Project 3
WENO reconstruction

Paride Passelli and Niko Masiero

Numerical methods for conservation laws
Swiss Federal Institute of Technology in Lausanne

Prof. J.S. Hesthaven

December 30, 2018

The objective of this project is to apply the WENO reconstruction. For this purpose, as in the previous projects, we consider the one-dimensional *shallow water* equation:

$$\begin{bmatrix} h \\ m \end{bmatrix}_{tt} + \begin{bmatrix} m \\ \frac{m^2}{h} + \frac{1}{2}gh^2 \end{bmatrix}_x = \mathbf{S}(x,t)$$

Where we define $h = h(x,t)$, $m = m(x,t)$, we set the acceleration due to gravity $g = 1$. $\mathbf{S}$ is a source term, we define also the horizontal velocity $u = \frac{m}{h}$ and the spatial domain $\Omega = [0,2]$.
We define $\mathbf{q} = [h,m]^T$, hence the previous equation can be rewritten as follows:

$$\mathbf{q}_t + f(\mathbf{q})_x = \mathbf{S}(x,t), \tag{1}$$

where

$$f(\mathbf{q}) = \begin{bmatrix} m \\ \frac{m^2}{h} + \frac{1}{2}gh^2 \end{bmatrix}$$

Hence we can redefine (1) as follow:

$$\mathbf{q}_t + \nabla_{\mathbf{q}} f(\mathbf{q})_x = \mathbf{S}(x,t) \tag{2}$$

After computation of the Jacobian

$$\nabla_{\mathbf{q}} f = \begin{bmatrix} 0 & 1 \\ -\frac{m^2}{h^2} + gh & \frac{2m}{h} \end{bmatrix}$$

we obtain

$$\begin{bmatrix} h \\ m \end{bmatrix}_t + \begin{bmatrix} 0 & 1 \\ -\frac{m^2}{h^2} + gh & \frac{2m}{h} \end{bmatrix} \begin{bmatrix} h \\ m \end{bmatrix}_x = \mathbf{S}(x,t)$$

The objective is to obtain an high-order accurate scheme, for this purpose we consider the semi-discrete finite volume scheme, that as suggested, is obtained after integration of the conservation law in each cell $I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$.

$$\frac{dq_i}{dt} = -\frac{1}{h}(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}) + S_i(t) \tag{3}$$

where q_i and S_i are cell averages and $F_{1\pm\frac{1}{2}}$ is a consistent numerical flux approximating $f(q_{i\pm\frac{1}{2}})$. The goal that we want to reach in this case is the creation of a method with order convergence as a parameter but that avoid the possible creation of oscillations in the numerical solution. For this purpose we analyze the ENO and WENO schemes

1 ENO

Consider a function u of which we know the cell averages $\{\bar{u}_{j-p}, ..., \bar{u}_{j+q}\}$ around a cell interface $x_{j+\frac{1}{2}}$ and suppose that we want to approximate $u(x_{j+\frac{1}{2}})$ with a method

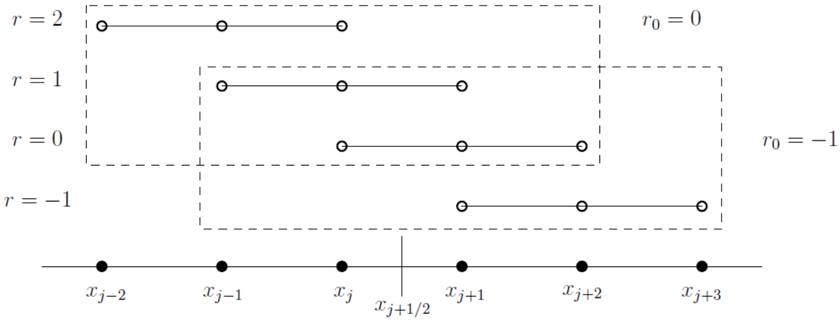

Figure 1: Three point stencil

that permits to choose the order of convergence as a parameter. As we can see in the figure 1 there are many possible choices of the cell averages that we want to use Fixing $m = p + q + 1$ as the convergence parameter and r as the shifting parameter, the basic idea of the ENO (Essentially Non Oscillatory) scheme is to find a polynomial π of order $m - 1$ such that

$$v(x_{j+\frac{1}{2}}) = \pi(x_{j+\frac{1}{2}}) + \mathcal{O}(h^m)$$

with the additional condition

$$\frac{1}{h}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \pi(x)dx = \bar{u}_i, \text{ for i} = j - p - r, ..., j + q - r$$

