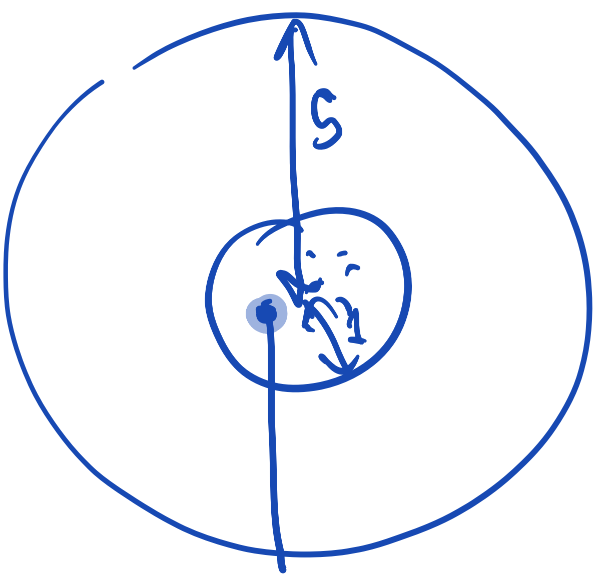
Number of zeroes vs growth.

f - holomorphic function in D.
$\{|z| \le 1\}$

$N =$ number of zeroes of f in D.

<u>Exercise.</u> $$\frac{\max\limits_{5D} |f|}{\max\limits_{D} |f|} \ge e^{cN}$$



$a_1, \dots, a_N$ - zeroes in D

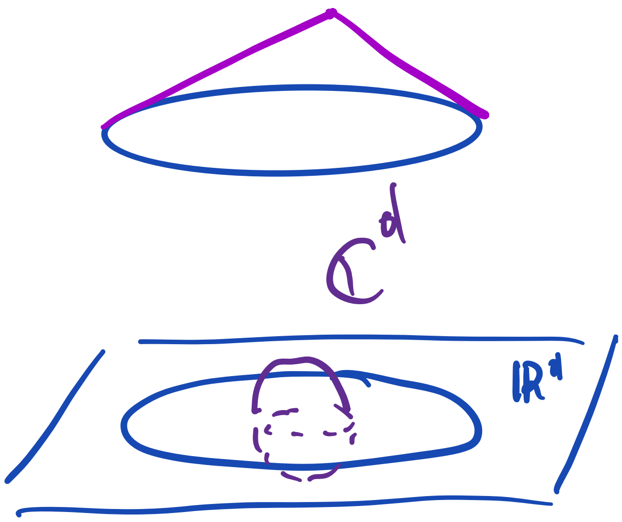
$$f(z) = g(z) \cdot \prod_{k=1}^{N} (z - a_k)$$

Corollary. One can estimate the number of zeroes in terms of growth.

Example. e^z, e^{e^z} don't have any zeroes

Thm (Liouville). $\Delta u = 0$, $u > 0$ in $\mathbb{R}^n$, then $u \equiv \text{const}$

Holomorphic extension of solutions to elliptic PDE



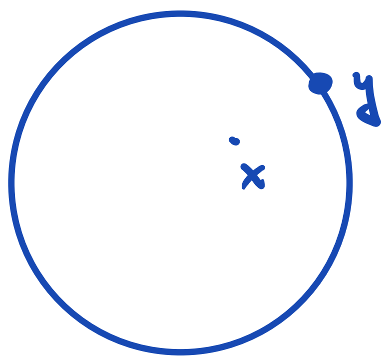
u - harmonic in $B_1 \subset \mathbb{R}^d$

There is $r_d < 1$:

u can be extended to a holomorphic function in $B_{r_d}^{\mathbb{C}^d} \subset \mathbb{C}^d$

with estimate :

$$\max_{B_{r_d}^{\mathbb{C}^d}} |u| \le C_d \max_{B_1^{\mathbb{R}^d}} |u|$$



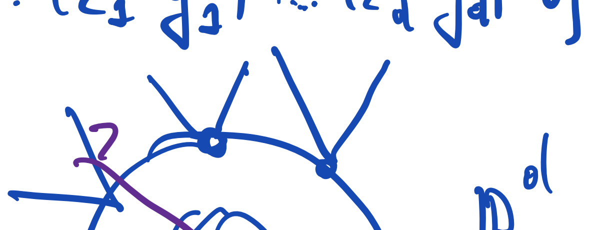
$B_1 \subset \mathbb{R}^d$

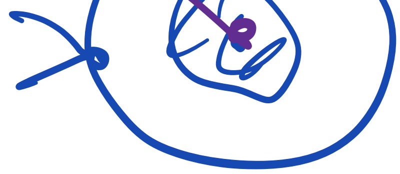
$$P(x,y) = c \cdot \frac{1-|x|^2}{|x-y|^d}$$

