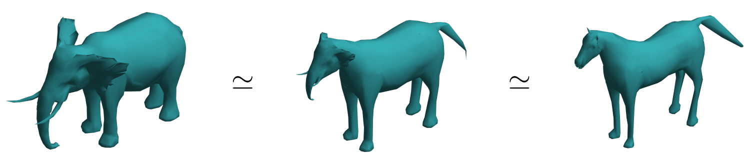
Summary

I am an algebraic topologist—more specifically, a homotopy theorist. My research focuses on problems dealing with **completion**, **localization**, and **functor calculus** in the context of **operadic algebras** in **spectra.** In particular, I have identified (Theorem 2.6) a new class of fibrations that are preserved by TQ-completion, and established conditions (Theorem 3.3) under which the Taylor tower of the identity functor in $\mathcal{O}$-algebras recovers TQ-completion. I am also interested in pursuing several related projects, discussed in Section 4, as well as a number of problems in graph theory—accessible to undergraduate students—described in Section 5.

1. Introduction

Much of my research focuses on **spectra**, but spectra are built from more fundamental objects, namely **topological spaces.** A more classical notion, spaces are used in many areas of mathematics, from category theory to data analysis; and whenever we encounter two spaces X and Y, we can ask the fundamental question, "Are they the same?" We should, however, first specify what we mean by "the same."

In homotopy theory, we study spaces up to a notion of equivalence that is weaker than that of homeomorphism. Instead, we think of two (nice) spaces as being "the same" if each can be continuously deformed into the other. For instance, computational differential geometers (and biologists) might want to consider a horse and an elephant to be different (because of their manifold structure, DNA sequences, etc.). Homotopically, however, they are the same: were we to imagine the surface of an elephant made out of Play-Doh, we could deform the Play-Doh elephant into a horse without tearing anything (and vice-versa).



There are also non-topological manifestations of this theme. Recall that a **chain complex** C is a sequence $\{C_i\}$ of abelian groups (or, more generally, R-modules) along with structure maps $\partial\colon C_i \to C_{i-1}$ which we use to construct the **homology groups** of C. In this setting, asking for a levelwise isomorphism is often restrictive; instead, we consider two chain complexes to be *homotopically* the same if there is a map (or zig-zag of maps) between them that induces an isomorphism on all homology groups.

In essence, we can study homotopy theory whenever we have a class of objects (e.g., spaces or chain complexes) and a notion of when two of these objects are the same, i.e., a collection of maps $X \xrightarrow{\sim} Y$ called **weak equivalences**. My research focuses on the settings of **spaces** and **structured ring spectra**.

One useful way to think about spectra is as a topological analogue of chain complexes: we replace "abelian groups" by "spaces" and have a different collection of structure maps which we use to define the **stable homotopy groups** of a spectrum. Due to the work of [EKMM97] and [HSS00], among others, we have a well-behaved tensor product of spectra, just like we have for chain complexes. In other words, given two spectra X and Y, their **smash product** $X \otimes_S Y$ is again a spectrum; and if a spectrum X has an associative and unital pairing $X \otimes_S X \to X$ we call it a **ring spectrum**, analogous to the fact that an abelian group R with the same kind of pairing $R \otimes_{\mathbb{Z}} R \to R$ is precisely a ring. (As the "$\otimes_S$" notation suggests, spectra can be equivalently described as modules over the sphere spectrum S.)

Operads in spectra can encode other, more diverse, flavors of algebraic structure—such as E_n, E_∞, or Lie algebra structures. More precisely, given an operad $\mathcal{O}$, the category $\mathsf{Alg}_{\mathcal{O}}$ of **algebras over** $\mathcal{O}$ comprises precisely those spectra with the extra structure prescribed by $\mathcal{O}$. Succinctly:

$$\mathsf{Alg}_{\mathcal{O}} = \{\text{"topological chain complexes" with algebraic "flavor" encoded by } \mathcal{O}\}$$

Although homotopy groups are the main invariant of both spaces and $\mathcal{O}$-algebras, they are also notoriously difficult to compute. It is therefore often beneficial to study their *homology* groups—a weaker, but more computable, invariant. In the setting of spaces, there are a number of deep theorems (see, for instance, [Bou75] and [BK72]) that relate homotopy to homology via **localization** and **completion**.

One direction (see Section 2) of my research program develops analogous relationships in the setting of $\mathcal{O}$-algebras. For instance, I describe in Theorem 2.6 a new class of $\mathcal{O}$-algebras that can be recovered from purely homological information.

