

Thm(Zygmund) : $\varphi_\lambda : \Delta f + \lambda f = 0$ on \mathbb{T}^2

$$\|\varphi_\lambda\|_{L^4} \leq C \cdot \|\varphi_\lambda\|_{L^2}$$

\uparrow
uniform in λ .

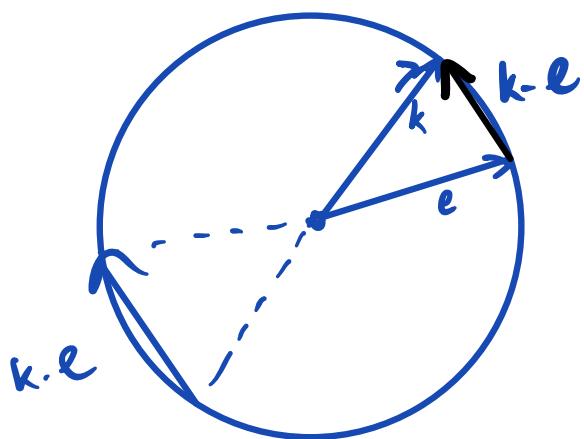
Proof. Nice exercise

Proof. $\ell_\lambda(x) = \sum_{|k|^2=\lambda} a_k e^{ikx}$

$$|\psi_\lambda(x)|^2 = \left(\sum_{|k|^2=\lambda} a_k e^{ikx} \right) \cdot \overline{\left(\sum_{|l|^2=\lambda} a_l e^{ilx} \right)} =$$

$$= \sum_{m=k-l} e^{imx} \cdot b_m, \text{ where } b_m = \sum_{\substack{k-l=m \\ |k|=|\ell|=\lambda^{\frac{1}{2}}}} a_k \bar{a}_\ell$$

Geometrical observation



$k-e$ connects
two integer points
on the circle.

For $m \neq 0$,
there are at most 2 ways to write m
as $k-e$, $|k|=|e|$

$$S|\varphi_\lambda|^4 = \|\varphi_\lambda^2\|_2^2 = \sum |b_m|^2$$

$$|b_0|^2 = |\sum |\alpha_k|^2|^2$$

$$\sum_{m \neq 0} |b_m|^2 \leq 2 \cdot \sum_{k-e=m} |\alpha_k|^2 |\alpha_e|^2 \leq 2 \cdot \sum |\alpha_k|^2 \sum |\alpha_e|^2$$

$$|b_m|^2 = |\alpha_k \bar{\alpha}_e + \alpha_{\bar{k}} \bar{\alpha}_{\bar{e}}|^2 \leq 2 |\alpha_k \alpha_e|^2 + 2 |\alpha_{\bar{k}} \alpha_{\bar{e}}|^2$$

$$S|\varphi_\lambda|^4 \leq C \left(S|\varphi_\lambda|^2 \right)^2$$

L^p bounds on \mathbb{T}^2

$$\Delta \varphi + \lambda \varphi = 0$$

$$\|\varphi_\lambda\|_{L^\infty} \leq C_\varepsilon \cdot \lambda^\varepsilon \cdot \|\varphi_\lambda\|_{L^2}$$

$$\|\varphi_\lambda\|_{L^4} \leq C \cdot \|\varphi_\lambda\|_{L^2}$$

Open question (1993 Bourgain)

$$\|\varphi_\lambda\|_{L^p} \leq C_p \cdot \|\varphi_\lambda\|_{L^2} \quad p < +\infty$$

Conjecture (Bourgain - Zeev)

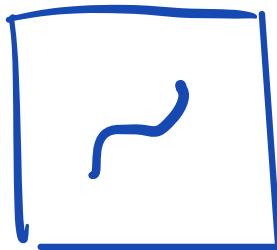
$$\Delta \varphi + \lambda \varphi = 0 \quad T^d \quad p = 2 + \epsilon_d$$

suff. small
1.

$$\|\varphi\|_{L^p} \leq C_p \cdot \|\varphi\|_{L^2}$$

Restriction problem.

smooth curve of finite length



$$\Gamma \subset \mathbb{H}^2$$

Γ can be a segment

$$\frac{\int |\varphi_\lambda|^2}{\Gamma} \stackrel{?}{\leq} C_\Gamma \cdot \frac{\int |\varphi_\lambda|^2}{\pi^d}$$

Thm (Bourgain - Zelev)

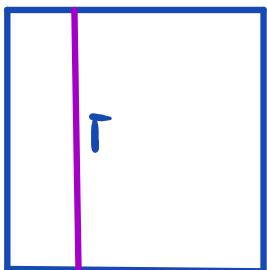
If T is (strictly) convex

$$\int_{\Gamma} |\ell_\lambda|^2 \leq C_T \cdot \int_{\mathbb{T}^2} |\ell_\lambda|^2$$

C_T independent of λ $\frac{\Delta f + \lambda f = 0}{\pi^2}$

The proof will take a while
and is very related to integer points
on the circle.

