6/29/18

Announcements

- HW 4 due Thurs
- Exam corrections due Thurs

Today

- Half-angle formulas
  - $\sin(\frac{\alpha}{2})$ and $\cos(\frac{\alpha}{2})$
  - $\tan(\frac{\alpha}{2})$

(lecture and examples, Tues. is 6.5)

Half-angle formulas

Motivation: We used sum formulas to figure out what, e.g. $\sin(2\alpha)$ is (double angle formulas). ($2 \cdot \alpha$)

Can we get a nice formula for $\sin(\frac{\alpha}{2})$? (half-angle)

$\sin(\frac{\alpha}{2})$ and $\cos(\frac{\alpha}{2})$

Recall that from double angle, we have

$$1-2\sin^2\theta = \cos 2\theta \quad (\text{for any } \theta).$$

Note: This is half of this, so LHS involves an angle that is half of the angle on RHS

Then $1-2\sin^2\theta = \cos(2\theta)$ $\Rightarrow 1 = \cos(2\theta) + 2\sin^2\theta$

$\Rightarrow 1-\cos(2\theta) = 2\sin^2\theta$

$$\Rightarrow \frac{1-\cos(2\theta)}{2} = \sin^2\theta$$ (power reducing)

$$\Rightarrow \sin\theta = \pm\sqrt{\frac{1-\cos(2\theta)}{2}}$$ For <u>any</u> angle θ.

Now suppose we're given an angle α. We were interested in $\sin(\frac{\alpha}{2})$. So, let $\theta = \frac{\alpha}{2}$. Then

This is just some angle.

$$\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos(2\cdot\frac{\alpha}{2})}{2}}$$

$$\Rightarrow \boxed{\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos(\alpha)}{2}}}$$

here is our half-angle formula for sine.

What about $\cos(\frac{\alpha}{2})$?

Proceed similarly. From double angle,

$$\cos(2\theta) = 2\cos^2\theta - 1 \quad \text{(alt. form)}$$

$$\Rightarrow 2\cos^2\theta = \cos(2\theta) + 1$$

$$\Rightarrow \cos^2\theta = \frac{\cos(2\theta) + 1}{2}$$

$$\Rightarrow \cos\theta = \pm\sqrt{\frac{\cos(2\theta) + 1}{2}}$$

Let $\theta = \frac{\alpha}{2}$ and we have

$$\cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{\cos(2\cdot\frac{\alpha}{2}) + 1}{2}}$$

so

$$\boxed{\cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos(\alpha)}{2}} \qquad \sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos(\alpha)}{2}}}$$

* Leave up *

(full table p. 638)

We'll decide + vs. − based on where (in which quadrant)

$\frac{\alpha}{2}$ is.

(Talk about tangent in a bit).

e.g. ① What is $\cos\left(\frac{\pi}{12}\right)$?

(Recall last time, we determined it was $\sqrt{\frac{\sqrt{3}}{4}+\frac{1}{2}}$.)

Note that $\frac{\pi}{12} = \frac{\overset{=\alpha}{\frac{\pi}{6}}}{2} = \frac{\alpha}{2}$. Take $\alpha = \frac{\pi}{6}$, and

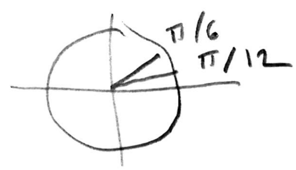
$$\cos\left(\frac{\pi}{12}\right) = \cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

$$= \pm\sqrt{\frac{1+\cos(\pi/6)}{2}}$$

$$= \pm\sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}}$$

$$= \pm\sqrt{\frac{1}{2}+\frac{\sqrt{3}}{4}}.$$

Since $\pi/12$ is acute,



take pos, So:

$$\boxed{\cos\left(\frac{\pi}{12}\right) = \sqrt{\frac{1}{2}+\frac{\sqrt{3}}{4}}}$$

(which agrees with what we had).

② If $\cos\alpha = \frac{-12}{37}$ and $\pi < \alpha < \frac{3\pi}{2}$, find exact values of $\sin(\frac{\alpha}{2})$ and $\cos(\frac{\alpha}{2})$.

