Darcy equations:

We consider the Darcy problem

$$\begin{cases} \frac{1}{k}\vec{u} + \nabla p = 0 & \text{in } \Omega \\ \operatorname{div}\vec{u} = f & \text{in } \Omega \\ p = d & \text{on } \partial\Omega \end{cases} \quad (1)$$

with $f \in L^2(\Omega)$ and $d \in H^{1/2}(\partial\Omega)$

For easiness of exposition we consider only the case of Neumann boundary conditions. The theory presented hereafter generalizes without difficulty to the case of Dirichlet boundary conditions. On the other hand, the case of <u>mixed</u> boundary conditions is more troublesome.

The weak formulation of (1) is:
find $\vec{u} \in H(\operatorname{div},\Omega)$ and $p \in L^2(\Omega)$ such that

$$\begin{cases} \int_\Omega \frac{1}{k}\vec{u}\cdot\vec{v} - \int_\Omega p \operatorname{div}\vec{v} = \int_{\partial\Omega} d\, \vec{v}\cdot\vec{n} & \forall \vec{v} \in H(\operatorname{div},\Omega) \\ \int_\Omega \operatorname{div}\vec{u}\, q = \int_\Omega f q & \forall q \in L^2(\Omega) \end{cases} \quad (2)$$

which can be put in the abstract form

find $(u,p) \in V\times Q$ s.t.
$$\begin{cases} a(u,v) + b(v,p) = F(v) & \forall v \in V \\ b(u,q) = G(q) & \forall q \in Q \end{cases}$$

with $V = H(\operatorname{div},\Omega)$, $Q = L^2(\Omega)$, $a(u,v) = \int_\Omega \frac{1}{k} uv$

$b(v,p) = -\int_\Omega \operatorname{div} v\, p$, $F(v) = \int_{\partial\Omega} d v\cdot n$, $G(q) = \int_\Omega f q$.

<u>Well posedness of (2)</u>

- the continuity of $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ and $G(\cdot)$ is straightforward
- the continuity of $F(\cdot)$ is a consequence of the trace theorem in $H(\operatorname{div},\Omega)$:

$$F(v) = \int_{\partial\Omega} d\, \vec{v}\cdot\vec{n} \le \|d\|_{H^{1/2}(\partial\Omega)} \|\vec{v}\cdot\vec{n}\|_{H^{-1/2}(\partial\Omega)}$$
$$\le \gamma \|d\|_{H^{1/2}(\partial\Omega)} \|v\|_{H(\operatorname{div},\Omega)}$$

- $a(\cdot,\cdot)$ is coercive on $V^0 = \{\vec{v} \in H(div, \Omega) ;\ div\, \vec{v} = 0\}$

this is immediate as

$$a(u,u) = \int_\Omega \frac{1}{k} u^2 \geq \frac{1}{k_{max}} \|u\|^2_{L^2(\Omega)} = \frac{1}{k_{max}} \|u\|^2_{H(div,\Omega)} \quad \text{if } u \in V^0.$$

notice, however, that $a(\cdot,\cdot)$ is <u>not</u> coercive on V !

- $b(\cdot,\cdot)$ satisfies the inf-sup condition. Indeed:

$\forall p \in L^2(\Omega)$ let us define φ : $\begin{cases} \Delta \varphi = p & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega \end{cases}$

$\|\varphi\|_{H^1} \leq C \|p\|_{L^2}$

let $\vec{v} = \nabla \varphi$

clearly, $\vec{v} \in L^2(\Omega)$ and $div\, \vec{v} = div \nabla \varphi = \Delta \varphi = p$

hence $\|\vec{v}\|_{H(div,\Omega)} \leq C \|p\|_{L^2}$ and

$$\frac{|b(p,v)|}{\|\vec{v}\|_{H(div,\Omega)}} = \frac{\|p\|^2_{L^2(\Omega)}}{\|\vec{v}\|_{H div}} \geq \frac{1}{C} \|p\|_{L^2(\Omega)}$$

and $$\inf_{p \in L^2} \sup_{v \in H(div,\Omega)} \frac{b(v,p)}{\|v\|_{H(div)} \|p\|_{L^2}} \geq \frac{1}{C}$$

We consider now a finite element approximation of (2).
~~A good choice~~ that uses non standard finite elements for the velocity $\vec{u} \in H(div, \Omega)$