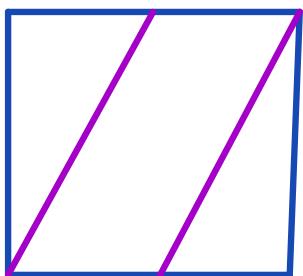
Exercise. a) Let Γ be a vertical segment on \mathbb{H}^2 .



Show that

$$\int_{\Gamma} |P_\lambda|^2 \leq C \cdot \int_{\mathbb{H}^2} |P_\lambda|^2$$

b)



Show that the same inequality is true when Γ is a closed geodesic.

Conjecture (Bourgain-Rudnick).

Upper bound should be true for any curves

Open even for geodesic segments and is related to open problems on integer points

Thm Bourgain - Rudnick)

Γ - convex smooth curve on \mathbb{T}^2

If λ is sufficiently large, then

$$\lambda > \lambda_0$$

$$\int_{\Gamma} |\varphi_\lambda|^2 \geq C_\Gamma \cdot \int_{\mathbb{T}^2} |\varphi_\lambda|^2$$

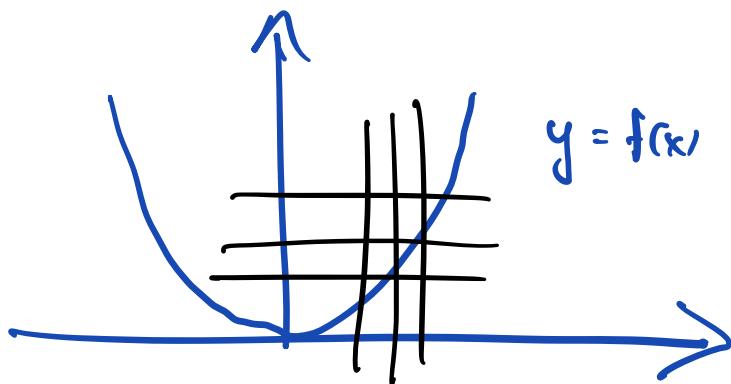
$$C_\Gamma > 0$$

Integer points on convex curves.

Thm (Jarnik, 1926) Every strictly convex arc $y = f(x)$ of length ℓ contains at most