The second major avenue (see Section 3) I have pursued is similar in spirit. In the context of spaces, the tools of **functor calculus**, developed in [Goo90], [Goo92], and [Goo03] have proven useful in a variety of ways; see, for instance [Beh12] and [Kuh06]. One of the goals of my research program is to develop and apply the tools of functor calculus in the setting of $\mathcal{O}$-algebras, and in particular to questions of localization and completion. For example, the conditions I identify in Theorem 3.3 suggest an intrinsic connection between functor calculus and TQ-completion in the context of $\mathcal{O}$-algebras.

When considering both homology completion and functor calculus, it can be useful to keep in mind the following question: When can we take as input a rough approximation to an object, and—through some formal process—recover the object itself, up to weak equivalence?

2. Completion of spaces and spectra

Given a pointed space X, its homology groups $\tilde{H}_n(X) \cong \pi_n \widetilde{\mathbb{Z}} X$ form a more computable, though coarser, invariant than its homotopy groups. In other words, the comparison map $X \to \widetilde{\mathbb{Z}} X$ provides a rough approximation of X, and the idea of completion is to improve this comparison by *iterating* it. That is, we cosimplicially resolve X as below, and glue together the pieces [BK72].

$$X \dashrightarrow \boxed{\widetilde{\mathbb{Z}} X \rightrightarrows \mathbb{Z}^2 X \Rrightarrow \mathbb{Z}^3 X \cdots} \tag{1}$$

Definition 2.1. The $\mathbb{Z}$-completion of X is $X^\wedge_{\mathbb{Z}} := \operatorname{holim}(\square)$.[1]

A natural question to ask is, "When does this process recover X itself?" In the context of spaces, Bousfield-Kan [BK72] show that if π_1 of a connected space X "takes care of itself," then $\mathbb{Z}$-completion recovers X, up to weak equivalence.

Theorem 2.2 ([BK72])**.** *If a connected space X is nilpotent (e.g., simply connected), then $X \xrightarrow{\sim} X^\wedge_{\mathbb{Z}}$.*

When working in the pointed setting $\mathsf{Alg}_{\mathcal{O}}$ of algebras over a reduced operad in S-modules (i.e., spectra) topological Quillen homology (or TQ-homology for short) takes the place of singular homology. Developed in [Bas99] and [BM05] (see also [Kuh06], [Har10], and [HH13]), TQ-homology is the spectral ring analogue of Quillen homology, which itself generalizes singular homology, and [KP17] is weakly equivalent to stabilization.[2] In the same way that we build the $\mathbb{Z}$-completion of a space by gluing together iterates $\widetilde{\mathbb{Z}}^k X$ of its $\mathbb{Z}$-homology, we construct the TQ-completion of an $\mathcal{O}$-algebra A by gluing together iterates $\mathsf{TQ}^k A$ of its TQ-homology spectrum [HH13].

Definition 2.3. The TQ-completion of an $\mathcal{O}$-algebra A is

$$A^\wedge_{\mathsf{TQ}} := \operatorname{holim}\Big(\mathsf{TQ} A \rightrightarrows (\mathsf{TQ})^2 A \Rrightarrow (\mathsf{TQ})^3 A \cdots\Big)$$

I am interested in the question, "When do we have a weak equivalence $A \xrightarrow{\sim} A^\wedge_{\mathsf{TQ}}$?" Ching-Harper [CH19] provide the following condition.

Theorem 2.4 ([CH19])**.** *If an $\mathcal{O}$-algebra A is 0-connected, then $A \xrightarrow{\sim} A^\wedge_{\mathsf{TQ}}$ is a weak equivalence.*

This theorem should be thought of as analogous to the result that simply connected spaces are $\mathbb{Z}$-complete. (Here, $\pi_0(A) = *$ rather than $\pi_1(X) = *$.) However, since Bousfield-Kan in fact prove that all *nilpotent* spaces are $\mathbb{Z}$-complete, this led me to ask the following.

1. If the reader is unfamiliar with the notion of "holim," it can be thought of as a black box used to piece together objects of a diagram, like the one boxed, in a homotopically meaningful way. Intuitively, holim's are easy to map *into*.
2. Hence, we have $\mathsf{TQ} \simeq \Omega^\infty \Sigma^\infty \simeq P_1(\mathrm{id})$, but more on this in the next section.

Question 2.5. Can the assumption that $\pi_0(A) = *$ be relaxed? Is there an interesting[3] class of TQ-complete $\mathcal{O}$-algebras that are *not* 0-connected?

