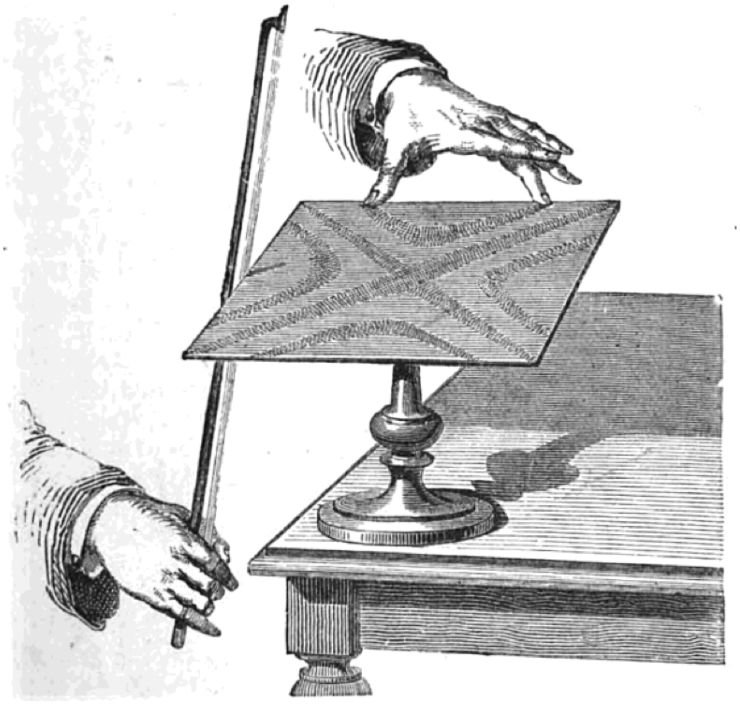
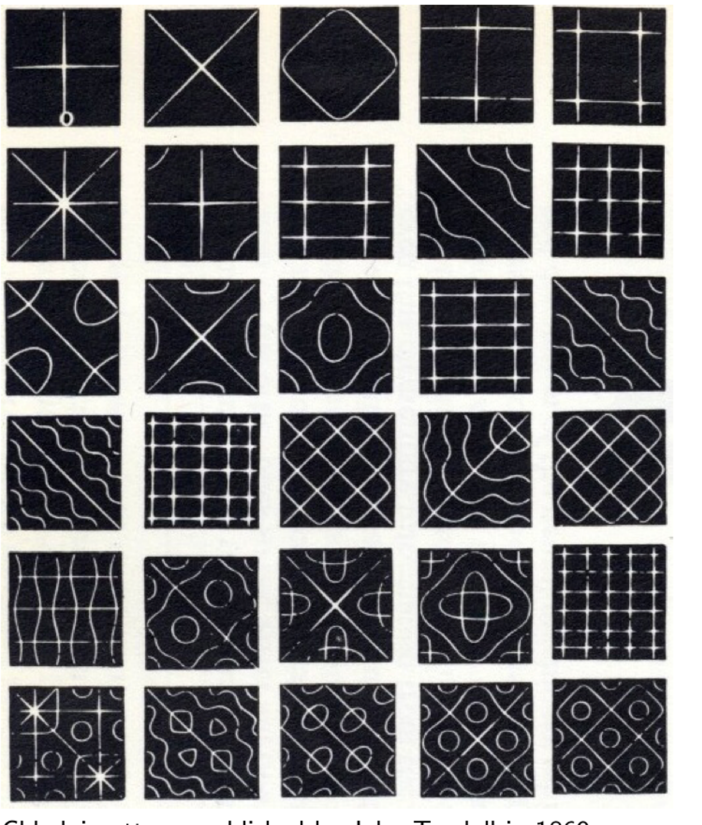
Nodal sets were observed in resonance experiments by Leonardo da Vinci, Galileo Galilei, Robert Hooke, Ernst Chladni, ...



William Henry Stone (1879), Elementary Lessons on Sound, Macmillan and Co., London, p. 26, fig. 12;



Chladni patterns published by John Tyndall in 1869.

What are nodal sets?

Equation for the vertical displacement of the metal plate:

$$\Delta^2 W + c \cdot W_{tt} = 0$$

(we will put $c=1$ for the sake of good notations)

Vibration modes

$$W(x,t) = \varphi(x) \cdot \sin(\lambda t)$$

$$\Delta^2 \varphi - \lambda^2 \varphi = 0$$

$$\Delta \Delta \varphi = \Delta^2 \varphi$$

Clamped boundary conditions

$$\varphi = 0 \quad , \quad \frac{\partial}{\partial n}\varphi = 0$$

Spectral thm.

