7/6/18

Announcements

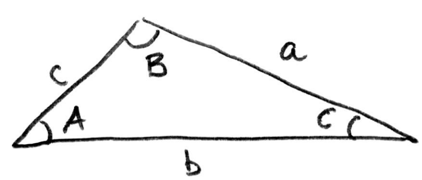
- HW 5 due 7/17 (it's long)
- Midterm 7/19
- Out of town next week (reachable by email, will have extra OH)

Today

- Law of Sines
  - AAS and ASA (easier)
  - SSA (hard)
    - 3 cases
- Area of a triangle given SAS.

Law of Sines

Given any triangle



(side a opposite angle A, etc.) the Law of Sines (LoS) says

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \left(\text{equivalently } \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}\right)$$

AAS and ASA

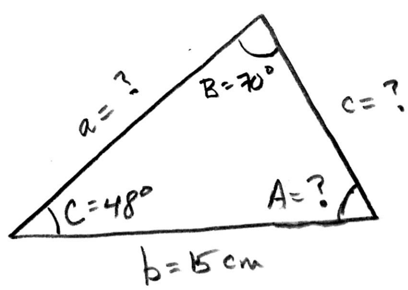
Idea: Given two angles and one side of a triangle, find the rest.

Good news: In these cases, the triangle is uniquely determined (only one possible).

e.g. Solve $\triangle ABC$ (find all angles and sides) with $B = 70°$, $C = 48°$, and $b = 15$ cm.

Sol

We have:



(case AAS)

① Find missing angle.

All angles must add to $180° \Rightarrow A + 70° + 48° = 180° \Rightarrow \boxed{A = 62°}$

② Use LoS to find missing sides:

$$\text{LoS} \Rightarrow \frac{b}{\sin B} = \frac{a}{\sin A} \Rightarrow \frac{15}{\sin(70°)} = \frac{a}{\sin(62°)} \Rightarrow a = \frac{15\sin(62°)}{\sin(70°)}$$

$$\Rightarrow \boxed{a \approx 14.1\text{cm}}$$

and $$\frac{c}{\sin 48°} = \frac{15}{\sin(70°)} \Rightarrow \boxed{c \approx 11.9 \text{ cm}}$$

SSA

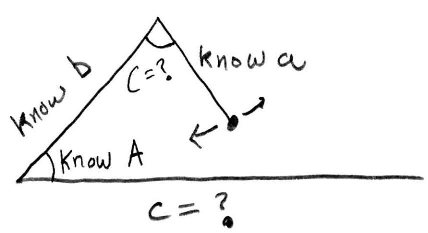
<u>Idea</u> Given one <u>angle</u> and <u>two sides</u> of a triangle, find the rest.

BAD news: In this case, there may be no triangle, one triangle, or two triangles possible.

(mnemonic: SSA is ASS)

<u>Three cases</u>: Suppose we are told A, a, and b

① No triangle:



No matter how we vary angle C and side c, cannot form a triangle.

e.g. Solve ΔABC with $A = 48°$, $a = 3$, $b = 6.2$.

If this "triangle" existed, LoS tells us:

$$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{3}{\sin(48°)} = \frac{6.2}{\sin(B)}$$

$$\Rightarrow \sin(B) \cdot \frac{3}{\sin(48°)} = 6.2$$

$$\Rightarrow \sin(B) = \frac{6.2 \sin(48°)}{3}$$

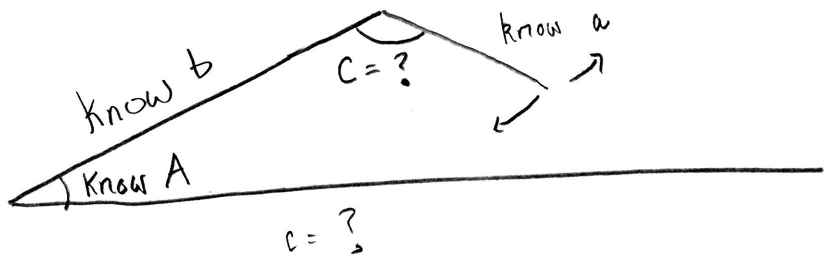
$$\Rightarrow \sin(B) = 1.54$$

But this is impossible! Range of $\sin(x)$ is $[-1, 1]$,

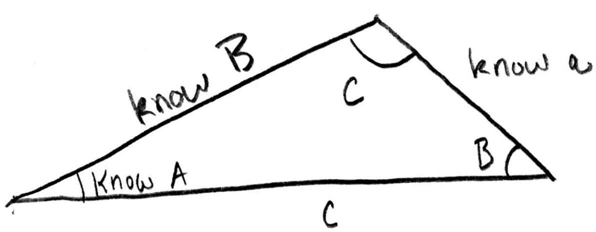
so there is no such B, so there is

$\boxed{\text{no triangle}}$

② Two triangles: Again, suppose we know A, a, and b.