$\underline{\sin(\frac{\alpha}{2})}$

Whatever else is true, $\sin(\frac{\alpha}{2}) = \pm\sqrt{\frac{1-\cos(\alpha)}{2}}$

and we know $\cos\alpha = \frac{-12}{37}$.

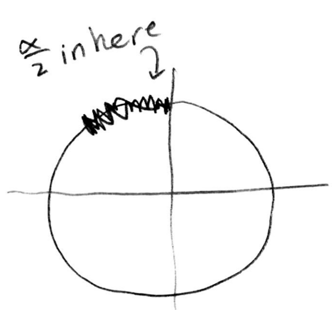
We just need to decide + vs. − .

Now, we do know $\pi < \alpha < \frac{3\pi}{2} \Rightarrow \alpha$ in Q III

BUT for + vs − , we care about $\frac{\alpha}{2}$, not α.

(We want to know if $\sin\frac{\alpha}{2}$ is pos or neg, not $\sin\alpha$).

Since $\pi < \alpha < \frac{3\pi}{2} \Rightarrow \frac{\pi}{2} < \frac{\alpha}{2} < \frac{3\pi}{4}$.



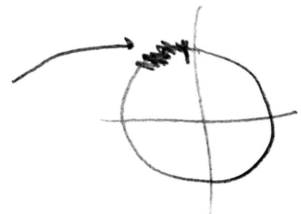
so $\sin(\frac{\alpha}{2})$ is pos.

(Note that $\sin(\alpha)$ is neg since α in Q III, so we really did need to look at $\frac{\alpha}{2}$)

$\underline{\text{So}}$ $\sin(\frac{\alpha}{2}) = +\sqrt{\frac{1-\frac{-12}{37}}{2}} = \sqrt{\frac{49}{74}} = \boxed{\frac{7}{\sqrt{74}}}$

$\underline{\cos\left(\frac{\alpha}{2}\right)}$

We've already done the work.

Know $\cos(\alpha) = -\frac{12}{37}$ and $\frac{\alpha}{2}$ in 

so $\cos\left(\frac{\alpha}{2}\right)$ is neg.

So $\cos\left(\frac{\alpha}{2}\right) = -\sqrt{\frac{1+\frac{-12}{37}}{2}} = -\sqrt{\frac{\frac{25}{37}}{2}}$

$= \boxed{-\frac{5}{\sqrt{74}}}$

Now let's talk about

$\underline{\tan\left(\frac{\alpha}{2}\right)}$.

We know $\tan\left(\frac{\alpha}{2}\right) = \frac{\sin\left(\frac{\alpha}{2}\right)}{\cos\left(\frac{\alpha}{2}\right)} = \frac{\pm\sqrt{\frac{1-\cos\alpha}{2}}}{\pm\sqrt{\frac{1+\cos\alpha}{2}}}$

$= \frac{\pm\frac{\sqrt{1-\cos\alpha}}{\sqrt{2}}}{\pm\frac{\sqrt{1+\cos\alpha}}{\sqrt{2}}} = \pm\frac{\sqrt{1-\cos\alpha}}{\cancel{\sqrt{2}}}\cdot\frac{\cancel{\sqrt{2}}}{\sqrt{1+\cos\alpha}}$

$$= \pm \frac{\sqrt{1-\cos\alpha}}{\sqrt{1+\cos\alpha}} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}.$$

add to others

So $$\boxed{\tan\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}}$$

With some clever algebra, you can show

$$\tan\left(\frac{\alpha}{2}\right) = \frac{1-\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1+\cos\alpha} \quad \text{(alt. forms)}$$

but I'd stick with (we know where it's coming from, and once we have $\cos\alpha$, just need to decide on sign by looking at where $\frac{\alpha}{2}$ is.

e.g. Suppose $\sin\alpha = \frac{33}{65}$ and $\frac{\pi}{2} < \alpha < \pi$.

what is $\sin(\frac{\alpha}{2})$, $\cos(\frac{\alpha}{2})$, $\tan(\frac{\alpha}{2})$?

