Econometrics Assignment 3

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Question 1

1

Population moment conditions:

$\mu_2 = E((X_1 - \mu)^2 - \theta_2) = 0$

$\mu_3 = E(X_1 - \mu)^3 - \theta_3\theta_2^{\frac{3}{2}}) = 0$

$\mu_4 = E(X_1 - \mu)^4 - \theta_4\theta_2^2) = 0$

Sample moment conditions:

$\frac{1}{n}\sum_{i=1}^{n}((X_1 - \mu)^2 - \theta_2) = 0$

$\frac{1}{n}\sum_{i=1}^{n}((X_1 - \mu)^3 - \theta_3\theta_2^{\frac{3}{2}}) = 0$

$\frac{1}{n}\sum_{i=1}^{n}((X_1 - \mu)^4 - \theta_4\theta_2^2) = 0$

2

$\bar{g}(\theta) = \frac{1}{n}\sum_{i=1}^{n} g(x_i;\theta) = \frac{1}{n}\sum_{i=1}^{n}\begin{bmatrix}(x_i-\mu)^2 - \theta_2\\ (x-\mu)_i^3 - \theta_3\theta_2^{\frac{3}{2}}\\ (x_i-\mu)^4 - \theta_4\theta_2^2\end{bmatrix}$

min $\bar{g}(\theta)^T W \bar{g}(\theta)$=$\{\frac{1}{n}\sum_{i=1}^{n}\begin{bmatrix}(x_i-\mu)^2 - \theta_2\\ (x_i-\mu)^3 - \theta_3\theta_2^{\frac{3}{2}}\\ (x_i-\mu)^4 - \theta_4\theta_2^2\end{bmatrix}\}^\top W\{\frac{1}{n}\sum_{i=1}^{n}\begin{bmatrix}(x_i-\mu)^2 - \theta_2\\ (x_i-\mu)^3 - \theta_3\theta_2^{\frac{3}{2}}\\ (x_i-\mu)^4 - \theta_4\theta_2^2\end{bmatrix}\}$ with W=I

It is not important to include a weighting matrix different from an identity matrix, I, because the case is exactly identified (number of moments equals the number of parameters).

3

(a)

Using the provided asymptotic variance formula, other conditions provided, and that the empirical moments are $m_k = \sum_{i=1}^{n}(x_i - \bar{x})^k/n$, we have that the variance matrix S is

$$S = \begin{bmatrix}\mu_4 - \mu_2^2 & 0 & \mu_6 - \mu_2\mu_4\\ 0 & \mu_6 - 9\mu_2^3 - 6\mu_4\mu_2 & 0\\ \mu_6 - \mu_2\mu_4 & 0 & \mu_8 - \mu_4^2\end{bmatrix} = \begin{bmatrix}2\sigma^4 & 0 & 12\sigma^6\\ 0 & 6\sigma^6 & 0\\ 12\sigma^6 & 0 & 96\sigma^8\end{bmatrix}$$

(b)

$G = \frac{g(\theta_2,\theta_3,\theta_4;x_i)}{\delta\theta_2\delta\theta_3\delta\theta_4} = \begin{bmatrix}-1 & 0 & 0\\ -\frac{3}{2}\theta_3\theta^{\frac{1}{2}} & -\theta_2^{\frac{3}{2}} & 0\\ -2\theta_4\theta_2 & 0 & -\theta_2^2\end{bmatrix} = \begin{bmatrix}-1 & 0 & 0\\ 0 & -\sigma^3 & 0\\ -6\sigma^2 & 0 & -\sigma^4\end{bmatrix}$

(c)

We know that $\sqrt{n}(\hat{\theta}_n - \theta) \to N(0, V)$, where $V = (G^\top W G)^{-1} G^\top W S W G (G^\top W G)^{-1}$. Since this is an exactly identified case $W = I$, so $V = G^{-1} S (G^\top)^{-1}$.

Using (a) and (b), we get:

$$V = \begin{bmatrix} 2\sigma^4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 24 \end{bmatrix}$$

4

We test $H_0 : \theta_3 = 0,\ \theta_4 = 3$ vs. $H_1 : \theta_3 \neq 0,\ \theta_4 \neq 3$

Then, we have that $R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and $q = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$.