$$3(4\pi)^{-\frac{1}{3}} \cdot \ell^{\frac{2}{3}} + O(\ell^{\frac{1}{3}})$$

integral lattice points



„The number of integral points
on arcs and ovals“

E. Bombieri & J. Pila

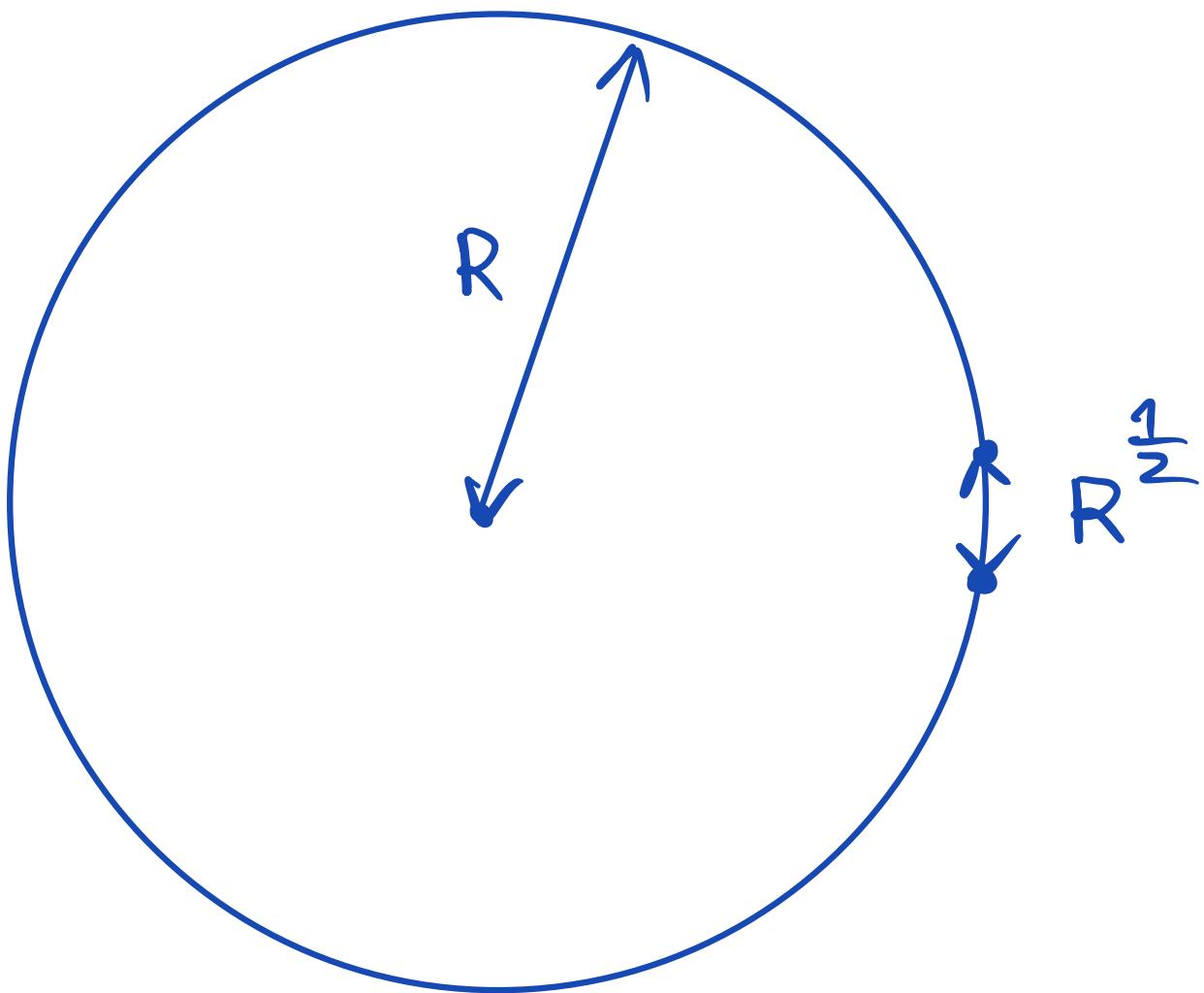
Thm* (Bombieri & Pila)

If $f \in C^\infty([0, 1])$ is strictly convex and $\Gamma = \{ (f(x), x) , x \in [0, 1] \}$, then

$$|t\Gamma \cap \mathbb{Z}^2| \leq c_{f,\varepsilon} \cdot t^{\frac{1}{2} + \varepsilon}$$

$$|\Gamma \cap \frac{1}{t} \cdot \mathbb{Z}^2| \leq c_{f,\varepsilon} \cdot t^{\frac{1}{2} + \varepsilon}$$

Open question.



The number of lattice points

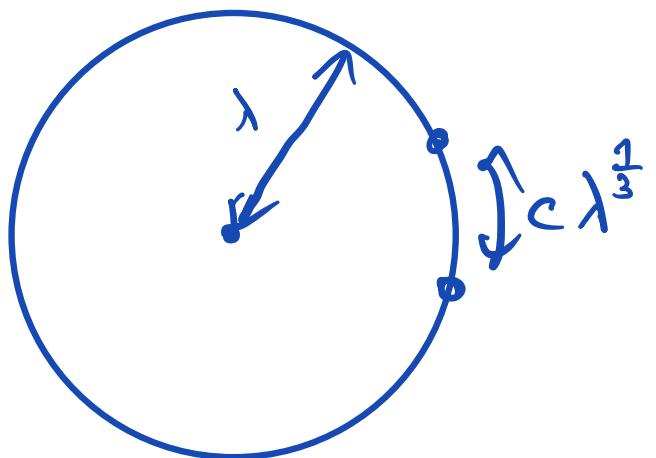
on an arc of length $R^{\frac{1}{2}}$

admits a uniform upper bound

Update

It is also conjectured that there is a uniform bound C_ε for the number of integer points on an arc of length $R^{1-\varepsilon}$ of the circle of radius R .