<u>Raviart - Thomas finite elements</u>

Given a ~~triangle~~ simplex K we define the space

$$RT_r = (\mathbb{P}_r)^d \oplus \vec{x}\, \mathbb{P}_r \subset (\mathbb{P}_{r+1})^d$$

$\vec{v} \in RT_r \iff \vec{v} = \vec{w} + \vec{x} z \quad \vec{w} \in (\mathbb{P}_r(K))^d, \ z \in \mathbb{P}_r(K)$

in 2 dimensions $(d=2)$

$$\vec{v} \in RT_r \iff \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix} z \quad \text{with } w_1, w_2, z \in \mathbb{P}_r(K)$$

in 3 dimensions

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} + \begin{pmatrix} x \\ y \\ z \end{pmatrix} z \quad \text{with } w_1, w_2, w_3, z \in \mathbb{P}_r(K)$$

Properties of $RT_r(K)$:

- <u>dimension</u>: notice that $\vec{x}\,\mathbb{P}_{r-1} \subset \mathbb{P}_r$

hence $\dim(RT_r(K)) = \dim((\mathbb{P}_r)^d) + \dim(\mathbb{P}_r \setminus \mathbb{P}_{r-1})$

now: $\dim \mathbb{P}_r = \binom{r+d}{d}$ and $\dim(\mathbb{P}_r \setminus \mathbb{P}_{r-1}) = \binom{r+d-1}{d-1}$

$$\Rightarrow \dim(RT_r) = d\binom{r+d}{d} + \binom{r+d-1}{d-1} = \frac{(r+d)!}{r!(d-1)!} + \frac{(r+d-1)!}{r!(d-1)!}$$

$$\Rightarrow \boxed{\dim(RT_r) = (r+d+1)\frac{(r+d-1)!}{r!(d-1)!}}$$

- Let $\vec{v} \in RT_r$, and Γ an edge of K. Then $\vec{v}\cdot n|_\Gamma \in \mathbb{P}_r(\Gamma)$

indeed let $\vec{a}$ be a vertex of Γ and $\vec{x} \in \Gamma$

$$\Rightarrow \vec{v}\cdot n = \vec{w}\cdot n + (\vec{x}-\vec{a})\cdot n z + \vec{a} z$$

notice now that $\vec{x}-\vec{a}$ is a vector on Γ so $(\vec{x}-\vec{a})\cdot\vec{n} = 0$

$$\Rightarrow \vec{v}\cdot n|_\Gamma = \vec{w}\cdot n|_\Gamma + z|_\Gamma \in \mathbb{P}_r(\Gamma)$$

- $\operatorname{div}\vec{v} \in \mathbb{P}_r(K)$

Indeed $\operatorname{div}\vec{v} = \operatorname{div}\vec{w} + \operatorname{div}\vec{x}\, z + \vec{x}\,\nabla z$

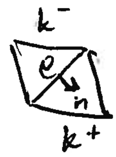
$$= \underbrace{\operatorname{div}\vec{w}}_{\in \mathbb{P}_{r-1}} + \underbrace{3z}_{\mathbb{P}_r} + \underbrace{\vec{x}\,\nabla z}_{\mathbb{P}_r} \in \mathbb{P}_r$$

Let now $\mathcal{T}_h$ be a regular triangulation of Ω. We define the global space

$$W_h^r = \{\vec{v} \in (L^2(\Omega))^d : \vec{v}|_K \in RT_r(K)\ \forall K \in \mathcal{T}_h,\ [\![\vec{v}\cdot n]\!]_e = 0 \quad \forall e \in \mathcal{E}_h^i\}$$

where $\mathcal{E}_h^i$ is the set of all internal edges (faces) of $\mathcal{T}_h$ and

given a face $e \in \mathcal{E}_h^i$ shared by two triangles K^+ and K^-



$[\![\vec{v}\cdot n]\!]_e = \vec{v}|_{K^+}\cdot n - \vec{v}|_{K^-}\cdot n$ is the jump of the normal component across the edge e.

Hence, functions in W_h^r are not globally continuous but have <u>normal component</u> continuous across each internal edge.