Once we found this polynomial the next step will be the choice of the m-point stencil to use between the $m + 1$ possibles. Since the accuracy of a numerical solution can be dramatically affected by the presence of a jump in the solution that we want to approximate, the ENO scheme will decide to choose the stencil that doesn't include a jump of the solution between two points, or at least it choose the stencil such that the polynomial as less oscillations, in other words we need to choose the smoothest stencil among the possible ones. In figure 2 we can see an example of the right/bad choice of a stencil.

The convergence result given using this type of stencil selection is

$$u(x_{j+\frac{1}{2}}) = \pi(x_{j+\frac{1}{2}}) + \mathcal{O}(h^m) \tag{4}$$

2 WENO

In the ENO scheme we need to evaluate $m + 1$ stencils, so we use $2m$ points. This is quite expensive in terms of running time, especially if the convergence rate is only of order $\mathcal{O}(h^m)$. Moreover we know that in the regions where the solution is sufficiently smooth we need only a linear scheme to approximate the solution. The idea of WENO

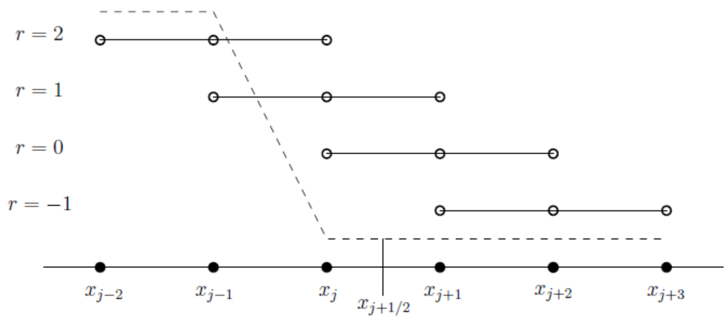

Figure 2: Three point stencil with a jump in the solution. In this case the first two stencils will be excluded due to the presence of a jump in the interval $[x_{j-1}, x_j]$

(Weighted Essentially Non Oscillatory) scheme is to use all the stencils to approximate the value of u at the cell interface and achieve a better convergence rate.
Let $u^{(r)}_{j+\frac{1}{2}}$ be the numerical solution at the interface computed with the stencil r. In first case we assume that the solution u is smooth in a neighborhood of the cell interface then, as said before, a linear scheme is sufficient so we can use a linear combination of the numerical solutions founded

$$u_{j+\frac{1}{2}} = \sum_{r=r_0}^{m-1+r_0} \omega_r^{r_0} u^{(r)}_{j+\frac{1}{2}} \tag{5}$$

with the coefficients $\omega_r^{r_0}$ chosen such that the order of convergence of the numerical solution is $\mathcal{O}(h^{2m-1})$. Moreover, in order to guarantee stability and convergence we need also that the sum of these weights is equal to 1.
Now we consider a region where there is a jump of the solution, in this case we know that the ENO scheme perform well so we can think to apply this scheme in this particular region. The idea of the WENO scheme is to add more weight on the solution computed with stencils that don't capture the jump and consider less the solution with stencils that consider the jump.

3 Lax-Friedrichs implementation

We now test our code with the Lax-Friedrichs flux (obtained in the same way as project 1). Remark that in this section and the next one, each time that we compute the error of a scheme we compute the infinity norm, thus we compute the max of the error of $h(x,t)$ and $m(x,t)$ and we take the maximum between the two.

