Testing numerical schemes for the shallow water equations

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Abstract

We consider the Riemann problem for the shallow water equations and show that its solution is in general non-uniqie. We propose a procedure of constructing exact solutions to the Riemann problem and use these exact solutions as test cases in order to assess the performance of numerical methods. It appears that depending on the method, the numerical solution can pick out different exact solutions for the same initial data. Moreover, the numerical solution does not necessarily converge towards an exact one.

1 Introduction

In this work, we are concerned with the system of one-dimensional shallow water equations, which can be written in the form

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = \mathbf{h}(\mathbf{u})\, z_x, \tag{1}$$

with

$$\mathbf{u} = \begin{bmatrix} z \\ h \\ hu \end{bmatrix}, \qquad \mathbf{f}(\mathbf{u}) = \begin{bmatrix} 0 \\ hu \\ hu^2 + gh^2/2 \end{bmatrix}, \qquad \mathbf{h}(\mathbf{u}) = \begin{bmatrix} 0 \\ 0 \\ -hg \end{bmatrix}. \tag{2}$$

Here z is the bottom topography, h the water height, u the water velocity, and g is the gravitational constant. Usually, the bottom topography is assumed to be given *a priori*. In (1), we consider $z = z(x)$ as an additional unknown and supply a trivial equation $z_t = 0$ for determining it.

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Consider the Riemann initial data for the system (1), i.e.,

$$\mathbf{u}(x,0) = \begin{cases} \mathbf{u}_L, & x \leq 0 \\ \mathbf{u}_R, & x > 0. \end{cases} \tag{3}$$

Our interest in the Riemann problem (1), (3) is motivated by two main factors. Firstly, understanding the structure of the solution to the Riemann problem (1), (3) is essential for constructing of efficient Godunov-type methods for the system (1). Secondly, the knowledge of the exact solution to (1), (3) provides valuable test cases for assessing the performance of numerical methods for the system (1).

The system (1) is derived by averaging the incompressible Navier–Stokes equations with free surface in a vertical direction. This averaging is done under the assumption that the vertical component of acceleration is negligible. Therefore, formally one is not allowed to take discontinuous initial data (3) for the system (1). Indeed, depending on the jump in bottom topography $z_R - z_L$, one can get large vertical component of acceleration. However, for the numerical solution to (1) one needs to discretize it, and therefore one is forced to consider the Riemann problems (1), (3) at each cell interface.

One of the main difficulties concerning the system (1) is the fact that it cannot be written in divergence form, i.e. it is *nonconservative*. As a consequence, one cannot use the definition of weak solution, used in the theory of conservation laws, see e.g. Smoller [12]. Also, there is no analog of the Lax–Wendroff theorem for the system (1), i.e. a convergent numerical solution to (1) will not necessarily be the correct one. Therefore, there is no fundamental guideline how to construct numerical schemes for the system (1), analogous to the conservation requirement for conservation laws. In this light one strives to ensure several desired properties of a numerical method. These properties include the ability of the method to solve the steady state solutions to (1) exactly, to be positively conservative with respect to the water height h, and be able to handle dry states $h = 0$. Some other desirable properties can be found in e.g. Bouchut [4].

The main goal of this work is to assess the performance of different numerical methods for the system (1). In this end, we consider exact solutions to the Riemann problem (1), (3). We use two methods for obtaining them, the so-called *inverse* solution to the Riemann problem, and the exact Riemann solver. The first method consists in prescribing the exact solution to the Riemann problem and determining the initial data, which correspond to this solution. This procedure is implemented in CONSTRUCT [1]. An advantage of this method is that we can easily obtain a wave configuration we are interested in, e.g. with coinciding waves, almost dry states, etc. While CONSTRUCTing the solution like this, we consider only classical waves in the solution to the Riemann problem, that is Lax shocks, rarefaction waves, and contacts. However, as it was pointed out by e.g. Chinnayya, LeRoux, and Seguin [5], some Riemann initial left and right states (3) cannot be connected by these waves only. In other words, there can be no solution to the Riemann problem (1), (3) if one is restricted to the classical waves. In order to overcome this difficulty, Chinnayya *et al.* [5] propose to introduce certain composite waves in the solution to the Riemann problem (1), (3). In this work, we use the exact Riemann solver of [5] to find the solution in a presence of these composite waves.

The paper is organized as follows. In Section 2 we discuss several properties of the system of shallow water equations (1) and of the associated Riemann solution. It appears that in general, the solution to the Riemann problem (1), (3) is not unique. In Section 3 we describe the procedure of obtaining exact solutions to the Riemann problem (1), (3). In Section 4 we propose several test problems, based on the exact solution to the Riemann problem (1), (3). Section 5 contain the results of several numerical schemes, proposed for the system (1). We consider the hydrostatic reconstruction method of Audusse *et al.* [3], the VFRoe method of Gallouët *et al.* [6], the relaxation method of Bouchut [4], and the kinetic method of Perthame and Simeoni [11]. We end up with some conclusions in Section 6.

2 Properties of the Riemann solution

In order to provide the characteristic analysis of the system (1), we rewrite it as follows,

$$\mathbf{u}_t + \mathbf{A}(\mathbf{u})\mathbf{u}_x = 0, \tag{4}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ gh & gh - u^2 & 2u \end{bmatrix}. \tag{5}$$

The eigenvalues of $\mathbf{A}(\mathbf{u})$ are

$$\lambda_0 = 0, \qquad \lambda_1 = u - \sqrt{gh}, \qquad \lambda_2 = u + \sqrt{gh}. \tag{6}$$

One can easily show that when $\lambda_{1,2} = \lambda_0 = 0$, i.e. $u \mp \sqrt{gh} = 0$, the corresponding eigenvectors become linearly dependent. Therefore, the system (1) is hyperbolic away from the critical points in the flow with $u \mp \sqrt{gh} = 0$, where it becomes parabolic degenerate.

In the solution to the Riemann problem (1), (3), each characteristic field associated with (6), can be either a shock, a rarefaction, or a contact wave, see e.g. Smoller [12]. The stationary 0-contact wave plays a special role in the solution to the Riemann problem (1), (3). Indeed, the nonconservative term $\mathbf{h}(\mathbf{u})\, z_x$ plays a role only across this wave. The Riemann invariants for the 0-wave are

$$\begin{aligned} hu &= \text{const} \\ \frac{u^2}{2} + g(h+z) &= \text{const}. \end{aligned} \tag{7}$$

Note that these relations are exactly the time-independent solution of the shallow water equations (1).

