Eigenfunctions of the Laplace operator

Let Ω be a smooth bounded domain in $\mathbb{R}^n$. There is a sequence of eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots, \quad \lambda_k \to \infty$$

and a sequence eigenfunctions φ_k:

$$\Delta \varphi_k = -\lambda_k \varphi_k \text{ in } \Omega, \quad \varphi_k = 0 \text{ on } \partial\Omega.$$

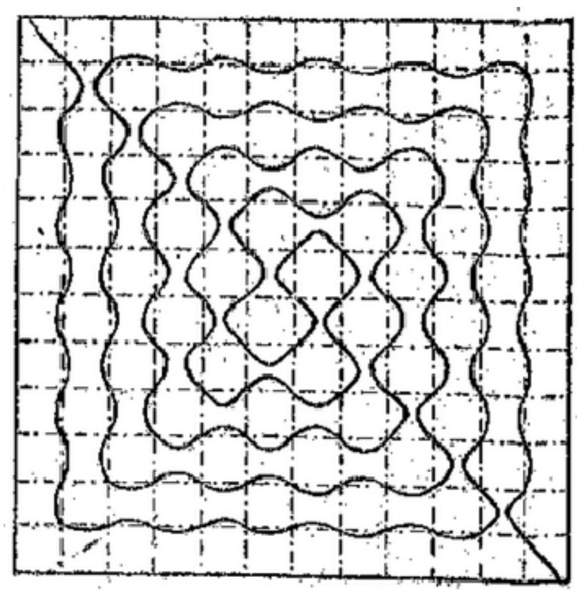


Figure 8 from Ph.D. thesis of A.Stern, 1925.

Nodal sets separate Ω into several connected components that are called nodal domains.

Courants' theorem. The number of nodal domains of the k-th eigenfunction φ_k is at most k.

Antonie Stern constructed examples of high frequency eigenfunctions with only two nodal domains and only one nodal curve.

Sobolev spaces

$\Omega \subset \mathbb{R}^n$ - bounded with smooth boundary.

$W^{1,2}(\Omega) = \{$ functions $f \in L^2(\Omega)$ with $\nabla f \in L^2(\Omega)\}$

$$\|f\| = \sqrt{\int_\Omega f^2 + |\nabla f|^2}$$

$W_0^{1,2}(\Omega) =$ Closure of $C_0^\infty(\Omega)$

w.r.to $\| \ \|$

Exercise.

(a) Show that $f \equiv 1$ is not in $W_0^{1,2}(B_1)$

(b) Show that $1-|x|$ is in $W_0^{1,2}(B_1)$

(c) If Ω is a Lipschitz bounded domain and $u \in C^1(\bar{\Omega})$ and $u=0$ on $\partial\Omega$, then $u \in W_0^{1,2}(B_1)$.

Variational principle

$$\lambda_1(\Omega) = \inf_{\substack{f \in W_0^{1,2}(\Omega) \\ f \neq 0}} \frac{\int |\nabla f|^2}{\int |f|^2} = \inf_{f \in C_0^\infty(\Omega)} \frac{\int |\nabla f|^2}{\int |f|^2}$$

Exercise:

Poincare/Steklov's inequality:

$$\int_\Omega |f|^2 \leq C \cdot \int_\Omega |\nabla f|^2 \quad f \in C_0^\infty(\Omega)$$

One can take $C = \dfrac{\operatorname{diam}^2(\Omega)}{\pi^2}$, but the best constant is the same as the first eigenvalue of Δ.

Proof of the variational principle.

φ_k form the basis in $L^2(\Omega)$.