To use formulas, we need to figure out:

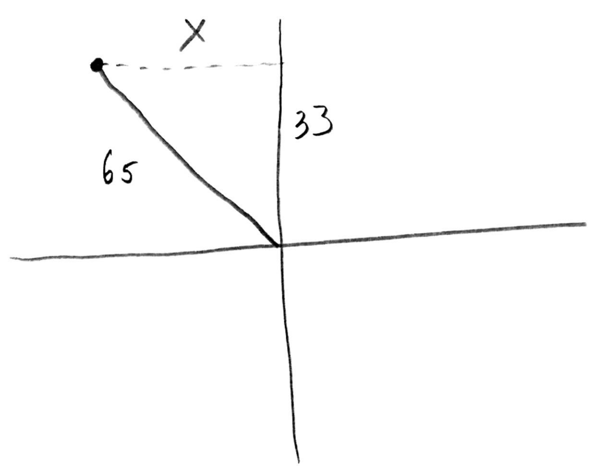
1. What is $\cos(\alpha)$?
2. Where is $\frac{\alpha}{2}$? (to decide + vs. −)

To find $\cos(\alpha)$ (not $\cos(\frac{\alpha}{2})$ yet) note

$\frac{\pi}{2} < \alpha < \pi \Rightarrow \alpha$ in Q II and

$\sin\alpha = \frac{33}{65} = \frac{y}{r}$

So



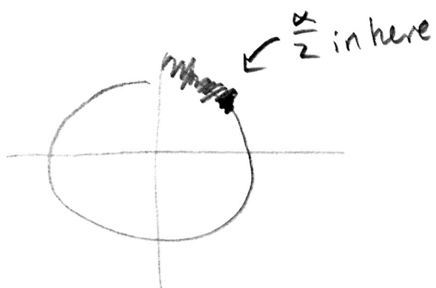
$x^2 + 33^2 = 65^2 \Rightarrow x^2 = 3136 \Rightarrow x = \pm 56$.

From picture, take $\underline{x = -56}$

So $\cos\alpha = \dfrac{x}{r} = \dfrac{-56}{65}$

Now, where is $\frac{\alpha}{2}$?

$$\frac{\pi}{2} < \alpha < \pi \implies \frac{\pi}{4} < \frac{\alpha}{2} < \frac{\pi}{2}$$



So $\sin(\frac{\alpha}{2})$ pos, $\cos(\frac{\alpha}{2})$ pos, $\tan(\frac{\alpha}{2})$ pos.

Now plug in to formulas:

- $\sin\left(\frac{\alpha}{2}\right) = \sqrt{\dfrac{1-\left(\frac{-56}{65}\right)}{2}} = \sqrt{\dfrac{121}{130}} = \boxed{\dfrac{11}{\sqrt{130}}}$

- $\cos\left(\frac{\alpha}{2}\right) = \sqrt{\dfrac{1+\left(\frac{-56}{65}\right)}{2}} = \sqrt{\dfrac{9}{130}} = \boxed{\dfrac{3}{\sqrt{130}}}$

- $\tan\left(\frac{\alpha}{2}\right) = \sqrt{\dfrac{1-\left(\frac{-56}{65}\right)}{1+\left(\frac{-56}{65}\right)}} = \sqrt{\dfrac{121}{9}} = \boxed{\dfrac{11}{3}}$

Notice: As usual, it would be easier to

think of $\tan\left(\frac{\alpha}{2}\right)$ as $\frac{\sin\left(\frac{\alpha}{2}\right)}{\cos\left(\frac{\alpha}{2}\right)}$ because then

$$\tan\left(\frac{\alpha}{2}\right) = \frac{\frac{11}{\sqrt{130}}}{\frac{3}{\sqrt{130}}} = \frac{11}{\cancel{\sqrt{130}}} \cdot \frac{\sqrt{130}}{3} = \boxed{\frac{11}{3}}$$

Recap (if time)

6.1: applying stuff we knew to simplify expressions/verify identities.

6.2: sum formulas:
$$\boxed{\begin{aligned} \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{aligned}}$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

These gave us the difference formulas as well.

6.3 • Double angle formulas ($\sin(2\theta)$, $\cos(2\theta)$, $\tan(2\theta)$) came from sum formulas by taking $\alpha = \beta = \theta$

• From double angle we got (power reducing and) half-angle formulas ($\sin\left(\frac{\alpha}{2}\right)$, $\cos\left(\frac{\alpha}{2}\right)$, $\tan\left(\frac{\alpha}{2}\right)$).