Distribution of integer points on $\partial B_\lambda(0)$



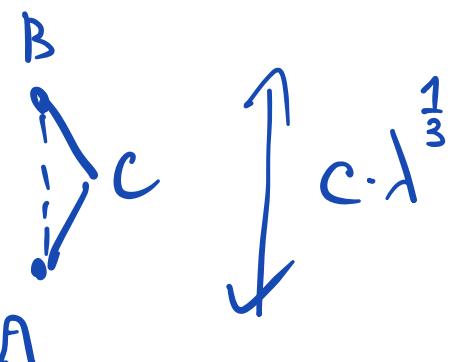
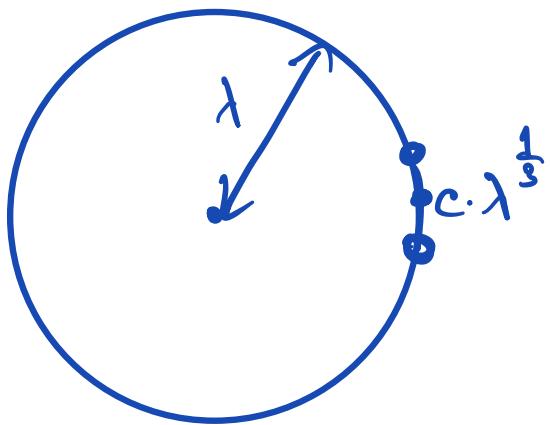
(Thm! Jarnik or earlier)

Every arc
of Length $c \cdot \lambda^{1/3}$
on circle of
radius λ

contains at most
two integer points.

Proof

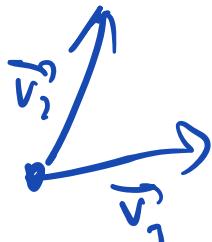
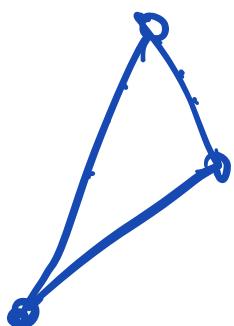
Assume the contrary



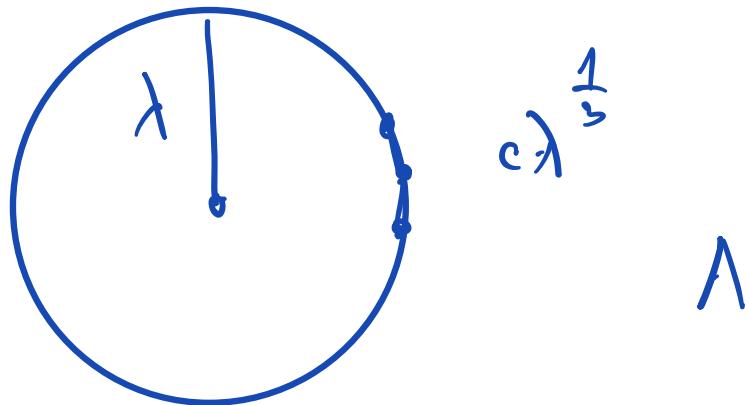
Claim. a) $\text{Area}(\triangle_{\text{BC}}) > 0$

