$\boxed{7/20/18}$

## Announcements

- HW 6 due next Wed
- Final Aug 1
  - Bonus points

## Today

- Dot products (8.5)
  - Definition(s)
  - Angle between vectors
  - Work

### Definition(s)

The dot product encodes some geometric content by turning two vectors into a scalar.

(Note: not really vector "multiplication" b/c get scalar, not vector)

Def If $\vec{v} = \langle a_1, b_1 \rangle$ and $\vec{w} = \langle a_2, b_2 \rangle$, their dot product

is $\vec{v} \cdot \vec{w} = a_1 a_2 + b_1 b_2$

vectors

numbers/scalars

⚠ $\vec{v} \cdot \vec{w}$ is a scalar, even though $\vec{v}$ and $\vec{w}$ are vectors e.g. Given $\vec{v} = \langle 3, -7 \rangle$ and $\vec{w} = \langle -2, 5 \rangle$, find

a) $\vec{v} \cdot \vec{w}$ b) $\vec{w} \cdot \vec{v}$ c) $\vec{v} \cdot \vec{v}$ d) $\|\vec{v}\|$

Sol a) $\vec{v} \cdot \vec{w} = \langle 3, -7 \rangle \cdot \langle -2, 5 \rangle = (3)(-2) + (-7)(5)$

$= -6 - 35 = \boxed{-41}$

Write the dot!

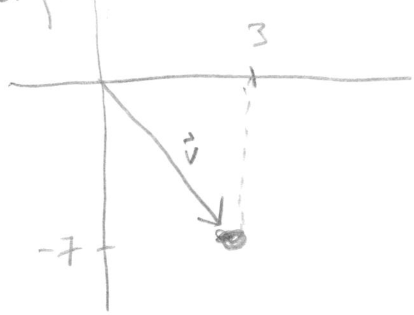
b) $\vec{w} \cdot \vec{v} = \langle -2, 5 \rangle \cdot \langle 3, -7 \rangle = (-2)(3) + (5)(-7)$

$= -6 - 35 = \boxed{-41}$

c) $\vec{v} \cdot \vec{v} = \langle 3, -7 \rangle \cdot \langle 3, -7 \rangle = (3)(3) + (-7)(-7)$

$= 9 + 49 = \boxed{58}$

d) $\|\vec{v}\|$ = length




$= \sqrt{3^2 + (-7)^2}$

$= \boxed{\sqrt{58}}$

Notice: At least in this example,

i) $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$

ii) $\|\vec{v}\|^2 = \vec{v} \cdot \vec{v}$

This is always true!

Claim For any $\vec{v} = \langle a_1, b_1 \rangle$ and $\vec{w} = \langle a_2, b_2 \rangle$,

$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$.

Proof: By def, $\vec{v} \cdot \vec{w} = \langle a_1, b_1 \rangle \cdot \langle a_2, b_2 \rangle$

$= a_1 a_2 + b_1 b_2$. But this is then equal to

$a_2 a_1 + b_2 b_1$, which by def is

$\langle a_2, b_2 \rangle \cdot \langle a_1, b_1 \rangle = \vec{w} \cdot \vec{v}$. $\square$

Claim For any $\vec{v} = \langle a, b \rangle$, $\|\vec{v}\|^2 = \vec{v} \cdot \vec{v}$.

Proof: We know $\|\vec{v}\| = \sqrt{a^2 + b^2}$, so

$\|\vec{v}\|^2 = a^2 + b^2$. Also, by def,

$\vec{v} \cdot \vec{v} = \langle a, b \rangle \cdot \langle a, b \rangle = (a)(a) + (b)(b) = a^2 + b^2$.

Thus, $\|\vec{v}\|^2 = \vec{v} \cdot \vec{v}$. $\Delta$

We can similarly show:

Properties of dot product; For any vectors $\vec{u}, \vec{v}$, and $\vec{w}$, and scalar $c$

① $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$

② $(c\vec{v}) \cdot \vec{w} = c(\vec{v} \cdot \vec{w}) = \vec{v} \cdot (c\vec{w})$

③ $\vec{v} \cdot (\vec{w} + \vec{u}) = \vec{v} \cdot \vec{w} + \vec{v} \cdot \vec{u}$