3.1 First case

We apply the following initial conditions:

$$h(x,0) = 1 + \frac{1}{2} sin(\pi x)$$

$$m(x,0) = uh(x,0)$$

and periodic boundary conditions $q_0 = q_{N-1}$ and $q_N = q_1$ Moreover we consider the following source

$$S(x,t) = \begin{bmatrix} \frac{\pi}{2}(u-1)cos(\pi(x-t)) \\ \frac{\pi}{2}cos(\pi(x-t))(-u+u^2+gh(x-t,0)) \end{bmatrix}$$

and we evaluate the time-step as

$$k = \text{CFL}\frac{\Delta x}{\max_i(|u_i| + \sqrt{gh_i})}$$

with CFL$= \frac{1}{2}$ and $T = 2$. We also choose $K = 2$.
In Figure 3 we can observe the plot of the numerical solution against the true solution at final time $T = 2$. Moreover we plot the error on a loglog graph at final time $T = 2$,

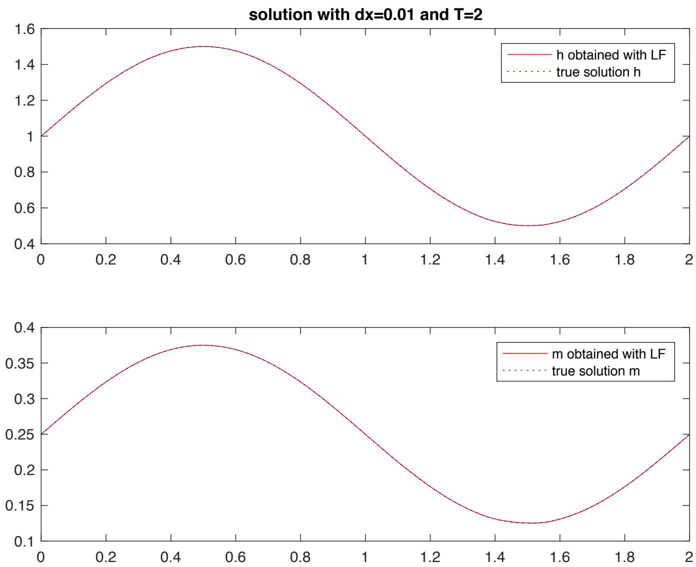

Figure 3: Numerical solution with LF at final time

since the solution is smooth we expect a order of convergence of 3 (because of the choice of $K = 2$. In Figure 4 we can observe that our expectation is satisfied.

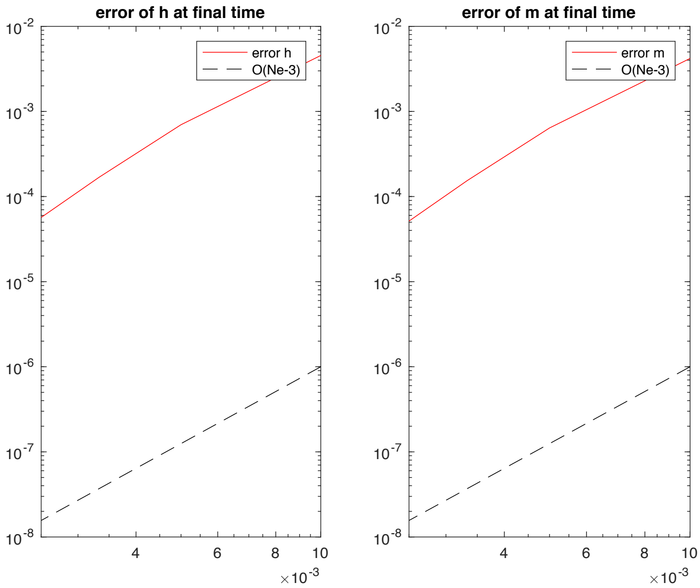

Figure 4: Numerical error with LF at final time

3.2 Second case

We test again our code with the following initial conditions: $h(x,0) = 1 - 0.1sin(\pi x)$, $m(x,0) = 0$ and source term equal to zero. We take again periodic boundary conditions and $K = 2$. Figure 5 shows the numerical solution against a reference one (computed on a fine grid $\Delta x = 0.001$). We can measure the error of the scheme at final time as function of Δx, this is shown by Figure 6, where we observe that the order of convergence is 3, as expected.