For a fixed y, one can plug $z=(z_1,\dots,z_d)\in \mathbb{C}^d$ in place of $x=(x_1,\dots,x_d)\in B_1$.

$$P(z,y) = \frac{1-(z_1^2+\dots+z_d^2)}{\left(\sqrt{(z_1-y_1)^2+\dots+(z_d-y_d)^2}\right)^d}$$

If $d:2$ $P(z,y)$ is a holomorphic function outside the cone $\Gamma_y=\{z:(z_1-y_1)^2+\dots+(z_d-y_d)^2=0\}$





$\Delta u = 0$

If $u \in C(\overline{B_1})$, then

$$u(z) = \int_{\partial B_1} P(z,y)\, u(y)\, d\sigma(y)$$

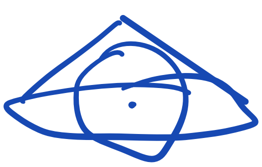
is a holomorpic function in a complex neighborhood of 0.

$$\sup_{B^{\mathbb{C}}_{r_d}} |u(z)| \le C \sup_{\partial B_1} |u(y)|$$

$$\Downarrow$$

$$C = \sup_{\substack{z \in B^{\mathbb{C}}_{r_d} \\ y \in \partial B_1}} P(z,y)$$

Question. What is the best r_d



Elliptic PDE.

$$Lu = \sum_{|\alpha| \le m} a_\alpha \partial^\alpha u$$ — linear elliptic differential operator

$\alpha = (\alpha_1, .., \alpha_d)$ $\qquad \partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}$

<u>Ellipticity</u> $\quad c \cdot |x|^m \le \sum_{|\alpha| = m} a_\alpha \cdot x^\alpha \le C \cdot |x|^m$

$c, C > 0$ — ellipticity constants.

Cauchy estimates.

Def. A function f is called <u>real-analytic</u> near point $0 \in \mathbb{R}^d$ if there exist $C, R > 0$:

$$|D^\alpha f| \leq C \cdot \alpha! \; R^{|\alpha|}$$

in some neighborhood of 0

for all multiindices α.

In particular, Taylor series of f converge in a neighborhood of zero

$$f(x) = \sum \partial_\alpha f(0) \cdot x^\alpha / \alpha!$$

Cauchy estimates.

Any solution to linear elliptic PDE with real-analytic coefficients is real-analytic.

Thm(folklor, Hormander*)

If $Lu=0$ in B_1, then there is $C, r, R>0$ depending on L only such that

$$\sup_{B_r} |\partial^\alpha u| \leq C \cdot R^\alpha \sup_{B_1} |u| \cdot \alpha!$$

Corollary (holomorphic extension with estimate).

Every real solution to $Lu=0$ in $B_1 \subset \mathbb{R}^d$ has holomorphic extension to $B_r^{\mathbb{C}} \subset \mathbb{C}^d$ with estimate:

$$\sup_{B_r^{\mathbb{C}}} |u| \leq C \cdot \sup_{B_1^{\mathbb{R}}} |u|$$

Eigenfunctions on manifolds.

In local coordinates the Laplace operator on any Riemannian mfd can be written as

$$\Delta_M u = \mathrm{div}_M \nabla_M u$$

$$\nabla_M u = g^{ij} \nabla u \qquad g^{ij} = (g_{ij})^{-1}$$

$$\mathrm{div}_M \vec{F} = \frac{1}{\sqrt{\det(g_{ij})}} \mathrm{div}\left(\sqrt{\det(g_{ij})} \cdot \vec{F}\right)$$

$$\Delta_M u = 0 \iff \frac{1}{\sqrt{|g|}} \cdot \mathrm{div}(A \nabla u) = 0$$

Exercise. $d = 2$ $\quad \det(A) = 1$

Harmonic functions on mfds are solutions to second order elliptic PDE in local coordinates.

$$\operatorname{div}(A \nabla u) = 0 \qquad A\text{-elliptic matrix function}$$

Exercise. For harmonic functions on 2D mfds A must have determinant one.

