École Polytechnique Fédérale de Lausanne

On the locking phenomena for fibered materials

Master thesis

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Abstract

The linear elastic problem is one of the simplest problems if the material that we use has common properties such as compressibility and extensibility. When one of these two properties is not granted, many of the numerical approximation schemes for boundary value problems (BVP) suffer from locking phenomena.
In this project we study the linear elastic BVP of an isotropic material reinforced with quasi-inextensible fibers.
In order to solve the locking phenomena on a material with such properties we split the strain energy function into the isotropic and anisotropic contributions, and we add one additional field in order to represent the material extension along the fibers direction. This procedure leads us to a saddle point problem.
To achieve the results we present later we studied different discretisation schemes and their optimal approximation properties and we recovered the hypothesis on the existence of a solution in different types of weak formulation of the problem. The last part was the implementation of these methods using the Matlab package GEOpdes along with some new other function created specifically for our examples.

Chapter 1

Introduction

In the last years many fields had to deals with fibered materials with quasi-inextensibility property, one of them is the biology. In particular biologist had to model the theoretical and numerical modeling of the mechanical behavior of soft biological tissues in different conditions. The creation of a numerical model of these cases allowed them to predict the behavior of organs or biological ensembles.

1.1 Continuum mechanics

We provide a summary of the continuum mechanics background for the formulation of problems involving anisotropic materials in linear elasticity. For this part we assume that the material is not extensible along the given fiber direction a.
The Saint Venant–Kirchhoff constitutive law help us to define the isotropic part of the strain energy function (Peter Wriggers 2008)

$$\psi^{iso} = \frac{1}{2}(2\mu(\nabla^s u)^2 + \lambda(div(\nabla^s u))^2) \tag{1.1}$$

where μ and λ are the Lamé constants and $\nabla^s u = \frac{1}{2}(\nabla u + (\nabla u)^T)$ is the symmetric gradient of the displacement u.
The anisotropic part of the strain energy function must take into account the direction of the fibers and the elastic coefficient E_f of the fibers. Let us define the tensor $M = a \otimes a$, it follows that

$$\psi^{aniso} = \frac{1}{2}E_f(\nabla^s u \colon M)^2 = \frac{1}{2}E_f(J_4(u))^2 \tag{1.2}$$

1.2 Problem

This work is focused on the creation of a computational framework for the simulation of the displacement of anistropic materials nearly inextensible due to the presence of fibers. The

application of this type of simulations can be found in fields such as bio-mechanics or fibre-reinforced rubber-like materials.
Let us define the Cauchy stress tensor for an isotropic material

$$\sigma(u) = 2\mu\varepsilon(u) + \lambda div(\varepsilon(u)). \tag{1.3}$$

To (1.3) we add the anisotropic part of the fibers to obtain

$$\sigma_{an}(u) = \sigma(u) + E_f J_4(u)M. \tag{1.4}$$

We are ready to define the weak formulation of our elasticity problem: find $u \in \mathcal{V} = \{v \in H^1(\Omega)$ such that $v\,|_{\Gamma_D} = g\}$ such that

$$\int_\Omega [(\sigma_{an}(u)) \colon \varepsilon(v)] = \int_\Omega (f \cdot v) - \int_{\Gamma_N} (dv), \forall v \in \mathcal{V} \tag{1.5}$$

where $g \in L^2(\Gamma_D)$ and $d \in L^2(\Gamma_N)$, $\Gamma_N \cup \Gamma_D = \delta\Omega$ and $\Gamma_N \cap \Gamma_D = \emptyset$, $n = n(x)$ is the orthonormal vector to each point $x \in \Gamma_N$ and $f \in L^2(\Omega)$.
There exists different approaches to formulate and solve a finite element method (FEM) for these types of material. The problems we find when we use these methods can be traced back to the ratio between stiffness coefficients of the fibers and the material (P. Wriggers, J. Schröder and Auricchio 2016). The one that we are going to study and fix is the locking phenomenon. In particular we will see that the solution of a classical low order FEM does not converge to the right solution of the discretised problem when we refine the mesh.

1.3 LOCKING PHENOMENON

In order to have a better comprehension of the locking phenomenon we study another problem that gives us a visual explanation of the problem (I. Babuska 1992).
Let us study a classical linear elasticity problem in 1D for a nearly incompressible material. We need to find the solution u that satisfies

$$\begin{cases} -div(\sigma(u)) = f \text{ , in } \Omega = [0,1] \\ u = u_0 \text{ , on } \partial\Omega \\ \sigma(u) = 2\mu\varepsilon(u) + \lambda div(\varepsilon(u))I_d \end{cases} \tag{1.6}$$

with the constant $\lambda \to +\infty$. If we solve the problem using a low order FEM we will find a solution u_h that does not converge to u when $h \to 0$. More generally we observe this behaviour when $\lambda/\mu >> 1$. This behaviour is caused by the choice of the degree of the polynomial approximation, supposing an inextensible material, we can suppose that the divergence of the symmetric gradient $div(\varepsilon(u))$ will be near to zero. This implies that the left side of (1.6) is $-div(\sigma(u)) = -div(2\mu\varepsilon(u))$. When we transform the problem in its weak formulation and we discretize it with polynomials of degree 1, the unique solution of this problem will be the null solution everywhere. Id the material is nearly inextensible the problem tend to model the material more rigid that in the reality, so it compute a smaller displacement respect to the real one.

If we want to use a classical FEM we can find a true approximation of the real solution increasing the order of the polynomial space, but this augmentation of complexity will reflect also in an augmentation of running time and this is not always feasible.
In our case the locking phenomenon shows up when we have very rigid fibers in diagonal direction and the displacement is not in the direction of the fibers.

CHAPTER 2

THE FINITE ELEMENT METHOD(FEM)

The FEM is a numerical method to solve partial differential equations (PDE)

$$L(u) = f \tag{2.1}$$

where L is an operator and f is a function. To solve a PDE using the FEM we need to transform (2.1) in its weak form: Find $u \in \mathcal{V}$ such that

$$a(u, v) = F(v)\ , \forall v \in \mathcal{V} \tag{2.2}$$

where the space $\mathcal{V}$ is an Hilbert space defined using the boundary conditions of (2.1), $a : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}$ is a continuous and coercive bilinear form and $F : \mathcal{V} \rightarrow \mathbb{R}$ is a continuous linear form. To apply the FEM we need another step: the discretisation of the space $\mathcal{V}$. Using the polynomial approximation of the functions laying in the space and a parmeter $h > 0$ we can define another space $\mathcal{V}_h \subset \mathcal{V}$ of finite dimension.
We use the FEM to solve a problem of the form: find $u_h \in \mathcal{V}_h$ such that

$$a(u_h, v_h) = F(v_h)\ , \forall v_h \in \mathcal{V}_h. \tag{2.3}$$

The existence of the solution as well as the convergence of the solution of (2.3) to the solution of (2.2) when $h \rightarrow 0$ are assured by theorems (Quarteroni 2009a).

2.1 THE MESH

Let $\Omega \subset \mathbb{R}^d$ be a domain, we call $\mathcal{T}_h$ a polyhedral conformal mesh if $\mathcal{T}_h$ is the union of a finite number of polyhedra K_i that respect two properties:

- $\bar{\Omega} = \bigcup_{K_i \in \mathcal{T}_h} K_i$
- $\mathring{K}_i \cap \mathring{K}_j = \emptyset$ if $i \neq j$

In this case we call K_i an element of the mesh. The most popular shapes of the elements used to divide the domain are triangle and quadrilateral in two dimensions, and hexahedra and tetrahedra in three dimensions.