b) $\text{Area}(\triangle_{\text{BC}}) \geq \frac{1}{2}$

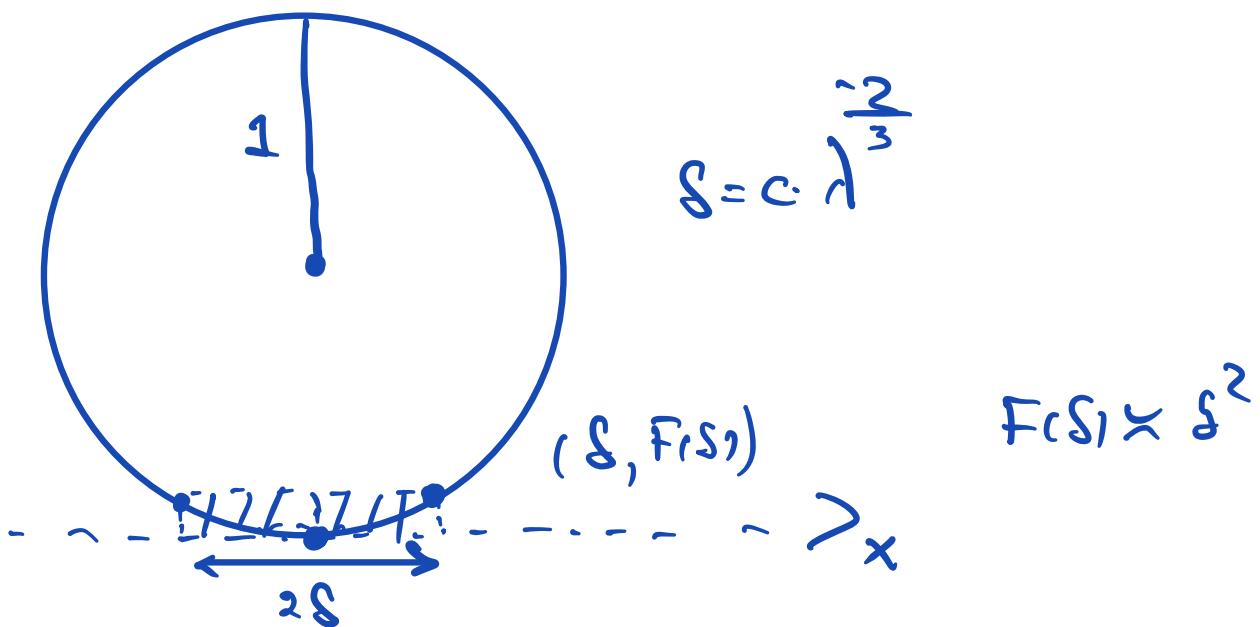
A, B, C -
have integer
coordinate



$$\det(\vec{v}_1, \vec{v}_2) = \pm \text{Area}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$$



$\downarrow \frac{1}{\lambda}$ rescaling



$$\text{Area} \left(\text{dashed rectangle} \right) \leq C \cdot \delta^2 \approx$$

$$\approx c \cdot \lambda^{-2}$$

After rescaling $\text{Area}(ABC) \leq c \cdot \lambda^{-2}$
 Before rescaling $\text{Area}(ABC) \leq c \cdot$

Distractive question.

Given a curve on \mathbb{H}^2 ,

can u_x vanish on it?

$\sin(ax + \overset{+c}{\cancel{By}})$ vanishes
on a geodesic curve.

Exercise

Thm of Bourgain and Rudnick implies that only finitely many of eigenfunctions on \mathbb{T}^2 can vanish on a curve, which is not a closed geodesic.

For the exercise it will be helpful to know.

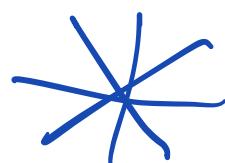
Zero sets of solutions

to $\Delta u + \lambda u = 0$ on T^2

are real-analytic curves.

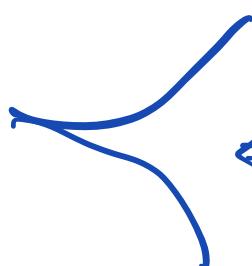
Thm (Bers).

The intersection of several nodal curves is equiangular.



$$|\operatorname{Re}(z^k)|$$

Remark.



$$x^3 - y^2 = 0$$

cannot happen

Demonstration.

$$\Delta u + \lambda u = 0$$

Taylor's expansion

$$u(z) = P_k(z) + P_{k+1}(z) + \dots$$

$P_e(z)$ -homogeneous polynomial
of order e

Exercise.

P_k is harmonic.

$$0 = \Delta u + \lambda u = \frac{\Delta P_k(x) + O(|x|^{k-1})}{|x|^{k-2}} \rightarrow \Delta P_k\left(\frac{x}{|x|}\right)$$