Equation for eigenfunctions in local coordinates.

$$\Delta_M u + \lambda u = 0$$

$$\Delta_M u = 0 \iff \operatorname{div}(\sqrt{|g|} \cdot g^{ij} \nabla u) = 0$$

$$|g| = \det(g_{ij})$$

$$\frac{1}{\sqrt{|g|}} \operatorname{div}(\sqrt{|g|}\, g^{ij} \nabla u) + \lambda u = 0$$

$$\operatorname{div}(\sqrt{|g|}\, g^{ij} \nabla u) + \lambda \cdot \sqrt{|g|} \cdot u = 0$$

In 2D the equation can be simplified.

Thm (Gauss) There are local coordinates (called isothermal) such that $(g_{ij}) = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{Euclidean metric}} \cdot q$

q is a real function: $c < q < C$

Names: conformal factor, metric density.

In isothermal coordinates the PDE for eigenfunctions is simplified to:

$$\Delta_M = \frac{1}{q}\Delta$$

$$\Delta u + \lambda q u = 0.$$

$\uparrow$ ordinary Euclidean Laplacian

In particular, harmonicity in isothermal coordinates is equivalent to harmonicity in Euclidean coordinates.

Remark. Isothermal coordinates – 2D only.

Thm (Donnelly - Fefferman)

(bounded and compact)

If (M^d, g) is a closed Riemannian manifold with real-analytic metric (g_{ij}), then there a complex neighborhood $M^{\mathbb{C}}$ of M such that such that all eigenfunctions φ_λ on M can be extended holomorphically to $M^{\mathbb{C}}$ with estimate:

$\Delta\varphi + \lambda\varphi = 0$

$$\sup_{M^{\mathbb{C}}} |\varphi_\lambda| \leq e^{c\sqrt{\lambda}} \cdot \sup_M |\varphi_\lambda|$$

Proof.

(simplified of the original proof due to F.H. Lin)

Harmonic lift: $M \times \mathbb{R} = \widetilde{M}$

$$u = \varphi_\lambda(x) \cdot e^{\sqrt{\lambda}\, t} \qquad \Delta_{\widetilde{M}} u = 0$$

Work in local coordinates on $\widetilde{M}$

$$x_{d+1} = t$$

$$(x_1, x_2, \ldots, x_d, x_{d+1})$$

$$L u = 0 \quad \text{in} \quad B \subset \mathbb{R}^{d+1}$$

u has holomorphic extension to $B_r^{\mathbb{C}} \subset \mathbb{C}^{d+1}$

with estimate:

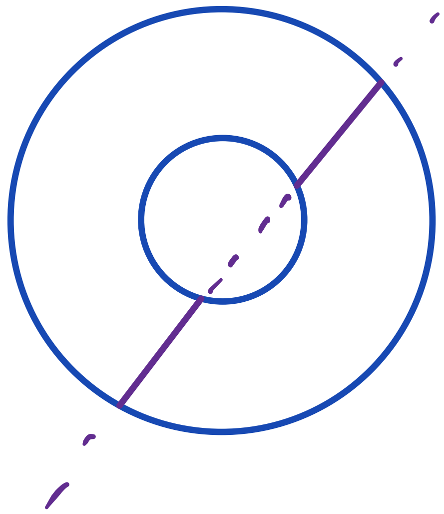
$$\sup_{B_r^{\mathbb{C}}} |u| \underset{\text{Cauchy estimate}}{\leq} C \cdot \sup_{B} |u| \underset{\text{harmonic extension}}{\leq} C \cdot e^{C\sqrt{\lambda}} \cdot \sup_{M} |\varphi|$$

$$u = \varphi \cdot e^{\sqrt{\lambda} t}$$

$$\varphi = u \quad \text{on} \quad \{x_{d+1} = 0\} \cap B$$

Holomorphic extension of u restricted to $\mathbb{C}_0^{d+1} = \{(z_1, \dots, z_d, 0)\}$ is holomorphic extension of φ.