3.3 Third case

Once again we solve the one-dimensional shallow water equation using the usual method with initial conditions:

$$h(x,0) = 1 - 0.2sin(2\pi x)$$

and

$$m(x,0) = 0.5$$

As in the previous case we compute a reference solution on a fine grid (Δx=0.005). We can observe the plot of the numerical solution at final time $T = 2$ against the reference one (Figure 7). We are still interested on the order of convergence. We plot the error as function of Δx. In opposition of the previous case we notice that the solution is not smooth, in fact there are some sharp corners, hence the solution is not even differentiable in some points, so we can not expect to get a global order 3. We can observe the plot in Figure 8.

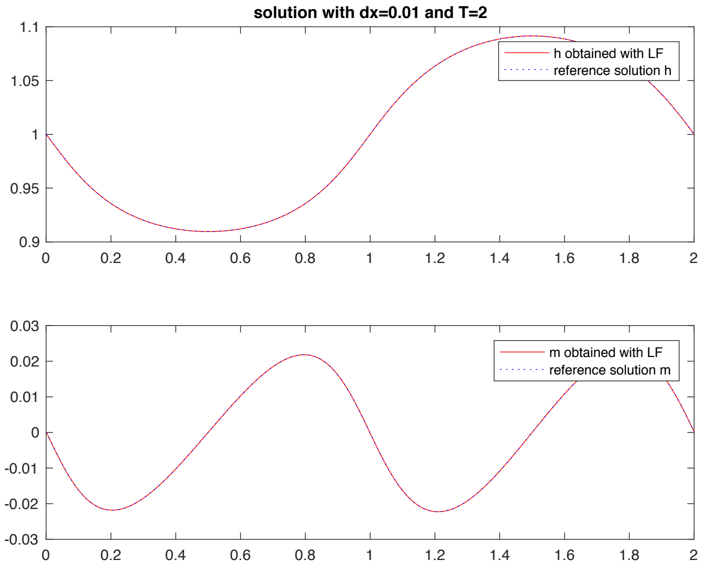

Figure 5: Numerical solution with LF at final time

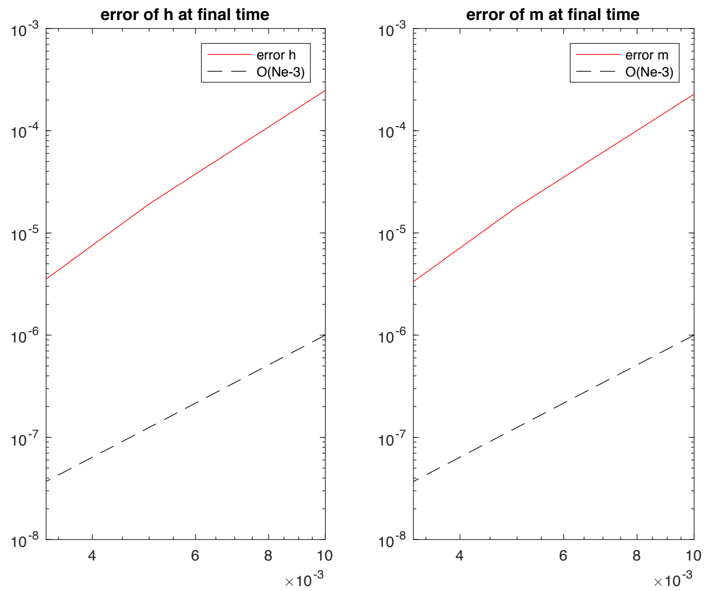

Figure 6: Numerical error with LF at final time

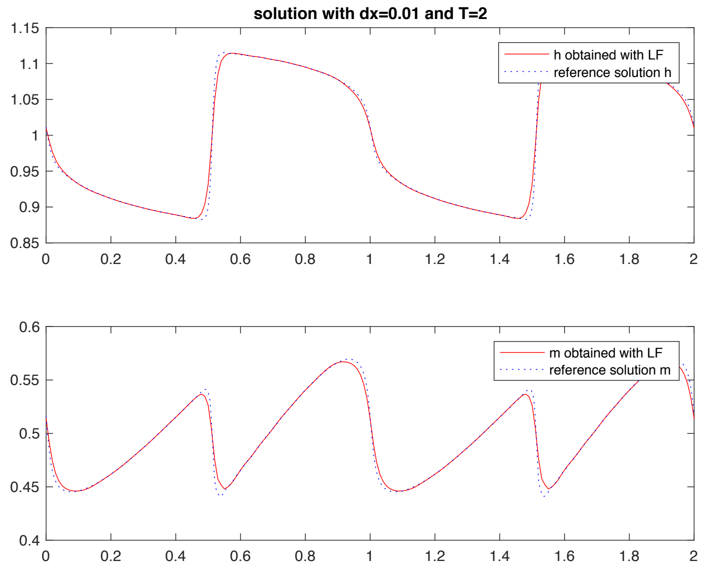

Figure 7: Numerical solution with LF at final time

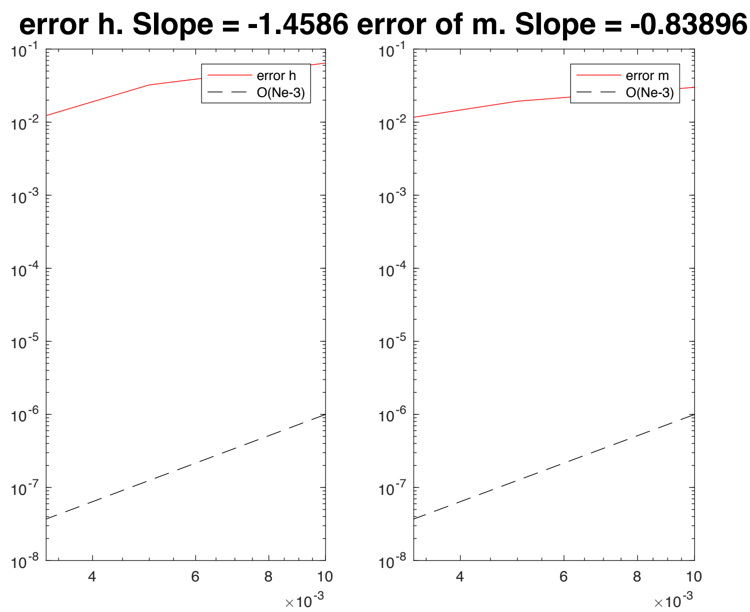

Figure 8: Numerical error with LF at final time

4 Roe implementation

We repeat all the experiment of section 3 using the Roe flux. Also in this case all the derivation of the Roe scheme can be find in project 1.

4.1 First case

As before we start by testing our implementation with the following initial conditions:

$$h(x,0) = 1 + \frac{1}{2}sin(\pi x)$$

$$m(x,0) = uh(x,0)$$

and periodic boundary conditions $q_0 = q_{N-1}$ and $q_N = q_1$ Moreover we consider the following source

$$S(x,t) = \begin{bmatrix} \frac{\pi}{2}(u-1)cos(\pi(x-t)) \\ \frac{\pi}{2}cos(\pi(x-t))(-u+u^2+gh(x-t,0)) \end{bmatrix}$$

and we evaluate the time-step as

$$k = \text{CFL}\frac{\Delta x}{\max_i(|u_i| + \sqrt{gh_i})}$$

with usual CFL$= \frac{1}{2}$ and $T = 2$. As always until now we take $K = 2$.
Figure 9 shows the numerical solution plotted against a reference one (computed with $\Delta x = 0.005$) at final time $T = 2$. We observe the measure of the error at final time as function of Δx (Figure 10) since the solution is smooth and $K = 2$ we obtain an order of 3 as expected.

