ON THE CHROMATIC LOCALIZATION OF THE HOMOTOPY COMPLETION TOWER FOR $\mathcal{O}$-ALGEBRAS

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Abstract. The completion tower of a nonunital commutative ring is a classical construction in commutative algebra. In the setting of structured ring spectra as modeled by algebras over a spectral operad, the analogous construction is the homotopy completion tower. The purpose of this brief note is to show that localization with respect to the Johnson-Wilson spectrum $E(n)$ commutes with the terms of this tower.

1. Introduction

We work in the context of symmetric spectra [13], and consider any algebraic structure described by algebras over a reduced operad $\mathcal{O}$; that is, $\mathcal{O}[0] = *$ and hence $\mathcal{O}$-algebras are nonunital . (See [4] and [16] for other well-behaved categories of spectra.) To any $\mathcal{O}$-algebra X, there is an associated homotopy completion tower [11], analogous to the R-adic completion tower of a nonunital commutative ring R. The aim of this short paper is to show that localization with respect to the Johnson-Wilson spectrum $E(n)$ commutes with the terms of this tower.

To keep this paper appropriately concise, we freely use the notation of [11] in discussing homotopy completion in $\mathsf{Alg}_{\mathcal{O}}$. The basic idea of the construction is that the map of operads $\mathcal{O} \to \tau_k\mathcal{O}$ induces a Quillen adjunction

$$\mathsf{Alg}_{\mathcal{O}} \underset{F}{\overset{\tau_k\mathcal{O}\circ_{\mathcal{O}}-}{\rightleftarrows}} \mathsf{Alg}_{\tau_k\mathcal{O}} \tag{1}$$

which, for any $\mathcal{O}$-algebra X, gives rise to a tower $X \to \{\tau_k\mathcal{O} \circ_{\mathcal{O}} (X)\}_k$. If X is cofibrant, this construction is homotopical and is known as the homotopy completion tower of X. As shown in [15, 2.21] and [18, 4.3], this tower can be identified with the Taylor tower [7] of the identity functor on $\mathsf{Alg}_{\mathcal{O}}$. We further discuss this comparison in Section 4.

Throughout the paper, we let $E(n)$ denote the Johnson-Wilson spectrum [14] for a fixed prime p. A construction of $E(n)$ in symmetric spectra is given in [23, I.6.63]. It is a classical result [19] that the corresponding $E(n)$-localization of [2], denoted L_E, is smashing; this fact is crucial to our analysis. Much of our technical work involves constructing a well-behaved model S_E of the E-local sphere spectrum (see Theorem 1.5). Using this model, we obtain our main result, proven in Section 3.

Theorem 1.1. *Let $\mathcal{O}$ be a Σ-cofibrant operad in Sp^{Σ} and $\mathcal{U}$ the forgetful functor from symmetric to ordinary spectra. If X is a fibrant $\mathcal{O}$-algebra, then there is a weak equivalence*

$$L_E\big(\mathcal{U}\tau_n\mathcal{O} \circ^{\mathsf{h}}_{\mathcal{O}} (X)\big) \simeq \mathcal{U}\tau_n\mathcal{O} \circ^{\mathsf{h}}_{\mathcal{O}} (S_E \wedge X) \tag{2}$$

with $\mathcal{U}(S_E \wedge X) \simeq L_E(\mathcal{U}X)$.

Remark 1.2. As in [11, 3.3], the superscript "h" above is added to denote a suitably derived version of $\tau_n \mathcal{O} \circ_{\mathcal{O}} -$. Furthermore, although we have assumed Σ-cofibrancy, any operad can be replaced by a weakly equivalent Σ-cofibrant operad, and the induced map of homotopy completion towers is a weak equivalence; see, for instance, [11, 3.26, 5.48].

Remark 1.3. The first layer $\tau_1 \mathcal{O} \circ^{\mathsf{h}}_{\mathcal{O}} (X)$ of the homotopy completion tower of an $\mathcal{O}$-algebra X is the TQ-homology of X (see, e.g., [3], [9], [20], and [21]). Thus, under appropriate fibrancy conditions, Theorem 1.1 shows that TQ-homology commutes with $E(n)$-localization.

Theorem 1.1 follows without too much difficulty from Theorems 1.4 and 1.5, which we turn to now.

