Exercise 4

$x\ [-1,1]$
$y\ [-1,1]$
$z\ [0,1]$

$[0,1]^2$

model space $\xrightarrow{\text{mMatrix}}$ world space $\xrightarrow{\text{vMatrix}}$ view space $\xrightarrow{\text{pMatrix}}$ NDC $\longrightarrow$ texture coordinates $\longrightarrow$

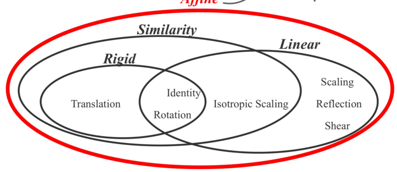
Rigid transformations (also called "euclidean") preserve distances and angles (a reflection for example)

Similarity transformations preserve angles but not distances

Linear transformations $L(p+q) = L(p) + L(q)$
$L(ap) = a \cdot L(p)$
$L(0) = 0$

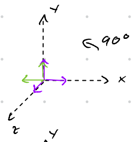
Affine transformations preserve parallel lines

Affine → Preserve paralell lines (linear transformation + translation)

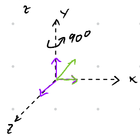


- Rotation (counterclockwise around x axis): $T_{rotx} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
- Rotation (counterclockwise around y axis): $T_{roty} = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
- Rotation (counterclockwise around z axis): $T_{rotz} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

How to remember →



$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} \cos\theta & -\text{sen}\,\theta & 0 \\ \text{sen}\,\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} \cos\theta & 0 & \text{sen}\,\theta \\ 0 & 1 & 0 \\ -\text{sen}\,\theta & 0 & \cos\theta \end{pmatrix}$$

- Scaling: $T_{scale} = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
- Reflection (here by yz plane): $T_{reflection,xy} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
- Shearing: $T_{shear} = \begin{pmatrix} 1 & \underline{\lambda} & \underline{\mu} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
- Translation: $T_{translation} = \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$

① Derive the matrix to rotate an object along z by 90°

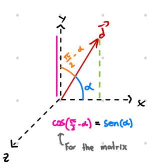
$$T = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\text{sen}\,\theta & 0 & 0 \\ \text{sen}\,\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\text{sen}\,\theta & 0 & -\cos\theta \\ \text{sen}\,\theta & \cos\theta & 0 & -\text{sen}\,\theta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\text{sen}\,\theta & 0 & -\cos\theta + 1 \\ \text{sen}\,\theta & \cos\theta & 0 & -\text{sen}\,\theta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \underset{=}{\theta = \frac{\pi}{2}} \begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

↳ we move the axis (origin of rotation) to (0,0,0), rotate, and the go back to the original position

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos & -\text{sen} & 0 & 0 \\ \text{sen} & \cos & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

②

a) Rotate 45° around axis (0.5, 0.5, 0.75)



$\cos(\frac{\pi}{2} - \alpha) = \text{sen}(\alpha)$
↑ For the matrix

$\vec{d}'$ is direction of $\vec{d}$ in xy plane $\vec{d}' = \begin{pmatrix} 0.5 \\ 0.5 \\ 0 \end{pmatrix}$
↳ The projection of the rotation axis

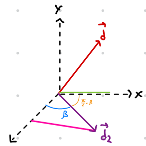
$$\begin{pmatrix} \cos & -\text{sen} & 0 & 0 \\ \text{sen} & \cos & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\vec{d}' \cdot e_x = \|\vec{d}'\| \, \|e_x\| \cos\alpha$$

$$\cos\alpha = \frac{\vec{d}' \cdot e_x}{\|\vec{d}'\| \, \|e_x\|} = \frac{\frac{1}{2}}{\sqrt{0.5^2 + 0.5^2}} = \frac{\frac{1}{2}}{\sqrt{\frac{1}{2}}} = \frac{\frac{1}{2}\sqrt{\frac{1}{2}}}{\frac{1}{2}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\vec{d}' \cdot e_y = \|\vec{d}'\| \, \|e_y\| \cos(\tfrac{\pi}{2} - \alpha)$$

$$\cos(\tfrac{\pi}{2} - \alpha) = \text{sen}(\alpha) = \frac{\vec{d}' \cdot e_y}{\|\vec{d}'\| \, \|e_y\|} = \frac{\frac{1}{2}}{\sqrt{\frac{1}{2}}} = \frac{\sqrt{2}}{2}$$



$sen(\beta) = \cos(\frac{\pi}{2} - \beta)$

$$\vec{d_2} = R_1 \cdot \vec{a} = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0.5 \\ 0.5 \\ 0.75 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ 0.75 \end{pmatrix}$$

$$R_2 = \begin{pmatrix} \cos\theta & 0 & sen\theta \\ 0 & 1 & 0 \\ -sen\theta & 0 & \cos\theta \end{pmatrix} = \begin{pmatrix} 3/\sqrt{17} & 0 & \frac{2\sqrt{2}}{\sqrt{17}} \\ 0 & 1 & 0 \\ \frac{-2\sqrt{2}}{\sqrt{17}} & 0 & 3/\sqrt{17} \end{pmatrix}$$