4.2 Second case

We apply once more our code to the following initial conditions: $h(x,0) = 1-0.1sin(\pi x)$, $m(x,0) = 0$ and source term equal to zero. We take again periodic boundary conditions and $K = 2$. We compute a reference solution with $\Delta x = 0.005$ and we plot it against the numerical solution at final time in Figure 11. Once again we measure the error on a loglog graph (Figure 12), as in the case of the Lax-Friedrich flux we obtain a third order convergence.

4.3 Third case

Last we do our test with the following initial conditions:

$$h(x,0) = 1 - 0.2sin(2\pi x)$$

and

$$m(x,0) = 0.5$$

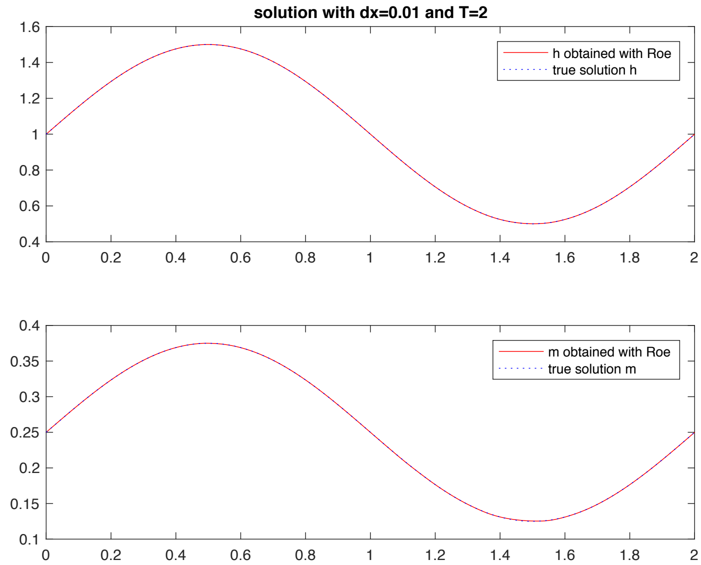

Figure 9: Numerical solution with Roe at final time

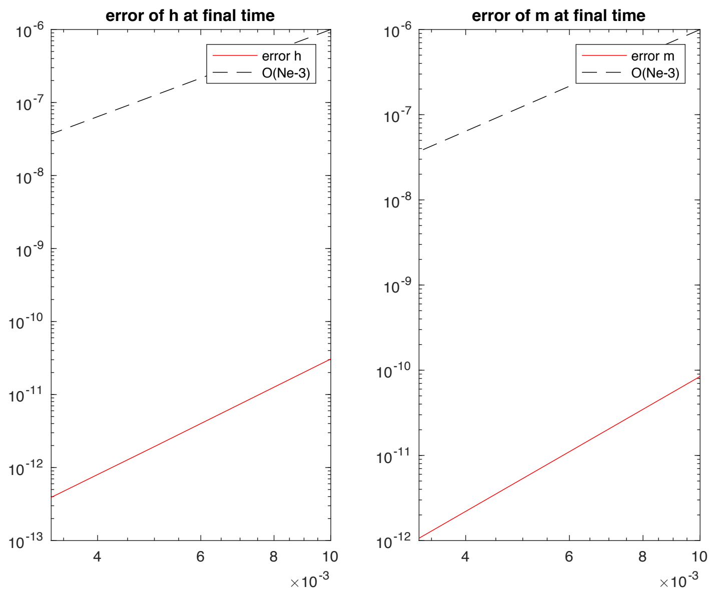

Figure 10: Numerical error with Roe at final time