Classically, each characteristic field determines a corresponding wave in the solution to the Riemann problem, separating the constant states, see e.g. Smoller [12]. However, one can point out the initial data (3) for which the solution to the Riemann problem with classical waves only does not exist. In order to get existence, one has to use certain *composite* waves, see [7, 10, 5] for details.

The system (1) belongs to the class of *resonant* systems, introduced by Isaacson and Temple [8, 9]. The Riemann problem for such systems was studied in [7, 10, 2]. Analogously to the analysis of [7, 10, 2] one can show that the solution to the Riemann problem (1), (3) is not unique. In Section 4 below we provide an example of the Riemann problem with the non-unique solution.

The choice of the physically relevant solution can be motivated by the comparison with the 2D or 3D incompressible free-surface code, analogously to how it is done in [2]. One can consider the 2D or 3D initial data, corresponding to the 1D data which produces the non-unique solution to the Riemann problem, and compare the 1D non-unique results with 2D or 3D averaged solution. This work is currently in progress.

3 Exact solution to the Riemann problem

In this work, we use the exact solutions to the Riemann problem (1), (3) in order to provide test cases for numerical methods for the shallow water equations (1). In these test cases, one is typically interested in particular flow configurations, which may be difficult to solve numerically. The common requirements to a numerical method for (1) are its ability to solve the steady state solutions (7) exactly, and to handle the dry states $h = 0$. Since the system (1) is nonstrictly hyperbolic, and the solution to the Riemann problem (1), (3) can be non-unique, one wishes to assess the performance of numerical methods in these cases, too. h An easy way to obtain a Riemann solution with certain properties is to solve the so-called *inverse* Riemann problem. It consists in prescribing the solution to the Riemann problem, and finding the corresponding initial data. This procedure is implemented in a software package CONSTRUCT [1], which is freely available on the web. With its help, one can easy obtain a Riemann problem with desired properties. Currently, CONSTRUCT handles only the Riemann solutions consisiting of classical waves. We have use the exact Riemann solver of [5] to find the solution in a presence of composite waves.

4 Test problems

In all tests below, the domain is $[0, 1]$ with the position of the initial discontinuity $x_0 = 0.5$, and the gravitational constant $g = 2$. All test cases were obtained using CONSTRUCT [1] except the last one, where the exact Riemann solver of [5] has been used. More test problems together with exact solutions can be found on the web page [1].

Dry state The exact solution consists of the 1-rarefaction, followed by the 0-contact, followed by the 2-shock. The water height between the 1-rarefaction and the 0-contact is $h = 0.01$. The initial data are

$$(z, h, u) = \begin{cases} (2, 0.222, -1), & x \leq x_0 \\ (0.1, 0.7246, -1.6359), & x > x_0. \end{cases} \tag{8}$$

Non-unique solution 1 The Riemann problem for (1) with the following initial data has two solutions

$$(z, h, u) = \begin{cases} (1.5, 0.15, 1), & x \leq x_0 \\ (1.1, 0.4, -2.2), & x > x_0. \end{cases} \tag{9}$$

Non-unique solution 2 The Riemann problem for (1) with the following initial data has two solutions

$$(z, h, u) = \begin{cases} (1.5, 1.3, -2), & x \leq x_0 \\ (1.1, 0.1, -2), & x > x_0. \end{cases} \tag{10}$$

Composite wave The Riemann problem for (1) with the following initial data contains the "triple wave", which conists of a sonic rarefaction, attached to a 0-contact, followed by a zero speed 1-shock, followed by another 0-contact.

$$(z, h, u) = \begin{cases} (1.5, 3.5, 1), & x \leq x_0 \\ (1.1, 0.5, -2), & x > x_0. \end{cases} \tag{11}$$

Due to the sonic rarefaction in the solution of this Riemann problem, the system looses hyperbolicity left to the stationary 0-contact.

5 Numerical results

In this section, we assess the performance of several numerical methods for the shallow water equations (1) on the test cases given in Section 4. Since there is no analog of the Lax–Wendroff theorem for the non-conservative system (1), there is no guarantee that a convergent numerical method will converge to the correct solution. Therefore, we the convergence to the correct solution will be

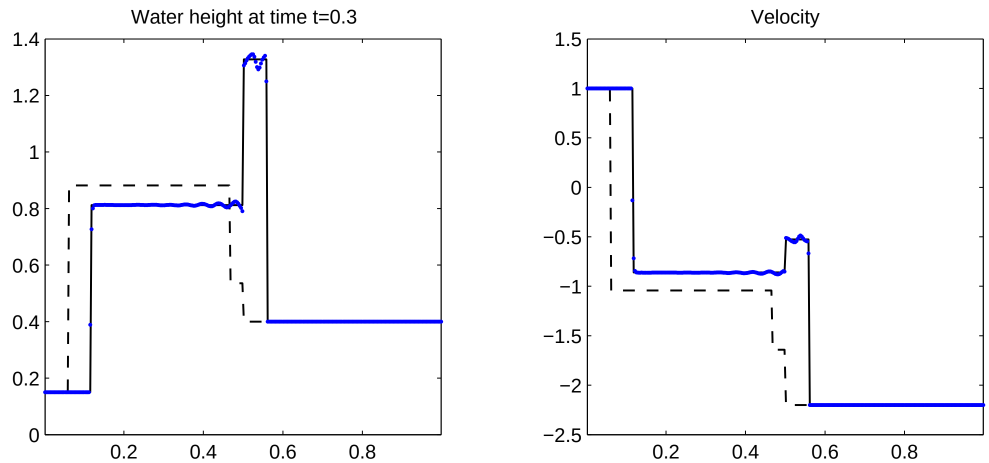

Figure 1: The results of VFRoe [6] for the non-unique solution test (9).

our primary question here. For simplicity, we will always use first order methods using a CFL number of 0.9, if not stated otherwise. The results shown here were obtained on 300 mesh points; the results on more fine grids do not differ significantly from the ones presented here. In Figs. 1–22, the numerical results are marked with dots, and exact solutions with solid or dashed line, in case of non-unique solutions.