2.1.1 REFERENCE ELEMENT

For every different element's shapes we can define the reference element $\widehat{K}$:

- for triangular elements: the vertices of $\widehat{K}$ are (0,0), (1,0) and (0,1)
- for quadrilateral elements: the vertices of $\widehat{K}$ are (0,0), (1,0), (1,1) and (0,1)
- for tetrahedral elements: the vertices of $\widehat{K}$ are (0,0,0), (1,0,0), (0,1,0) and (0,0,1)
- for hexahedral elements: $\widehat{K}$ is the unite cube $[0,1]^3$

The utility of these elements will be clarified in the next part of this chapter but for our cases we use only the first two types of elements and for this reason we focus only on triangular and quadrilateral elements.

2.2 BASIS FUNCTIONS

2.2.1 CONTINOUS FINITE ELEMENTS ON A TRIANGULAR MESH

Let suppose we want to find a numerical solution of a problem using the FEM of order r using a triangular mesh $\mathcal{T}_h$. We denote by $\mathbb{P}_r$ the space of polynomials of global degree less than or equal to r, for $r = 1, 2, ...$.The definition of this space is:

$$\mathbb{P}_r = span\{x_1^{r_1} \cdots x_d^{r_d} \colon \sum_{j=1}^{d} r_j \leq r, r_j \geq 0\}. \tag{2.4}$$

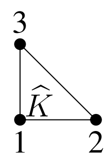
It follows that the dimension of $\mathbb{P}_r$ is

$$\binom{r+d}{d}.$$

In the FEM we look for a numerical solution u_h in a mesh $\mathcal{T}_h$ that is a piecewise polynomial approximation of u. In particular, if we use a method of order r we want

$$u_h \in X_h^r = \{v \in C^0(\Omega) \colon v|_K \in \mathbb{P}_r(K)\ \forall K \in \mathcal{T}_h\}. \tag{2.5}$$

It is easy to see that this new space is finite dimensional and we can recover the basis functions of it but another important property is that $X_h^r \subset H^1(\Omega)$.
Let us focus on the recovering of the basis functions of the space X_h^r, since u_h must be in polynomial form in every element K of the mesh it follows that the basis of the space is the union of the basis of each K. This method presents some issue, so we use another type of approach. We fix $r = 1$ and define the basis functions on the reference element $\widehat{K}$:



in this case we need to define a set of functions $\{\widehat{\phi}\}_{i=1}^{3}$ such that for each vertex P_i of $\widehat{K}$ we have $\widehat{\phi}_i(P_j) = \delta_{ij}$. The functions are:

$$\begin{cases} \widehat{\phi}_1(x,y) = (1-x)(1-y) \\ \widehat{\phi}_2(x,y) = x \\ \widehat{\phi}_3(x,y) = y. \end{cases}$$

Moreover, we introduce a simple discretised problem as example:
Find $u_h \in \mathcal{V}_h = \{v_h^1h$ such that $v_h|_{\delta\Omega} = 0\}$

$$\int_\Omega \nabla u_h \nabla v_h = \int_\Omega f v_h \text{ , } \forall v_h \in \mathcal{V}_h \tag{2.6}$$

To solve (2.6) we need to compute different integrals on each element $K \in \mathcal{T}_h$ but we also know that for every K there exists a unique map F_K such that

$$\forall x \in K \; x = F_K(\widehat{x}) = B_K\widehat{x} + b_K \text{ with } \widehat{x} \in \widehat{K}, \tag{2.7}$$

for a matrix $B_K \in \mathbb{R}^{d\times d}$ and a vector $b_K \in \mathbb{R}^d$. It follows that $K = F_K(\widehat{K})$. Combining the two previous results we can express the integral in (2.6) using only the basis in the reference element and the map F_K. Let $\{\phi_j^K\}j = 1^3$ be the basis functions of the space X_h^r non-null in the element K, it follows

$$\int_K \nabla\phi_i^K \nabla\phi_j^K = \int_{\widehat{K}} B_K^{-T}\nabla\widehat{\phi}_i B_K^{-T}\nabla\widehat{\phi}_j \cdot |det(B_K)|.$$

If we go back to (2.6) we find that the integral on a particular element K can be defined as:

$$\int_K \nabla u_h \nabla v_h = \sum_{i=1}^{3}\sum_{j=1}^{3} u_i v_j \int_{\widehat{K}} B_K^{-T}\nabla\widehat{\phi}_i B_K^{-T}\nabla\widehat{\phi}_j \cdot |det(B_K)|. \tag{2.8}$$

2.2.2 DISCONTINUOUS FINITE ELEMENTS ON A TRIANGULAR MESH

In some cases the continuity requirement is not necessary and we want to remove it in order to use a wider space. For this reason we can construct a new space of discontinuous finite element functions

$$X_{h,dc}^r = \{v \in L^2(\Omega) : v|_K \in \mathbb{P}_r(K) \; \forall K \in \mathcal{T}_h\}. \tag{2.9}$$

We have two important differences between this space and the previous one: the first is the number of basis functions needed to define it. As a matter of fact the space of discontinues functions has a bigger dimension

$$dim(X_{h,dc}^r) = \binom{r+d}{d} N_{el}. \tag{2.10}$$

The second is that this space is no more a subspace of $H^1(\Omega)$ but it contains only square integrable functions, i.e. $X_{h,dc}^r \subset L^2(\Omega)$.
In the examples we will use a space of this type with $r = 0$, this means that we will approximate some type of function by piecewise constant functions. In this case the number of basis functions will be same as the number of elements in the mesh.

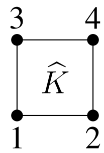
2.2.3 CONTINUOUS FINITE ELEMENTS ON A QUADRILATERAL MESH

Let $\mathcal{T}_h$ be a quadrilateral mesh, we can define $\mathbb{Q}_r(K)$ as the tensor product of polynomials of degree $\leq r$ along each direction in an element $K \in \mathcal{T}_h$.
Definition 1. *The space of continuous Finite Elements of degree r over a quadrilateral mesh $\mathcal{T}_h$ is defined for $r \geq 1$ as*

$$Y_h^r = \{v \in C^0(\Omega)\colon v|_K \circ F_K \in \mathbb{Q}_r(\widehat{K})\ \forall K \in \mathcal{T}_h\}.$$

As before, the fact that $Y_h^{r1}(\Omega)$ is easy to prove, and also the property of finite dimension space is still true.

Let us suppose we are on the reference element $\widehat{K}$=$[0,1]^2$ and we want to define the space $\mathbb{Q}_1(\widehat{K})$.