Exercise (favorite question of D. Khavinson)



Suppose u is harmonic in $B_R \setminus B_r$

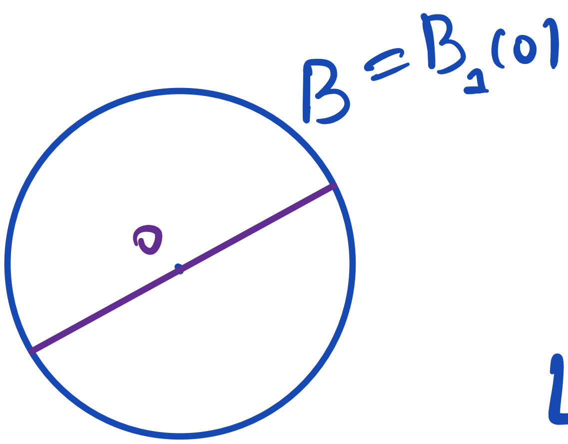
A line L intersect $B_R \setminus B_r$ and the intersection consists of two segments L_1 and L_2

$$u = 0 \text{ on } L_1 \overset{?}{\Longrightarrow} u = 0 \text{ on } L_2.$$

Show that $\Longrightarrow$ is true if $R/r \gg 1$.

Application of holomorphic extension.

<u>Lemma</u> Let $\Delta u = 0$ in $B \subset \mathbb{R}^n$



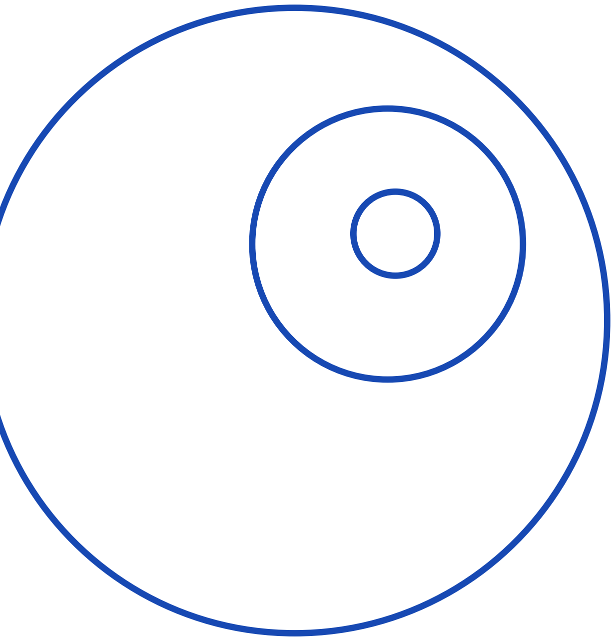
$$\sup_B |u| \leq e^N \cdot |u(0)|$$

L - one dimensional line passing passing through 0.

Then the number of zero points on L within $\frac{1}{2}B$ is smaller than $C \cdot N$.

$u \in \mathrm{Harm}(\Omega_3)$

$$\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_3$$

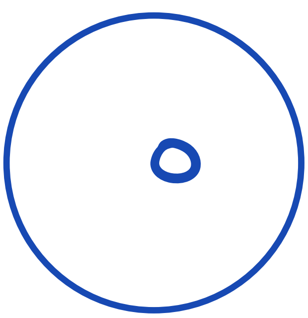


$$C_\alpha \left(\frac{\sup\limits_{\Omega_3}|u|}{\sup\limits_{\Omega_2}|u|}\right)^\alpha \geq \frac{\sup\limits_{\Omega_2}|u|}{\sup\limits_{\Omega_1}|u|}$$