Theorem 1.4. *Let R be a commutative ring spectrum, i.e., a commutative monoid in Sp^Σ, for which the pairing map $R \wedge R \xrightarrow{\sim} R$ is a weak equivalence and suppose that R is cofibrant as a symmetric spectrum. If $\mathcal{O}$ is a Σ-cofibrant operad in Sp^Σ, then there is a zigzag of weak equivalences*

$$R \wedge (\tau_n \mathcal{O} \circ^{\mathsf{h}}_{\mathcal{O}} X) \simeq \tau_n \mathcal{O} \circ^{\mathsf{h}}_{\mathcal{O}} (R \wedge X) \tag{3}$$

in the underlying category Sp^Σ.

The proof of Theorem 1.4 is the content of Section 2. In Section 3, we construct S_E and prove the following, along with Theorem 1.1. Here, $\mathcal{RU}$ denotes the right-derived functor of $\mathcal{U}$.

Theorem 1.5. *There exists a commutative ring spectrum S_E with $S_E \wedge S_E \xrightarrow{\sim} S_E$ a weak equivalence, and which is cofibrant as a symmetric spectrum, with the following property: for any $X \in \mathsf{Sp}^\Sigma$, there is a natural isomorphism*

$$\mathcal{RU}(S_E \wedge X) \cong \mathcal{RU}(S_E) \wedge \mathcal{RU}(X) \cong L_E(\mathcal{RU}X) \tag{4}$$

in the stable homotopy category.

It is worth pointing out that, throughout this paper, we are considering the homotopy completion of $R \wedge X$ *as an $\mathcal{O}$-algebra.* It is known (see, e.g., [15, 2.1]) that the objectwise smash of an operad with a commutative ring spectrum is again an operad. Thus, one could also consider the homotopy completion of $R \wedge X$ as an algebra over the operad $R \wedge \mathcal{O}$ which has values in R-modules (where the symmetric monoidal product is $\wedge_R$). In this case, the result analogous to Theorem 1.4 follows from [11, 4.10] and [15, 2.11(d)].

Assumption 1.6. *Throughout this paper, $\mathcal{O}$ will denote a reduced operad in Sp^Σ. We assume, for all $n \geq 0$, that $\mathcal{O}[n]$ is (-1)-connected and that each map $I[n] \to \mathcal{O}[n]$ of the unit morphism $I \to \mathcal{O}$ is a flat stable cofibration between flat stable cofibrant objects in Sp^Σ. (Weaker than the Σ-cofibrancy condition needed for some of our results, this assumption guarantees that the forgetful functor $\mathsf{Alg}_{\mathcal{O}} \to \mathsf{Sp}^\Sigma$ preserves cofibrant objects* [11, 4.11]*.) Unless otherwise stated, we consider Sp^Σ and $\mathsf{Alg}_{\mathcal{O}}$ with their positive flat stable model structures* [11, Section 7]*. Here, we have followed Schwede's terminology* [23] *for what Shipley refers to* [24] *as the "positive S-model structure."*

Acknowledgements. The authors would like to thank John E. Harper, Duncan Clark, David White, and Yu Zhang for many useful conversations, and Mark

Behrens for several helpful correspondences. We are also indebted to Nick Kuhn for alerting us to the comparison between the homotopy completion tower and the Taylor tower of the identity functor in the absence of connectivity assumptions, as well as an enlightening discussion of this point. The second author was supported in part by Simons Foundation: Collaboration Grants for Mathematicians #638247 and AFOSR grant FA9550-16-1-0212.

2. Proof of Theorem 1.4

The purpose of this section is to prove the first technical result of the paper, Theorem 1.4. However, there are a few points that we need to address first. To begin, note that the statement of Theorem 1.4 implicitly uses the following fact. We suspect this is known to experts in the field (see, e.g., [15, 2.1]) but have included the statement and proof for the sake of completeness.