Given a C^∞ smooth bounded domain $\Omega \subset \mathbb{R}^n$,
There is a sequence of functions $\{\varphi_k\}_{k=1}^{\infty}$
and numbers $\lambda_k \to \infty$ such that

$$\Delta^2 \varphi_k - \lambda_k^2 \varphi_k = 0 \quad \text{in } \Omega \subset \mathbb{R}^2.$$

with clamped boundary conditions

$$\varphi = 0, \ \frac{\partial}{\partial n}\varphi = 0 \quad \text{on } \partial\Omega$$

φ_k form orthonormal basis in $L^2(\Omega)$.

Nodal sets are zero sets of solutions to elliptic PDE.

Vibration modes of a metal plate:

$$\Delta^2 \varphi - \lambda^2 \varphi = 0$$

Vibration modes of a membrane

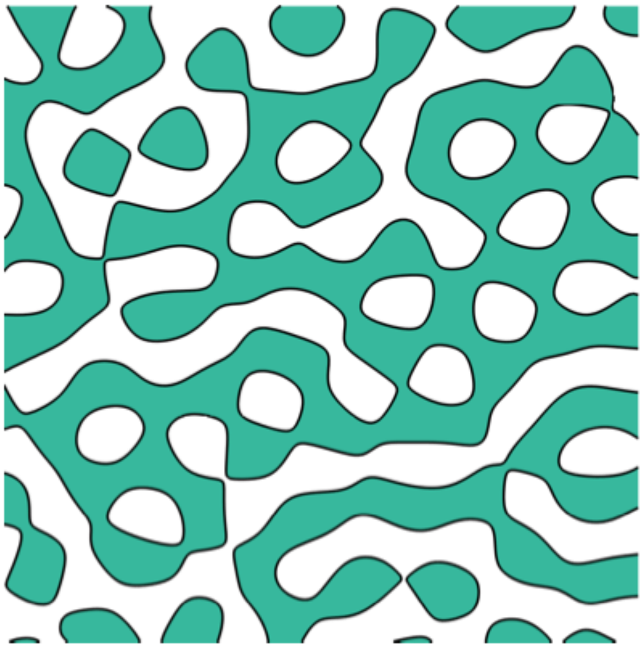
(simpler problem) (thin elastic material)

$$\Delta \varphi + \lambda \varphi = 0 \quad \text{in } \Omega$$

with Dirichlet boundary conditions

$$\varphi = 0 \quad \text{on } \partial\Omega$$

$\frac{1}{\sqrt{\lambda}}$

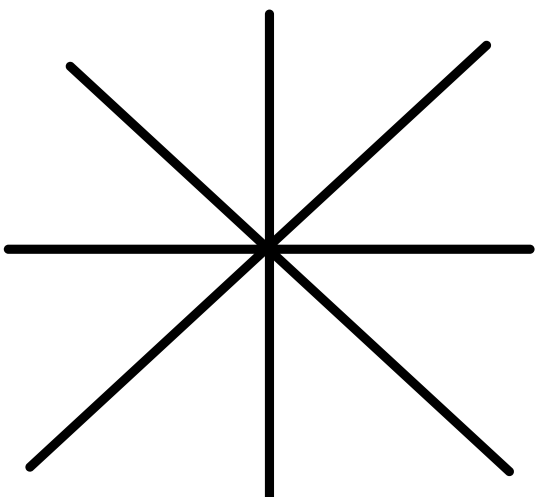


The sign of a solution to $\Delta\varphi + \lambda\varphi = 0$ on the plane (no boundary conditions).
Picture credits: Dmitry Belyaev.

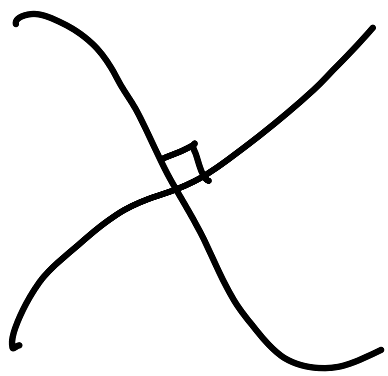
Nodal set of any solution to $\Delta u + Vu = 0$ on $\mathbb{R}^2$ where V is a bounded function, is a union of smooth curves.

Bers' thm (2D only)

Intersections of nodal curves are equiangular.



$u = \mathrm{Im}(z^4)$



$\Delta u = 0$

Exercise. Any solution φ to $\Delta^2\varphi - \lambda^2\varphi = 0$ can be written

as $\varphi = \varphi_+ + \varphi_-$, where

$\Delta\varphi_+ + \lambda\varphi_+ = 0$, $\Delta\varphi_- - \lambda\varphi_- = 0$.
$\lambda > 0$.

$$(\Delta^2 - \lambda^2 I) = (\Delta - \lambda I)(\Delta + \lambda I)$$

What is the difference between positive and negative coefficients in the PDE?