$\varphi_k \in W_0^{1,2}(\Omega)$

$$\int \varphi_k \cdot \varphi_j = 0 \qquad k \neq j$$

$$\int \nabla \varphi_k \cdot \nabla \varphi_j \overset{?}{=} \begin{cases} 0, & k \neq j \\ \lambda_j & k = j \end{cases}$$

$$\int_\Omega \nabla \varphi_k \cdot \nabla \varphi_j + \int_\Omega \Delta \varphi_k \cdot \varphi_j = \int_{\partial\Omega} \partial_n \varphi_k \cdot \varphi_j$$

$$\int_\Omega \Delta \varphi_k \cdot \varphi_j = \int -\lambda_k \cdot \varphi_k \cdot \varphi_j = -\lambda_k \cdot \delta_{k,j}$$

$$\int_{\partial\Omega} \partial_n \varphi_k \cdot \varphi_j = 0$$

$$\frac{\int |\nabla f|^2}{\int f^2}$$

$$f = \sum c_k \cdot \varphi_k \qquad \int f^2 = 1$$

$$\Updownarrow$$

$$\int \left(\sum c_k \nabla \varphi_k\right)^2 \qquad \sum c_k^2 = 1$$

$$\int |\nabla f|^2 = \sum c_k^2 \cdot \int |\nabla \varphi_k|^2 =$$

$$= \sum c_k^2 \cdot \lambda_k \geq \lambda_1 \cdot \sum c_k^2 = \lambda_1$$

Similarly,

Raleigh quotient

$$\lambda_k = \inf_{\substack{f \in W_0^{1,2}(\Omega) \\ f \perp \varphi_1, \dots, \varphi_{k-1}}} \frac{\int_\Omega |\nabla f|^2}{\int_\Omega f^2}$$

The proof is the same as for λ_1.

Remark If $\overset{W_0^{1,2}(\Omega)}{f} \perp \varphi_1, \dots, \varphi_{k-1}$ and

$$\int |\nabla f|^2 = \lambda_k \int f^2,$$

then $\Delta f + \lambda_k f = 0$ in Ω.

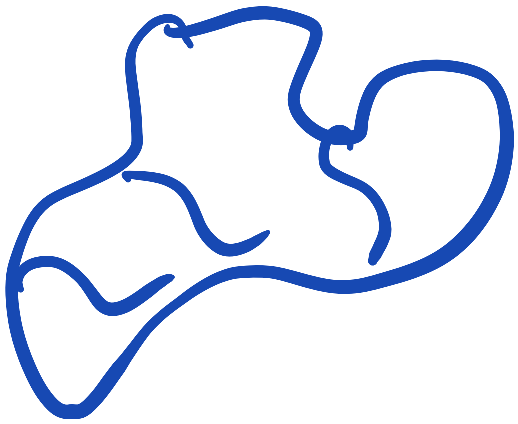
f is in the eigenspace for λ_k.

We will discuss Courant's original proof. The idea is short and elegant, but technical difficulties appear (justification of Green's formula, etc)

Proof.

Proof (Courant's theorem)

Assume the contrary



If φ_k has at least $k+1$ nodal domains $\Omega_1, \Omega_2, \dots, \Omega_{k+1}, \dots$

$\varphi_k = 0$ on $\partial \Omega_j$, $j = 1 \dots k+1$

$\varphi_k \in W_0^{1,2}(\Omega_j)$

Green's formula holds here

$$\Delta \varphi_k + \lambda_k \varphi_k = 0 \qquad \varphi_k \in W_0^{1,2}(\Omega_j)$$

$$\int \nabla \varphi_k \cdot \nabla \varphi_k = \lambda_k \int \varphi_k^2 .$$

$$\int_{\Omega_j} |\nabla \varphi_k|^2 = \lambda_k \int_{\Omega_j} \varphi_k^2$$

$$\varphi_k \cdot \mathcal{I}_{\Omega_j} = f_j$$

$$\mathrm{Span}\{ f_j , j = 1 \ldots k \}$$

For every $f \in \mathrm{Span}\{f_j\}$

$$\int |\nabla f|^2 = \lambda_k \int |f|^2$$

$f = 0$ on Ω_{k+1}

There is a non-zero

$f \in \mathrm{Span}\{f_j\}$ with

$f \perp \varphi_1, \dots, \varphi_{k-1}$

$\int f_j \cdot \varphi_1 = a_{1,j}$ $\quad$ $\int f_j \cdot \varphi_k = a_{k,j}$