A key step in Bousfield-Kan's proof is to understand when $(-)^\wedge_{\mathbb{Z}}$ preserves certain fibration sequences. Since there were no known analogues for $\mathcal{O}$-algebras, I began looking for classes of fibration sequences in $\mathsf{Alg}_{\mathcal{O}}$ that TQ-completion preserves, and proved [Sch] the following.

Theorem 2.6 (**S**, 2020)**.** *TQ-completion preserves fibration sequences $F \to E \to B$ of $\mathcal{O}$-algebras in which E, B are 0-connected. Furthermore, a similar statement holds if "fibration sequence" is replaced by "homotopy pullback diagram."*

My strategy of proof is to show that, under the conditions of the theorem, the map $F \xrightarrow{\sim} F^\wedge_{\mathsf{TQ}}$ is a weak equivalence; this gives an affirmative answer to Question 2.5. To argue that $F \xrightarrow{\sim} F^\wedge_{\mathsf{TQ}}$, we first consider the TQ-resolutions of E, B and then construct the coaugmented cosimplicial diagram $F \to \tilde{F}$ that is built by taking homotopy fibers at each cosimplicial degree. We then objectwise resolve $F \to \tilde{F}$ with respect to TQ-homology to construct the bicosimplicial diagram $(\mathsf{TQ})^{\bullet+1}\tilde{F}$. An extra codegeneracy argument implies that taking holim of this diagram "vertically" and then "horizontally" recovers F. To finish the argument, we use uniform cartesian estimates and the higher dual Blakers-Massey theorem of [CH16] to show that, on the other hand, taking holim "horizontally" and then "vertically" gives $F^\wedge_{\mathsf{TQ}}$. In fact, we are able to show that the map of abelian group towers

$$\{\pi_n F\} \to \left\{\pi_n \mathrm{holim}_{\Delta^{\leq s}} (\mathsf{TQ})^{\bullet+1} F\right\}_s$$

is a pro-isomorphism for each fixed n. The same overall strategy, with some extra cubical analysis, lets us replace "fibration sequence" by "homotopy pullback square."

Theorem 2.6 is a step in the right direction, but does not provide a complete characterization of TQ-complete $\mathcal{O}$-algebras, as Bousfield-Kan are able to give [BK72, III.5.3] for spaces. I would like to further pursue this avenue of research:

Question 2.7. Given an $\mathcal{O}$-algebra A, what are necessary and sufficient conditions for the map from A to its TQ-completion tower to induce a pro-isomorphism on all homotopy groups?

3. Functor calculus and connections to TQ-completion

The basic, beautiful, idea of functor calculus is this: in the same way that we can approximate a large class of functions $f\colon \mathbb{R} \to \mathbb{R}$ by a sequence of simpler Taylor polynomials $p_n f$, we can approximate sufficiently nice functors $F\colon \mathcal{C} \to \mathcal{C}$ (or, more generally, $\mathcal{C} \to \mathcal{D}$) by a sequence of simpler functors $P_n F$, which we think of as "Taylor approximations."

Let's make this a little more precise. If $f\colon \mathbb{R} \to \mathbb{R}$ is analytic at $x = 0$, we can expand it into a power series within a certain radius of convergence. With a little re-writing, we have

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \lim_n \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = \lim_n p_n f(x) \tag{2}$$

where $p_n f$ denotes the n^{th} Taylor polynomial of f.

Suppose now that rather than considering functions $f\colon \mathbb{R} \to \mathbb{R}$, we want to analyze functors[4] $F\colon \mathcal{C} \to \mathcal{D}$. To begin with, stable homotopy $\pi_n^s(X) \cong \pi_n(\Omega^\infty \Sigma^\infty X)$ provides a natural example of a **linear functor**, in that just as a linear *function* f satisfies $f(x+y) = f(x) + f(y)$, stable homotopy satisfies **excision**, and consequently has the following property.

$$\pi_n^s(X \amalg Y) \;\cong\; \pi_n^s(X) \oplus \pi_n^s(Y) \tag{3}$$

3. For instance, it is true for formal reasons that any S-module given the trivial $\mathcal{O}$-algebra structure is TQ-complete.
4. Strictly speaking, we consider *homotopy* functors, i.e., those which respect the notions of equivalence in their domain and target categories.