Examples.

1) $u(x,y) = e^x$ solves $\Delta u - u = 0$

2) $u(x,y) = \sin(y)$ solves $\Delta u + u = 0$

3) $u(x,y) = e^x \cdot \sin(y)$ solves $\Delta u = 0$

Some facts about harmonic functions.

Mean value property.

$$u(x) = \fint_{B_r(x)} u := \int_{B_r(x)} u \Big/ \mathrm{Volume}(B_r)$$

Exercise. Show that the mean-value property implies

a) Harnack's inequality:

If u is positive and harmonic in $2B$, then

$$\sup_B u \le C_d \cdot \inf_B u$$

b) Liouville theorem.

If $u > 0$ and $\Delta u = 0$ in $\mathbb{R}^d$,

then $u \equiv \text{const}$.

Oscilation inequality

There is $T > 1$ such that

$$\operatorname{Osc}_{2B} u \geq T \cdot \operatorname{Osc}_{B} u$$

where $\operatorname{Osc}_{B} u := \sup_{B} u - \inf_{B} u$

and u is any harmonic function.

*

Oscilation inequality

(Di Giorgi, Landis)

$\operatorname{div}(A\nabla u)=0$ $\qquad A=(a_{ij})$

matrix-valued function

elliptic, symmetric

$c\cdot|x|^2 \leq (Ax,x) \leq C\cdot|x|^2$

$c, C>0$ – ellipticity constants

There is $T>1$:

$$\operatorname*{Osc}_{2B} u \geq T\cdot \operatorname*{Osc}_{B} u$$

$T=T(c,C)$ – depends on ellipticity constants

<u>Exercise</u>. Using the oscilation inequality show that u: $\operatorname{div}(A\nabla u) = 0$ is α-Holder continuous, where α depends on the ellipticity constants of u.

Mean-value property for u: $\Delta u + \lambda u = 0$.

There is a special function. J_λ

$$J_\lambda(r)\, u(0) = \int_{\partial B_r} u(y)\, dy$$

Maximum principle does not hold for solutions to $\Delta u + \lambda u = 0$

Exercise. $\Delta u + u = 0$ in $\mathbb{R}^d$

There is r: $\int_{\partial B_r(x)} u = 0$ for any x.

Exercise.

① Solutions to $\Delta u + \lambda u = 0$ in $\mathbb{R}^d$ like to oscilate:

The nodal set $Z_u = \{x : u(x) = 0\}$ is $C/\sqrt{\lambda}$-dense in $\mathbb{R}^d$.

② Solutions (non-zero) to $\Delta u - u = 0$ in $\mathbb{R}^n$ grow exponentially fast

$$\max_{\partial B_R} |u| \gtrsim e^{cR}$$

Hint. Use the mean-value property for $u(x)\cdot\cos(t)$ to show that

$$\sup_{B_R} |u| \Big/ \sup_{B_1} |u| > C_R > 1 \,, \quad R > 1$$

Show that

$$\max_{B_{k+1}(0)} |u| \geq C \cdot \max_{B_k(0)} |u| \,, \quad k \in \mathbb{N}$$

Open questions about nodal sets of a vibrating metal plate.

$$\Delta^2 \varphi - \lambda^2 \varphi = 0 \quad \text{in } \Omega \subset \mathbb{R}^2$$

$$\varphi,\ \partial_n \varphi = 0 \quad \text{on } \partial\Omega$$

① Ernst Chladni:

\# nodal curves

② Polterovich:

density of Z_φ as $\lambda \to +\infty$

Exercise. $\varphi = \varphi_+ + \varphi_-$, $\lambda > 0$

$$\Delta \varphi_+ + \lambda \varphi_+ = 0 \qquad \Delta \varphi_- - \lambda \varphi_- = 0$$

$$\left.\begin{array}{l} \varphi_+ = -\varphi_- \\ \partial_n \varphi_+ = -\partial_n \varphi_- \end{array}\right. \quad \text{on } \partial\Omega$$

Exercise (improved maximum principle)

$\tilde{\Omega} \Subset \Omega$

$$\sup_{\tilde{\Omega}} \varphi_- \lesssim e^{-c\sqrt{\lambda}} \sup_{\Omega} \varphi_-$$

Spherical Laplacian:

$$\Delta_{\mathbb{R}^d} = \partial_r^2 + \frac{n-1}{r}\partial_r + \frac{1}{r^2}\Delta_{S^{d-1}}$$

Exercise. Describe harmonic functions in $\mathbb{R}^n$ that are rotationally invariant.