5.1 Hydrostatic reconstruction method

In the hydrostatic reconstruction method [3], the first step is to compute the numerical flux of some scheme for the homogeneous shallow water equations (1), i.e. without the non-conservative term $\mathbf{h}(\mathbf{u})\, z_x$. Then, the numerical flux is modified in a way that the rest steady states of (1), i.e. the relations (7) with $u = 0$, are preserved on discrete level. In the numerical tests below we will use the VFRoe [6], relaxation [4], and kinetic [11] methods in order to get the flux of the homogeneous problem (1).

5.2 VFRoe method

The VFRoe method [6] uses an approximate solution to the Riemann problem (1), (3). Depending on the variables, in which this approximate Riemann solution is found, one distinguishes between different versions of the method. Here we have used the variables $(z, 2\sqrt{gh}, u)$.

The method produces negative water heights on the dry state test (8). For the non-unique test (9), one needs to diminish the CFL number down to 0.05 in order to preserve positiveity of the water height. The numerical results for are

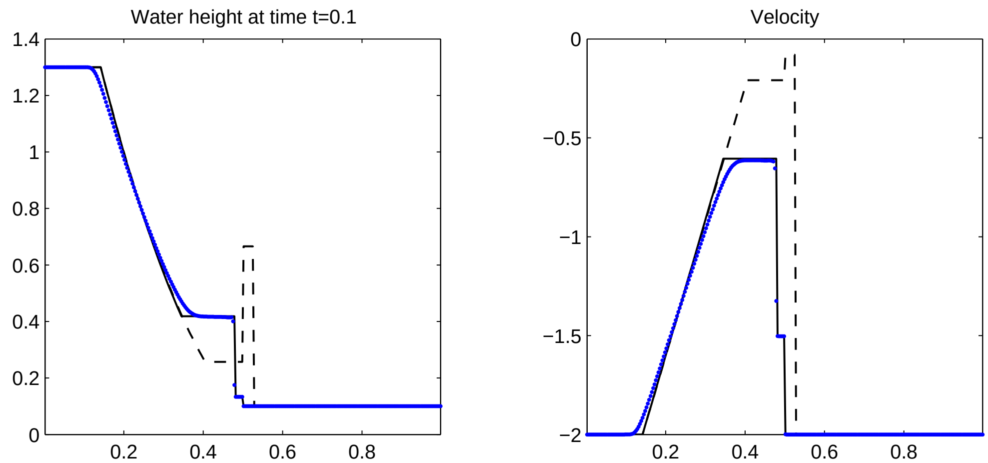

Figure 2: The results of VFRoe [6] for the non-unique solution test (10).

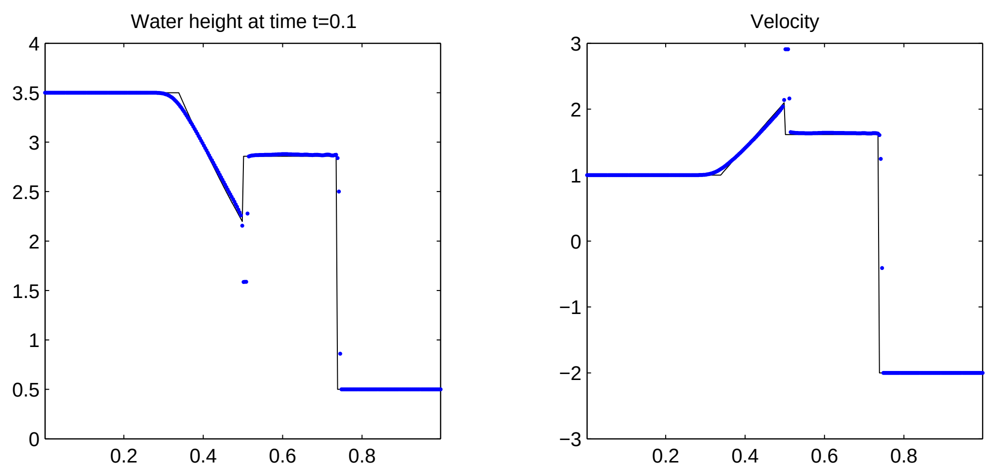

Figure 3: The results of VFRoe [6] for the composite wave test (11).

presented in Figs. 1, 2, 3. The results for the test (9) suffer from oscillations, see Fig. 1. Observe that the method converges to the exact solution marked with solid line for both problems with non-unique solutions (9), (10), see Figs. 1 and 2. For the composite wave test (11) the method produces a non-physical peak at the location of the stationary discontinuity, where the system becomes parabolic degenerate, see Fig. 3.

The remarkable difference between the VFRoe method and VFRoe with hydrostatic reconstruction and is they pick out *different* exact solutions for the non-unique test (10), cf. Figs. 2 and 5. A judgement which of these solutions is physical can be done by comparison with averaged 2D or 3D incompresiible

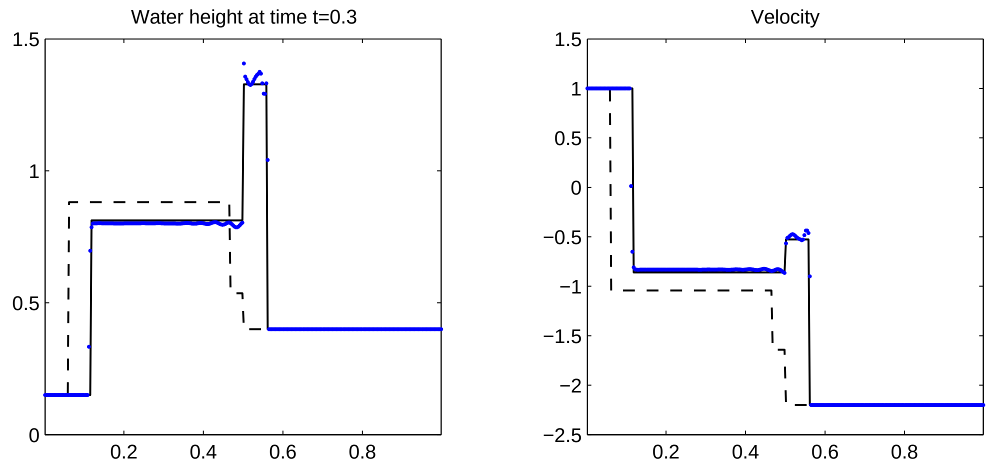

Figure 4: The results of hydrostatic reconstruction [3] VFRoe for the non-unique solution test (9).