The key insight to generalize this idea to higher "degrees" is this: excision, and hence the "linearity" of stable homotopy, is a consequence of how it plays with certain 2-dimensional cubical diagrams,[5] i.e., that stabilization is *1-excisive* in the language of [Goo92]. To define n-**excisive functors**, we consider higher dimensional cubical diagrams—rather than just 2-cubes—and think of a functor as being "polynomial of degree n" (that is, n-excisive) if it has an effect on $(n+1)$-cubical diagrams analogous to that of stabilization on 2-cubical diagrams.

For any homotopy functor F, Goodwillie [Goo03] constructs universal n-excisive (or "n^{th}-degree polynomial") approximations P_nF of F which arrange in a tower as below.

$$\begin{tikzcd} & F \arrow[dl] \arrow[d] \arrow[dr] & & \\ P_1F & P_2F \arrow[l] & P_3F \arrow[l] & \cdots \arrow[l] \end{tikzcd} \tag{4}$$

The remarkable fact is that, just as in (2), we have $F(X) \simeq \lim_n P_nF(X)$ within a certain "radius of convergence." A similar story in $\mathsf{Alg}_{\mathcal{O}}$ is given in, for instance, [Per13a] and [Per13b].

The following contrast, however, stands out to me: in the *stable* setting of spectra, all of the structure maps in the Taylor tower of the identity functor are weak equivalences—analogous to the fact that, in classical calculus, the identity *function* and its Taylor series are indistinguishable. However, in the *unstable* setting of, e.g., pointed spaces, the Taylor tower of the identity encodes a wealth of nontrivial information; see, for instance, [AM99] and [Beh12]. For example, if the Taylor tower of the identity *does* converge on a space X, it interpolates between a rough, more computable approximation to X, namely its stabilization $\Omega^\infty\Sigma^\infty X$, and X itself. Goodwillie [Goo03] shows that the Taylor tower of the identity converges on 1-connected spaces, and it is a consequence of [CH16] that the Taylor tower of the identity converges on 0-connected $\mathcal{O}$-algebras.

I would like to answer, "What happens without these connectivity assumptions, e.g., outside of the radius of convergence?" If we "miss" our original target, what *is* the Taylor tower telling us about? For example, in the context of spaces, Arone and Kankaanrinta [AK98] show the following.

Theorem 3.1 ([AK98])**.** *If a space X is connected, then the Taylor tower of the identity functor on X always converges to $X^\wedge_{\Omega^\infty\Sigma^\infty}$. That is, $P_\infty(\mathrm{id})(X) \simeq X^\wedge_{\Omega^\infty\Sigma^\infty}$.*

I am very interested in the analogous story for $\mathcal{O}$-algebras, where it is known that $\mathsf{TQ} \simeq \Omega^\infty\Sigma^\infty \simeq P_1(\mathrm{id})$. In other words, I would like to provide an answer to:

Question 3.2. In the context of $\mathcal{O}$-algebras, what can we say about $P_\infty(\mathrm{id})X$ without assuming that X is 0-connected? For instance, does an analogue of Theorem 3.1 hold?

While this is the subject of ongoing work, I have proven the following result in this direction.

Theorem 3.3 (S)**.** *If an $\mathcal{O}$-algebra A is (-1)-connected and $\mathsf{TQ}(A)$ is 0-connected, then $P_\infty(\mathrm{id})A \simeq A^\wedge_{\mathsf{TQ}}$. Furthermore, without any connectivity assumptions on A, $P_\infty(\mathrm{id})(A)$ can always be recovered from its TQ-homology via a long TQ-completion tower.*

My argument for the first statement is motivated by [AK98]; the basic idea is that the Taylor tower of $(\mathsf{TQ})^k$ is easier to analyze than that of the identity functor. Taking advantage of this, we resolve the identity functor by TQ-homology, and consider the diagram $\mathcal{D}$ of cosimplicial Taylor towers $\{P_n(\mathsf{TQ})^k\}_{n,k}$. It follows from [Blo19, 7.1] that $\mathrm{holim}\mathcal{D} \simeq P_\infty(\mathrm{id})$, while the assumption that $\mathsf{TQ}(A)$ is 0-connected lets us analyze each tower in $\mathcal{D}$ individually to conclude that $\mathrm{holim}\mathcal{D} \simeq A^\wedge_{\mathsf{TQ}}$.[6] This gives a zig-zag of weak equivalences $P_\infty(\mathrm{id})(A) \simeq A^\wedge_{\mathsf{TQ}}$. Using the spectral sequence associated to the bar construction

5. More specifically, that stabilization sends homotopy pushouts to homotopy pullbacks.
6. The assumption that $\mathsf{TQ}(A)$ is 0-connected can be thought of as analogous to assuming, for a connected space X, that $\tilde{\mathbb{Z}}X$ is simply connected, i.e., that $\pi_1(X)$ is a perfect group.