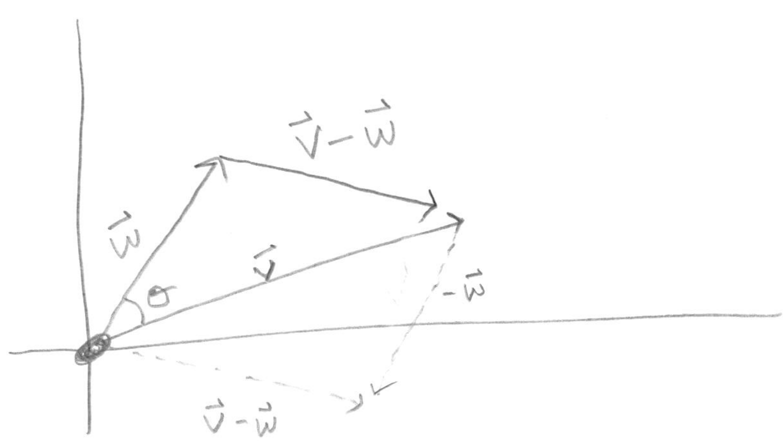
Good exercise!

④ $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$

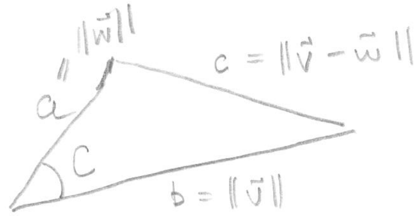
⑤ $\vec{0} \cdot \vec{v} = 0$

## Angle between vectors

Idea The dot product encodes the angle between two vectors.



Use Law of Cosines




$$\|\vec{v} - \vec{w}\|^2 = \|\vec{w}\|^2 + \|\vec{v}\|^2 - 2\|\vec{w}\|\,\|\vec{v}\| \cos\theta$$

$$\Rightarrow (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = RHS$$

$$\Rightarrow \overset{\|\vec{v}\|^2}{\vec{v} \cdot \vec{v}} - \vec{v} \cdot \vec{w} - \vec{w} \cdot \vec{v} + \overset{\|\vec{w}\|^2}{\vec{w} \cdot \vec{w}} = RHS$$

$$\Rightarrow \cancel{\|\vec{v}\|^2} - 2(\vec{v} \cdot \vec{w}) + \cancel{\|\vec{w}\|^2} = \cancel{\|\vec{w}\|^2} + \cancel{\|\vec{v}\|} - 2\|\vec{w}\|\,\|\vec{v}\| \cos\theta$$

$$\Rightarrow -\cancel{2}(\vec{v} \cdot \vec{w}) = -\cancel{2}\|\vec{w}\|\,\|\vec{v}\| \cos\theta$$

$$\Rightarrow \boxed{\vec{v} \cdot \vec{w} = \|\vec{v}\|\,\|\vec{w}\| \cos\theta}$$

This is another way to define dot product

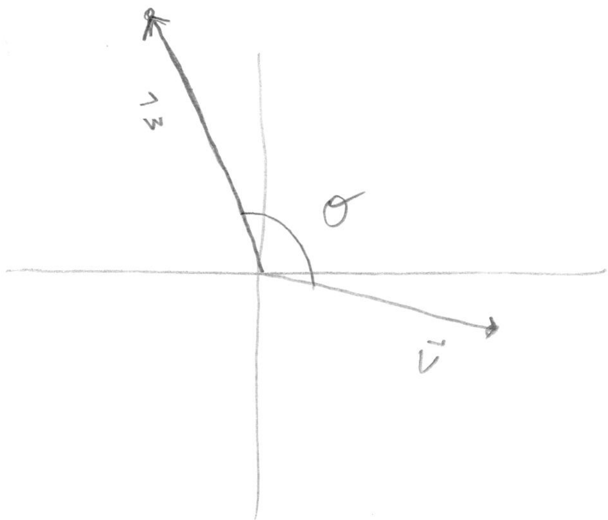
$$\Rightarrow \cos\theta = \frac{\vec{v} \cdot \vec{u}}{\|\vec{v}\|\,\|\vec{w}\|} \Longrightarrow \theta = \cos^{-1}\left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\,\|\vec{w}\|}\right)$$

as long as

$0 \le \theta \le \pi$

e.g. Find the angle between $\vec{v} = \langle 5, -1 \rangle$ and $\vec{w} = \langle -5, 12 \rangle$

Sketch



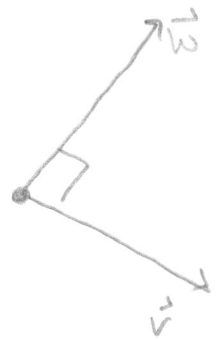

We know $\theta = \cos^{-1}\left( \dfrac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \, \|\vec{w}\|} \right)$ and