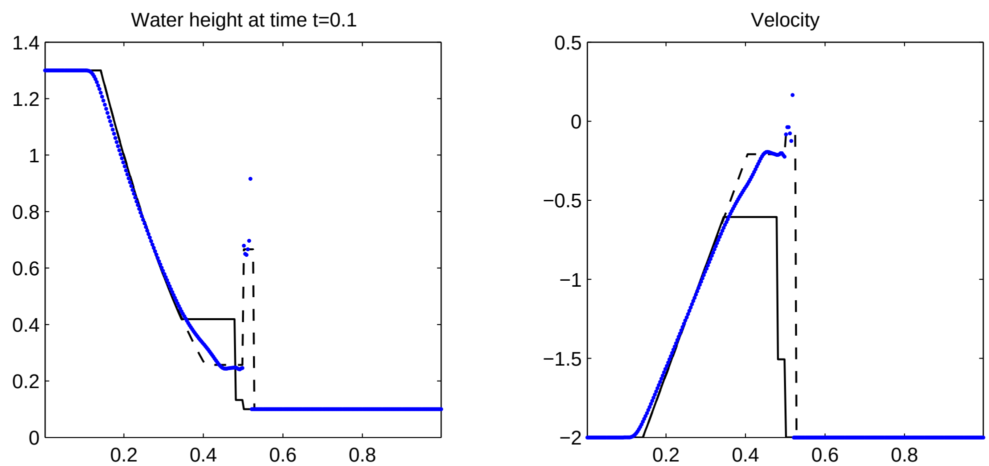

Figure 5: The results of hydrostatic reconstruction [3] VFRoe for the non-unique solution test (10).

Navier–Stokes equations with free surface in spirit of [2]. The results for the other tests are qualitatively similar.

5.3 Relaxation solver

In the framework of the relaxation method [4], the system of shallow water equations is replaced by a modification of Suliciu's relaxation system (see e.g. Tzavaras [13]). The results are presented in Figs. 7, 8, 9, 10. The results of the hydrostatic reconstruction are presented in Figs. 11, 12, 13, 14.

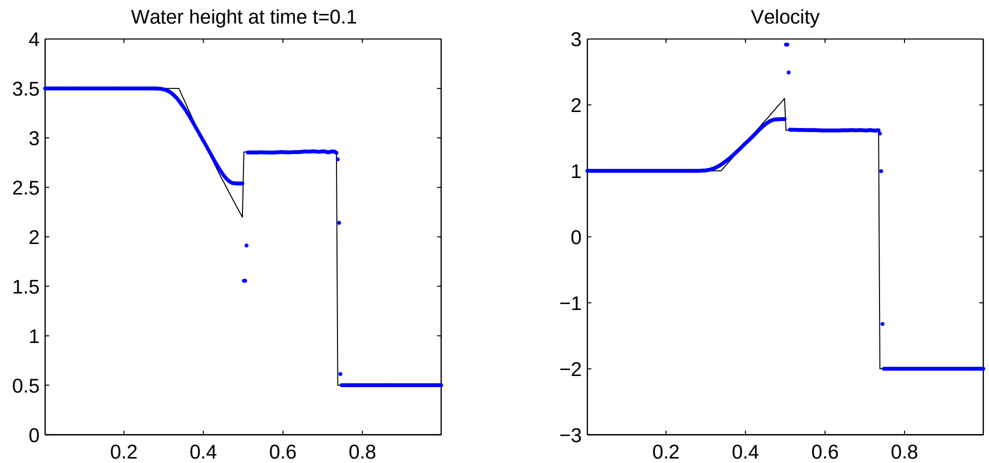

Figure 6: The results of hydrostatic reconstruction [3] VFRoe for the composite wave test (11).

The relaxation method [4] is able to solve the dry state case (8), however the results for the velocity appear to slightly incorrect. The numerical results pick out the exact solution marked with solid line for both problems with non-unique solutions (9), (10), see Figs. 8 and 9. However, we witness distinct discrepancies between the chosen exact solution and the numerical one. For the composite wave test (11) we observe a non-physical peak at the location of the stationary discontinuity, where the system becomes parabolic degenerate, see Fig. 10.

As in case with VFRoe, the relaxation method and the relaxation method with hydrostatic reconstruction [3] pick out *different* exact solutions for the non-unique test (10), cf. Figs. 9 and 13. The results for the other tests are qualitatively similar.

5.4 Kinetic solver

In the kinetic approach [11] one solves a kinetic equation for the particle density, constructed in such a way, that the moments of this equation are exactly the shallow water equations (1). The results of the kinetic solver are presented in Figs. 15, 16, 17, 18. The results of the hydrostatic reconstruction are presented in Figs. 19, 20, 21, 22.

The numerical results are quite similar to these of the relaxation solver [4]. Again, we observe that the kinetic method and the kinetic method with hydrostatic reconstruction [3] pick out different numerical solutions for the non-unique test (10), cf. Figs. 17 and 21. Also, we note several discrepancies between the exact solution and the numerical one.

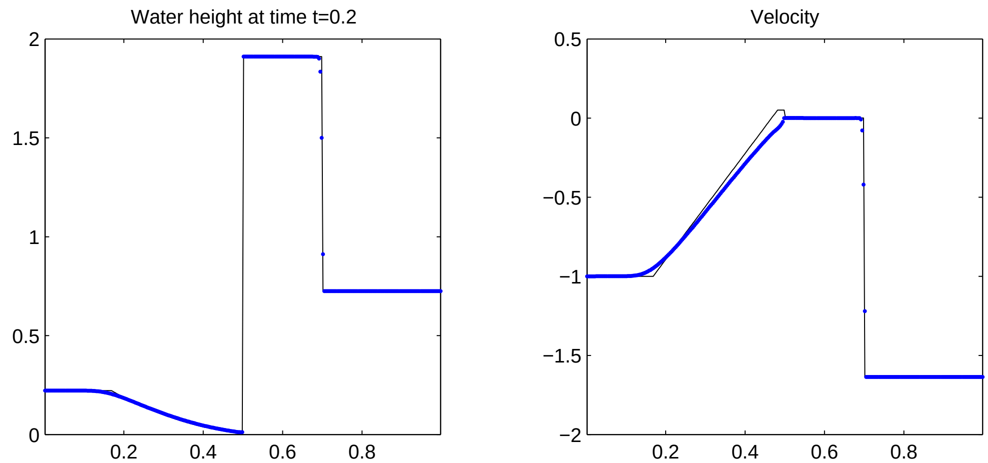

Figure 7: The results of the relaxation method [4] for the dry state test (8).