[Har10, 1.1] that computes $\mathsf{TQ}(A)$, I also establish that $\mathsf{TQ}(A)$ is 0-connected if, for some $k \geq 2$, the map $\pi_0(\mathcal{O}[k] \wedge_{\Sigma_k} A^{\wedge k}) \to \pi_0(A)$ is a surjection.

To make precise the second statement, I consider a "long completion" construction inspired by [DD77]. The idea is that just as we iterate the map $A \to \mathsf{TQ}A$ to construct $A^{\wedge}_{\mathsf{TQ}}$, we can iterate the map $A \to A^{\wedge}_{\mathsf{TQ}} =: (\hat{T}_1\mathsf{TQ})A$. That is, we define

$$A \longrightarrow (\hat{T}_2\mathsf{TQ})A := \operatorname{holim}\Big((\hat{T}_1\mathsf{TQ})A \rightrightarrows (\hat{T}_1\mathsf{TQ})^2A \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} (\hat{T}_1\mathsf{TQ})^3A\cdots\Big) \tag{5}$$

and continue inductively. My strategy to prove the second part of Theorem 3.3 is to use the description (see [KP17] and [HH13]) of the Taylor tower of the identity in $\mathsf{Alg}_{\mathcal{O}}$ as a tower of nilpotent $\mathcal{O}$-algebras, $\tau_n\mathcal{O} \circ_{\mathcal{O}} (X)$. I show, using [CH18], that $\tau_n\mathcal{O} \circ_{\mathcal{O}} (X) \xrightarrow{\sim} (\hat{T}_2\mathsf{TQ})(\tau_n\mathcal{O} \circ_{\mathcal{O}} (X))$. Given this, I construct an on-the-nose retraction of $P_\infty(\mathrm{id})(A) \to (\hat{T}_2\mathsf{TQ})P_\infty(\mathrm{id})(A)$ and by an extra codegeneracy argument, this implies that $P_\infty(\mathrm{id})(A) \xrightarrow{\sim} (\hat{T}_3\mathsf{TQ})P_\infty(\mathrm{id})(A)$.

4. Future directions

In addition to the questions raised above, there are a number of projects on which I am currently working and would like to pursue in the future. The first of these is to establish a fibration theorem for $P_\infty(\mathrm{id})$, i.e., provide an answer to the following.

Question 4.1. Under what conditions on a fibration sequence $F \to E \to B$ in $\mathsf{Alg}_{\mathcal{O}}$ can we conclude that $P_\infty(\mathrm{id})F \to P_\infty(\mathrm{id})E \to P_\infty(\mathrm{id})B$ is again a fibration sequence?

The arguments I give for the case of TQ-completion do not immediately carry over to this setting; since it is unknown precisely when $P_\infty(\mathrm{id}) \simeq (-)^{\wedge}_{\mathsf{TQ}}$, the conditions on fibration sequences may well be different. However, the tools I use to prove Theorem 2.6 are powerful and remain available to attack Question 4.1.

The second project stems from the observation that TQ-completion can be equivalently thought of as completion with respect to $P_1(\mathrm{id})$. I am curious about what happens when we vary the "parameters."

Question 4.2. In both spaces and $\mathcal{O}$-algebras, what can we say (using the notation of (5)) about completion and localization with respect to $(\hat{T}_nP_m(F))$, for various choices of n, m, and F?

One would expect that, e.g., completion with respect to $P_n(\mathrm{id})$ "sees more" than completion with respect to $P_1(\mathrm{id})$, since $P_n(\mathrm{id})$ "sees more" than $P_1(\mathrm{id})$. Bousfield [Bou03] provides a general framework in which to study completion constructions, and in particular to compare different completions. This would provide a strategy of attack for spaces, and generalizing Bousfield's results to $\mathcal{O}$-algebras is another project I hope to pursue.

As I enter the next phase of my career, I also plan to diversify my research program and explore different areas of homotopy theory, such as the **chromatic point of view**. As evidenced by, e.g., [Kuh06] and [Kuh04], functor calculus interacts very nicely with chromatic homotopy theory. For instance, we would often like to know "how to build" P_nF from $P_{n-1}F$. The simplest case is if each P_nF splits as a product of its homogeneous layers, i.e., $P_nF \simeq \prod_{i=1}^n D_iF$, but this does not, generally, happen. However, Kuhn shows that one *does* obtain such a splitting for functors from spectra to spectra *after localization* with respect to, for example, $K(n)$. More succinctly: functor calculus plays nicely with chromatic localization. I am interested in the following.