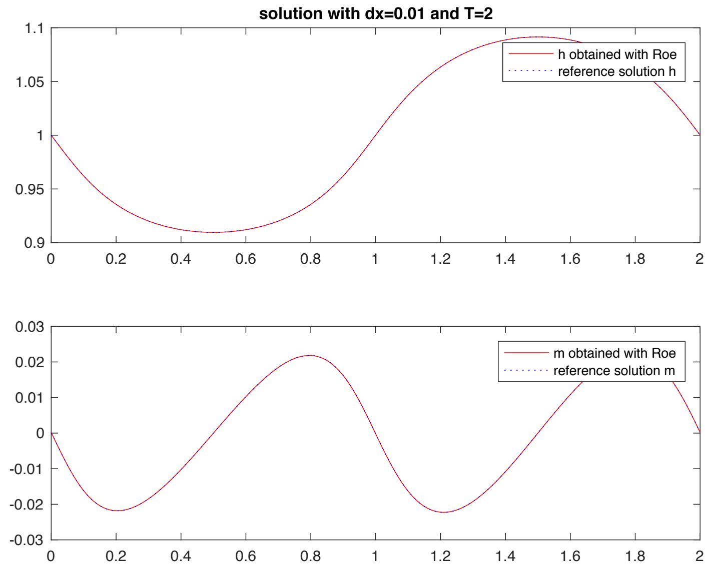

Figure 11: Numerical solution with Roe at final time

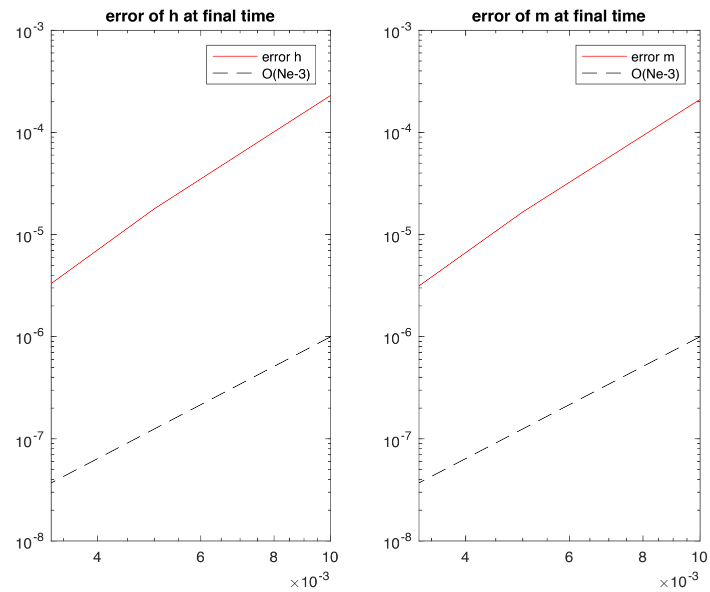

Figure 12: Numerical error with Roe at final time

with periodic boundary conditions and zero source term. We compute, as usual, a reference solution with $\Delta x = 0.005$ that we will use to compute the error of the scheme and also to make a comparison with the computed solution. In Figure 13 we observe the plot of the numerical solution against the reference one at $T = 2$. In Figure 14, instead,

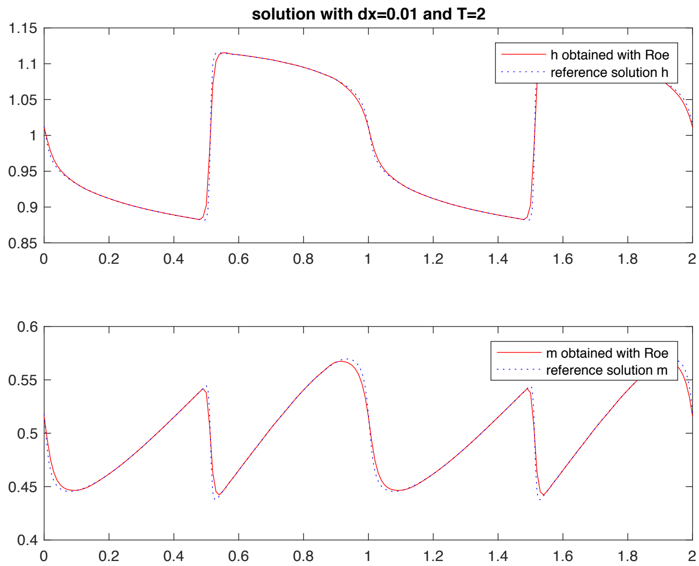

Figure 13: Numerical solution with Roe at final time

we can observe the measure of the error at final time. As before the error is not expected to be of order 3, because of the non smoothness of the solution.