C_α, $\alpha > 0$ depend on $\Omega_1, \Omega_2, \Omega_3$

Exercise. Doubling index

$$N_u(B) = \log \frac{\sup\limits_{2B}|u|}{\sup\limits_{B}|u|}$$



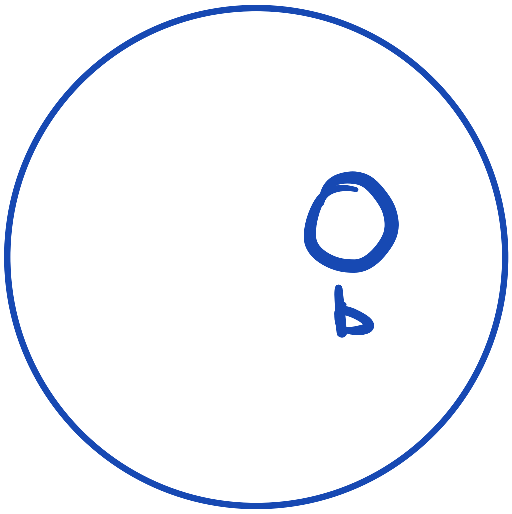
$$b < \frac{1}{4}B \overset{?}{\Rightarrow} N_u(b) \leq C \cdot N_u(B) \quad (+C)$$

a) C depends on b

b) C does not depend on b.

Exercise. $\Delta u = 0$ $\quad N_u(B) = N$

$\sup_B |u| = 1$



$b \subset \frac{1}{4} B$

$$\sup_b |u| \geq \left(\frac{|b|}{|B|}\right)^{CN}$$

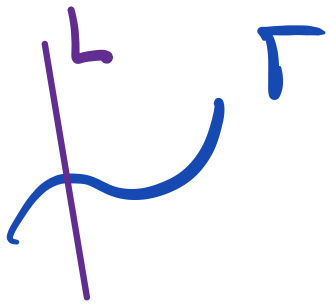
$\Delta u = 0$ in $2B$

Thm. $H^{n-1}(Z_u \cap \frac{1}{2}B) \le C \cdot N_u(B)$

Idea of the proof.

Lemma + Crofton's formula.

Simplest version of Krofton's formula.

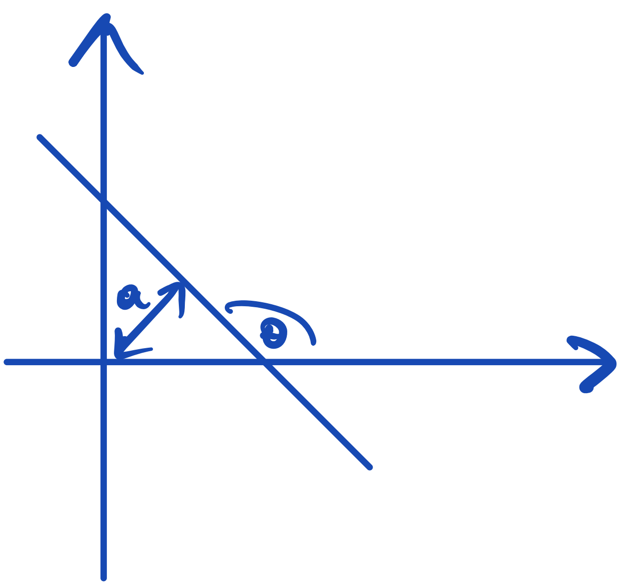


Γ - curve in $\mathbb{R}^2$

uniform measure on lines

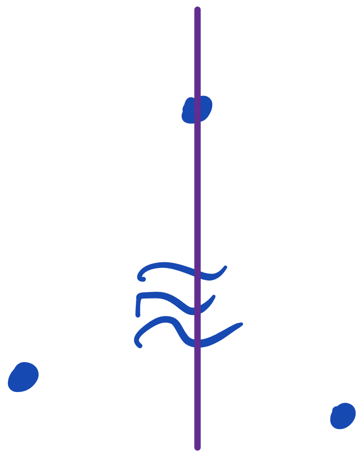
$$\text{length}(\Gamma) = \int \#\{\Gamma \cap L\}\, d\mu(L)$$

$$d\mu(L) = d\theta \cdot da$$



Advanced version.

$x_1, \dots, x_{n+1}$ - observation points in $\mathbb{R}^n$



S - bounded surface
or union of

If every line passing through one of x_i has no more than A intersection points with S,

Then $H^{n-1}(S) \le C \cdot A$

C - depends on $x_1, \dots, x_{n+1}$, $\operatorname{diam}(S)$
distance from S to $x_1, \dots, x_{n+1}$