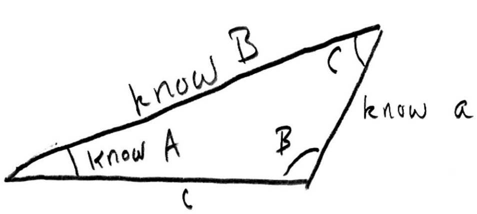


If we let angle C get smaller (and adjust side c):



Here's one possible triangle.

And we can let C get even smaller:



and here's another.

e.g. Solve $\triangle ABC$ with $A = 40°$, $a = 22$, and $b = 25$

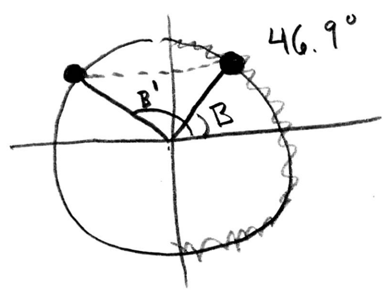
LoS $\Rightarrow \dfrac{a}{\sin A} = \dfrac{b}{\sin B} \Rightarrow \dfrac{22}{\sin(40°)} = \dfrac{25}{\sin(B)}$

$\Rightarrow 22 \sin(B) = 25 \sin(40°) \Rightarrow \sin(B) = \dfrac{25 \sin(40°)}{22}$

$\Rightarrow \sin(B) \approx 0.73$.

Calculator: $B = \sin^{-1}(0.73) \approx 46.9°$

But remember, calc. will only give you angles between $-\pi/2$ and $\pi/2$.



All we needed was $\sin(B) = 0.73$, so B' might also work, since $\sin(B') = 0.73$.

What is B'? From picture, it is $180° - 46.9° = 133.1°$.

So there are two possibilities for B, $46.9°$ and $133.1°$

Triangle 1

Take $B = 46.9°$. Since know $A = 40°$, have

$C = 180° - 46.9° - 40° = 93.1°$

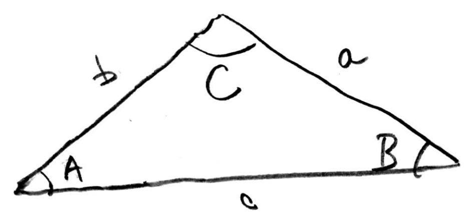
so $\boxed{B = 46.9°,\ A = 40°,\ C = 93.1°}$

Then use LoS to find side c:

We were given $\boxed{a = 22,\ b = 25}$ so $\dfrac{c}{\sin(93.1°)} = \dfrac{22}{\sin(40°)}$

$\Rightarrow \boxed{c \approx 34.2}$

sketch:



Triangle 2

Take $B = 133.1°$. Since know $A = 40°$ have

$C = 180° - 133.1° - 40° = 6.9°$, so

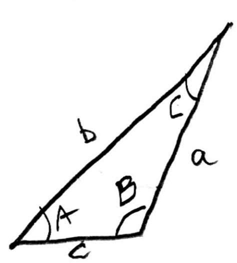
$\boxed{B = 133.1°,\ A = 40°,\ C = 6.9°}$

Then use LoS to find c:

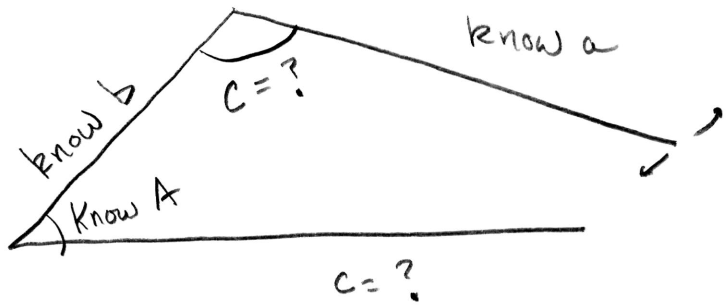
We were given $\boxed{a = 22,\ b = 25}$ so $\dfrac{c}{\sin(6.9°)} = \dfrac{22}{\sin(40°)}$