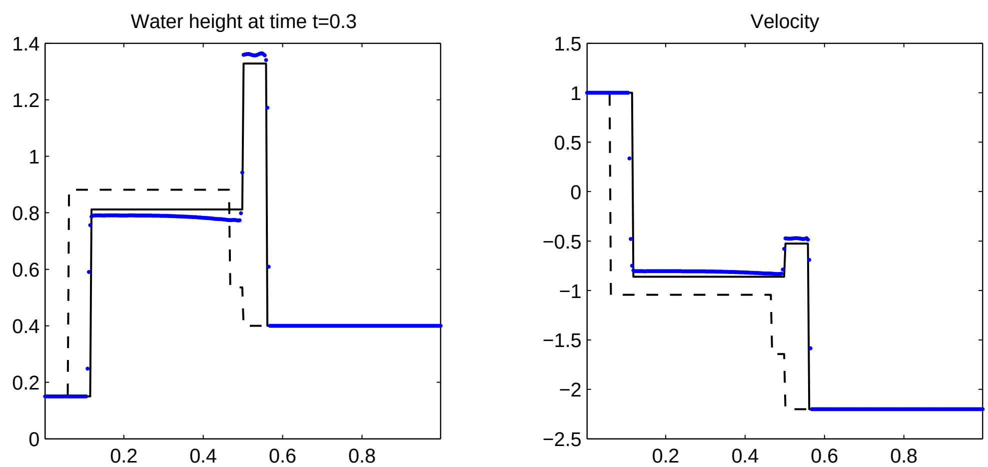

Figure 8: The results of the relaxation method [4] for the non-unique solution test (9).

6 Conclusions

Exact solutions to the Riemann problem for the shallow water equations provide valuable test cases for numerical methods. A convenient way to obtain the exact solutions with desired properties is the so-called inverse solution to the Riemann problem, which is provided in [1]. It appears that the solution to the Riemann problem is in general non-unique and it is not clear which exact solution will be picked out by a numerical one. We show that several numerical methods for shallow water equations can apparently pick out different exact solutions for the same initial data. Moreover, grid convergence studies show that the numerical

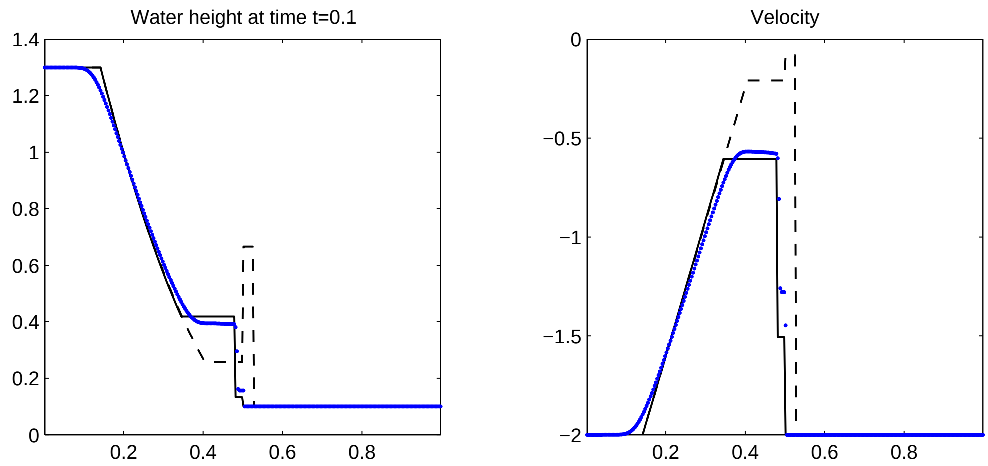

Figure 9: The results of the relaxation method [4] for the non-unique solution test (10).

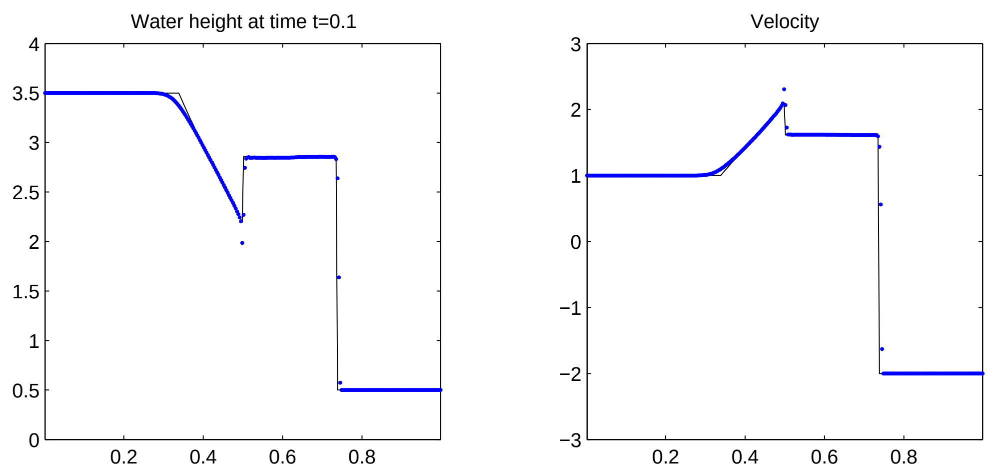

Figure 10: The results of the relaxation method [4] for the composite wave test (11).

solutions do not necessarily converge towards an exact solution.