- $\vec{v} \cdot \vec{w} = \langle 5, -1 \rangle \cdot \langle -5, 12 \rangle = (5)(-5) + (-1)(12) = -37$
- $\|\vec{v}\| = \sqrt{5^2 + (-1)^2} = \sqrt{26}$
- $\|\vec{w}\| = \sqrt{(-5)^2 + (12)^2} = 13$

$\Rightarrow \theta = \cos^{-1}\left( \dfrac{-37}{13 \cdot \sqrt{26}} \right) = \boxed{123.9°}$

The dot product also tells us when two vectors form a right angle.

<u>Def</u>: Two vectors are <u>orthogonal</u> if they form a right angle, i.e.




Note: This would mean $\theta = 90°$, so

$$\vec{v} \cdot \vec{w} = \|\vec{v}\|\,\|\vec{w}\| \cos\theta = \|\vec{v}\|\,\|\vec{w}\| \underbrace{\cos(90°)}_{=0} = 0.$$

Prop: Two vectors $\vec{v}$ and $\vec{w}$ are orthogonal if and only if $\vec{v} \cdot \vec{w} = 0$.

e.g. Are the following vectors orthogonal, parallel, neither?

a) $\vec{v} = 4\hat{i} + 3\hat{j}$ and $\vec{w} = 6\hat{i} - 10\hat{j}$ b) $\vec{s} = -2\hat{i} + \hat{j}$ and $\vec{t} = 11\hat{i} + 22\hat{j}$

Sol Recall $\vec{v}$ is parallel to $\vec{w}$ if $\vec{v} = c\vec{w}$ for some $c$.

a) orthogonal?

$\vec{v} \cdot \vec{w} = \langle 4, 3 \rangle \cdot \langle 6, -10 \rangle = 24 - 30 = -6 \neq 0 \Rightarrow$ Not orthogonal.

Parallel?

We would need $\vec{v} = c\vec{w}$, so $\langle 4, 3 \rangle = c\langle 6, -10 \rangle$

i.e. $\langle 4, 3 \rangle = \langle 6c, -10c \rangle$.

If $6c = 4$, then $c = 2/3$. But then $-10c = -10\left(\frac{2}{3}\right) \neq 3$.

So there is no such $c$, so $\boxed{\text{neither}}$

b) orthogonal?

$\vec{s} \cdot \vec{t} = \langle -2, 1 \rangle \cdot \langle 11, 22 \rangle = -22 + 22 = 0$ ✓

so $\boxed{\text{orthogonal}}$

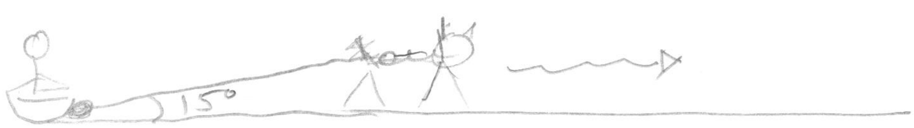
# Work

Work is defined as $\vec{F} \cdot \vec{D}$ (so is a scalar!)

More specifically

Def: If a constant force $\vec{F}$ (vector) moves an object in a straight line w/ displacement vector $\vec{D}$, then the work done on the object is $W = \vec{F} \cdot \vec{D}$.

e.g. A husky pulls a sled along a snowy field for 200ft w/ a constant force of 25 lbs directed 15° upward from the horizontal, how much work did she exert?




200 ft to the right

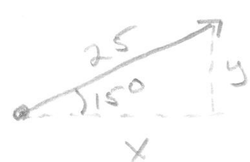
$\Downarrow$

$\vec{D} = \langle 200, 0 \rangle$

One way : $W = \vec{F} \cdot \vec{D} = \|\vec{F}\| \, \|\vec{D}\| \cos\theta$

$= 25(200)\cos(15°) \approx \boxed{4830 \text{ ft}\cdot\text{lb}}$

another way :




$x = 25\cos(15°) \approx 24.15$

$y = 25\sin(15°) \approx 6.47$

so $\vec{F} = \langle 24.15, 6.47 \rangle$

Then $\vec{W} = \vec{F} \cdot \vec{D} = \langle 24.15, 6.47 \rangle \cdot \langle 200, 0 \rangle$

$= 24.15(200) + 6.47(0) = \boxed{4830 \text{ ft}\cdot\text{lb}}$