In this case we need to define a set of functions $\{\widehat{\phi}\}_{i=1}^4$ such that for each vertex P_i of $\widehat{K}$ we have $\widehat{\phi}_i(P_j) = \delta_{ij}$. In this case the functions are:

$$\begin{cases} \widehat{\phi}_1(x,y) = (1-x)(1-y) \\ \widehat{\phi}_2(x,y) = x(1-y) \\ \widehat{\phi}_3(x,y) = (1-x)y \\ \widehat{\phi}_4(x,y) = xy. \end{cases}$$

Also in this case we want to find a method that uses the basis functions on the reference element $\widehat{K}$ to compute the integral on every element K of a mesh $\mathcal{T}_h$. For every point $(x,y) \in \Omega$ we can find a linear combination of $\{\widehat{\phi}\}_{i=1}^4$ evalued in $(\widehat{x},\widehat{y}) \in \widehat{K}$ that express this point.

$$\begin{cases} x(\widehat{x},\widehat{y}) = \sum_{i=1}^4 x_i \phi_i(\widehat{x},\widehat{y}) \\ y(\widehat{x},\widehat{y}) = \sum_{i=1}^4 y_i \phi_i(\widehat{x},\widehat{y}). \end{cases} \tag{2.11}$$

For $x_i, y_i \in \mathbb{R}$ for $i = 1, ..., 4$. The equations (2.11) allows us to define the derivative on the reference element basis functions using the points (x,y)

$$\begin{cases} \frac{\partial \widehat{\phi}_i}{\partial \widehat{x}} = \frac{\partial \widehat{\phi}_i}{\partial x}\frac{\partial x}{\partial \widehat{x}} + \frac{\partial \widehat{\phi}_i}{\partial y}\frac{\partial y}{\partial \widehat{x}} \\ \frac{\partial \widehat{\phi}_i}{\partial \widehat{y}} = \frac{\partial \widehat{\phi}_i}{\partial x}\frac{\partial x}{\partial \widehat{y}} + \frac{\partial \widehat{\phi}_i}{\partial y}\frac{\partial y}{\partial \widehat{y}}. \end{cases} \tag{2.12}$$

From (2.12) it is easy to define the Jacobian matrix that we will use when we integrate the basis functions like in the previous part.

$$J = \begin{pmatrix} \frac{\partial x}{\partial \widehat{x}} & \frac{\partial y}{\partial \widehat{x}} \\ \frac{\partial x}{\partial \widehat{y}} & \frac{\partial y}{\partial \widehat{y}} \end{pmatrix} \tag{2.13}$$

2.3 CREATION OF THE LINEAR SYSTEM

So far we created new spaces with some interesting properties, it remains to use them to solve a problem with the FEM. We will use the example (2.6) once again to explain the procedure used. Let the space $\mathcal{V}_h = span\{\phi_1, ...\phi_n\}$, the idea is to construct a linear system $A\tilde{u} = F$ easy to be solved. The entries A_{ij} are defined as

$$A_{ij} = \int_\Omega \nabla\phi_i \cdot \nabla\phi_j = \sum_{K\in\mathcal{T}_h} \int_K \nabla\phi_i \cdot \nabla\phi_j \tag{2.14}$$

As explained before we do not compute the integrals on each element K but we transform each of these integrals to an integral on the reference element using the fact that we found an invertible map F_K for every element and

$$\int_K g(x) = \int_{\widehat{K}} g(F_K^{-1}(x))|\det(J_{F_K})|. \tag{2.15}$$

Assuming for simplicity that no Neumann BC is present, the right-hand side (RHS) of the linear system is a vector whose entries are computed as follows

$$F_i = \int_\Omega f \cdot \phi_i = \sum_{K\in\mathcal{T}_h} \int_K f \cdot \phi_i. \tag{2.16}$$

The solution of the linear system $\tilde{u} = A^{-1}F = (u_1, ..., u_n)^T$ is composed by the coefficients that we use to recover our FEM solution u_h

$$u_h(x,y) = \sum_{i=1}^{n} u_i\phi_i(x,y) \tag{2.17}$$

CHAPTER 3

VARIATIONAL FORMULATION

As said before, we want to solve numerically the linear elasticity problem

$$-div(\sigma_{an}(u)) = f \text{ , in } \Omega \tag{3.1}$$
$$u = h \text{ , on } \Gamma_D \tag{3.2}$$
$$\sigma_{an} \cdot n = g \text{ , on } \Gamma_N \tag{3.3}$$

with

$$\sigma_{an}(u) = 2\mu\varepsilon(u) + \lambda div(\varepsilon(u))I + E_f(\varepsilon(u) : (a \otimes a))(a \otimes a)$$

3.1 ELASTIC ENERGY AND LAGRANGIAN FUNCTIONAL

In the first part we introduced the concept of strain energy function, and we found that it is composed by two different parts:

$$\psi(u) = \psi^{iso}(u) + \psi^{aniso}(u) \tag{3.4}$$

with the first part depending on the material isotropic properties and the anisotropic part depending on properties of the fibers. Let us define more explicitly this function

$$\psi(u) = \frac{1}{2}(2\mu(\nabla^s u)^2 + \lambda(div(\nabla^s u))^2 + E_f(\nabla^s u\colon M)^2) \tag{3.5}$$

where $\nabla^s u = \frac{\nabla u + (\nabla u)^T}{2}$ is the symmetric gradient of u. In order to have a more compact form we use from now another notation of the same quantity $\varepsilon(u)$. With this change the form of the elastic energy is

$$\psi(u) = \frac{1}{2}(2\mu(\varepsilon(u))^2 + \lambda(div(\varepsilon(u)))^2 + E_f(\varepsilon(u)\colon M)^2) \tag{3.6}$$

Our goal is to find a function u that minimize the energy of the complete system, including also the external form. This concept can be translated using the notion of Lagrangian functional

$$\mathcal{L}(v) = \int_\Omega \psi(v) - F^{ext}(v). \tag{3.7}$$

We look for $u = \arg\min_{v\in\mathcal{V}} \mathcal{L}(v)$.
In order to solve this problem we need to define the Fréchet derivative. Let $F : U \to \mathbb{R}$ be a functional and U be a subspace of a Banach space B, we define the Fréchet derivative of F in the direction v as

$$\frac{\partial F(u)}{\partial u}[v] = \lim_{\epsilon\to 0} \frac{F(u+\epsilon v) - F(u)}{\epsilon} \tag{3.8}$$

3.2 WEAK FORM

In order to minimize the Lagrangian (3.7) the idea is to find a function u such that the Fréchet derivative of $\mathcal{L}$ is null.
Supposing, as in the examples in the next parts, that the external force is defined as

$$F^{ext}(u) = \int_\Omega fu \tag{3.9}$$

for some square integrable function f. The computation of the derivative gives us

$$\frac{\partial \mathcal{L}(u)}{\partial u}[v] = \int_\Omega 2\mu\varepsilon(u)\varepsilon(v) + \lambda div(\varepsilon(u))div(\varepsilon(v)) + E_f(\varepsilon(u) : M)(\varepsilon(v) : M) - \int_\Omega fv. \tag{3.10}$$

It is now easy to define our weak formulation of the problem: find $u \in \mathcal{V}$ such that

$$\frac{\partial \mathcal{L}(u)}{\partial u}[v] = 0\,, \forall v \in \mathcal{V}. \tag{3.11}$$

In our case it is clear that u must be at least differentiable, so the largest function space that we can take is $\mathcal{V} = [H^1(\Omega)]^d$.
This formulation will theoretically give us a weak solution of the problem (3.1), but we will discover later that the discretised problem deriving from this formulation has convergence problems when we refine the mesh in the case of high values of the coefficient E_f. For this reason we reformulate the energy (3.6) introducing new fields representing the fibers elongation and an associate Lagrange multiplier (Jörg Schröder et al. 2016).