Lemma: $W_h^r \subset H(\mathrm{div}, \Omega)$

we have to check that we can define $\mathrm{div}\,\vec{v} \in L^2(\Omega)$ $\forall \vec{v} \in W_h^r$

let φ be a smooth function that vanishes on $\partial\Omega$

$$\int_\Omega \mathrm{div}\,\vec{v}\,\varphi = -\int_\Omega \vec{v}\cdot\nabla\varphi = \Sigma_K -\int_K \vec{v}\cdot\nabla\varphi = \Sigma_K \left(+\int_K \mathrm{div}\,\vec{v}|_K\,\varphi - \int_{\partial K} (\vec{v}\cdot n)\varphi \right)$$

$$= \Sigma_K \int_\Omega \mathrm{div}\,\vec{v}|_K\,\varphi \;+ \cancel{\sum_{e\in\mathcal{E}_h^i} \int_e [\![\vec{v}\cdot n]\!]\varphi}$$

if we define $w \in L^2(\Omega)$: $w|_K = \mathrm{div}\,\vec{v}|_K$

then $\mathrm{div}\,\vec{v} = w \in L^2(\Omega)$. Hence $\vec{v} \in H(\mathrm{div}, \Omega)$

We want now to define a set of degrees of freedom.

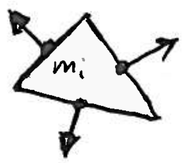
Let us consider the bi-dimensional case $d=2$

$\dim(\mathbb{RT}_r) = (r+3)\frac{(r+1)!}{r!}$

$\boxed{r=0}$ (lowest order Raviart-Thomas $\mathbb{RT}_0$ space)

$\dim(\mathbb{RT}_0) = 3$ $\qquad \vec{v}|_e \cdot n = \mathbb{P}_0$

Since $\vec{v}\cdot n$ has to be <u>continuous</u> across edges and is constant on each edge, the natural choice is to define $\vec{v}\cdot n|_e$ as dof



The degree of freedom can be associate to the value of $\vec{v}\cdot n$ in the midpoint of the edge

$\mu_i(v) = \vec{v}(m_i)\cdot n$

or to the ~~flax~~ flux of $\vec{v}$ across the edge $\mu_e(\vec{v}) = \int_e \vec{v}\cdot n$

the second choice is more common.

Basis of W_h^r: we can take the "Lagrangian" basis

$\{\vec{\varphi}_i\}_{i=1}^{Ne}$ such that $\mu_i(\vec{\varphi}_j) = \delta_{ij}$

i.e. the basis function $\vec{\varphi}_i$ has unitary flux on e_i

$\int_{e_i} \vec{\varphi}_i \cdot n = 1$ and zero flux across all other edges

$\int_{e_j} \vec{\varphi}_i \cdot n = 0 \quad i \neq j$.

Once we have defined the dofs and associated basis functions, we can define an interpolant operator: for a smooth function $\vec{v} : \Omega \to \mathbb{R}^d$

$$I_h^{RT_r} \vec{v} = \sum_{e \in \mathcal{E}_h} \mu_e(\vec{v}) \vec{\varphi}_e = \sum_{e \in \mathcal{E}_h} \left(\int_e \vec{v} \cdot n \right) \vec{\varphi}_e$$

~~Lemma: the interpolant operator $I_h^{RT_r}$ is continuous on $(H^1(\Omega))^d$~~

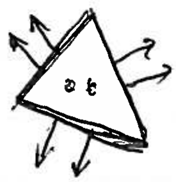
$$\| I_h^{RT} \vec{v} \|_{H(\mathrm{div},\Omega)} \leq C_I \| \vec{v} \|_{H^1(\Omega)} \qquad \forall \vec{v} \in H^1(\Omega)$$

~~Follows from Trace inequality: if K is a Triangle having edge e, then $\int_e \vec{v} \cdot n \leq \gamma \| \vec{v} \|_{H^1(K)}$~~

- Local interpolant operator on a single element: $I_K^{RT}(\vec{v}) = \sum_{e \in \partial K} \int_e \vec{v} \cdot n \, \vec{\varphi}_e|_K$

$\boxed{r = 1}$ $\dim(RT_1) = 4 \cdot \frac{2!}{1!} = 8$ and $\vec{v} \cdot n|_e \in \mathbb{P}_1(e)$

we can choose 2 values on each edge to uniquely determine the flux. We are left with 2 dofs.