4.4 Fourth case

We are now studying the one-dimensional shallow water equation with open boundary conditions, zero source term $S = 0$, final time $T = 0.5$ and the following initial conditions:

$$h(x, 0) = 1$$

and

$$m(x, 0) = \begin{cases} -1.5 & \text{if } x < 1 \\ 0.0 & \text{if } x > 1 \end{cases}$$

We use Lax-Friedrichs scheme in order to compute a reference solution on a very fine grid mesh ($\Delta x = 0.05$). Then we compute a numerical solution with the Roe flux. We can observe the plot at final time in Figure 15. In this case we actually observe that there is an entropy violation given by the small obscillation in the neighborhood of $x = 1$.

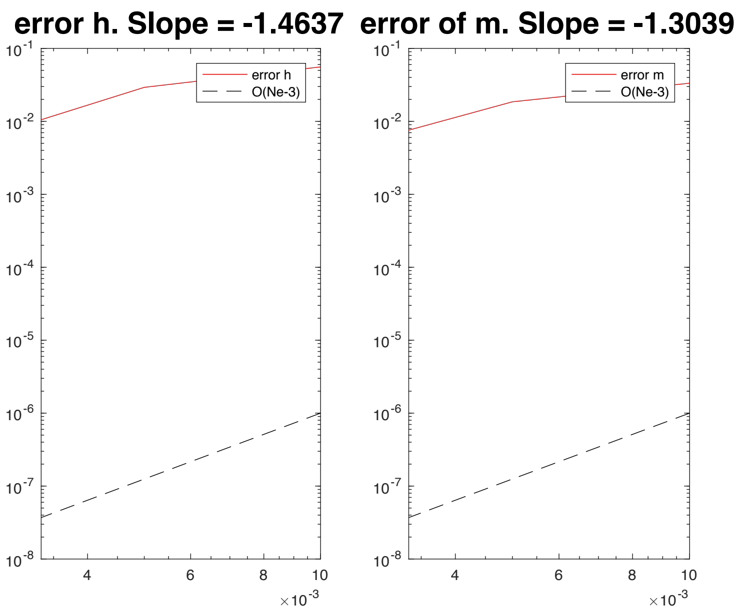

Figure 14: Numerical error with Roe at final time

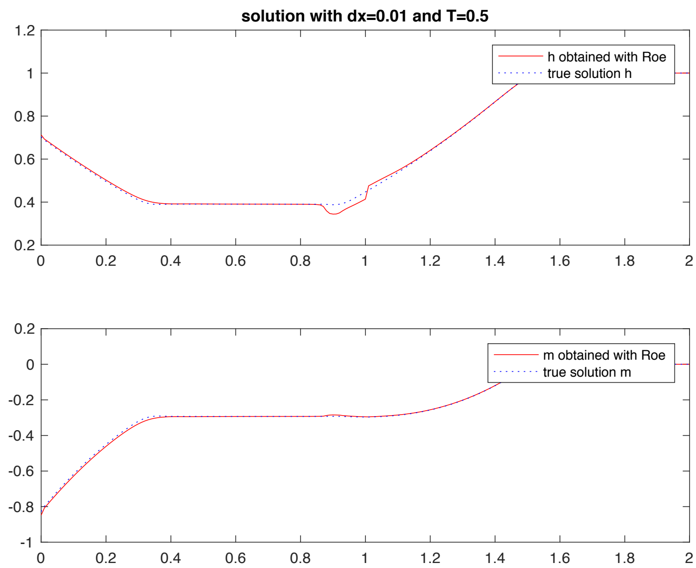

Figure 15: Numerical solution with Roe at final time