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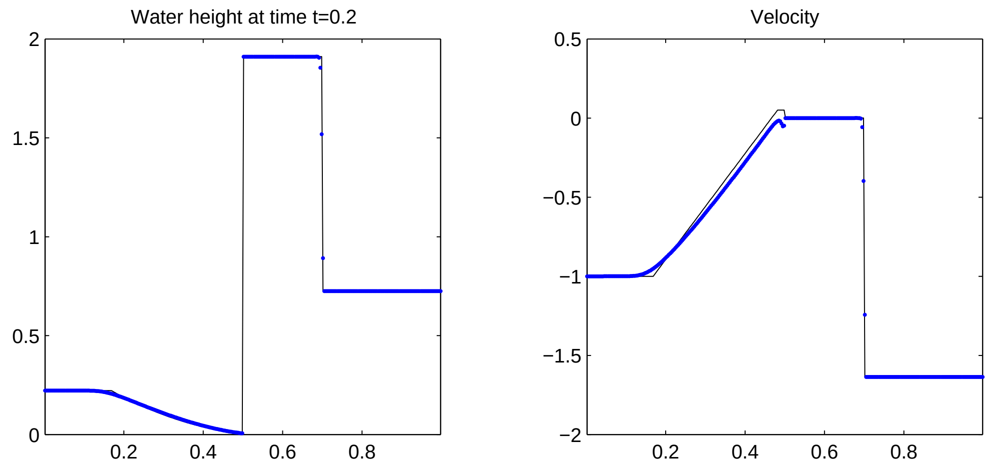

Figure 11: The results of relaxation with hydrostatic reconstruction [3] for the dry state test (8).

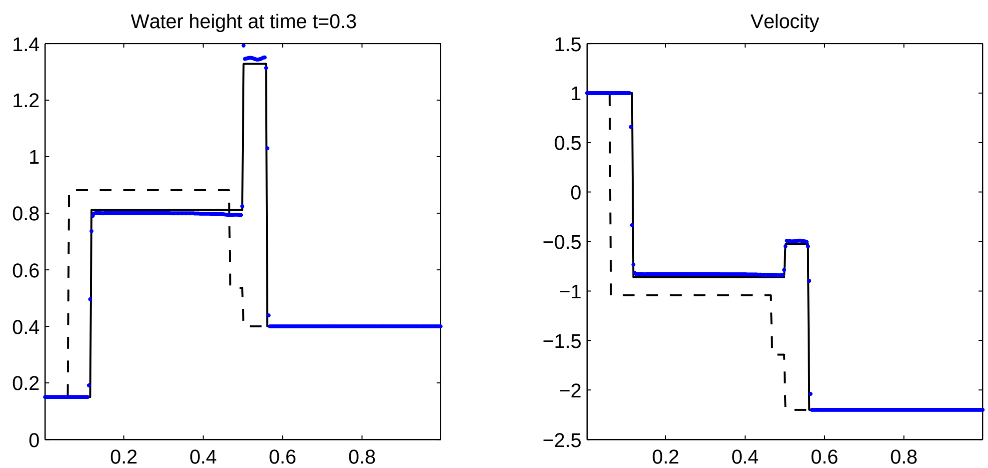

Figure 12: The results of relaxation with hydrostatic reconstruction [3] for the non-unique solution test (9).

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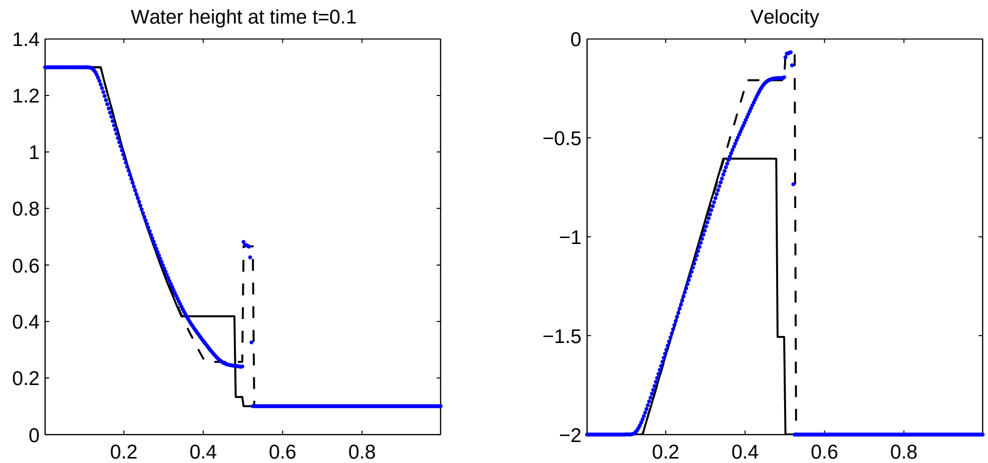

Figure 13: The results of relaxation with hydrostatic reconstruction [3] for the non-unique solution test (10).

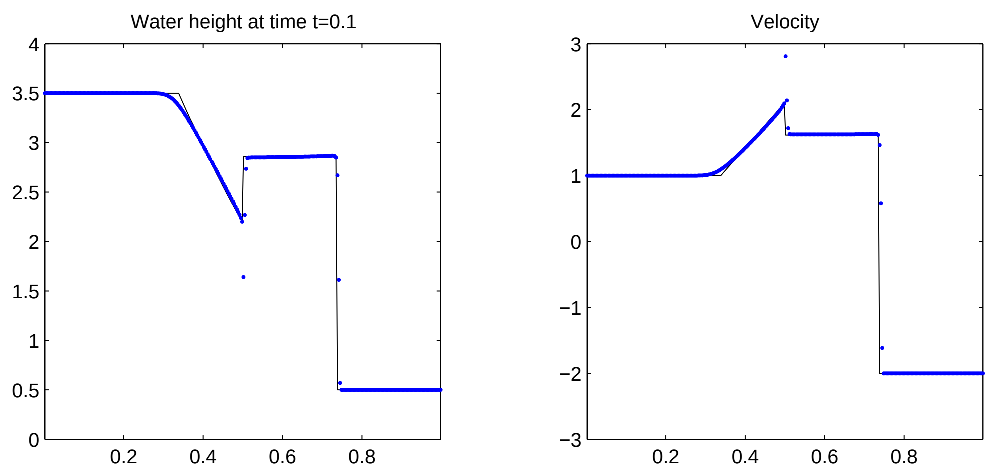

Figure 14: The results of relaxation with hydrostatic reconstruction [3] for the composite wave test (11).