Question 4.3. Are there conditions under which the Taylor tower of a functor from spectra to spectra splits after *completion* with respect to $K(n)$?

One starting point would be to first consider *long* completion, as in (5). That is, Dror and Dwyer [DD77] are able to show that transfinitely iterating the R-completion of a space eventually recovers its R-localization. If there are conditions under which a long $K(n)$-completion tower recovers $K(n)$-localization, this would, by Kuhn's result, give us the desired splitting for long completion. The goal then would be to find conditions under which we can obtain the splitting after, for instance, a finite number of iterations—or after just one iteration, namely, $K(n)$-completion.

5. Undergraduate research

A number of other problems in which I am interested are very much accessible to undergraduate students. The first of these is the game of Cops and Robbers on graphs, and stems from my own work as an undergraduate. The second deals with graph coloring, and arises from my experience volunteering as a mentor for an Ohio State REU in the summer of 2020.

Given a graph G, the game of Cops and Robbers is played as follows. Player 1 first places several "cops" on different vertices of G; Player 2 then places one robber on an unoccupied vertex. Each turn, all of the cops can move to an adjacent vertex, and then the robber can move to an adjacent vertex. As one might expect, the goal of the robber is to evade the cops, while on the other hand the cops win if one cop can move onto the vertex occupied by the robber. All parties involved play with perfect information. Among other questions in this area, I am interested in the following.

Question 5.1. Given a family (possibly infinite) of graphs G, how many cops are needed to guarantee that they will, in a finite number of turns, capture the robber?

The precise number of cops needed is known as the **cop number** of a graph, and there are many results that bound the cop number of different families of graphs. For instance, Aigner and Fromme [AF84] show that 3 cops always suffice for planar graphs. While I was an undergraduate, I participated in Michigan State University's SURIEM program, and we proved [BBG+17] the following.

Theorem 5.2 (Ball–Bell–Guzman–Hanson-Colvin–**S**, 2017)**.** *Given a generalized Petersen graph G, four cops always suffice to capture the robber.*

We argue by playing a modified version of the game on an infinite cyclic covering of G, where the goal is not to capture, but rather force the robber to move indefinitely in one direction. This strategy projects to a capture on G. To my knowledge, this is the first use of such a strategy, and I am interested in using it on other families of graphs.

Question 5.3. Which kinds of graphs G admit such a modified game? For what graphs G will forcing the robber to move indefinitely in one direction project to a capture on G? How many cops are needed to do so?

I am also interested in combining the combinatorics of Cops and Robbers with abstract algebra (for more advanced undergraduate students). More specifically:

Question 5.4. What is the cop number of different Cayley graphs G? How does changing the chosen generators of the associated group (and hence the Cayley graph) affect the cop number?

In addition to Cops and Robbers, I would like to further explore **graph coloring.** Given a set of colors $C := \{1, 2, \ldots, n\}$ and a graph G, a **coloring** of G is a function which assigns a color to each vertex of G. It is said to be **proper** if adjacent vertices are colored differently.

Definition 5.5. For a fixed graph G, let $\chi_G(q)$ be the number of proper colorings of G with q colors. It is known that $\chi_G(q)$ is a polynomial in q, and is called the *chromatic polynomial* of G.

During the REU for which I volunteered as a mentor, we studied a number of generalizations of the chromatic polynomial, such as [Sta95] Stanley's **symmetric chromatic polynomial** and [Tut54] the **Tutte-Whitney polynomial**. There are many avenues of potential research in this area, but one that interests me in particular is as follows. If two graphs G and H have the same chromatic polynomial, they are said to be **chromatically equivalent**. (Isomorphic graphs are necessarily chromatically equivalent, but not conversely.) We may also define a similar notion for the Stanley symmetric function and the Tutte-Whitney polynomial. I would like to explore how these different equivalence classes of graphs interact with each other. For instance, are there conditions under which two graphs being chromatically equivalent implies that they also have the same Stanley symmetric chromatic polynomial? What about their Tutte-Whitney polynomial? I suspect this is, in general, a very difficult problem, but restricting attention to certain infinite families of graphs could provide a foothold to begin work.

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