Coloring the Map of U.S. Counties

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MATHEMATICAL BACKGROUND

Maps are used in the modern world on a daily basis, and coloring maps is a practice that's been around for centuries. Historically, colors in maps were used in early exploration or to map out occupied territories in conflicts. Borders have been used since ancient times to separate two general areas under different rulers and cultures. In mapmaking, color is used to distinguish one known area from the other. The critical use of coloring a border's area is to make sure the neighboring areas are not the same color, otherwise the map will be misleading.

In 1852, Francis Guthrie came up with the idea that maps of any size can consist of four colors. He realized this fact when he was coloring a map of England's counties, and found that he had only used four colors in the process. Throughout the following decades, the concept got the attention of Arthur Cayley and Alfred Kempe, who grew the four color concept into the Four Color Theorem. The theorem's proof was widely disputed by mathematicians, until 1967 when it was proved by a computer. The Four Color Theorem is also known as the first verified mathematical proof to be completed by a computer.

TERMS & CONCEPTS

> **Territory:** An area of land under a jurisdiction.

> **Border:** A line separating two sovereign areas.

> **Map Coloring:** Assigning colors to maps for the purpose of distinguishing separate areas.

> **Theorem:** A widely accepted proven fact that does not require elaboration.

> **Proof:** An argument through observations and inferences to logically validate a fact.

> **Graph:** A finite set of points (vertices) connected with a finite set of lines (edges).

> Planar **Graph:** A graph where its lines (edges) do not intersect.

MATHEMATICAL TECHNIQUES

In graph theory, we can convert the area of a territory to be represented as a point, where each neighboring points are connected to each other with a line. This will always result in a planar graph.

The Four Color Theorem

> Any planar graph can be colored with four or fewer colors.

> No adjacent area may be the same color

> The algorithm is a repetitive task, where you move along each adjacent vertex and fill it in with a different color.

> Pick a set of four colors in a prioritized list and start at any point...
- Color the starting point with Color 1, and move onto the next adjacent vertex, then color that vertex with Color 2.
- As you move along each point, if the adjacent points are of Color 1 or 2, use Color 3.
- If Color 1, 2, and 3 are taken on the next adjacent vertices, use Color 4.
- In the case of a point's adjacent neighbors are of all Colors 1, 2, 3, and 4, go back and make a change. There must be an ordering mistake in one of the previous points. This is to properly apply the theorem.
- Repeat until all vertices are properly colored.

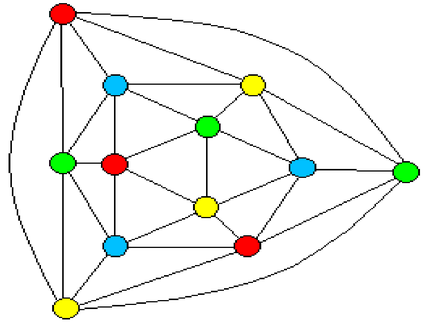

Figure 1: Example of a planar graph using the Four Color Theorem [4].

APPLICATION

The map of United States counties may be one of the most complex applications of the Four Color Theorem. Each U.S. State is composed of a certain amount of counties, that has a smaller-scale government authority over its people. Politically, the map of U.S. counties are used to help visualize how a certain area tends to vote in elections. There are a total of 3,143 counties in the United States.

> Please Note that some states like Alaska refers to their compositional areas as Boroughs, where Louisiana refers to its counties as Parishes. For the sake of this application, we will refer to those areas also as counties.

> Consider Washoe County in Nevada. There are 13 neighboring counties that surround Washoe.

> There are six other counties in the U.S. that have at least 10 neighbors.

> If you look at the eastern United States, notice how the counties get much smaller. This fact makes the application fragile, as you do not want to exceed four colors.

Map of U.S. Counties

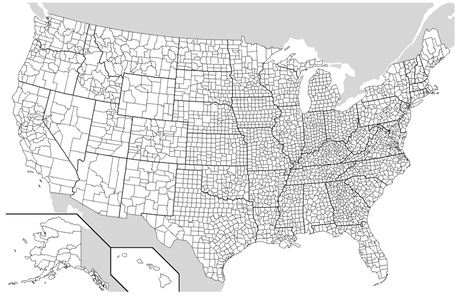

Figure 2: Blank map of the United States counties [3].

THE MODEL

There are 3,143 possible starting points in this model. Start out by taking the blank map of U.S. counties and drawing a point in every county, and connect each of the adjacent points to each other. The setup is visualized below in Figure 3, where the West Coast of the United States is mapped. Note that interstate connections are also made.

Figure 3: Planar graph of the counties of the U.S. states on the West Coast.

> No two neighboring counties may be the same color.

> To achieve the solution, since there are so many counties, either work on one U.S. State at a time or use a computer to generate results. Once every county is converted into a vertex, and are all adjacently connected to the other vertices, apply the algorithm exactly as stated in the Mathematical Techniques section.

> The solution is displayed below in Figure 4.

Colored Map of U.S. Counties

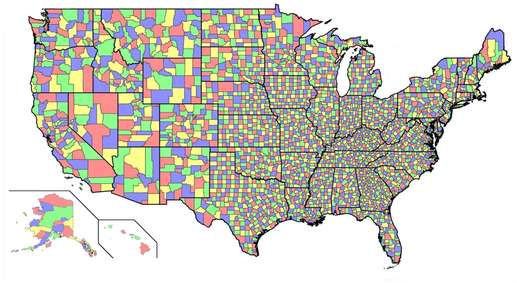

Figure 4: Colored Map of all the U.S. counties in compliance with the Four Color Theorem [6].

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