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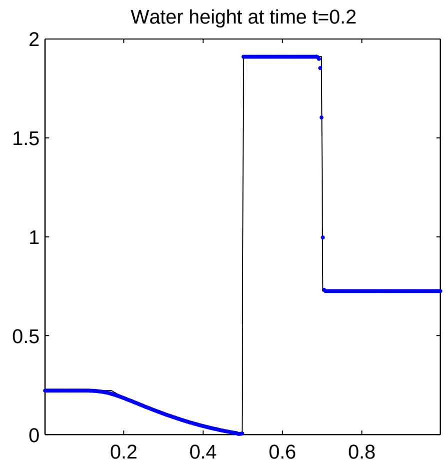

Figure 15: The results of the kinetic method [11] for the dry state test (8).

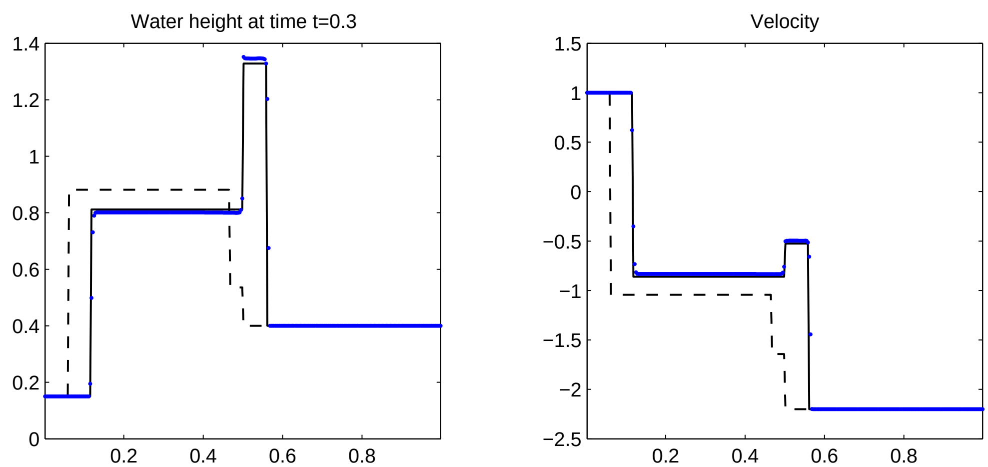

Figure 16: The results of the kinetic method [11] for the non-unique solution test (9).

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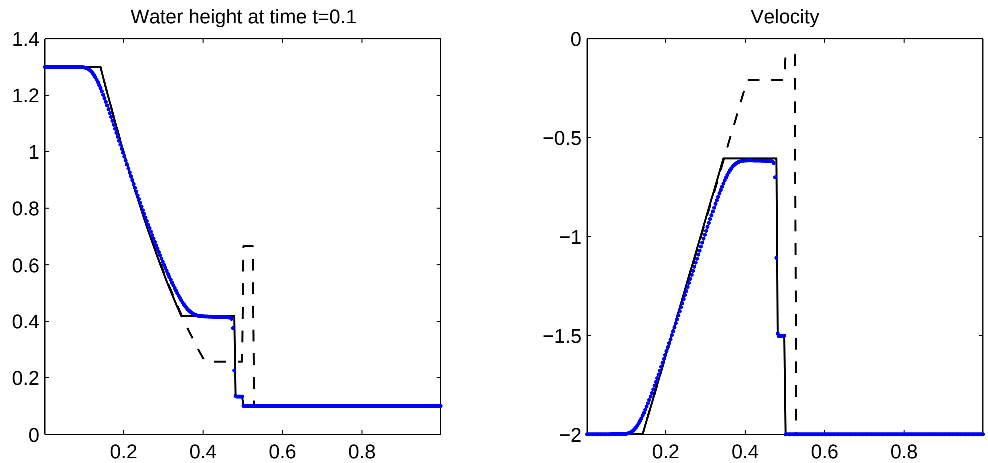

Figure 17: The results of the kinetic method [11] for the non-unique solution test (10).

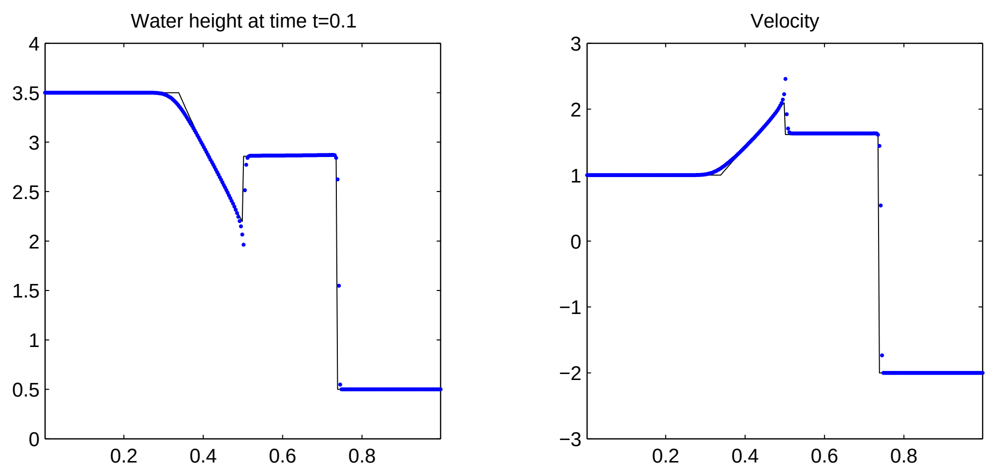

Figure 18: The results of the kinetic method [11] for the composite wave test (11).