$f_1, \dots, f_{k+1}$ are linearly independent

$f = \sum c_j \cdot f_j$ with $\int f \cdot \varphi_j = 0$

non-zero

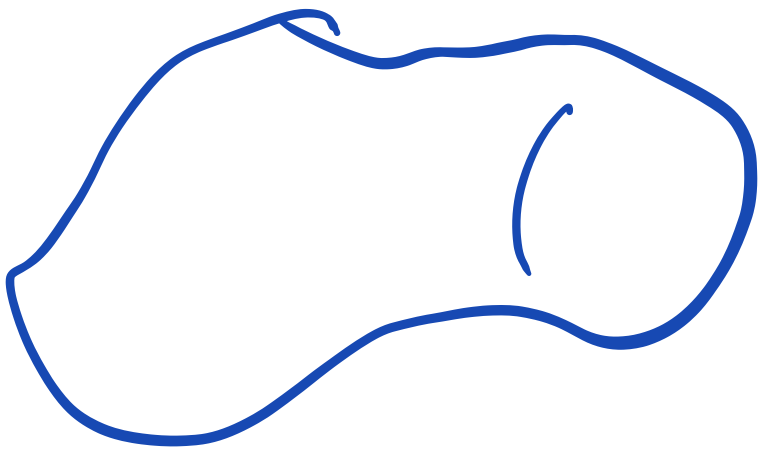
$j = 1 \dots k-1$

$\Delta f + \lambda_k f = 0$ by the remark

f - should be real analytic

$f = 0$ on Ω_{k+1}

$\Rightarrow f \equiv 0$



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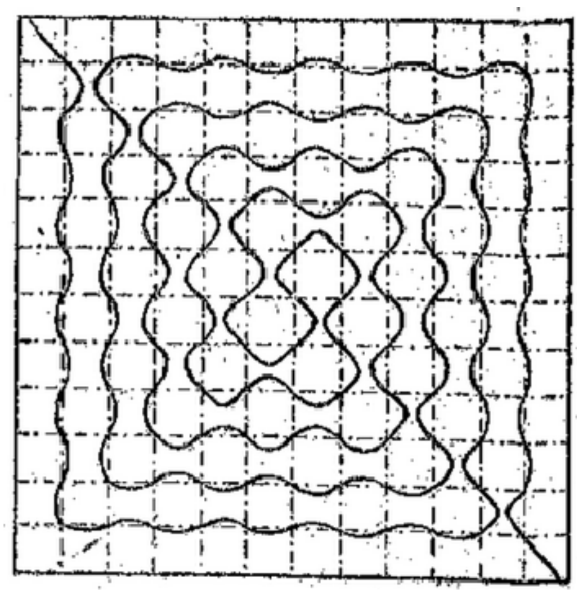

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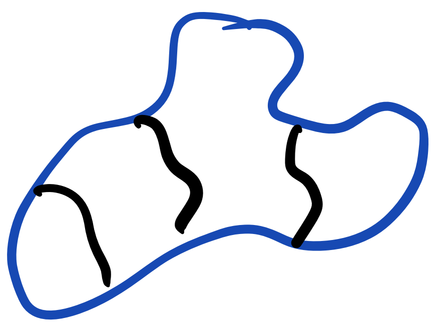
Example. Eigenfunctions on $(0,1)$

$\varphi_k = \sin(\pi k x)$ has exactly k nodal domains.

Remark. Courant's theorem claims that φ_1 does not change sign.

The first eigenvalue is of multiplicity 1.

Useful remark.



If Ω_i are nodal domains of the eigenfunction φ: $\Delta\varphi + \lambda\varphi = 0$, then

$\lambda_1(\Omega_i)$ are the same and equal to λ.

If $\tilde{\varphi}$ is first eigenfunction in Ω_i

$\tilde{\varphi} > 0$ in Ω_i, $\int \varphi\tilde{\varphi} = 0$

Courant's theorem holds in extreme generality.