Proposition 2.1. *If X is an $\mathcal{O}$-algebra and R is a commutative ring spectrum, then $R \wedge X$ inherits an $\mathcal{O}$-algebra structure and the natural map $X \cong S \wedge X \to R \wedge X$ induced by the unit map of R is a map of $\mathcal{O}$-algebras.*

Proof. Let μ be the multiplication map of R and $\mathcal{O} \circ (X) \xrightarrow{\alpha} X$ the algebra structure map of X. The maps μ and α induce the following.

$$\begin{aligned} &\coprod_{n\geq 0} \mathcal{O}[n] \wedge (R \wedge X)^{\wedge n} \cong \coprod_{n\geq 0} \mathcal{O}[n] \wedge (R^{\wedge n} \wedge X^{\wedge n}) \xrightarrow{\mu_*} \\ &\xrightarrow{\mu_*} \coprod_{n\geq 0} \mathcal{O}[n] \wedge (R \wedge X^{\wedge n}) \cong R \wedge \coprod_{n\geq 0} \mathcal{O}[n] \wedge X^{\wedge n} \xrightarrow{\alpha_*} R \wedge X \end{aligned} \tag{5}$$

The fact that R is (strictly) commutative implies that this composite is Σ_n-equivariant and hence induces a map

$$\mathcal{O} \circ (R \wedge X) = \coprod_{n\geq 0} \mathcal{O}[n] \wedge_{\Sigma_n} (R \wedge X)^{\wedge n} \to R \wedge X \tag{6}$$

One then checks that this map is associative, unital, and compatible with the $\mathcal{O}$-algebra structure on X. $\square$

Corollary 2.2. *Given a commutative ring spectrum R, there is a natural map*

$$\mathcal{O} \circ (R \wedge X) \to R \wedge \big(\mathcal{O} \circ (X)\big) \tag{7}$$

Proof. The composite of the first three (Σ_n-equivariant) maps of (5) induces the desired map. $\square$

Though discovered independently, a result similar to the following appears in unpublished work by John E. Harper in collaboration with the first author and Yu Zhang.

Lemma 2.3. *Suppose $\mathcal{O}$ is as in Theorem 1.4. Let Y be a cofibrant $\mathcal{O}$-algebra and R a commutative ring spectrum for which the pairing map $R \wedge R \xrightarrow{\sim} R$ is a weak equivalence and which is cofibrant in Sp^{Σ}. Then the map in Corollary 2.2 is a weak equivalence.*

Proof. It follows from our assumptions and [11, 7.12] that the multiplication map $R^{\wedge n} \xrightarrow{\sim} R$ is a weak equivalence for any $n \geq 0$. Similarly, since Y was assumed to be cofibrant, we know that $Y^{\wedge n}$ is as well and so the map $R^{\wedge n} \wedge Y^{\wedge n} \xrightarrow{\sim} R \wedge Y^{\wedge n}$ is a weak equivalence. The assumption that $\mathcal{O}$ is Σ-cofibrant implies that $\mathcal{O}[n]$ is projectively cofibrant and so the functor $\mathcal{O}[n] \wedge_{\Sigma_n} -$ preserves weak equivalences. By considering the injective stable model structure (see [13, 5.3] and [23, III.4.13]) on Sp^Σ, in which every object is cofibrant, it follows that the map

$$\coprod_{n\geq 0} \mathcal{O}[n] \wedge_{\Sigma_n} (R^{\wedge n} \wedge Y^{\wedge n}) \xrightarrow{\mu_*} \coprod_{n\geq 0} \mathcal{O}n \wedge_{\Sigma n} (R \wedge Y^{\wedge n}) \tag{8}$$

is a weak equivalence. □

Remark 2.4. The previous proof remains valid if $\mathcal{O}$ is replaced by $\tau_n\mathcal{O}$.

We can now give the proof of Theorem 1.4.

Proof of Theorem 1.4. By replacing if necessary, it suffices to consider the case of a cofibrant $X \in \mathsf{Alg}_{\mathcal{O}}$. We construct a weak equivalence

$$|\mathrm{Bar}(\tau_n\mathcal{O}, \mathcal{O}, R \wedge X)| \simeq R \wedge |\mathrm{Bar}(\tau_n\mathcal{O}, \mathcal{O}, X)| \tag{9}$$

The proof is completed by appealing to [11, 4.10].