3.3 MIXED FORMULATION

As we have seen before, the energy function is composed by two parts, this means that it could be useful to model the behaviour of our solution dividing it into two parts.

$$\psi(u, \bar{e}) = \psi^{iso}(u) + \psi^{aniso}(\bar{e}) \tag{3.12}$$

where $\psi^{iso}(u) = \frac{1}{2}(2\mu\varepsilon(u)^2 + \lambda div(u)^2)$ and $\psi^{aniso}(u) = \frac{1}{2}E_f(J_4(u))^2$.
For this reason we enrich our Lagrangian functional (3.7) adding also a Lagrange multiplier to link the kinematic quantities:

$$\mathcal{L}(u, \bar{e}, \tilde{\lambda}) = \int_\Omega \psi^{iso}(u)dV + \int_\Omega \psi^{aniso}(\bar{e})dV + \int_\Omega \tilde{\lambda}\colon (\nabla^s u - \bar{e}) + F^{ext}(u) \tag{3.13}$$

This time we look for $(u, \bar{e}, \tilde{\lambda}) = \arg\min_{(v,\bar{g},\eta)\in\mathcal{V}\times\Lambda\times\Lambda} \mathcal{L}(v, \bar{g}, \eta)$. The choice of the space $\mathcal{V}$ is motivated by the same reasons as before. Analysing (3.13) we can notice that the new space Λ must be a space of symmetric tensors where each component must be at least square integrable, since the behaviour of $\bar{e}$ must be equal to the behaviour of the symmetric gradient of u:

$$\Lambda = \{\eta \in [\mathcal{L}^2(\Omega)]^{d\times d} : \eta \text{ is symmetric}\}. \tag{3.14}$$

We choose the same space also for the Lagrange multiplier since it is connected to the second-order tensor function $\varepsilon(u)$, which denotes a kinematic quantity like the deformation gradient.

3.4 MIXED WEAK FORM

To find the mixed weak form we follow the same procedure as before, computing the Fréchet derivative of the enriched Lagrangian in the direction of each component and equating them to 0.

$$\frac{\partial\mathcal{L}}{\partial u}(u, \bar{e}, \tilde{\lambda})[v] = 0, \forall v \in \mathcal{V} \tag{3.15}$$

$$\frac{\partial\mathcal{L}}{\partial \bar{e}}(u, \bar{e}, \tilde{\lambda})[\bar{g}] = 0, \forall \bar{g} \in \Lambda \tag{3.16}$$

$$\frac{\partial\mathcal{L}}{\partial \tilde{\lambda}}(u, \bar{e}, \tilde{\lambda})[\eta] = 0, \forall \eta \in \Lambda \tag{3.17}$$

The explicit computation of these derivatives gives us the weak formulation: find $(u, \bar{e}, \tilde{\lambda}) \in \mathcal{V}\times\Lambda\times\Lambda$ such that

$$\int_\Omega \sigma(u)\colon \varepsilon(v)dV + \int_\Omega \tilde{\lambda}\colon \varepsilon(v)dV = F(v), \qquad \forall v \in [\mathcal{H}^1(\Omega)]^d \tag{3.18}$$

$$\int_\Omega E_f J_4(\bar{e})J_4(\bar{g})dV - \int_\Omega \tilde{\lambda}\colon \bar{g}dV = 0, \qquad \forall \bar{g} \in \Lambda \tag{3.19}$$

$$\int_\Omega \eta\colon (\varepsilon(u) - \bar{e})dV = 0, \qquad \forall \eta \in \Lambda \tag{3.20}$$

3.5 DISCRETISATION AND LINEAR PROBLEM

From (3.18) we can extract six bilinear forms, that we can group into four different types considering symmetries:

$$a(u, v) = \int_\Omega \sigma(u)\colon \epsilon(v)dV \tag{3.21a}$$

$$b(\tilde{\lambda}, v) = \int_{\Omega} \tilde{\lambda} \colon \varepsilon(v) dV \tag{3.21b}$$

$$c(\tilde{\lambda}, \bar{g}) = \int_{\Omega} \tilde{\lambda} \colon \bar{g} dV \tag{3.21c}$$

$$d(\bar{e}, \bar{g}) = \int_{\Omega} E_f J_4(\bar{e}) J_4(\bar{g}). \tag{3.21d}$$

The spaces we introduced for the solution are of infinite dimension and not treatable by us. We can discretize these spaces in order to obtain finite dimensional spaces dense in the previous ones. From now on we will refer to these discretized spaces with the subscript letter h.
Let be $V_h = span\{v_1, v_2, ..., v_n\}$ and $\Lambda_h = span\{\lambda_1, \lambda_2, ..., \lambda_m\}$ then the discretized unknowns $u_h, \bar{e}_h, \lambda_h$ are linear combinations of the basis that span those discrete spaces.
If we evaluate the basis bilinear forms previously defined in the basis functions we obtain four matrices, with entries:

$$A_{ij} = a(v_i, v_j) \text{ with } A \in \mathbb{R}^{n \times n} \tag{3.22a}$$

$$B_{ij} = b(\lambda_i, v_j) \text{ with } B \in \mathbb{R}^{m \times n} \tag{3.22b}$$

$$C_{ij} = c(\lambda_i, \lambda_j) \text{ with } C \in \mathbb{R}^{m \times m} \tag{3.22c}$$

$$D_{ij} = d(\lambda_i, \lambda_j) with D \in \mathbb{R}^{m \times m} \tag{3.22d}$$

The full system matrix is :

$$K = \begin{bmatrix} A & B^T & 0 \\ B & 0 & -C \\ 0 & -C^T & D \end{bmatrix} \in \mathbb{R}^{n+2m \times n+2m}.$$

The right-hand side (RHS) is created by testing the forcing term against each basis function of the space $\mathcal{V}_h$.

$$(f_h)_i = \int_{\Omega} f v_i$$

To this term we need to add the boundary condition $\tilde{B}$ and we obtain the total RHS

$$F = \begin{bmatrix} f_h + \tilde{B} \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^{n+2m}$$

The discretised unknowns $u_h, \bar{e}_h$ and $\tilde{\lambda}_h$ are the solution of the linear system $K \cdot x = F$, $x = (u_h, \tilde{\lambda}_h, \bar{e}_h) \in \mathbb{R}^{n+2m}$.

3.5.1 DISCRETIZED SPACES

In the previous chapter we defined continuous and discontinuous basis functions, now we will use them to define the spaces that we need.
The first space we take into account is $\mathcal{V}_h$, in this space we want to have the discretized approximation of differentiable functions $u : \mathbb{R}^2 \to \mathbb{R}^2$ in a quadrilateral mesh $\mathcal{T}_h$. Consider $r \geq 0$ and

suppose we want to use the space of continuous polynomials of degree r for the approximation. Let $\{\phi_i\}_{i=1}^q$ be the basis of the space Y_h^r, then we can define

$$\mathcal{V}_h = span\{\begin{pmatrix}\phi_i\\0\end{pmatrix}, \begin{pmatrix}0\\\phi_i\end{pmatrix}, i = 1, 2, \cdots, q\} \tag{3.23}$$

Therefore we can affirm that $dim(\mathcal{V}_h) := n = 2 \cdot q$. The choice of continuous function is necessary to guarantee that the gradient of these functions is integrable, too.
The second and last space is Λ_h. We want this space to be composed by approximations of continuous and symmetric functions $\lambda : \mathbb{R}^2 \to \mathbb{R}^{2\times2}$ in a quadrilateral mesh $\mathcal{T}_h$. In this case we do not need the differentiability of the function so we can use the discontinuous polynomial space $Y_{h,dc}^r$ for each component of the space Λ_h. Let $\{\psi_i\}_{i=1}^p$ be the basis functions of $Y_{h,dc}^r$, then

$$\Lambda_h = span\{\begin{pmatrix}\psi_i & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & \psi_i\\\psi_i & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & \psi_i\end{pmatrix}, i = 1, \cdots, p\} \tag{3.24}$$

This choice of basis guarantees the symmetry of each function. It is easy to conclude also that $dim(\Lambda_h) := m = 3 \cdot p$

CHAPTER 4

EXISTENCE AND UNIQUENESS

Supposing that a solution of the initial problem (1.5) exists, we will now focus on the existence of a solution of the variational problem in the classic form and in the mixed formulation.