So, our test is $H_0 : R\hat{\theta} - q = 0$ vs. $H_1 : R\hat{\theta} - q \neq 0$

We define $\hat{\theta}_2 = m_2$, $\hat{\theta}_3 = \frac{\mu_3}{\mu_2^{1.5}} = \frac{m_3}{m_2^{1.5}}$, and $\hat{\theta}_4 = \frac{\mu_4}{\mu_2^2} = \frac{m_4}{m_2^2}$. Also, using (b) define $\hat{V}_n = \begin{bmatrix} 2m_2^2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 24 \end{bmatrix}$

In this case, the Wald test statistic is

$$W_n = n(R\hat{\theta} - q)^\top (R\hat{V}_n R^\top)^{-1} (R\hat{\theta} - q) \to \chi^2_2$$

5

```
data = read.csv("C:/Users/eitas/OneDrive/Stalinis kompiuteris/Econometrics/Tutorial 1/daxrets.csv",)

x = data$stdres
n = length(x)

m2 = 0
m3 = 0
m4 = 0

for (i in 1:n) {
  m2 = m2 + (x[i]-mean(x))^2
  m3 = m3 + (x[i]-mean(x))^3
  m4 = m4 + (x[i]-mean(x))^4
}

m2 = 1/n*m2
m3 = 1/n*m3
m4 = 1/n*m4

theta2 = m2
theta3 = m3/(m2)^(3/2)
theta4 = m4/(m2)^2

R = rbind(c(0, 1, 0), c(0, 0, 1))
q = c(0, 3)
```

```r
theta = rbind(theta2, theta3, theta4)
V = rbind(c(6, 0), c(0, 24))

W = n*t(R%*%theta-q)%*%solve(V)%*%(R%*%theta-q)
chi = qchisq(p = .99, df = 2)
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We have that $\chi^2_{2,\ p=0.99} = 9.21$, $W = 755$. Since $W > \chi^2_{2,\ p=0.99}$ we reject the null hypothesis at 99% significane level.

Question 2

1

It's given that $y_t = \beta_0 + \beta_1 y_{t-1} + \epsilon_t$.

Since $\mu = E(y_t) = E(\beta_0 + \beta_1 y_{t-1} + \epsilon_t) = \beta_0 + \beta_1 E(y_{t-1}) + E(\epsilon_t) = \beta_0 + \beta_1 \mu$ as $E(\epsilon_t) = 0$. Then is follows that $\mu(1-\beta_1) = \beta_0$, which gives us $\mu = \frac{\beta_0}{1-\beta_1}$ as $|\beta_1| < 1$ (we divide by a nonzero number).

$E(y_t) = \mu = \frac{\beta_0}{1-\beta_1}$

2

An instrument is valid if an only if it's correlated with a variable that we would use an instrument for and if the instrument is not correlated with the error term. Since correlation is defined to be a covariance of the instrument and the variable we're using the instrument for divided by the product of their standard deviation. Therefore, for y_{t-s} to be a valid instrument we need: $Cov(y_{t-s}, y_t) \neq 0$ and $Cov(y_{t-s}, \epsilon_t) = 0$.

$Cov(y_t, \epsilon_t) = Cov(y_t, u_t + \alpha_1 u_{t-1} + \alpha_2 u_{t-2}) = Cov(y_{t-s}, u_t) + \alpha_1 Cov(y_{t-s}, u_{t-1}) + \alpha_2 Cov(y_{t-s}, u_{t-2})$. By definition, a covariance of 2 variables is equal to the expectation of the product minus the product of expectations. Since the expectations of u_t, u_{t-1} and u_{t-2} are all 0, the product of expectations is 0 for each covariance. Therefore, $Cov(y_t, \epsilon_t) = E(y_{t-s}u_t) + \alpha_1 E(y_{t-s}u_{t-1}) + \alpha_2 E(y_{t-s}u_{t-2})$. $Cov(y_t, \epsilon_t)$ will always be equal to 0 if $s \geq 3$, because the past results are not correlated with future error terms.

$Cov(y_{t-s}, y_t) = Cov(y_{t-s}, \beta_0 + \beta_1 y_{t-1} + \epsilon_t) = 0 + \beta_1 Cov(y_{t-s}, y_{t-1}) + 0 = \beta_1 Cov(y_{t-s}, y_{t-1})$. Since the time series are correlated, the covariance between the variable will be different from 0. Therefore, $\beta_1 \neq 0$ will guarantee the second condition of instrument's validity.

3

We know given population moments:

$E(\epsilon_t^2) = Var(\epsilon_t) = Var(u_t + \alpha_1 u_{t-1} + \alpha_2 u_{t-2}) = \sigma^2 + \alpha_1^2\sigma^2 + \alpha_2^2\sigma^2$

$E(\epsilon_t\epsilon_{t-1}) = E((u_t + \alpha_1 u_{t-1} + \alpha_2 u_{t-2})(u_{t-1} + \alpha_1 u_{t-2} + \alpha_2 u_{t-3})) = \alpha_1\sigma^2 + \alpha_1\alpha_2\sigma^2$

$E(\epsilon_t\epsilon_{t-2}) = E((u_t + \alpha_1 u_{t-1} + \alpha_2 u_{t-2})(u_{t-2} + \alpha_1 u_{t-3} + \alpha_2 u_{t-4})) = \alpha_2\sigma^2$

Population moment equation:

$$0 = E(g(\epsilon_t; \alpha_1, \alpha_2, \sigma^2)) = E\begin{bmatrix} \epsilon_t^2 - (\sigma^2 + \alpha_1^2\sigma^2 + \alpha_2^2\sigma^2) \\ \epsilon_t\epsilon_{t-1} - (\alpha_1\sigma^2 + \alpha_1\alpha_2\sigma^2) \\ \epsilon_t\epsilon_{t-2} - \alpha_2\sigma^2 \end{bmatrix}$$