Our cofibrancy assumption on $\mathcal{O}$ ensures that the forgetful functor $\mathsf{Alg}_{\mathcal{O}} \to \mathsf{Sp}^\Sigma$ preserves cofibrant objects (see, e.g., [11, 4.11]). Hence, the fact that X is cofibrant implies that $\mathcal{O}^{\circ k} \circ (X)$ is cofibrant for any $k \geq 0$ by [9, 1.2]. Inductive application of Lemma 2.3, with $Y = \mathcal{O}^{\circ k} \circ (X)$, then shows that there is a levelwise weak equivalence

$$\mathrm{Bar}(\tau_n\mathcal{O}, \mathcal{O}, R \wedge X) \xrightarrow{\sim} R \wedge \mathrm{Bar}(\tau_n\mathcal{O}, \mathcal{O}, X) \tag{10}$$

Applying geometric realization and commuting with $R\wedge -$ completes the proof. □

3. Constructing S_E and the proof of Theorem 1.1

The purpose of this section is to construct a convenient model for the E-local sphere spectrum and prove the main result of the paper. In particular, we use [24] and [25] to find a symmetric spectrum S_E that satisfies the desirable properties listed in Propositions 3.5. With this model in hand, we then prove Theorem 1.5 and conclude with the proof of Theorem 1.1.

Remark 3.1. There is a subtlety in this section that we wish to highlight. In his original work [2], Bousfield did not have the luxury of working in a highly structured category of spectra. His localization construction, which we denote by L_E, is therefore slightly different than the localization in the sense of [12], which we implicitly use in constructing S_E. We denote by L the localization functor of [12] in Sp^Σ at the $E(n)$-equivalences. Using the comparisons established in [16], one can show that $\mathcal{U}L(S)$ is a model for $L_E(S)$, and hence the two are canonically weakly equivalent.

To construct S_E, first let $S^c \xrightarrow{\sim} S$ be a functorial cofibrant replacement of the symmetric sphere spectrum S in the model structure on commutative symmetric ring spectra established by [24, 3.2]. The following two lemmas are needed to apply the relevant result of [25] ultimately used to construct S_E.

Lemma 3.2. *L is a monoidal Bousfield localization in the sense of* [25, 4.4]

Proof. We appeal to [25, 4.6], noting that, when endowed with the positive flat stable model structure, the category Sp^Σ is a cofibrantly generated monoidal model category in which cofibrant objects are flat; see [24, 3.1] and [11, 7.12], respectively.

Let $f\colon A \to B$ be an $E(n)$-equivalence and K a cofibrant symmetric spectrum. Cofibrantly replacing A and B, we have that the map

$$E(n) \wedge A^c \xrightarrow{\sim} E(n) \wedge B^c \tag{11}$$

is a weak equivalence. Smashing with the cofibrant spectrum K preserves this weak equivalence and it follows that the map

$$E(n) \wedge^L (A \wedge K) \xrightarrow{\sim} E(n) \wedge^L (B \wedge K) \tag{12}$$

is a weak equivalence, i.e., that $A \wedge K \to B \wedge K$ is an $E(n)$-equivalence. □

Lemma 3.3. *The category of symmetric spectra with the positive flat stable model structure satisfies the rectification axiom of* [26, 4.5].

Proof. By considering symmetric sequences concentrated at 0, this follows from [10, 4.29*(b)]. □

Lemmas 3.2 and 3.3 now show that the conditions of [25, 7.2] are satisfied; hence, there is a commutative diagram

$$\begin{tikzcd} S^c \arrow[r, "(*)"] \arrow[d] & \widetilde{S^c} \\ L_E(S^c) \arrow[ur, "\sim"'] & \end{tikzcd} \tag{13}$$

in Sp^Σ, where $(*)$ is a map of commutative monoids. Replacing if necessary, we may also assume that $\widetilde{S^c}$ is fibrant.

Definition 3.4 (Construction of S_E)**.** We define S_E by factoring the map $(*)$

$$S^c \longrightarrow S_E \xrightarrow{\sim} \widetilde{S^c} \tag{14}$$

as a cofibration followed by an acyclic fibration in the model structure of [24, 3.2].

The following Proposition details the advantageous properties of this construction.

Proposition 3.5. *As defined, S_E is*

(i) is a strictly commutative monoid in Sp^Σ;
(ii) positive flat stable cofibrant and fibrant;
(iii) naturally weakly equivalent to the E-local sphere $L(S)$.

Proof. That S_E is a strictly commutative monoid and fibrant is immediate from its construction. It follows from [24, 4.1] that S^c is positive flat stable cofibrant and so S_E is as well. It is worth noting that, by [11, 7.12], the functor $S_E \wedge -$ preserves weak equivalences.

