

(25) $\quad P(n,m)$: The functor $(UQ)^mU\colon \mathsf{Alg}_J\to\mathsf{Alg}_{\mathcal{O}}$ satisfies $E'_n(-1,0)$.

Lemmas 7.1, 7.2, and 7.3 show that $P(n,m)$ holds for all $m\geq 0$ and $n\geq 1$, and this proves Proposition 5.3. We have included an extra base case to better illustrate, in a low dimension, the more general inductive step. Throughout the arguments that follow, we make frequent use of the fact [22, 4.4] that the forgetful functor $U\colon\mathsf{Alg}_J\to\mathsf{Alg}_{\mathcal{O}}$ preserves cartesianness of all cubes, and that [22, 4.5] the functor $Q\colon\mathsf{Alg}_{\mathcal{O}}\to\mathsf{Alg}_J$ preserves cocartesianness of objectwise cofibrant and (-1)-connected cubes.

Lemma 7.1. *$P(1,m)$ holds for all $m\geq 0$.*

Proof. Let $\mathcal{Y}$ be an objectwise cofibrant, objectwise (-1)-connected, cocartesian 2-cube

$$\begin{tikzcd} Y_\emptyset \arrow[r, "k_1"] \arrow[d, "k_2"] & Y_{\{1\}} \arrow[d] \\ Y_{\{2\}} \arrow[r] & Y_{\{1,2\}} \end{tikzcd} \tag{26}$$

in Alg_J, with initial maps of the indicated connectivities, and $k_1,k_2\geq 0$. Since $\mathcal{Y}$ is a cocartesian square in the stable category Alg_J, it is also cartesian. Hence, so is $U\mathcal{Y}$; this proves $P(1,0)$.

To see that $P(1,1)$ holds, note that $Y_{\{2\}} \to Y_{\{1,2\}}$ is k_1-connected and $Y_{\{1\}} \to Y_{\{1,2\}}$ is k_2-connected, by [6, 3.8]. It follows from [6, 1.8] that $U\mathcal{Y}$ is $(2+k_1+k_2)$-cocartesian and, hence, $QU\mathcal{Y}$ is as well. By [6, 3.10], this means that $QU\mathcal{Y}$ is $(1+k_1+k_2)$-cartesian and, hence, so is $UQU\mathcal{Y}$; this verifies $P(1,1)$.

Inductively, suppose that $P(1,m)$ holds for fixed $m \geq 1$. Let's show that $P(1,m+1)$ is true. By the inductive hypothesis, the cube $(UQ)^m U\mathcal{Y}$ is $(1+k_1+k_2)$-cartesian. Using now the dual Blakers-Massey theorem [6, 1.9] for $\mathsf{Alg}_{\mathcal{O}}$, we have that this cube is $(2+k_1+k_2)$-cocartesian and, hence, so is $Q(UQ)^m U\mathcal{Y}$. By [6, 3.10], this means $Q(UQ)^m U\mathcal{Y}$ is $(1+k_1+k_2)$-cartesian, so $UQ(UQ)^m U\mathcal{Y} = (UQ)^{m+1} U\mathcal{Y}$ is as well. $\square$

Lemma 7.2. *$P(2,m)$ holds for all $m \geq 0$.*

Proof. Let $\mathcal{Y}$ be an objectwise cofibrant, objectwise (-1)-connected, strongly cocartesian 3-cube

(27)
$$\begin{tikzcd}
\mathcal{Y}_\emptyset \arrow[rr, "k_1"] \arrow[dr, "k_3"] \arrow[dd, "k_2"] & & \mathcal{Y}_{\{1\}} \arrow[dr] \arrow[dd, dotted] & \\
& \mathcal{Y}_{\{3\}} \arrow[rr] \arrow[dd] & & \mathcal{Y}_{\{1,3\}} \arrow[dd] \\
\mathcal{Y}_{\{2\}} \arrow[rr] \arrow[dr] & & \mathcal{Y}_{\{1,2\}} \arrow[dr, dotted] & \\
& \mathcal{Y}_{\{2,3\}} \arrow[rr] & & \mathcal{Y}_{\{1,2,3\}}
\end{tikzcd}$$

in Alg_J, with $k_1, k_2, k_3 \geq 0$. Arguing as in Lemma 7.1 and replacing [6, 1.8] by [6, 1.10], shows that $P(2,0)$ and $P(2,1)$ hold.