4.1 EXISTENCE AND UNIQUENESS OF A SOLUTION IN THE CLASSICAL VARIATIONAL PROBLEM

In this part we focus on the problem: find $u \in \mathcal{V}$ such that

$$a(u, v) = F(v) \, , \forall v \in \mathcal{V} \tag{4.1}$$

with $\mathcal{V}$ defined as an Hilbert space.
In this case the existence and uniqueness of the solution is guaranteed by the following lemma
Theorem 1 (Lax-Milgram lemma)**.** *Let $\mathcal{V}$ be a Hilbert space equipped with its scalar product $\langle \cdot, \cdot \rangle$. Let $a : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ be a bilinear continuous and coercive form, i.e.*

$$\exists M > 0 \text{ such that } a(u, u) \leq M ||u||^2_{\mathcal{V}} \, , \forall u \in \mathcal{V}$$

$$\exists \alpha > 0 \text{ such that } |a(u, v)| \geq \alpha ||u||_{\mathcal{V}} ||v||_{\mathcal{V}} \, , \forall u, v \in \mathcal{V}$$

and $F : \mathcal{V} \to \mathbb{R}$ a linear bounded form, then $\exists! u \in \mathcal{V}$ such that

$$a(u, v) = F(v), \forall v \in \mathcal{V}$$

Moreover we have the following a priori estimate of the solution u

$$||u||_V \leq \frac{1}{\alpha} ||F||_{V'} \tag{4.3}$$

where $||F||_{V'} = \sup_{v \in \mathcal{V}} \frac{|F(v)|}{||v||_V}$ (Quarteroni 2009b).

When we discretize the problem we can use the Lax-Milgram lemma to infer the existence and uniqueness of the discretized solution, moreover the Céa Lemma guarantees the quasi-optimality of the solution.
Remark 1. *The Locking can be observed in cases when the ratio between the continuity constant and the coercivity constant is large, i.e. $M/\alpha >> 1$.*

4.2 EXISTENCE AND UNIQUENESS OF A SOLUTION IN THE MIXED VARIATIONAL PROBLEM

The goal of a mixed problem is to find two functions $u \in \mathcal{V}$ and $p \in \mathcal{Q}$ ($\mathcal{V}$ and $\mathcal{Q}$ are Hilbert spaces) that satisfy the problem: find $(u, p) \in (\mathcal{V} \times \mathcal{Q})$ such that

$$a(u, v) + b(v, p) = F(v) \;, \forall v \in \mathcal{V} \tag{4.4a}$$

$$b(u, q) = G(q) \;, \forall q \in \mathcal{Q} \tag{4.4b}$$

4.2.1 REFORMULATION OF THE PROBLEM

The first step is to rewrite the problem in the operator form. Thanks to the Riesz theorem we can infer that there exist two operators

$$A : \mathcal{V} \to \mathcal{V}' \tag{4.5}$$

and

$$B : \mathcal{V} \to \mathcal{Q}' \tag{4.6}$$

such that

$$\begin{cases} a(u, v) = {}_{\mathcal{V}'}\langle Au, v\rangle_{\mathcal{V}} \\ b(u, q) = {}_{\mathcal{Q}'}\langle Bu, q\rangle_{\mathcal{Q}} \end{cases} \tag{4.7}$$

This allows us to rewrite the problem in the following form

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \tag{4.8}$$

with the operator $B^T : \mathcal{Q} \to \mathcal{V}'$

4.2.2 ON THE HILBERT SPACES

In the context of Hilbert spaces we replace the concept of orthogonal spaces with the polar spaces.

Definition 2. *Let $\mathcal{V}$ be an Hilbert space, let $K \subset \mathcal{V}$. The polar space is*

$$K^\circ = \{L \in \mathcal{V}' : {}_{\mathcal{V}'}\langle L, v\rangle_{\mathcal{V}} = 0 \;\forall v \in K\} \tag{4.9}$$

Theorem 2. *Let $Z \subset \mathcal{V}$, then Z° is closed. Moreover, for every subspace $Z \subset \mathcal{V}$ we have*

$$Z^\circ = (\bar{Z})^\circ$$

and

$$Z = (Z^\circ)^\circ \iff Z \text{ is closed}$$

Theorem 3. *Given an operator $B : \mathcal{V} \to \mathcal{Q}'$ then*

$$(\ker B)^\circ = \overline{Im(B^T)} \text{ and } (\ker B^T)^\circ = \overline{Im(B)}$$

Theorem 4. *The following properties are equivalent*

1. $\exists \beta > 0$ *such that* $\sup_{v\in\mathcal{V}} \frac{b(v,q)}{||v||_{\mathcal{V}}} \geq \beta||q||_{\mathcal{Q}}\ \forall q \in \mathcal{Q}$

2. B^T *is an isomorphism between* $\mathcal{Q}$ *and* $(\ker B)^{\circ}$, *i.e.* $\exists \beta > 0$ *such that* $||B^T q||_{\mathcal{V}'} \geq \beta||q||_{\mathcal{Q}}$ $\forall q \in \mathcal{Q}$

3. B *is an isomorphism between* $(\ker B)^{\perp}$ *and* Q', *i.e.* $\exists \beta > 0$ *such that* $||Bv||_{\mathcal{Q}'} \geq \beta||v||_{\mathcal{V}}$ $\forall v \in (\ker B)^{\perp}$

Theorem 5 (Banach Closed Range Theorem)**.** *(Daniele Boffi 2013) Let $\mathcal{V}$ and $\mathcal{Q}$ be two Hilbert spaces and let B be a continuous linear operator from $\mathcal{V}$ to $\mathcal{Q}'$. Then, the six following statements are equivalent:*

- $Im(B)$ *is closed in* $\mathcal{Q}'$
- $Im(B^T)$ *is closed in* $\mathcal{V}'$
- $(\ker B)^{\circ} = Im(B^T)$
- $(\ker B^T) = Im(B)$
- $\exists L_B \in \mathcal{L}(Im(B), K^{\perp})$ *and* $\beta > 0$ *such that* $B(L_B(g)) = g \forall g \in Im(B)$. *Moreover* $\beta||L_B(g)||_{\mathcal{V}} \leq ||g||_{\mathcal{Q}'} \forall g \in Im(B)$
- $\exists L_{B^T} \in \mathcal{L}(Im(B^T), K^{\perp})$ *and* $\beta > 0$ *such that* $B(L_{B'}(f)) = f \forall f \in Im(B^T)$. *Moreover* $\beta||L_{B^T}(f)||_{\mathcal{Q}} \leq ||f||_{\mathcal{V}'} \forall f \in Im(B)$

4.2.3 EXISTENCE AND UNIQUENESS OF A SOLUTION

The section before is very useful to prove the main theorem of this chapter

Theorem 6. *Let $a : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ and $b : \mathcal{V} \times \mathcal{Q} \to \mathbb{R}$ be continuous (with constants M for $a(\cdot,\cdot)$ and γ for $b(\cdot,\cdot)$). Then:*

- *If* $\exists \alpha > 0$ *such that* $a(u,u) \geq \alpha||u||^2_{\mathcal{V}}\ \forall u \in \ker B$ *(Coercivity on the kernel)*
- *If* $\exists \beta > 0$ *such that* $\inf_{q\in\mathcal{Q}} \sup_{v\in\mathcal{V}} \frac{|b(v,q)|}{||v||_{\mathcal{V}}||q||_{\mathcal{Q}}} \geq \beta$ *(Inf-Sup condition)*