Sample moment equation: $0 = \frac{1}{T}\sum_{t=3}^{T} g(\epsilon_t; \alpha_1, \alpha_2, \sigma^2)$.

It's not important to include a weighting matrix different from an identity matrix, I, because the case is exactly identified (number of moments equals the number of parameters). The GMM estimator is then:
$\begin{bmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \\ \hat{\sigma}^2 \end{bmatrix} = \underset{\alpha_1,\alpha_2,\sigma^2}{argmin}(\frac{1}{T}\sum_{t=3}^{T} g(\epsilon_t;\alpha_1,\alpha_2,\sigma^2))^T I(\frac{1}{T}\sum_{t=3}^{T} g(\epsilon_t;\alpha_1,\alpha_2,\sigma^2))$

4

Testing for serial autocorrelation:

1. H_0: $\alpha_2 = 0$
2. H_0: $\alpha_1 = 0$ and $\alpha_2 = 0$

For case 1. $H_0 : R \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \sigma^2 \end{bmatrix} - q = 0$ vs $H_1 : R \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \sigma^2 \end{bmatrix} - q \neq 0$, with $R = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$ and $q = 0$.
Test statistic: $W_n = n \left(R \begin{bmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \\ \hat{\sigma}^2 \end{bmatrix} - q \right)^T \left(R\hat{V}_n R^T \right)^{-1} \left(R \begin{bmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \\ \hat{\sigma}^2 \end{bmatrix} - q \right) = \frac{\hat{\alpha}_2^2}{\widehat{Var(\alpha_2)}}$
Under H_0, $W_n \xrightarrow{d} \chi^2_1$
Reject H_0 if $W_n \geq \chi^2_{1;1-\alpha}$

For case 2. $H_0 : R \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \sigma^2 \end{bmatrix} - q = 0$ vs $H_1 : R \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \sigma^2 \end{bmatrix} - q \neq 0$, with $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $q = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
Test statistic: $W_n = n \left(R \begin{bmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \\ \hat{\sigma}^2 \end{bmatrix} - q \right)^T \left(R\hat{V}_n R^T \right)^{-1} \left(R \begin{bmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \\ \hat{\sigma}^2 \end{bmatrix} - q \right) = \begin{bmatrix} \hat{\alpha}_1 & \hat{\alpha}_2 \end{bmatrix} \left(\begin{bmatrix} \widehat{Var}(\hat{\alpha}_1) & \widehat{Cov}(\hat{\alpha}_1,\hat{\alpha}_2) \\ \widehat{Cov}(\hat{\alpha}_1,\hat{\alpha}_2) & \widehat{Var}(\hat{\alpha}_2) \end{bmatrix} \right)^{-1} \begin{bmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \end{bmatrix}$
Under H_0, $W_n \xrightarrow{d} \chi^2_2$
Reject H_0 if $W_n \geq \chi^2_{2;1-\alpha}$

5

Testing for second order autocorrelation (case 1 from 2.4). Reject H_0 if $W_n \geq \chi^2_{1;0.95} = 3.841$

$W_n : \frac{(-0.133756)^2}{0.072426^2} = 3.411 < 3.841$. We don't reject H_0. Therefore, we can't find presence of the second order autocorrelation at 5% significance level.

It's not possible to test the presence of any autocorrelation (case 2 from 2.4) since we don't have the estimated variance-covariance matrix for α_1 and α_2.

From the output provided, we reject the $H_0 : \alpha_1 = 0$, meaning that $\alpha_1 \neq 0$. It indicates that y_{t-1} (the explanatory variable) is correlated with the error term, ϵ_t. The OLS model would be inconsistent in this case and the GMM would have to be used.

6

From output and question 5, we concluded that at 5% significance level we reject $\alpha_1 = 0$ and we don't reject $\alpha_2 = 0$. From question 2, the $Cov(y_t, \epsilon_t)$ is correlated with α_2. Since we don't reject $\alpha_2 = 0$, we can assume that α_2 equals 0 and therefore, there is no second order autocorrelation. As a result, y_{t-2} can be used as an instrument. Since we reject for $\alpha_1 = 0$, we can't use y_{t-1} as an instrument. Hence, no autocorrelation holds for $s \geq 2$, instead of $s \geq 3$ as in question 2.