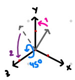
$$\vec{d_2} \cdot e_z = \|\vec{d_2}\| \cdot \|e_z\| \cdot \cos(\beta)$$

$$\cos(\beta) = \frac{\vec{d_2} \cdot e_z}{\|\vec{d_2}\| \ \|e_z\|} = \frac{0.75}{\sqrt{(\frac{\sqrt{2}}{2})^2 + 0.75^2}} = \frac{0.75}{\sqrt{\frac{1}{2} + \frac{9}{16}}} = \frac{\frac{3}{4}}{\sqrt{\frac{17}{16}}} = \frac{\frac{3}{4}}{\frac{\sqrt{17}}{4}} = \frac{12}{4\sqrt{17}} = \frac{3}{\sqrt{17}}$$

$$\vec{d_2} \cdot e_y = \|\vec{d_2}\| \cdot \|e_y\| \cdot \cos(\frac{\pi}{2} - \beta) = sen(\beta)$$

$$sen(\beta) = \frac{\frac{\sqrt{2}}{2}}{\sqrt{\frac{17}{16}}} = \frac{\sqrt{2}/2}{\sqrt{17}/4} = \frac{4\sqrt{2}}{2\sqrt{17}} = \frac{2\sqrt{2}}{\sqrt{17}}$$

So the whole rotation is: $R_1^{-1} R_2^{-1} R^z_{45^\circ} R_2 R_1$



$$R_1 = \begin{pmatrix} \cos & -sen & 0 & 0 \\ sen & \cos & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \theta = \frac{\pi}{4} \rightarrow \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

New vector is $\vec{d_2} = \begin{pmatrix} 0.5 \\ 0.5 \\ 0.75 \end{pmatrix} \cdot R_1 = \begin{pmatrix} 0 \\ \sqrt{2}/2 \\ 0.75 \end{pmatrix}$

$$R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos & -sen & 0 \\ 0 & sen & \cos & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3/\sqrt{17} & \frac{-2\sqrt{2}}{\sqrt{17}} & 0 \\ 0 & \frac{2\sqrt{2}}{\sqrt{17}} & 3/\sqrt{17} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

New for θ: $\vec{d_2} \cdot \vec{e_z} = \|\vec{d_2}\| \ \|\vec{e_z}\| \cos\theta$

$$0.75 = \left[\sqrt{(\tfrac{\sqrt{2}}{2})^2 + (\tfrac{3}{4})^2} \cdot 1\right] \cos\theta$$

$$\cos\theta = \frac{3}{4} \Big/ \sqrt{\frac{17}{16}} = \frac{3}{\sqrt{17}}$$

$R = R_1^{-1} R_2^{-1} R^z_{45} R_2 R_1$

$$\vec{d_1} \cdot \vec{e_y} = \|\vec{d_1}\| \ \|\vec{e_y}\| \cos(\frac{\pi}{2} - \theta)$$

$$\frac{\sqrt{2}}{2} = \left[\frac{\sqrt{17}}{4} \cdot 1\right] \cdot sen\theta$$

$$sen\theta = \frac{\sqrt{2}}{2} \Big/ \frac{\sqrt{17}}{4} = \frac{4\sqrt{2}}{2\sqrt{17}} = \frac{2\sqrt{2}}{\sqrt{17}}$$

b) Increase the object size by 50%.

$$S = \begin{pmatrix} 1.5 & 0 & 0 & 0 \\ 0 & 1.5 & 0 & 0 \\ 0 & 0 & 1.5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

c) Translate the object to $(0,5,0)$ followed by a 30° around z

$$M = T R^z_{30^\circ}, \quad T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \text{ and } R^z_{30^\circ} = \begin{array}{c} \begin{matrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} & & \end{matrix} \\ \begin{pmatrix} \cos(30) & -sen(30) & 0 & 0 \\ sen(30) & \cos(30) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

d) Reflect an object through a plane defined by its $\vec{n} = \begin{pmatrix} 0,7071 \\ 0,7071 \\ 0 \end{pmatrix}$.

First we align the plane with any of the known world planes using

$$R = \begin{pmatrix} 0,7071 & 0,7071 & 0 & 0 \\ -0,7071 & 0,7071 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and now we reflect over yz, which is where the plane now lies on

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

So $M = R^{-1} \cdot A \cdot R$

e) Shear the object along the x axis to a general parallelepiped so that the top left of the cube is in $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 1 & \lambda & \mu & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}}_{\text{Top left}} = \left(\begin{array}{c|c} -1 + \lambda + \cancel{\overset{0}{\mu}} & 1 \\ 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{array}\right)$$

$\lambda = 2$

$\cancel{\lambda = 1, \ \mu = 1}$

$\cancel{\mu = 2}$ Because $z = 0$

so $S = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & \lambda & \mu & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \rightarrow \left(\begin{array}{c|c} -1+\lambda & 1 \\ 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{array}\right)$$

$-1 + \lambda = 1$ then $\lambda = 2$ and $\mu = 0$

f) Play with the field of view

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/f & 0 \end{pmatrix}$$

f ↑ Bigger FOV ↓ Smaller FOV

③ Derive the matrix we need for transforming normal vectors ($(M^{-1})^T$ for the transformation matrix M)

a) $M = R_x(\frac{\pi}{2})$ Rotation matrices are part of SO(n), so $T^{-1} = T^T$

- $(M^{-1})^T = (M^T)^T = M$

b)

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M^{-1} = \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$(M^{-1})^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$