$\mu_{e,1}(\vec{v}) = \int_e \vec{v} \cdot n \qquad \mu_{e,2}(v) = \int_e (\vec{v} \cdot n)(\vec{x} \cdot \vec{t})$ $\quad \vec{t}$: tangent vector

$e \in \partial K$

$\mu_{K,1}(v) = \int_K v_1 \qquad \mu_{K,2}(v) = \int_K v_2$

$\boxed{r = 2}$ $\dim(RT_2) = 5 \, \frac{3!}{2!} = 15$, $\quad \vec{v} \cdot n|_e \in \mathbb{P}_2$

we need 3 dofs per edge to fix the flux on the edge + 6 dofs internal to the triangle

$\forall e \in \partial K \qquad \mu_{e,1} = \int_e \vec{v} \cdot n \qquad \mu_{e,2} = \int_e (\vec{v} \cdot n)(\vec{x} \cdot \vec{t}) \qquad \mu_{e,3} = \int_e (\vec{v} \cdot n)(\vec{x} \cdot \vec{t})^2$

inside K $\qquad \mu_{K,1} = \int v_1 \qquad \mu_{K,2} = \int v_2 \qquad \mu_{K,3} = \int v_1 x_1 \qquad \mu_{K,4} = \int v_1 x_2$

$\mu_{K,5} = \int v_2 x_1 \qquad \mu_{K,6} = \int v_2 x_2$

~~Let~~ More generally, the dofs are

$\forall e \in \partial K \quad \mu_{e,j} = \int_e (\vec{v}\cdot n)\psi_j \qquad \{\psi_j\}_{j=1}^{r}$ basis of $\mathbb{P}_r(e)$

$\mu_{K,\ell} = \int_K \vec{v}\cdot\vec{\varphi}_\ell \qquad \{\vec{\varphi}_\ell\}$ basis of $(\mathbb{P}_{r-1})^2$

- The functionals $\mu_{e,j}(\vec{v})$ and $\mu_{K,\ell}(\vec{v})$ are bounded in $(H^1)^d$
this follows from Trace inequalities

therefore, the interpolant operator $I_h^{RT}:(H^1)^d \to W_h^r$ is bounded.

Let $Y_h^r = \{v \in L^2(\Omega) : v|_K \in \mathbb{P}_r(K) \ \forall K \in \mathcal{T}_h\}$ be the finite element space of discontinuous piecewise polynomials of degree r.

We have seen that $\forall \vec{v} \in W_h^r$, $\operatorname{div}\vec{v} \in Y_h^r$.
Moreover, let us define the L^2 projection operator
$\Pi_h^r : L^2(\Omega) \to Y_h^r$,

$$\int_K \Pi_h^r v|_K \psi = \int_K v\psi \qquad \forall \psi \in \mathbb{P}_r(K)$$

__Lemma__: $\operatorname{div} I_h^{RT_r}\vec{v} = \Pi_h^r \operatorname{div}\vec{v}$

Indeed, $\forall \psi \in \mathbb{P}_r(K)$

$$\int_K \psi \operatorname{div} I_K^{RT_r}\vec{v} = -\int_K \underbrace{\nabla\psi}_{\mathbb{P}_{r-1}} I_K^{RT_r}\vec{v} + \int_{\partial K}(I_K^{RT_r}\vec{v}\cdot n)\underbrace{\psi}_{\mathbb{P}_r}$$

by construction of $I_K^{RT_r}$ $\quad \int_e I_K^{RT_r}\vec{v}\cdot n\,\psi = \int_e \vec{v}\cdot n\,\psi \quad \forall\psi \in \mathbb{P}_r$

$\int_K I_K^{RT_r}\vec{v}\cdot\vec{\psi} = \int_K \vec{v}\cdot\vec{\psi} \quad \forall\psi \in \mathbb{P}_{r-1}$

hence $\int_K \psi \operatorname{div} I_K^{RT_r}\vec{v} = -\int_K \nabla\psi\cdot\vec{v} + \int_{\partial K}(\vec{v}\cdot n)\psi = \int_K \psi \operatorname{div}\vec{v}$

$\Rightarrow \quad \Pi_h^r \operatorname{div} v = \operatorname{div} I_K^{RT_r}\vec{v}$.

