6/27/18

Announcements

- HW 3 due Thurs

Today

- Sum/diff formulas } 6.2
- Double angle formulas } 6.3
- Power reducing formulas } 6.3

Sum/diff formulas

Recall:

- $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$
- $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$
- $\tan(a+b) = \dfrac{\tan(a)+\tan(b)}{1-\tan(a)\tan(b)}$

To get diff. formulas, use $a-b = a+(-b)$

$\Rightarrow$ in formulas, $+ \rightsquigarrow -$ and $- \rightsquigarrow +$.

e.g.

① We can use sum/diff to verify identities.

e.g. Verify $\cos(x-\pi) = -\cos(x)$ is an identity

Proof: LHS $= \cos(x-\pi)$

$$\overset{\text{diff. form.}}{=} \cos(x)\underbrace{\cos(\pi)}_{-1} + \sin(x)\underbrace{\sin(\pi)}_{0}$$

$$= -\cos(x)+0 = -\cos(x) = \text{RHS}. \quad \square$$

e.g. (similar to HW)

(2) Find the exact value of $\cos\left[\sin^{-1}\left(\frac{-12}{37}\right) + \tan^{-1}\left(\frac{5}{12}\right)\right]$

just a big "("

We don't know right away what $\sin^{-1}\left(\frac{-12}{37}\right)$ or $\tan^{-1}\left(\frac{5}{12}\right)$ are.

Let $\alpha = \sin^{-1}\left(\frac{-12}{37}\right)$ and $\beta = \tan^{-1}\left(\frac{5}{12}\right)$

(replacing symbols w/ easier symbols).

Then we're looking for $\cos(\alpha + \beta)$. We know

(whatever α and β are) that

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta).$$

LEAVE UP

Analyze α

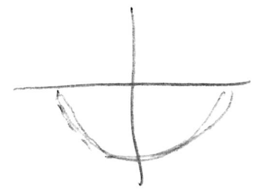
Well $\sin(\alpha) = \sin\left(\sin^{-1}\left(\frac{-12}{37}\right)\right) = \frac{-12}{37}$

(don't have to worry about range b/c "sin" is on outside, not "$\sin^{-1}$").

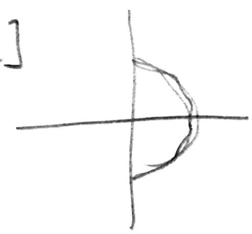
Also, α is $\sin^{-1}$(something), so α is in range of $\sin^{-1}(x)$ which is $[-\pi/2, \pi/2]$. So we know two things about α

① $\sin(\alpha) = \frac{-12}{37}$

- Since neg, α in Q III or Q IV

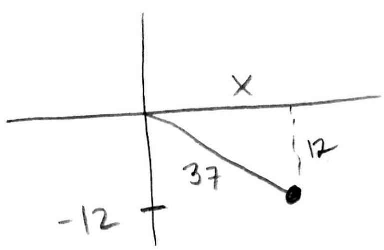


② α in range of $\sin^{-1}(x) \Rightarrow \alpha$ in $[-\pi/2, \pi/2]$



α has to be in both $\Rightarrow \alpha$ in Q IV.

and $\sin\alpha = \frac{-12}{37} = \frac{y}{r}$



Then $x^2 + 12^2 = 37^2 \Rightarrow x^2 = 1225 \Rightarrow x = \pm 35$

*Since we know α in Q IV * x is pos $\Rightarrow x = 35$.

Thus, $\cos\alpha = \frac{x}{r} = \frac{35}{37}$

Analyze β

Proceed similarly, find $\sin\beta = \frac{5}{13}$, $\cos\beta = \frac{12}{13}$

So

$$\cos(\alpha+\beta) = \frac{35}{37}\cdot\frac{12}{13} - \left(\frac{-12}{37}\right)\left(\frac{5}{13}\right)$$

$$= \frac{420+60}{37\cdot 13} = \boxed{\frac{480}{481}}$$

$\boxed{\text{Double angle formulas}}$

Motivating question(s): What are $\sin(2\theta)$, $\cos(2\theta)$, $\tan(2\theta)$?

We can figure out w/ sum formulas

(and then have a shortcut).

$\underline{\sin(2\theta)}$

Note that $2\theta = \theta + \theta$. Thus,

$$\sin(2\theta) = \sin(\overset{\text{"a"}}{\theta} + \overset{\text{"b"}}{\theta}) = \sin(a)\cos(b) + \cos(a)\sin(b)$$

$$= \sin\theta\cos\theta + \cos\theta\sin\theta$$

$$= 2\sin\theta\cos\theta.$$

So $\boxed{\sin(2\theta) = 2\sin\theta\cos\theta}$

$\underline{\cos(2\theta)}$

Using sum formula,

$$\cos(2\theta) = \cos(\theta+\theta) = \cos\theta \cdot \cos\theta - \sin\theta \cdot \sin\theta$$

$$= \cos^2\theta - \sin^2\theta.$$

$\underline{\text{So}}$ $\boxed{\cos(2\theta) = \cos^2\theta - \sin^2\theta}$

and since "$\cos^2\theta$" and "$\sin^2\theta$" appear, we can use Pyth. identities to get alternate expressions.