The problem (4.4) *admits a unique solution* $(u,p) \in \mathcal{V} \times \mathcal{Q}$. *Moreover, the solution depends continuously on the data:*

$$\begin{cases} ||u||_{\mathcal{V}} \leq \frac{1}{\alpha}||f||_{\mathcal{V}'} + \frac{1}{\beta}(1 + \frac{M}{\alpha})||g||_{\mathcal{Q}'} \\ ||p||_{\mathcal{Q}} \leq \frac{1}{\beta}(1 + \frac{M}{\alpha})||f||_{\mathcal{V}'} + \frac{M}{\beta^2}(1 + \frac{M}{\alpha})||g||_{\mathcal{Q}'} \end{cases} \tag{4.10}$$

Proof. Let us suppose that $Im(B) = \mathcal{Q}'$. First of all we prove the existence of the solution: the operator B is surjective, thanks to the theorem 5 we can affirm that there exists an operator L_B such that $B(L_B g) = g \forall g \in \mathcal{Q}'$. Consider $u_g = L_B g$ and $u_0 = u - u_g \in \ker(B)$. Now we have $Bu_g = g$ and $Bu = g$.
This implies that

$$a(u_0, v_0) = {}_{\mathcal{V}'}\langle f, v_0\rangle_{\mathcal{V}} - a(u_g, v_0) \forall v_0 \in \ker(B) \tag{4.11}$$

The theorem 1 ensures the existence of a solution in $\ker(B)$ because of the coercivity on the kernel.

For the uniqueness of the solution we restrict to the case $f = 0$ and $g = 0$. In this case the solution u is in $\ker B$. If we take the first part of (3.18) and we test it against u we obtain

$$a(u,u) + b(u,p) = a(u,u) = 0 \tag{4.12}$$

Thanks to the coercivity on the kernel we can affirm that $u = 0$. If we return to the full form of (3.18) using $u = 0$, we also obtain $B^T p = 0$ and consequently $p = 0$. The unique solution is $(u,p) = (0,0)$.
Lastly, we prove the inequalities:
Let $u = u_g + u_0$ and p be the solution of (3.18), then for $v_0 \in \ker B$ we have

$$a(u,v_0) + b(v_0,p) = a(u,v_0) = a(u_0,v_0) + a(u_g,v_0) = \langle f, v_0 \rangle$$

The coercivity on the kernel and the continuity of $a(\cdot,\cdot)$ tell us that

$$\alpha ||u_0||^2_{\mathcal{V}} \leq a(u_0,u_0) = \langle f, u_0 \rangle - a(u_g,u_0) \leq (||f||_{\mathcal{V}'} + M||u_g||_{\mathcal{V}})||u_0||_{\mathcal{V}}$$

From this we can conclude that

$$||u||_{\mathcal{V}} = ||u_0 + u_g||_{\mathcal{V}} \leq \frac{1}{\alpha}||f||_{\mathcal{V}'} + (1 + \frac{M}{\alpha})\frac{1}{\beta}||g||_{\mathcal{Q}'}$$

The second inequality will use the Inf-Sup condition

$$\beta||p||_{\mathcal{Q}} \leq \sup_{v\in\mathcal{V}} \frac{b(v,p)}{||v||_{\mathcal{V}}} = \sup_{v\in\mathcal{V}} \frac{\langle f, v\rangle - a(u,v)}{||v||} \leq ||f||_{\mathcal{V}'} + M||u||_{\mathcal{V}}$$

To recover the final form we use the bound found before for the norm of u. □

4.3 WELL POSEDNESS OF OUR PROBLEM

Our problem is not in the classical setting of a saddle point problem but we can transform it in order to find a formula similar to (4.4).
Let us reorder our operator in this form

$$\tilde{K} = \begin{pmatrix} A & 0 & B^T \\ 0 & D & -C^T \\ B & -C & 0 \end{pmatrix} \tag{4.13}$$

We define two other matrices $\mathbb{A}$ and $\mathbb{B}$:

$$\mathbb{A} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \text{ and } \mathbb{B} = \begin{pmatrix} B & -C \end{pmatrix} \tag{4.14}$$

It is now clear that

$$\tilde{K} = \begin{pmatrix} \mathbb{A} & \mathbb{B}^T \\ \mathbb{B} & 0 \end{pmatrix}$$

We want to verify under which conditions the operator $\mathbb{A}$ is coercive, to do that we need to define the operators more precisely

$$\mathbb{A}((u,\bar{e});(v,\bar{g})) = \int_\Omega 2\mu\nabla^s u\nabla^s v + \int_\Omega E_f J_4(\bar{e})J_4(\bar{g}) \tag{4.15}$$

$$\mathbb{B}(\tilde{\lambda};(v,\bar{g})) = \int_\Omega \tilde{\lambda}\colon (\nabla^s v - \bar{g}) \tag{4.16}$$

The next step is to find a kernel for the operator $\mathbb{B}$, so we want to find a couple $(v,\bar{g})$ such that $\mathbb{B}(\tilde{\lambda};(v,\bar{g})) = 0\forall\tilde{\lambda} \in \mathcal{V}$. It is easy to see that if $\nabla^s v = \bar{g}$ the equality is respected, but this condition seems to be too restrictive. We can relax this condition using the fact that the operator $\mathbb{B}$ is a difference between two scalar products in L^2. It follows that $\ker(\mathbb{B})$ is defined as

$$\ker(\mathbb{B}) = \{(v,\bar{g}) \in \mathcal{V}\times\mathcal{Q} | \bar{g} = \Pi_{L^2}(\nabla^s v)\} \tag{4.17}$$

Now we consider $\bar{e}$, and we can split this element into two parts:

$$\bar{e} = \Pi_{L^2}(\nabla^s u) + \bar{e}_\perp$$

Moreover we can define the L^2-projection as $a\otimes a$ times a constant $\bar{e}_s$ and this gives us the final formulation of our tensor

$$\bar{e} = \bar{e}_s\cdot a\otimes a + \bar{e}_\perp \tag{4.18}$$

4.3.1 COERCIVITY ON THE KERNEL

Given $(u,\bar{e}) \in \ker\mathbb{B}$, it follows that

$$\mathbb{A}((u,\bar{e});(u,\bar{e})) = \int_\Omega 2\mu|\nabla^s u|^2 + \int_\Omega E_f|\bar{e}\colon (a\otimes a)|^2 = 2\mu||\nabla^s u||^2_{L^2} + E_f|\bar{e}_s|^2 \tag{4.19}$$

Using the Poincaré inequality we have the coercivity on the operator. In order to have a bound as small as possible we need a value $\bar{e}_s$ which is not too big, since the constant E_f will be large enough to ensure inextensibility.
It appears that to have a well posed problem we do not need to bound the norm of the complete tensor but simply bound its symmetric part.

CHAPTER 5

RESULTS

In this chapter we compare the results obtained solving a classical variational form using a $\mathbb{Q}_1$ discretisation, that suffers from the locking phenomenon, and the one obtained using the mixed formulation using a $\mathbb{Q}_1/\mathbb{P}_0$ discretisation, both for quadrangular meshes. We test these approaches on different examples in $2D$. In order to be consistent between one example and another we fix the fibers strain coefficient $E_f = 10^5$, $\lambda = 0.5769$ and $\mu = 0.3846$ for all the numerical experiments of this chapter.

5.1 CONVERGENCE

We will study different cases with different boundary conditions and the presence of fibers with different orientations.
We need to keep in mind that if we use an approximation of degree $r \geq 1$ we can prove that there exits two constants C_0^r and C_1^r such that the error e between the true solution u and the numerical one u_h is bounded (Quarteroni 2009c), in particular

$$||e||_{L^2} \leq C_0^r h^{r+1} ||u||_{H^{r+1}} \tag{5.1a}$$

$$||e||_{H^1} \leq C_1^r h^r ||u||_{H^{r+1}} \tag{5.1b}$$

In an ideal case the convergence will be of order h^{r+1} in norm L^2 and of order h^r in norm H^1.