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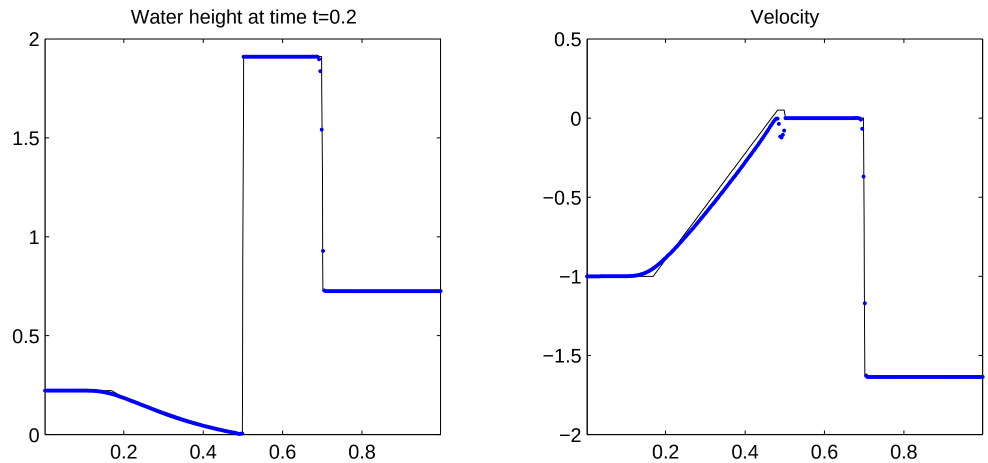

Figure 19: The results of the kinetic method with hydrostatic reconstruction [3] for the dry state test (8).

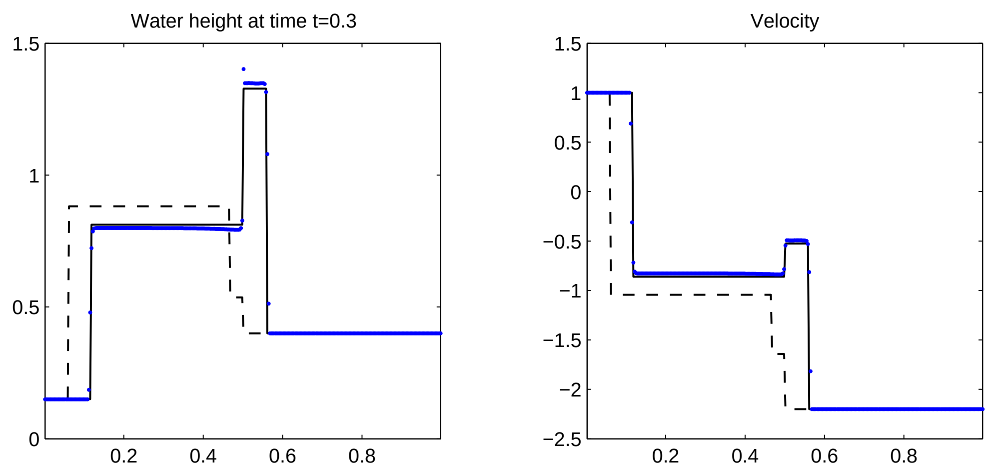

Figure 20: The results of the kinetic method with hydrostatic reconstruction [3] for the non-unique solution test (9).

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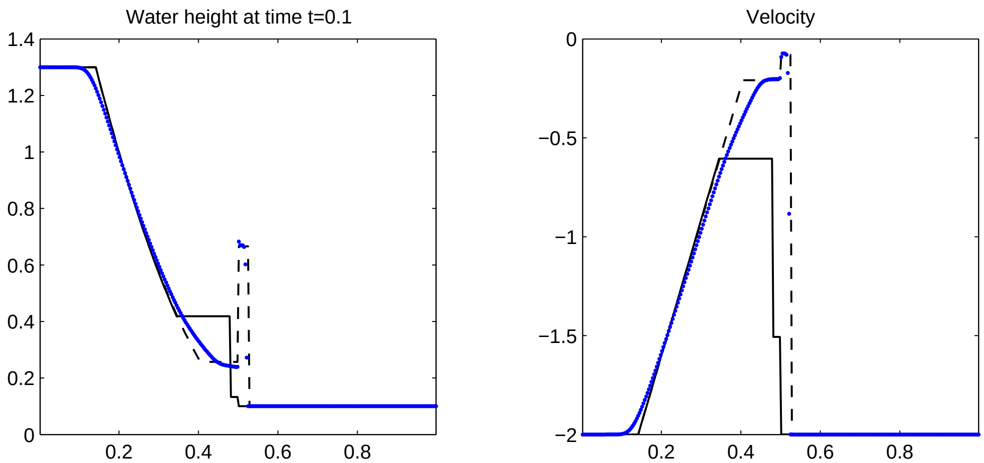

Figure 21: The results of the kinetic method with hydrostatic reconstruction [3] for the non-unique solution test (10).

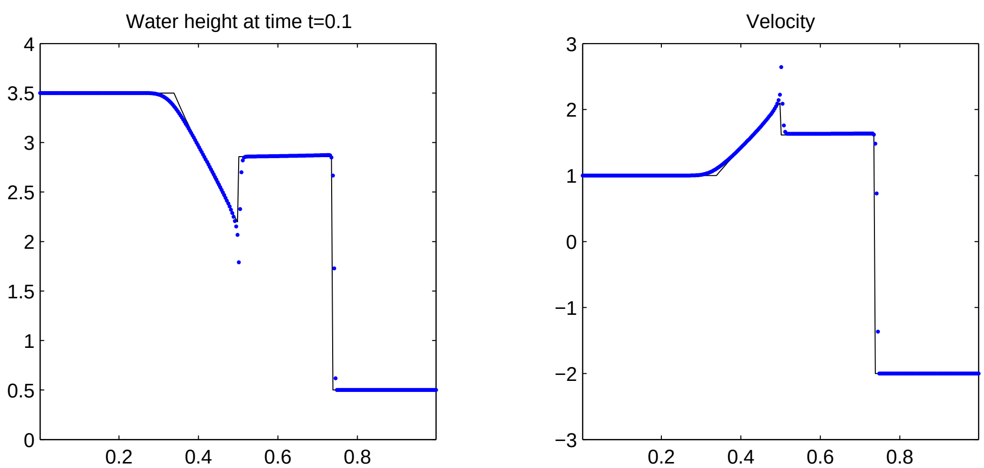

Figure 22: The results of the kinetic solver with hydrostatic reconstruction [3] for the composite wave test (11).

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