Recap:

Weyl's law:

Dirichlet eigenvalues

$$\lambda_k(\Omega) \underset{k\to\infty}{\sim} c_d \cdot \left(\frac{k}{|\Omega|}\right)^{\frac{2}{d}}$$

$$\sum_{k=1}^{\infty} \varphi_k(x)\cdot\varphi_k(y)\cdot e^{-\lambda_k t} = p(x,y,t)$$

Part I

$$\sum e^{-\lambda_k t} = \int p(x,x,t) \sim (4\pi t)^{-\frac{d}{2}}\cdot|\Omega|$$

Part II Karamata's Tauberian theorem.

$$\sum e^{-\lambda_k t} \sim (4\pi t)^{-d/2}\cdot|\Omega| \Rightarrow \text{Weyl's law}$$

Part II.

$$N(\lambda) := \{ \lambda_i \in (0, \lambda) \} \overset{?}{\sim} \frac{|\Omega|}{(4\pi)^{d/2}} \cdot \lambda^{d/2} \Big/ \Gamma\left(\frac{d}{2}+1\right)$$

dN - Lebesgue - Stieltjes measure

$dN[a, b) = N(b) - N(a) \qquad b > a.$

$$\int e^{-xt} \, dN(x) = \sum e^{-\lambda_i t}$$

Karamata's theorem. Assume $a > 0$.

Let $N(x)$ be $\nearrow$ on $(0, +\infty)$ and

assume $\int_0^\infty e^{-xt}\, dN(x)$ converges and

$f(t) = \int_0^\infty e^{-xt}\, dN(x) \sim \frac{1}{t^a}$ as $t \to 0$, then

$$N(t) \sim t^a / \Gamma(a+1) \quad t \to \infty.$$

Remark. If $N(x) = x^a$, then

$$\int_0^{+\infty} e^{-xt}\, dx^a = \frac{1}{t^a} \cdot \Gamma(a+1)$$

Proof. We know that

$$\int_0^{+\infty} e^{-tx}\, dN(x) \sim \frac{1}{t^a} = \frac{1}{\Gamma(a+1)} \int_0^{+\infty} e^{-xt} \cdot 1 \cdot dx^a$$

$$\Downarrow$$

$$\int_0^{+\infty} e^{-ktx}\, dN(x) \sim \frac{1}{(kt)^a} = \frac{1}{\Gamma(a+1)} \int_0^{+\infty} (e^{-xt})^k\, dx^a$$

If $P(x)$ is a polynomial, then

$$\int_0^{+\infty} e^{-tx} \cdot P(e^{-tx})\, dN(x) \overset{t\to 0}{\sim} \frac{1}{\Gamma(a+1)} \int_0^{+\infty} e^{-xt}\, P(e^{-xt})\, dx^a$$

$$\int_0^{+\infty} e^{-x}\, P(e^{-x}) \cdot dN\left(\frac{x}{t}\right) \overset{t\to 0}{\sim} \frac{1}{t^a} \cdot \frac{1}{\Gamma(a+1)} \int_0^{+\infty} e^{-x}\, P(e^{-x})\, dx^a$$

$$\int_0^{+\infty} e^{-x} P(e^{-x}) \cdot dN\left(\frac{x}{t}\right) \overset{t\to 0}{\sim} \frac{1}{t^a} \cdot \frac{1}{\Gamma(a+1)} \int_0^{+\infty} e^{-x} P(e^{-x})\, dx^a$$

We know that for polynomials P

If we were allowed to take

$P(e^{-x}) = e^{x} \cdot \mathcal{X}_{[0,1]}(x)$, we would be

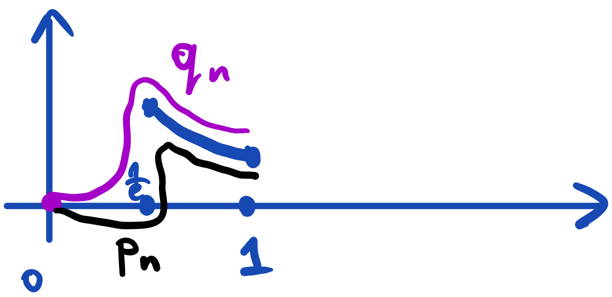
done.

$$N\left(\frac{1}{t}\right) = \int_0^{+\infty} \cdot \mathcal{X}_{[0,1]} \cdot dN\left(\frac{x}{t}\right) \sim \frac{1}{t^a} \cdot \Gamma(a+1)$$

Careful approximation

$P(e^{-x}) \sim e^{x} \cdot \chi_{[0,1]}(x)$

$e^{-x} = \xi$



Two sequences of polynomials

① $p_n(\xi) \le \frac{1}{\xi} \cdot \chi_{[\frac{1}{e},1]} \le q_n(\xi)$ on $[0,1]$