5.1.1 FIRST CASE

In this first case we consider a domain $\Omega = [0, 1]^2$ with fibers oriented at 60 degrees respect to the horizontal line. Moreover in three out of four boundaries we use the Dirichlet conditions, and in the last one we have a Neumann condition.

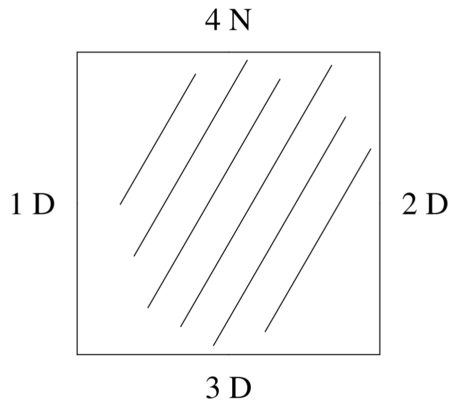


FIGURE 5.1
Domain for the first case

We use this case as a test to check the performance of our mixed formulation. We consider the analytical solution $u : \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$u(x, y) = \begin{pmatrix} \sin(x)\cos(y) \\ \sin(x - y) \end{pmatrix} \tag{5.2}$$

and we compute the RHS function $f = -div(\sigma(u))$ and the Dirichlet and Neumann conditions accordingly. For the computation of the convergence we used tensor product meshes (see Fig. 5.2) that are refined in a diadic way, considering 2^k elements, with $k = 1, 2, ..., 5$, for each boundary.

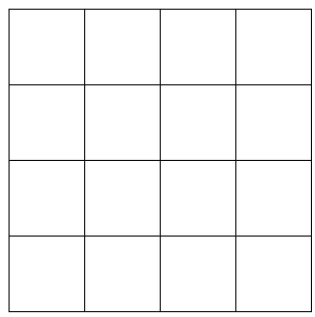


FIGURE 5.2
Example of a mesh with 4×4 elements used to compute the convergence of the different methods

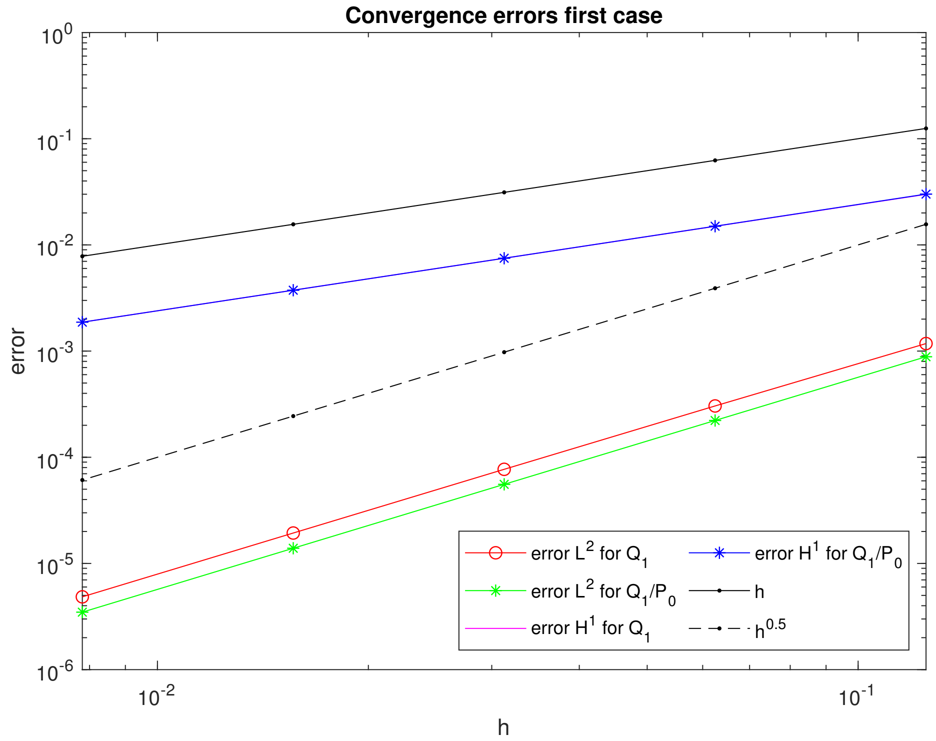


FIGURE 5.3
Convergence of the error in the first case

In the figure 5.4 we can see that the convergence rates are the same for both methods. The only difference we can appreciate is that the mixed method shows a smaller L^2 error for all the refining of the mesh.
Also the numerical computation of the convergence rates confirms that the optimal convergence rate is obtained for both methods in the L^2 and H^1 norms.

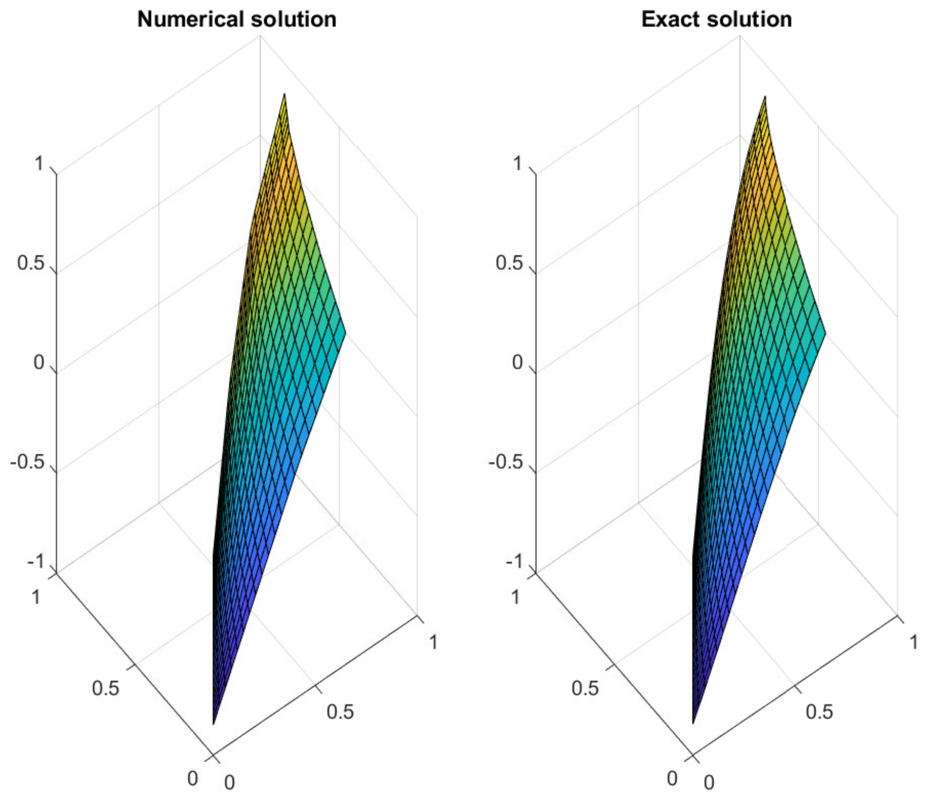


FIGURE 5.4
At the left the numerical solution computed with the mixed formulation Q_1/P_0, at the right the exact solution of the first case.

5.1.2 SECOND CASE

This is the last case we analyse in the configuration of a quasi-inextensible material. The domain is still the unit square with fibers positioned with an angle of $45°$ and a perpendicular load applied with an angle of $135°$, whose value is:

$$f = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{5.3}$$

The right and top boundaries present homogeneous Neumann conditions, whereas the other two present symmetry boundary conditions: the displacement is blocked perpendicularly to the faces but it is free to move along the other direction. See figure 5.5.