Homogeneous harmonic polynomials of 2-variables

$$u(z) = u\left(\frac{z}{|z|}\right) \cdot |z|^k.$$

$$z = r \cdot e^{i\varphi}$$

$$u(z) = r^k \cdot \sin(k(\varphi + \varphi_0))$$

$$u(z) \in \text{span} \{ \operatorname{Re}(z^k), \operatorname{Im}(z^k) \}$$

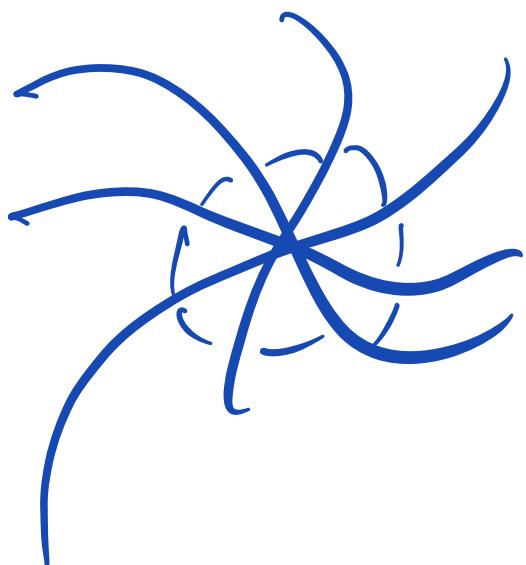
Exercise There are no others.

Exercise. Let $u \in C^\infty(\mathbb{R}^2)$

IF $u = |\operatorname{Re}(z^k)| + O(z^{k+1})$,
 $\nabla u = \nabla |\operatorname{Re}(z^k)| + O(|z|^k)$

then the zero set of u

looks like the zero



set of $|\operatorname{Re}(z^k)|$

$$u = |\operatorname{Im}(z^k)| + \text{Error}$$

$$\begin{aligned} |\nabla u| &= k \cdot |z|^{k-1} \\ |\nabla \text{Error}| &= O(|z|^k) \end{aligned}$$

Remark. It is also true

but non-trivial that

solutions to

(*)

$$\operatorname{div}(A \nabla u) + b \cdot \nabla u + c \cdot u = 0, \quad \text{in } \Omega \subset \mathbb{R}^d$$

$0 \in \Omega.$

$A(0) = \operatorname{Id}$, A - Lipschitz
 b, c - bounded

can be approximated

by harmonic polynomials.

Thm*. For every non-zero solution u to *
there is an integer $k > 0$
and a non-zero homogeneous harmonic polynomial
 P_k of degree k :

$$u(x) = P_k(x) + O(|x|)^{k+1}$$

Back to nodal sets on \mathbb{T}^2 .

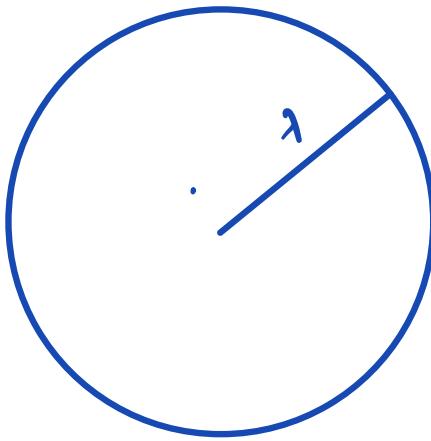
Thm (Bourgain-Rudnick).

For every C^∞ -smooth curve $\Sigma \subset \mathbb{T}^2$ with non-vanishing curvature there is $\Lambda = \Lambda(\Sigma)$ such that for $\lambda > \Lambda$ no eigenfunction $\varphi : \Delta \varphi + \lambda \varphi = 0$ on \mathbb{T}^2 vanishes on Λ .

Moreover, $\lambda > \Lambda_\Sigma$

$$\sum_{\lambda} C_{\Sigma} \cdot \|\varphi_{\lambda}\|_2 \leq \|\varphi_{\lambda}\|_{L^2(\Sigma)} \leq C_{\Sigma} \cdot \|\varphi_{\lambda}\|_2$$

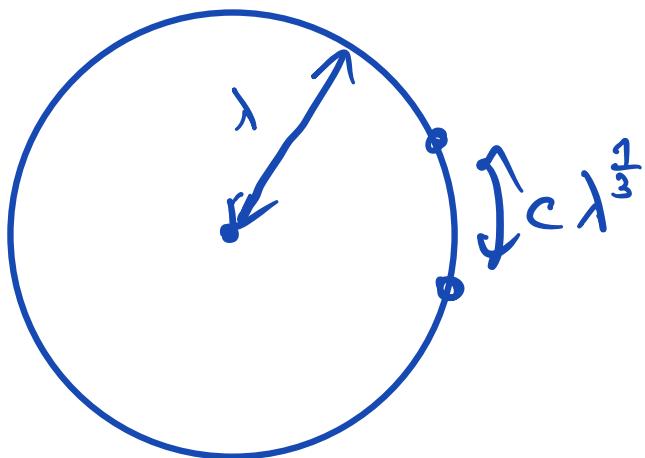
Tools: ①



$$\varepsilon > 0$$

integer points on
 $\{ |z| = \lambda \}$
 is $\leq C_\varepsilon \cdot \lambda^\varepsilon$

②



$$c = \frac{1}{10^{20}}$$

Every arc
 of Length $c \cdot \lambda^{\frac{1}{3}}$
 on circle of
 radius λ

contains at most
 two points.