To see that S_E is naturally weakly equivalent to $L(S)$, consider the following commutative diagram

$$\begin{tikzcd} S \arrow[r, equal] \arrow[d, equal] & S \arrow[r, equal] \arrow[d] & S \arrow[r, equal] \arrow[d] & S \arrow[d] \\ S \arrow[d] & S^c \arrow[l, "\sim"'] \arrow[r] \arrow[d] \arrow[drr, "\sim"'] & S_E \arrow[r, "\sim"] & \widetilde{S^c} \\ L(S) & L(S^c) \arrow[l, "(\#)"'] & & \end{tikzcd} \tag{15}$$

in Sp^Σ, where the upper vertical maps are the unit maps associated to each commutative monoid comprising the second row. The fact that $S^c \xrightarrow{\sim} S$ is a weak equivalence implies that $(\#)$ is a weak equivalence as well, and this gives the zigzag of weak equivalences $S_E \simeq L(S)$. $\square$

Proof of Theorem 1.5. To begin, note that S_E is cofibrant by Proposition 3.5. We now show the equivalence (4) claimed in Theorem 1.5. Since S_E and $L(S)$ are both fibrant symmetric spectra, the weak equivalence $S_E \simeq L(S)$ induces an isomorphism on stable homotopy groups. (We remind the reader that, in general, weak equivalences of symmetric spectra do *not* necessarily induce isomorphisms on stable homotopy groups.) Hence, the weak equivalence remains such after forgetting to ordinary spectra. Together Remark 3.1, this shows that the we have natural isomorphisms

$$\mathcal{U}S_E \cong \mathcal{R}\mathcal{U}S_E \cong \mathcal{R}\mathcal{U}L(S) \cong \mathcal{U}L(S) \cong L_E(S) \tag{16}$$

in the stable homotopy category. Using the comparisons of [16, 0.3], the fact that L_E is smashing, and the naturality shown in (15), we then obtain a commutative diagram

$$\begin{tikzcd} \mathcal{R}\mathcal{U}X \arrow[r, "\eta\wedge\mathrm{id}"] \arrow[d] & \mathcal{R}\mathcal{U}S_E \wedge \mathcal{R}\mathcal{U}X \arrow[dl, "\cong"] \\ L_E(\mathcal{R}\mathcal{U}X) & \end{tikzcd} \tag{17}$$

in the stable homotopy category, where η is induced by the unit map $S \to S_E$ and the vertical map is the natural localization map [2] of $\mathcal{U}X$. This establishes (4).

Lastly, we show that the pairing map $S_E \wedge S_E \xrightarrow{\sim} S_E$ is a weak equivalence. By taking $X = S_E$ in (17), we see that the unit map $\mathcal{R}\mathcal{U}S_E \to \mathcal{R}\mathcal{U}S_E \wedge \mathcal{R}\mathcal{U}S_E$ is equivalent to $\mathcal{R}\mathcal{U}S_E \to L_E(\mathcal{R}\mathcal{U}S_E)$ and, hence, is a weak equivalence. It follows from [16, 0.3] that the corresponding map $S_E \xrightarrow{\sim} S_E \wedge S_E$ is a weak equivalence and the fact that there is a retract

$$\begin{tikzcd} S_E \arrow[r] \arrow[dr, "\mathrm{id}"'] & S_E \wedge S_E \arrow[d] \\ & S_E \end{tikzcd} \tag{18}$$

from the commutative monoid structure of S_E then completes the proof. $\square$

Proof of Theorem 1.1. It follows from Theorem 1.4 that there is a weak equivalence

$$\mathcal{RU}\big(S_E \wedge (\tau_n \mathcal{O} \circ^{\mathsf{h}}_{\mathcal{O}} X)\big) \simeq \mathcal{RU}\big(\tau_n \mathcal{O} \circ^{\mathsf{h}}_{\mathcal{O}} (S_E \wedge X)\big) \tag{19}$$

By [16, 0.3] and Theorem 1.5, the left hand side of (19) is weakly equivalent to $L_E\big(\mathcal{RU}(\tau_n \mathcal{O} \circ^{\mathsf{h}}_{\mathcal{O}} X)\big)$ which, since $\tau_n \mathcal{O} \circ^{\mathsf{h}}_{\mathcal{O}} X$ is fibrant, is weakly equivalent to $L_E(\mathcal{U}\tau_n \mathcal{O} \circ^{\mathsf{h}}_{\mathcal{O}} X)$. For the same fibrancy reason, the right hand side of (19) is weakly equivalent to $\mathcal{U}\tau_n \mathcal{O} \circ^{\mathsf{h}}_{\mathcal{O}} (S_E \wedge X)$. This establishes the first weak equivalence of Theorem 1.1.