② $\int_0^{+\infty} e^{-x}\left(q_n(e^{-x}) - p_n(e^{-x})\right) dx^a \to 0$ $\quad \left(\int_0^1 \xi\,(q_n - p_n)\, d(-\ln \xi)^a \to 0 \right)$

$$\int_0^{+\infty} e^{-x} p_n(e^{-x})\, dx^a \le \int_0^{+\infty} \chi_{[0,1]}\, dx^a \le \int e^{-x} q_n(e^{-x})\, dx^a$$

So $\int_0^{+\infty} e^{-x} \underset{\substack{\updownarrow \\ q_n}}{p_n}(e^{-x})\, dx^a \longrightarrow 1$

On the other hand,

$$\int_0^{+\infty} e^{-x} p_n(e^{-x}) \cdot dN\left(\frac{x}{t}\right) \le N\left(\frac{1}{t}\right) \le \int_0^{+\infty} e^{-x} q_n(e^{-x})\, dN\left(\frac{x}{t}\right)$$

$$\wr\ t \to 0 \qquad\qquad \wr$$

$$\frac{1}{t^a} \cdot \frac{1}{\Gamma(a+1)} \cdot \underbrace{\int_0^{+\infty} e^{-x} p_n(e^{-x})\, dx^a}_{\text{tends to 1 as } n \to +\infty} \qquad \frac{1}{t^a} \cdot \frac{1}{\Gamma(a+1)} \cdot \underbrace{\int_0^{+\infty} e^{-x} q_n(e^{-x})\, dx^a}_{\text{tends to 1 as } n \to +\infty}$$

$$\sum |\varphi_k(x)|^2 \cdot e^{-\lambda_k t} = p(x,x,t) \sim \frac{1}{(4\pi t)^{d/2}}$$

Exercise. Use Caramata's thm to show

Local Weyl's law

Fix $x, y \in \Omega$

a) $$\sum_{\lambda_k \le \lambda} |\varphi_k(x)|^2 \sim C_d \cdot \lambda^{d/2}$$

b)* $$\sum_{\lambda_k \le \lambda} \varphi_k(x) \cdot \varphi_k(y) \Big/ \#\{\lambda_k \le \lambda\} \xrightarrow{\lambda \to +\infty} 0$$

On average $\varphi_k(x) \cdot \varphi_k(y)$ gives 0.

For smooth bounded domains $\Omega \subset \mathbb{R}^2$

$$\sum_{k=1}^{\infty} e^{-\lambda_k t} = \int_\Omega p(x,x,t)$$

$$= \frac{1}{4\pi t}\left[\text{area}(\Omega) - \sqrt{4\pi t}\ \text{length}(\partial\Omega) + \frac{2\pi t}{3}(1-\gamma(\Omega)) + o(1)\right]$$

the number of holes in Ω

Weyl's conjecture

Ω - smooth, bounded domain in $\mathbb{R}^d$.

The number of Dirichlet's eigenvalues λ_i in the interval $(0, \lambda)$ behaves as follows

$$(2\pi)^{-d} \cdot \lambda^{d/2} |B_1| \cdot |\Omega| - \frac{1}{4} (2\pi)^{1-d} \cdot |\partial B_1| \cdot \lambda^{\frac{d-1}{2}} \cdot |\partial\Omega| + o(\lambda^{\frac{d-1}{2}})$$

—,,— for Neumann eigenvalues with +

$$(2\pi)^{-d} \cdot \lambda^{d/2} |B_1| \cdot |\Omega| + \frac{1}{4} (2\pi)^{1-d} \cdot |\partial B_1| \cdot \lambda^{\frac{d-1}{2}} \cdot |\partial\Omega| + o(\lambda^{\frac{d-1}{2}})$$

Polya's conjecture.

For Euclidean domains

$$\mu_k \leq c_d \cdot \left(\frac{k}{|\Omega|}\right)^{\frac{2}{d}} \leq \lambda_k$$

Exercises.

Eigenfunctions on the torus: $[0,2\pi)^2 = \mathbb{T}^2$

ⓐ Show that $e^{ikx} \cdot e^{imy}$, $k, m \in \mathbb{Z}$ form orth. L^2-basis on $\mathbb{T}^2$ (use that e^{ikx} form orth basis on $\mathbb{T}^1$)

ⓑ Classify periodic functions in $\mathbb{R}^2$ ($\mathbb{Z}^2$-translations) solving $\Delta u + \lambda u = 0$.

For which λ solutions exist?

ⓒ Prove Weyl's law in the case of torus.