Recall $\sin^2\theta + \cos^2\theta = 1 \Rightarrow \cos^2\theta = 1 - \sin^2\theta$.

so $\underline{\text{also}}$, $\cos(2\theta) = (1-\sin^2\theta) - \sin^2\theta = \boxed{1 - 2\sin^2\theta}$

and similarly, $\cos(2\theta) = \cos^2\theta - (1-\cos^2\theta)$

$$= \boxed{2\cos^2\theta - 1}$$

$\underline{\tan(2\theta)}$

$$\tan(2\theta) = \tan(\theta + \theta) = \frac{\tan\theta + \tan\theta}{1 - \tan\theta \cdot \tan\theta}$$

$$= \boxed{\frac{2\tan\theta}{1 - \tan^2\theta} = \tan(2\theta)}$$

Notice: These formulas are a consequence of the sum formulas. And, in fact,

$$\underbrace{\text{sum form. for } \sin(x) \text{ and } \cos(x) \Rightarrow \text{sum form. for } \tan(x)}$$

$\Downarrow$

diff. formulas and double-angle formulas.

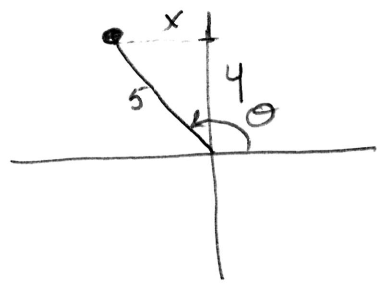
So they are all conceptually related.

(table p. 635)

e.g. ① Given that $\sin\theta = \frac{4}{5}$ for θ in Q II, find

$\sin(2\theta)$, $\cos(2\theta)$, and $\tan(2\theta)$.

Picture!



sin(2θ)

We know $\sin(2\theta) = 2\sin\theta\cos\theta$.

Find x: $x^2 + 4^2 = 5^2 \Rightarrow x^2 + 16 = 25 \Rightarrow x^2 = 9$

$\Rightarrow x = \pm 3$. From picture, $x = -3$.

So $\sin\theta = 4/5$ and $\cos\theta = \frac{x}{r} = \frac{-3}{5}$.

Therefore, $\sin(2\theta) = 2\sin\theta\cos\theta$

$= 2\left(\frac{4}{5}\right)\left(\frac{-3}{5}\right) = \boxed{\frac{-24}{25}}$

$\underline{\cos(2\theta)}$

We know $\cos(2\theta) = \cos^2\theta - \sin^2\theta$.

Since $\cos\theta = -3/5$ and $\sin\theta = 4/5$,

$$\cos(2\theta) = \left(-\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2 = \frac{9}{25} - \frac{16}{25} = \boxed{\frac{-7}{25}}$$

$\underline{\tan(2\theta)}$

We could use ~~fmula~~, but

$$\tan(2\theta) = \frac{\sin(2\theta)}{\cos(2\theta)} = \frac{-\frac{24}{25}}{-\frac{7}{25}}$$

$$= \frac{24}{\cancel{25}} \cdot \frac{\cancel{25}}{7} = \boxed{\frac{24}{7}}$$

(2) What is $\cos\left(\frac{\pi}{12}\right)$? (skip if needed)

Well, we know $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ and can write

"$\frac{\pi}{6}$" with "$\frac{\pi}{12}$"s (clever trick).

$$\underset{\substack{\uparrow \\ \text{something} \\ \text{we know} \\ \text{about}}}{\frac{\pi}{6}} = \frac{\pi}{12} + \frac{\pi}{12} = 2\Big(\underset{\substack{\uparrow \\ \text{what we want}}}{\frac{\pi}{12}}\Big)$$

$$\frac{\sqrt{3}}{2} = \cos\left(\frac{\pi}{6}\right) = \cos\left(2\left(\frac{\pi}{12}\right)\right) \overset{\text{alt. form}}{=} 2\cos^2\left(\frac{\pi}{12}\right) - 1$$

$$\Rightarrow \frac{\sqrt{3}}{2} + 1 = 2\cos^2\left(\frac{\pi}{12}\right)$$

$$\Rightarrow \frac{\sqrt{3}}{4} + \frac{1}{2} = \cos^2\left(\frac{\pi}{12}\right) \Rightarrow \cos\left(\frac{\pi}{12}\right) = \pm\sqrt{\frac{\sqrt{3}}{4} + \frac{1}{2}}$$

$\pi/12$ acute $\Rightarrow$ take pos.

So $$\boxed{\cos\left(\frac{\pi}{12}\right) = \sqrt{\frac{\sqrt{3}}{4} + \frac{1}{2}}}$$

Power reducing formulas

(won't focus on heavily, but good to know for calc)

In calc, might have $\int \sin^2 x \, dx = \int \sin x \cdot \sin x \, dx$

does not play well w/ multiplication of functions

Goal: re-write $\sin^2 x$ w/ lower power.

We know $\cos(2x) = 1 - 2\sin^2(x)$ (alt. form)

$\Rightarrow 2\sin^2(x) = 1 - \cos(2x)$

$\Rightarrow \boxed{\sin^2(x) = \frac{1-\cos(2x)}{2}} = \frac{1}{2} - \frac{\cos(2x)}{2}$

Then $\underbrace{\int \sin^2(x) dx}_{\text{hard}} = \underbrace{\int \frac{1}{2} - \frac{\cos(2x)}{2} dx}_{\text{easy.}}$

(full table p. 637)