Interpolation estimates $\quad \|\vec{v} - I_h^{RT}\vec{v}\|_{H(\mathrm{div},\Omega)} \leq Ch^{r+1}\left(|\vec{v}|_{H^{r+1}} + |\mathrm{div}\, v|_{H^{r+1}}\right)$

~~Observe that I_K^{RT} is exact on $\mathbb{P}_{r+1}$~~

Use exactness of I_K^{RT} on $\mathbb{P}_r$ + scaling argument to conclude that $\|\vec{v} - I_h^{RT} v\|_{L^2(\Omega)} \leq Ch^{r+1}|\vec{v}|_{H^{r+1}}$

Moreover, use that $\mathrm{div}\, I_h^{RT} v = \Pi_h^r \mathrm{div}\, v$ and the fact that Π_h^r is exact on $\mathbb{P}_r$ to conclude that $\|\mathrm{div}\,\vec{v} - \Pi_h^r \mathrm{div}\,\vec{v}\|_{L^2(\Omega)} \leq Ch^{r+1}|\mathrm{div}\, v|_{H^{r+1}}$

Approximation of the Darcy problem

We choose the spaces $V_h = W_h^r$ for the velocity and $Q_h = Y_h^r$ for the pressure.

Observe that $W_h^r \subset H(\mathrm{div},\Omega) \qquad Y_h^r \subset L^2(\Omega)$

Moreover $\mathrm{div}\, W_h^r \subset Y_h^r$

Therefore, the space $V_h^0 = \{ v \in V_h : b(v_h, q_h) = 0 \ \forall q_h \in Q_h \}$ is a subspace of V^0

Indeed $\quad b(v_h, q_h) = 0 \quad \forall q_h \in Q_h$

$\Rightarrow \int_\Omega \mathrm{div}\, v_h \, q_h = 0 \qquad \forall q_h \in Y_h^r \quad$ since $\mathrm{div}\, v_h \in Y_h^r$

we can take $q_h = \mathrm{div}\, v_h \Rightarrow \int_\Omega |\mathrm{div}\, v_h|^2 = 0 \Rightarrow \mathrm{div}\, v_h = 0$

hence $V_h^0 \subset V^0$

The approximation is therefore conforming!

In this spaces we have automatically
- continuity of $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ and $G(\cdot)$
- coercivity of $a(\cdot,\cdot)$ on V_h^0 (since $V_h^0 \subset V^0$)

Inf-sup condition

We want to prove now the discrete inf-sup condition:

$$\inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|q_h\|_Q \|v_h\|_V} \geq \beta_h > 0$$

which can be written as:

$$\forall\, q_h \in Y_h^r, \quad \exists\, \vec{v}_h \in W_h^r : \int_\Omega \operatorname{div} \vec{v}_h\, q_h \geq \beta_h \|\vec{v}_h\|_{H(\operatorname{div})} \|q_h\|_{L^2(\Omega)}$$

To prove this we exploit the continuous version of the proof:
$\forall\, q_h \in Y_h^r$ let φ be such that $\begin{cases} \Delta \varphi = q_h & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega \end{cases}$ and $v = \nabla \varphi$

Assuming Ω convex ~~then~~ we have $\varphi \in H^2$
We now construct $v_h = I_h^{RT_r} v$

Hence $\operatorname{div} v_h = \operatorname{div} I_h^{RT_r} v = \Pi_h^r \operatorname{div} v = \Pi_h^r q_h = q_h$

Moreover $\|v_h\|_{H(\operatorname{div})} = \| I_h^{RT_r} v \|_{H(\operatorname{div})} \leq c \|\vec{v}\|_{H^1} =$
$= c \|\nabla \varphi\|_{H^1} \leq c \|\varphi\|_{H^2} \leq c' \|q_h\|_{L^2(\Omega)}$

and $$\frac{\int_\Omega \operatorname{div} v_h\, q_h}{\|v_h\|_{H(\operatorname{div})}} \geq \frac{\|q_h\|_{L^2}^2}{c' \|q_h\|_{L^2}} \geq \frac{1}{c} \|q_h\|_{L^2(\Omega)}$$

Since all assumptions of the theorem (...) are satisfied, we conclude well posedness of the discrete problem and optimal convergence

$$\|u-u_h\|_{H(\mathrm{div},\Omega)} + \|p-p_h\|_{L^2(\Omega)} \leq C\Big(\inf_{v_h \in W_h^r} \|u-v_h\|_{H\mathrm{div}} + \inf_{q_h \in Y_h^r} \|p-q_h\|_{L^2(\Omega)}\Big)$$

by taking $v_h = I_h^{RT_r} u$ and $q_h = \Pi_h^r p$ we conclude

$$\|u-u_h\|_{H(\mathrm{div},\Omega)} + \|p-p_h\|_{L^2(\Omega)} \leq C h^{r+1}\big(|u|_{H^{r+1}} + |\mathrm{div}\, u|_{H^{r+1}} + |p|_{H^{r+1}}\big)$$