To see that $\mathcal{U}(S_E \wedge X) \simeq L_E(\mathcal{U}X)$, recall that both S_E and X are fibrant, and that the former is also cofibrant. By [22, 4.10], $S_E \wedge X$ is semistable (see [13, 5.6.1]). It follows that $\mathcal{U}(S_E \wedge X)$ is weakly equivalent to $\mathcal{RU}(S_E \wedge X)$, the latter of which is weakly equivalent to $L_E(\mathcal{RU}X)$. Since X is fibrant, $L_E(\mathcal{RU}X) \simeq L_E(\mathcal{U}X)$, and this completes the proof. □

4. Identification of the Taylor tower

The purpose of this section is to show the following, which identifies the homotopy completion tower with the Taylor tower of the identity functor on $\mathcal{O}$-algebras.

Theorem 4.1. *For any cofibrant $\mathcal{O}$-algebra X, there is a weak equivalence*

$$P_n(\mathrm{id})(X) \simeq \tau_n \mathcal{O} \circ_{\mathcal{O}} (X) \tag{20}$$

A proof of this result is given by Pereira in [18, 4.3]. Another proof, partially based on Pereira's argument, also appears in [15, 2.21]. As [18] has not yet been published, we have included an alternative proof for the sake of completeness. The authors gratefully thank Nick Kuhn for discussing and outlining this different strategy. It is also worth noting that, if X is 0-connected, Theorem 4.1 can be obtained as a consequence of the connectivity estimates used to prove [11, 1.12].

To keep this section appropriately brief, we assume the reader is familiar with standard constructions in Goodwillie calculus, both in the context of spaces and $\mathcal{O}$-algebras (see, for instance, [5], [6], [7], [17], and [18]). In particular, when working in $\mathcal{O}$-algebras, one often implicitly (pre)composes with functorial (co)fibrant replacements to keep things homotopically meaningful.

The strategy of our proof is as follows. To begin, because P_n behaves particularly well when applied to spectrum-valued functors, it is advantageous to reduce Theorem 4.1 to proving the result when we consider the functors id and $\tau_n \mathcal{O} \circ_{\mathcal{O}} (-)$ as landing in Sp^{Σ}, which is accomplished by Lemma 4.3. Next, we consider the free-forgetful adjunction

$$\mathsf{Sp}^{\Sigma} \underset{U}{\overset{\mathcal{O}\circ(-)}{\rightleftarrows}} \mathsf{Alg}_{\mathcal{O}} \tag{21}$$

and analyze the n^{th} degree Taylor approximation to the free $\mathcal{O}$-algebra functor in Lemma 4.4. Lastly, to prove Theorem 4.1, we use the fact [9, 1.8] that the identity functor on $\mathsf{Alg}_{\mathcal{O}}$ can be resolved as the homotopy colimit of iterates of the free $\mathcal{O}$-algebra functor.

Remark 4.2. It is common to apply $\mathcal{O} \circ (-)$ to an $\mathcal{O}$-algebra, in which case one is implicitly precomposing with the forgetful functor U.

Lemma 4.3. *Suppose F and G are homotopy functors defined on, and with values in, $\mathcal{O}$-algebras. If a weak equivalence $F \simeq G$ induces an equivalence $P_n(UF) \simeq P_n(UG)$, then it also induces an equivalence $P_nF \simeq P_nG$.*

Proof. Since the positive flat stable model structure on $\mathsf{Alg}_{\mathcal{O}}$ is induced by the forgetful functor U, we know that homotopy limits in $\mathsf{Alg}_{\mathcal{O}}$ are calculated in the underlying category Sp^{Σ}. It follows from [8, 3.27] and [11, 4.11] that filtered homotopy colimits in $\mathsf{Alg}_{\mathcal{O}}$ are also calculated in the underlying category of spectra. Hence, the forgetful functor U commutes with the construction of P_n. $\square$