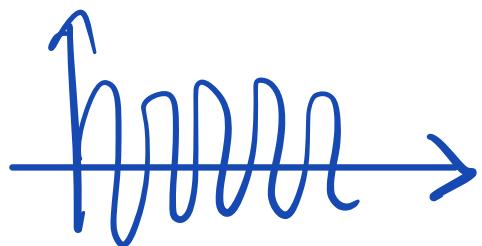
③

Useful tool: oscillatory integrals

real under some assumptions on f

$$I(\lambda) = \int_{-1}^1 e^{\lambda f(x)} dx$$

$$I(\lambda) \xrightarrow{\lambda \rightarrow +\infty} 0$$



The end of lecture 9.

Exponential sums

$$f = \sum a_k e^{inx}$$

1 dim. case

$$E \subset [0, 2\pi]$$

d-dim case

$$E \subset [0, 2\pi]^d$$

E can be an open set, a set of positive measure, or a set of lower dimension.

When $\int_E f^2 d\pi^d \leq \int_{\mathbb{R}^d} |f|^2 ?$

$$\int_E f^2 d\pi^d \approx \int_{\mathbb{R}^d} |f|^2 ?$$

Idea. Let E be a subinterval of $[0, 2\pi]$

If the distances between n_k are large, the e^{inx} are almost orthogonal on E .

$$\int_a^b |f|^2 = \int_a^b f \cdot \bar{f}$$

Demonstration

$$\int_a^b e^{inx} \cdot e^{\overline{imx}} = \frac{1}{n-m} \cdot i \cdot \int_a^b e^{i(n-m)x}$$

$$\int_a^b |e^{inx}|^2 = b-a$$

$$\int_a^b \left(\sum a_k e^{inx} \right)^2 = \int_a^b \sum_{k \neq l} a_k \cdot a_l \cdot e^{i(n_k - n_l)x} + (b-a) \cdot \sum |a_k|^2$$

Sum of non-diagonal terms $\leq (\# \text{ terms})^2 \cdot \frac{1}{\min_{n_k \neq l} |n_k - n_l|}$

Example $f = \sum_{k=1}^n c_k \cdot e^{i\lambda_k x}$ δ - size
of the interval

If $\min_{k \neq e} |\lambda_k - \lambda_e| >> n^2 \cdot \frac{1}{\delta}$, then

$$\int_0^\delta |f|^2 \sim \delta \sum c_k^2.$$

Remark. The argument works in highe dim.s.

Remark. If n_k are not separated, then there cannot be any lower bound, and the upper bound is trivial:

$$\left\| \sum c_k \cdot e^{i\lambda_k x} \right\| \leq \sum |c_k| \cdot \|e^{i\lambda_k x}\| = \delta \cdot \sum |c_k|$$

$L^2[0, \delta]$

Exercise. Let $\varphi_\lambda = \sum_{|k|^2=\lambda} c_k \cdot e^{ikx}$

be an eigenfunction on \mathbb{T}^2 , $\|\varphi_\lambda\|_{L^2} = 1$

a) Show that for a smaller square

$$Q_a = [0, a]^2 \subset \mathbb{T}^2 \quad a - \text{fixed}$$

$$\int_{Q_a} |\varphi_\lambda|^2 \leq 4a^2 + o(1) \quad \text{as } \lambda \rightarrow \infty$$

b) Quantify $o(1)$ in terms of a and λ .

c) Any weak limit of measures

$|\varphi_\lambda|^2 dx$ as $\lambda \rightarrow \infty$ is absolutely continuous w.r.t Lebesgue measure.

Back to them by Bourgain & Zeer.