$\Rightarrow \boxed{c \approx 4.1}$

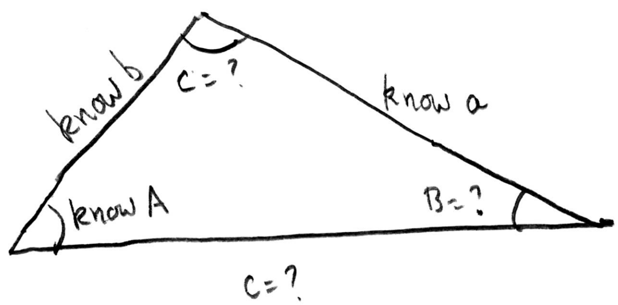
sketch



③ One triangle: Again, suppose we know A, a, and b.
"special case of 'two triangles' where one doesn't exist"

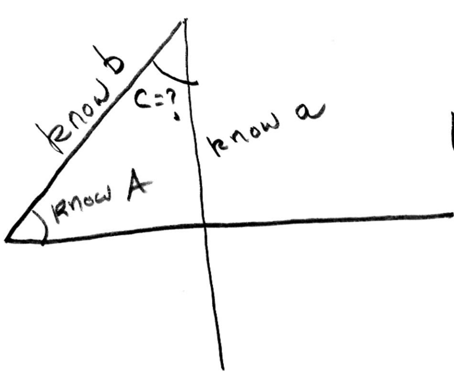


If we let angle C get smaller (and adjust side c)



Here's one possible triangle.

But if we let C get even smaller:



No triangle.

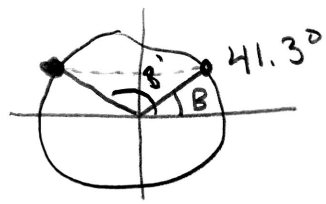
e.g. Solve $\triangle ABC$ with $A = 49°$, $a = 32$, and $b = 28$.

LoS $\Rightarrow \dfrac{32}{\sin(49°)} = \dfrac{28}{\sin(B)} \Rightarrow \sin(B) = \dfrac{28\sin(49°)}{32}$

$\Rightarrow \sin(B) \approx 0.66$

Calc $\Rightarrow B = \sin^{-1}(0.66) \approx 41.3°$

But again we have to check if there's another angle that would work:



$B' = 180° - 41.3° = 138.7°$

So our two possible B vals are $41.3°$ and $138.7°$

But hey! If $B = 138.7°$ and $A = 49° \Rightarrow A + B = 187.7°$ which is impossible (three angles add to $180°$).

So $\boxed{\text{B must be } 41.3°}$, and we know $\boxed{A = 49°,\ a = 32,\ b = 28}$

Then $C = 180° - 41.3° - 49° = \boxed{89.7° = C}$

and $\dfrac{c}{\sin(89.7°)} = \dfrac{32}{\sin(49°)} \Rightarrow \boxed{c = 42.4}$

Recap

- AAS and ASA = ☺ there is one unique triangle

- SSA = ☹ there are three possibilities:

Suppose given A, a, and b.

(1) Try to find B: $\frac{b}{\sin B} = \frac{a}{\sin A}$

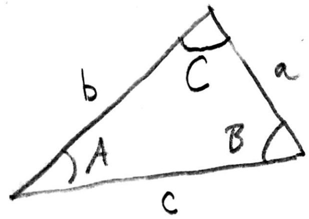
If get $\sin B$ = something impossible $\Rightarrow$ no triangle.

(2) If $B = 41°$, e.g. then $180° - 41° = 139°$ is another possibility.

If $A + 41°$ and $A + 139°$ both make sense ($< 180°$) you have two triangles. Otherwise, one is nonsense ($A + 139°$) and you have one triangle.

Compute area given SAS

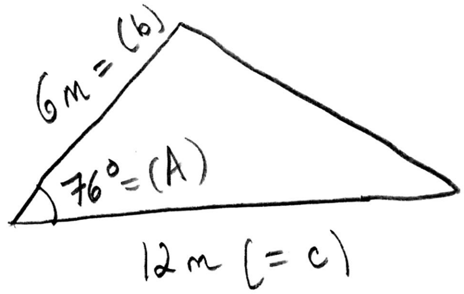
Consequence of LoS: For



$$\text{Area} = \frac{1}{2} bc \sin A = \frac{1}{2} ab \sin C = \frac{1}{2} ac \sin B.$$

e.g. A triangle has sides of 12m and 6m, and the angle between them is 76°. Find its area:

We have



(name them something)

$$\text{Area} = \frac{1}{2}(6)(12)\sin(76°) = \boxed{34.9\,m^2}$$