It is true for compact Riemannian mfds (with or without boundary)

In particular, it is true for the sphere.

Technical issues.

1) Justification of Green's formula (some structure theorems about nodal sets are needed)

Why a restriction of an eigenfunction onto a nodal domain is still in $W_0^{1,2}(\Omega)$?

2) Unique continuation property.

PDE $Lu=0$ has UCP if all solutions that vanish on an open set must vanish everywhere.

Laplace operator on manifolds.

In local coordinates Δ_M is an elliptic operator of second order.

The formal definition requires some time and involves the notion of Riemannian metric, but in local coordinates it can be written as

$$\Delta_M u = \frac{1}{\sqrt{|g|}} \operatorname{div}\left(\sqrt{|g|} \cdot (g_{ij})^{-1} \nabla f\right)$$

$$|g| = |\det g_{ij}|$$

$$\Delta_M = \frac{1}{\sqrt{|g|}} \operatorname{div}(A \nabla u)$$

Aronszajn (1954):

If $\operatorname{div}(A\nabla u) + b\cdot\nabla u + c\cdot u = 0$, where $A - C^1$, b, c – bounded, then UCP holds.

Muller (1954). Examples of PDE and solutions: $\operatorname{div}(A\nabla u) = 0$

A – elliptic, Holder,

u – is zero on an open set, but not everywhere.

Filonov. Solutions to $\operatorname{div}(A\nabla u) + \lambda u = 0$ with compact support.

Monotonicity property.

If $\Omega \subset \tilde{\Omega}$, then $\lambda_1(\Omega) \geq \lambda_1(\tilde{\Omega})$

$$\inf_{f \in W_0^{1,2}} \frac{\int |\nabla f|^2}{\int |f|^2} = \lambda_1 \qquad W_0^{1,2}(\Omega) \subset W_0^{1,2}(\tilde{\Omega})$$

It is also true

$$\lambda_k(\Omega) \geq \lambda_k(\tilde{\Omega})$$

$$\lambda_k = \inf_{f \perp \varphi_1 \ldots \varphi_{k-1}} \frac{\int |\nabla f|^2}{\int f^2}$$

Min-max principle

$$\lambda_k = \inf_{\substack{\dim(L)=k \\ L \subset W_0^{1,2}(\Omega)}} \sup_{f \in L} \frac{\int |\nabla f|^2}{\int f^2}$$

$\uparrow$

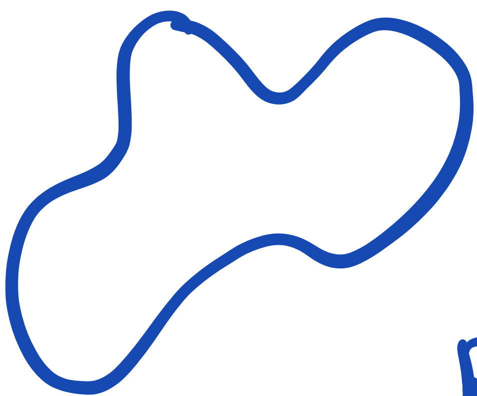
<u>Exercise.</u>

$$\Omega \subset \tilde{\Omega}$$

$$\lambda_k(\Omega) \geq \lambda_k(\tilde{\Omega})$$

$$W_0^{1,2}(\Omega) \subset W_0^{1,2}(\tilde{\Omega})$$

Remark. Monotonicity holds for Dirichlet eigenvalues, but not for Neumann eigenvalues



$\Delta f + \lambda f = 0 \quad \Omega$

$\partial_n f = 0 \quad \text{on } \partial\Omega$

Regularity for $\partial\Omega$

$\lambda_0 = 0 \qquad f_0 = \text{const}$

$$\lambda_1 = \inf_{\substack{f \in W^{1,2}(\Omega) \\ f \perp 1}} \frac{\int |\nabla f|^2}{\int f^2}$$

Exercise.