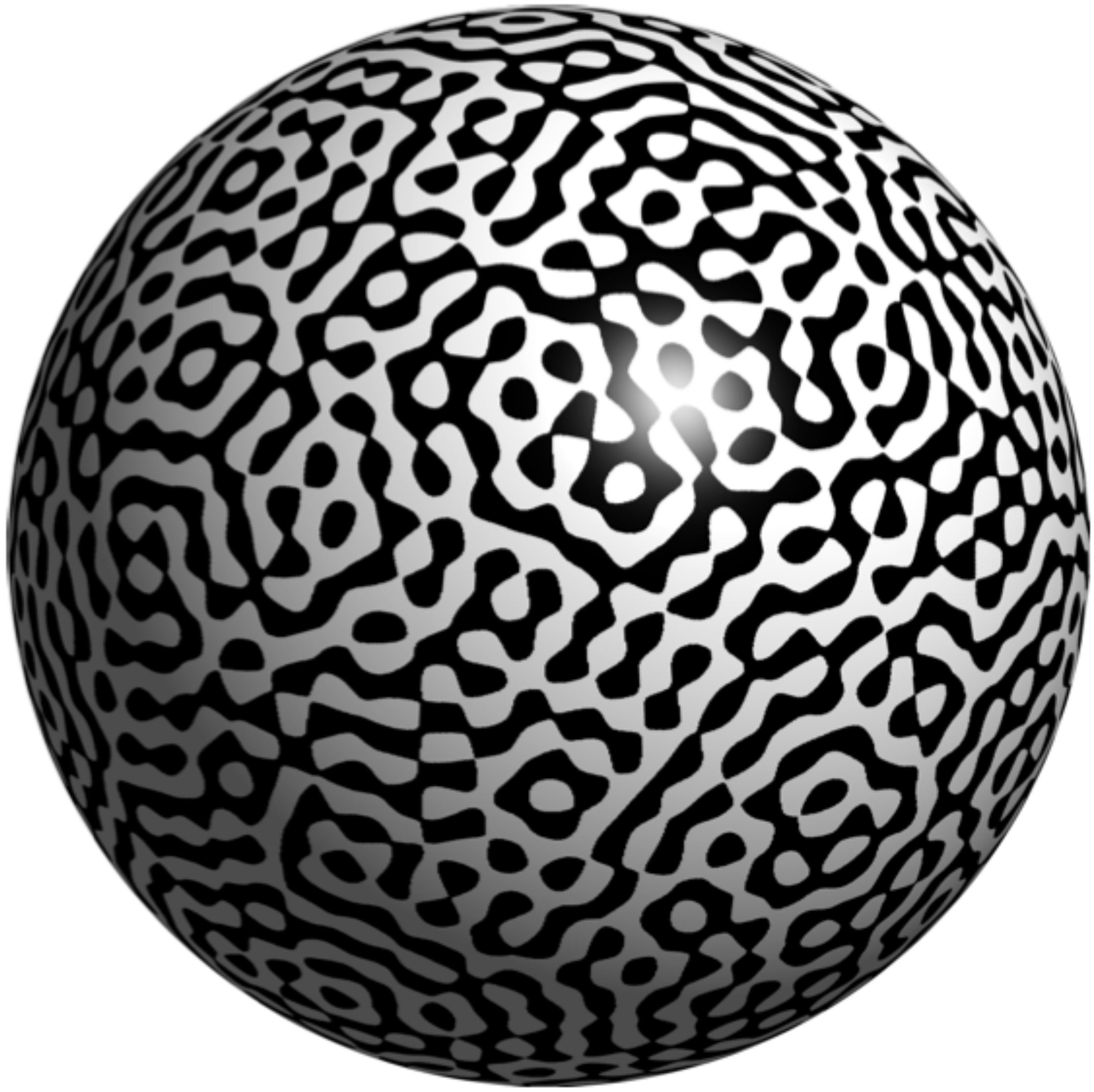
$$\Delta_{\mathbb{R}^d} = \partial_r^2 + \frac{d-1}{r}\partial_r + \frac{1}{r^2}\Delta_{S^{d-1}}$$

Eigenfunctins on S^{d-1}.

If $u(x) = |x|^n \cdot \varphi\left(\frac{x}{|x|}\right)$ is a homogeneous harmonic polynomial of degree n, then its restriction onto the unit sphere is an eigenfunction on the sphere

$$\varphi = u|_{S^{d-1}}$$

$$\Delta_{S^{d-1}}\varphi + \lambda\varphi = 0 \quad , \quad \lambda = n(n+d-1)$$



The sign of a spherical harmonic.

Picture credits: Dmitry Belyaev.

$\lambda = k \cdot (k+1)$
has multiplicity
$2k+1$.

Eigenspace for λ
is a linear space
of dimension $2k+1$.

Questions from the picture.

- The number of nodal curves.
- The length of the nodal curves
- The number of critical points.