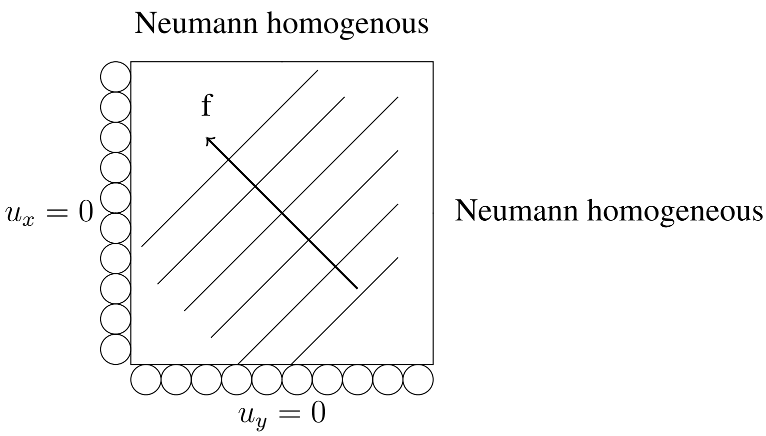


FIGURE 5.5
Domain for the second case

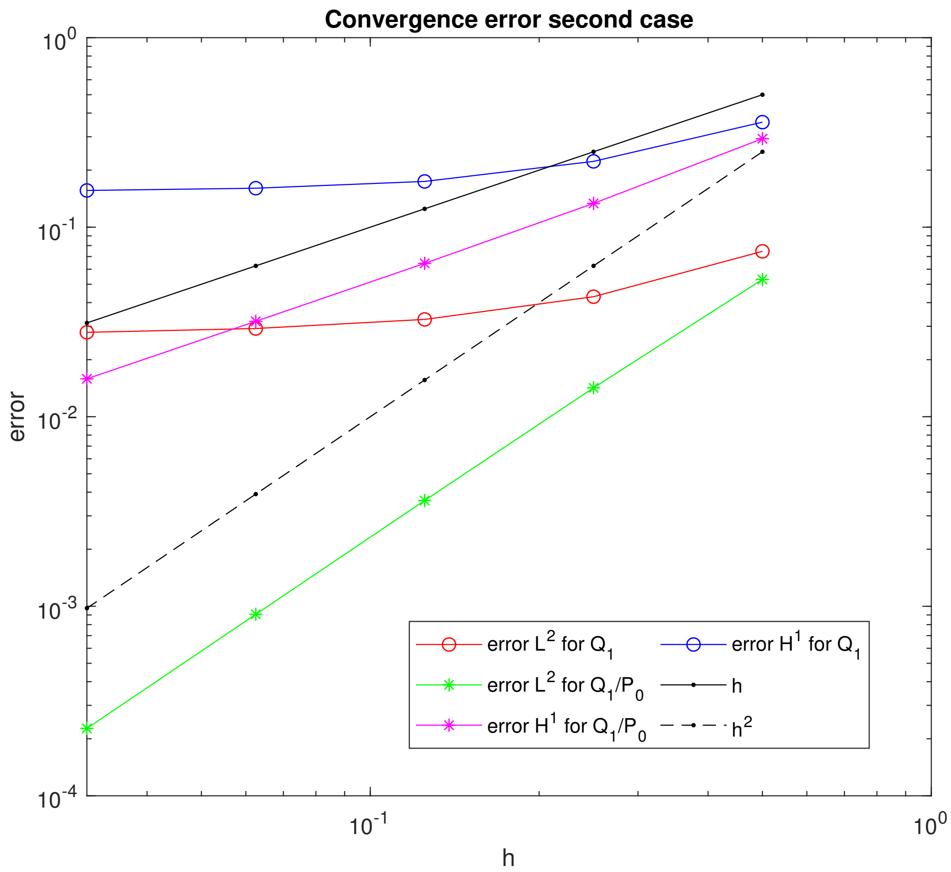


FIGURE 5.6
Convergence of the error in the second case for Q_1 and Q_1/P_0

As it can be seen in Figure 5.6, the mixed method reaches optimal convergence rates for errors in the L^2 and H^1 norms, while the classical method shows a slow convergence for the two first refinement of the mesh but after that the locking phenomenon completely blocks the solution.

Chapter 6

Conclusion

The method we developed throughout this paper fixes the locking phenomenon that appears in the presence of strong fibers. To achieve this result we introduced a mixed method splitting the energy function into the isotropic and anisotropic contribution. Moreover, we dedicated a part of our work to discuss some basic stability properties of the method.

The possible extensions of the present work involve a deeper study of the stability properties of the chosen discrete spaces for the mixed method and the extension to the case of materials that are, both, quasi-inextensible and quasi-incompressible. Other discretisation possibilities for the mixed problem, specially for higher degrees, should be also studied.

Bibliography

Wriggers, Peter (2008). 'Basic Equations of Continuum Mechanics'. In: *Nonlinear Finite Element Methods*. Berlin, Heidelberg: Springer Berlin Heidelberg, pp. 19–102. ISBN: 978-3-540-71001-1. DOI: 10.1007/978-3-540-71001-1_3. URL: https://doi.org/10.1007/978-3-540-71001-1_3.

Wriggers, P., J. Schröder and F. Auricchio (2016). 'Finite element formulations for large strain anisotropic material with inextensible fibers'. In: *Advanced Modeling and Simulation in Engineering Sciences* 3.1, p. 25. ISSN: 2213-7467. DOI: 10.1186/s40323-016-0079-3. URL: https://doi.org/10.1186/s40323-016-0079-3.

I. Babuska, M. Suri (1992). 'Locking effects in the finite element approximation of elasticity problems'. In: *Numerische Mathematik* 62, pp. 439–463.

Quarteroni, Alfio (2009a). *Numerical Models for Differential Problems*. Modeling, Simulation and Applications. 2. Written for students of bachelor and master courses in scientific disciplines: engineering, mathematics, physics, computational sciences, and information science. Heidelberg, DE: Springer, pp. 60–66. DOI: 10.1007/978-88-470-1071-0. URL: http://infoscience.epfl.ch/record/140685.

Schröder, Jörg et al. (2016). 'A novel mixed finite element for finite anisotropic elasticity; the SKA-element Simplified Kinematics for Anisotropy'. In: *Computer Methods in Applied Mechanics and Engineering* 310, pp. 475–494. ISSN: 0045-7825. DOI: https://doi.org/10.1016/j.cma.2016.06.029. URL: http://www.sciencedirect.com/science/article/pii/S0045782516306478.

Quarteroni, Alfio (2009b). *Numerical Models for Differential Problems*. Modeling, Simulation and Applications. 2. Written for students of bachelor and master courses in scientific disciplines: engineering, mathematics, physics, computational sciences, and information science. Heidelberg, DE: Springer, pp. 50–51. DOI: 10.1007/978-88-470-1071-0. URL: http://infoscience.epfl.ch/record/140685.

Daniele Boffi Franco Brezzi, Michel Fortin (2013). *Mixed Finite Element Methods and Applications*. Springer.

Quarteroni, Alfio (2009c). *Numerical Models for Differential Problems*. Modeling, Simulation and Applications. 2. Written for students of bachelor and master courses in scientific disciplines: engineering, mathematics, physics, computational sciences, and information science. Heidelberg, DE: Springer, pp. 95–96. DOI: 10.1007/978-88-470-1071-0. URL: http://infoscience.epfl.ch/record/140685.