Inductively, suppose that $P(2,m)$ is true for fixed $m \geq 1$. Let's show that $P(2,m+1)$ is true. By the inductive hypothesis, the cube $(UQ)^m U\mathcal{Y}$ is $(1+k_1+k_2+k_3)$-cartesian. As in Lemma 7.1, we first establish a cocartesian estimate on $(UQ)^m U\mathcal{Y}$, but now use the higher dual Blakers-Massey theorem [6, 1.11] of Ching-Harper. Note that applying Lemma 7.1 to the subcubes of $(UQ)^m U\mathcal{Y}$ guarantees that the conditions of this theorem are satisfied.

We will now adopt the notation of [6, 1.11]. Since $P(1,k)$ holds for all $k \geq 0$, the minimum of $\sum_{V \in \lambda} k_V$ over all partitions λ of $\{1,2,3\}$ by nonempty sets not equal to $\{1,2,3\}$ is achieved by the partition of singleton sets. It then follows from [6, 1.11] that $(UQ)^m U\mathcal{Y}$ is $(3+k_1+k_2+k_3)$-cocartesian and, hence, so is $Q(UQ)^m U\mathcal{Y}$. By [6, 3.10], this means $Q(UQ)^m U\mathcal{Y}$ is $(1+k_1+k_2+k_3)$-cartesian, so $UQ(UQ)^m U\mathcal{Y} = (UQ)^{m+1} U\mathcal{Y}$ is as well. $\square$

Lemma 7.3. *Fix $n \geq 2$. Assume, for all $1 \leq j \leq n$, that $P(j,m)$ holds for all $m \geq 0$. Then $P(n+1,m)$ also holds for all $m \geq 0$.*

Proof. Let $\mathcal{Y}$ be an objectwise cofibrant, objectwise (-1)-connected, strongly cocartesian W-cube in Alg_J with $W = \{1,2,\ldots,n+2\}$ such that for each $s \in W$, the map $Y_\emptyset \to Y_{\{s\}}$ is k_s-connected, with $k_s \geq 0$. Arguing as in Lemma 7.2 shows that $P(n+1,0)$ and $P(n+1,1)$ hold.

Inductively, suppose that $P(n+1,m)$ holds for fixed $m \geq 1$. To show that $P(n+1,m+1)$ holds as well, the strategy is the same as in Lemma 7.2. That is, we first establish a cocartesian estimate on $(UQ)^m U\mathcal{Y}$, which we know by the inductive hypothesis is $(1+\sum_{s \in W} k_s)$-cartesian.

The assumption that $P(j,m)$ holds for all $1 \leq j \leq n$ and $m \geq 0$ guarantees that the conditions of [6, 1.11] are satisfied. Furthermore, as in Lemma 7.2, this assumption implies that the minimum of $\sum_{V\in\lambda} k_V$ over all partitions λ of W by nonempty sets not equal to W is achieved by the partition of singleton sets. It then follows from [6, 1.11] that $(UQ)^m U\mathcal{Y}$ is $(|W| + \sum_{s\in W} k_s)$-cocartesian and, hence, so is $Q(UQ)^m U\mathcal{Y}$. By [6, 3.10], this means $Q(UQ)^m U\mathcal{Y}$ is $(1+\sum_{s\in W} k_s)$-cartesian, so $UQ(UQ)^m U\mathcal{Y} = (UQ)^{m+1}U\mathcal{Y}$ is as well. $\square$

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