Let u be a harmonic function in $\mathbb{R}^n \setminus \{0\}$ and $|u| \leq 1$ in $B_1 \setminus \{0\}$. Show that u can be extended to a harmonic function in B_1.

Hint. Use the fundamental solution to represent the values inside of the ball in terms of the Cauchy data.

Kelvin transform.

Given a function u in $\mathbb{R}^3$, the Kelvin transform of u is defined by

$$u^*(x) = \frac{1}{|x|} u\left(\frac{x}{|x|^2}\right)$$

Exercise. $\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3} \frac{1}{|x|} =: u(x)$

Show that u^* is a harmonic polynomial of degree $\alpha_1 + \alpha_2 + \alpha_3$

The end

of lecture 3!

Exercises on spherical harmonics

H_n is the space of homogeneous harmonic polynomials in $\mathbb{R}^3$ of degree n.

$$\varphi \in H_n \iff \begin{cases} \varphi(x) = |x|^n \varphi\left(\frac{x}{|x|}\right) \\ \Delta \varphi = 0 \end{cases}$$

<u>Def.</u> spherical harmonics of degree n are restrictions of H_n onto the unit sphere S_2

a) Show that spherical harmonics of different degrees are orthogonal in $L^2(S_2)$

b) Show that the map

$$q \longrightarrow \Delta\left[(1-|x|^2)q\right]$$

is a bijection from the space of polynomials of degree $\leq n$ to itself.

b) Let $p \in P_n$ - the space of homogeneous polynomials of degree n.

$$p = h + (1-|x|^2)\cdot q$$

$$\deg(q) \leq n-2 \quad , \quad \Delta h = 0$$

Taking the homogeneous part, show that

$$P_n = H_n + |x|^2 \cdot P_{n-2}$$

$$P_n = H_n + |x|^2 \cdot H_{n-2} + |x|^4 \cdot H_{n-4} + \dots$$

c) Spherical harmonics form a basis in $L^2(S_2)$.

d) $H_n \perp P_{n-1} + P_{n-2} + \dots + P_0$

Laplace eigenfunctions

(smooth)

On any compact Riemannian mfd (M^d, g) (with or without boundary) there is an orthonormal basis for $L^2(M)$ of eigenfunctions φ_{λ_k}:

$$\Delta_M \varphi_{\lambda_k} + \lambda_k \varphi_{\lambda_k} = 0, \quad \varphi_{\lambda_k} = 0 \text{ on } \partial M$$

$$0 < \lambda_1 \le \lambda_2 \dots, \quad \lambda_k \uparrow \infty.$$

Weyl's law and min-max principles still hold.

$$\lambda_k(M) \sim c_d \cdot \left(\frac{k}{|M|}\right)^{\frac{2}{d}}$$

d- dimension of M.

Example 0. $I=(0,1)$ $\varphi_k=\sin(k\pi x)$

Example 1. $\sin(ax)\cdot\sin(by)$ is the eigenfunction on $\underset{\substack{\| \\ [0,\pi]^2}}{\pi^2}$ with $\lambda=a^2+b^2$

($\cos(ax)\cdot\sin(by)$, $\sin(ax)\cdot\cos(by)$, $\cos(ax)\cdot\cos(by)$ work too)

$$u(x,y)=\sum_{a_k^2+b_k^2=\lambda} c_k \sin(a_k x)\cdot\sin(b_k y)$$ is

also the eigenfunction.

Problem from Bourgain's list.

Given an eigenfunction on $\mathbb{T}^n$

$$u = \operatorname{Re} \sum c_\alpha \cdot e^{i\alpha_1 x_1} \cdot \ldots e^{i\alpha_n x_n}$$

$$\Delta u + \lambda u = 0 \qquad \lambda = \sum \alpha_k^2$$

Open problem. $\|u\|_{L^p} \leq C_p \cdot \|u\|_{L^2}$

For $p = 2 + \varepsilon$ - small constant

True for $\mathbb{T}^2$ and $\mathbb{T}^3$ (Rudnick & Bourgain)