Lemma 4.4. *There is an equivalence of functors $P_n(U\mathcal{O} \circ (-)) \simeq U\tau_n\mathcal{O} \circ (-)$.*

Proof. Because finite products and coproducts agree in Sp^{Σ}, we have the following calculation

$$P_n\big(U\mathcal{O} \circ (-)\big) \cong P_n\Big(\coprod_{k=0}^{\infty} \mathcal{O}[k] \wedge_{\Sigma_k} (-)^{\wedge k}\Big) \simeq \coprod_{k=0}^{n} \mathcal{O}[k] \wedge_{\Sigma_k} (-)^{\wedge k} \cong U\tau_n\mathcal{O} \circ (-) \tag{22}$$

which gives the desired equivalence. $\square$

Proof of 4.1. It is explicitly shown in [15, 2.21] that $\tau_n\mathcal{O} \circ_{\mathcal{O}} (-)$ is n-excisive. The desired identification $P_n(\mathrm{id}) \simeq \tau_n\mathcal{O} \circ_{\mathcal{O}} (-)$ will therefore follow by showing an equivalence

$$P_n(\mathrm{id}) \simeq P_n\big(\tau_n\mathcal{O} \circ_{\mathcal{O}} (-)\big) \tag{23}$$

of functors $\mathsf{Alg}_{\mathcal{O}} \to \mathsf{Alg}_{\mathcal{O}}$.

The strategy now is to resolve the identity functor by iterates of the free $\mathcal{O}$-algebra functor $\mathcal{O} \circ (-)$. More explicitly, it follows from [9, 1.8] that we have a weak equivalence of functors

$$\mathrm{id} \simeq \operatorname{hocolim} \mathrm{Bar}(\mathcal{O}, \mathcal{O}, -) \colon \mathsf{Alg}_{\mathcal{O}} \to \mathsf{Alg}_{\mathcal{O}} \tag{24}$$

This equivalence holds after composing with the forgetful functor and applying P_n, i.e.,

$$P_n(U) \simeq P_n\big(U \operatorname{hocolim} \mathrm{Bar}(\mathcal{O}, \mathcal{O}, -)\big) \tag{25}$$

We then have the following weak equivalences of functors.

$$\begin{aligned}
&P_n(U) \simeq P_n\big(U \operatorname{hocolim} \mathrm{Bar}(\mathcal{O}, \mathcal{O}, -)\big) \overset{(1)}{\simeq} P_n\big(\operatorname{hocolim} U\, \mathrm{Bar}(\mathcal{O}, \mathcal{O}, -)\big) \\
&\overset{(2)}{\simeq} \operatorname{hocolim} P_n U\, \mathrm{Bar}(\mathcal{O}, \mathcal{O}, -) \overset{(3)}{\simeq} \operatorname{hocolim} P_n U\, \mathrm{Bar}(\tau_n\mathcal{O}, \mathcal{O}, -) \\
&\overset{(4)}{\simeq} P_n \operatorname{hocolim} U\, \mathrm{Bar}(\tau_n\mathcal{O}, \mathcal{O}, -) \overset{(5)}{\simeq} P_n\big(U \operatorname{hocolim} \mathrm{Bar}(\tau_n\mathcal{O}, \mathcal{O}, -)\big) \\
&\overset{(6)}{\simeq} P_n\big(U\tau_n\mathcal{O} \circ_{\mathcal{O}} (-)\big)
\end{aligned} \tag{26}$$

Equivalences (1) and (5) follow from [9, 1.6], while equivalences (2) and (4) follow from the fact [7, 1.7] that P_n commutes with homotopy colimits of spectrum-valued functors. Equivalence (3) follows from Lemma 4.4 and a straightforward generalization of [1, 3.1] to $\mathcal{O}$-algebras (see also [18, 4.9]). Lastly, equivalence (6) follows from [11, 4.10]. One can check that this zigzag is compatible with the natural map $\mathrm{id} \to \tau_n\mathcal{O} \circ_{\mathcal{O}} (-)$. Lemma 4.3 then gives the desired equivalence of (23), completing the proof. $\square$

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