σ - normalized
are-length measure on Σ .

$$G(\Sigma) = \widehat{G}(0) = 1$$

Claim.

$$\widehat{|G(z)|} < 1 \quad \text{for } z \neq 0.$$

$$\int e^{ikx} d\sigma(z) = \widehat{G}(k)$$

$$|\int e^{ikx} d\sigma| \leq \sup |e^{ikx}| \cdot |\sigma|$$

Σ - a convex curve on \mathbb{T}^2

Claim* (method of stationary phase)
(to be explained a bit later)

$$|\hat{G}(z)| \leq \frac{C_\varepsilon}{\sqrt{|z|}}, \quad |z| > 1.$$

Corollary. $|\hat{G}(z)| \leq 1 - \delta, \quad \delta > 0$

for $z \in \mathbb{C}^2 \setminus \{0\}$

Back to eigenfunctions.

$$\varphi(x) = \sum_{|n|=R} \widehat{\varphi}_{n1} \cdot e^{2\pi i n x}$$

$$R \cdot S^+ = \cup I_\alpha$$

$$|I_\alpha| \asymp R^{\frac{1}{3}}$$

Each I_α contains

one or two integer

points.

$$\begin{aligned} (\dagger) \quad & n \in I_\alpha, m \in I_\beta & n, m \in \mathbb{Z}^2 \\ (\ddagger) \quad & \Rightarrow |n-m| \gtrsim c \cdot R^{\frac{2}{3}} \end{aligned}$$

$$\varphi_{(x)}^{\alpha} = \sum_{n \in I_{\alpha}} \widehat{\varphi}(n) \cdot e^{2\pi i n x}$$

$$\varphi = \sum \varphi^{\alpha}$$

$$\sum \int |\varphi|^2 d\sigma = \sum_{I_{\alpha}, I_{\beta}} \int \varphi^{\alpha} \cdot \overline{\varphi^{\beta}} d\sigma$$

$$\alpha \neq \beta$$

$$\sum \int \varphi^{\alpha} \cdot \overline{\varphi^{\beta}} d\sigma$$

Oscillatory integral

$$\sum \int e^{2\pi i n_{\alpha} \cdot x} \cdot \overline{e^{-2\pi i n_{\beta} \cdot x}} d\sigma \leq C \cdot R^{-\frac{1}{6}}$$

$$|n_{\alpha} - n_{\beta}| \geq c \cdot \lambda^{\frac{1}{3}}$$

$$\widehat{G}(n_{\alpha} - n_{\beta})$$

$$\left| \int e^{i \lambda x} d\sigma \right| \leq \frac{C}{\sqrt{|\lambda|}}$$

$$\sum \left| \int f^\alpha \cdot \overline{f^\beta} d\sigma \right| \leq C \cdot R^{-\frac{1}{6} + 2\epsilon} \cdot R$$

^{non-diagonal} terms in the sum $\leq C \cdot R^{2\epsilon}$

integer points on $\{|z|=R\}$
is $\leq C \cdot R^{\epsilon}$

$$\sum \left| \text{non-diagonal terms} \right| \leq C R^{-\frac{1}{6} + 2\epsilon}$$

$f = \sum c_k \cdot e^{ikx}$

$$\sum |f^\alpha|^2 \asymp \sum \text{diagonal terms} \asymp \sum c_k^2$$

$$\int |e^{inx}|^2 d\sigma = 1$$

$$n_1 \neq n_2$$

$$n_1, n_2 \in I_\alpha \quad \int |c_1 \cdot e^{in_1 x} + c_2 \cdot e^{in_2 x}|^2 =$$

$$= |c_1|^2 + |c_2|^2 + c_1 c_2 \cdot \underbrace{\int e^{i(n_1-n_2)x} + e^{i(n_2-n_1)x} dx}_{= 0}$$

$$\boxed{2 - 2\delta \geq |\hat{G}(n_1, n_2)| + |\hat{G}(n_2, n_1)|}$$

$$\boxed{2 \cdot (1 - \delta) \cdot |c_1, c_2| \leq (1 - \delta)(|c_1|^2 + |c_2|^2)}$$

$$\int |c_1 \cdot e^{in_1 x} + c_2 \cdot e^{in_2 x}| \geq \delta \cdot (|c_1| + |c_2|)$$

$$\frac{1}{2} \cdot |c_1|^2 + |c_2|^2$$

$$\rho = \sum \rho^\alpha \quad \rho = \sum c_k \cdot e^{i n_k x}$$

$$\sum \int |\psi|^2 d\sigma = \sum_{\text{diag. terms}} + O(1)$$

$$S \cdot \sum |c_{kl}|^2 \leq \sum_{\text{diag. terms}} \quad 1 \leq 2 \cdot \sum |c_k|^2$$