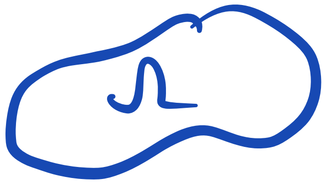
Courant's thm.



$\Omega \subset \mathbb{R}^n$

$u=0$ on $\partial\Omega$

$\Delta \varphi_k + \lambda_k \varphi_k = 0$

The k-th eigenfunction has at most k nodal domains.

Plejel's bound.

$$\overline{\lim} \ \frac{\#\text{ nodal domains of } \varphi_k}{k} \le c_d < 1.$$

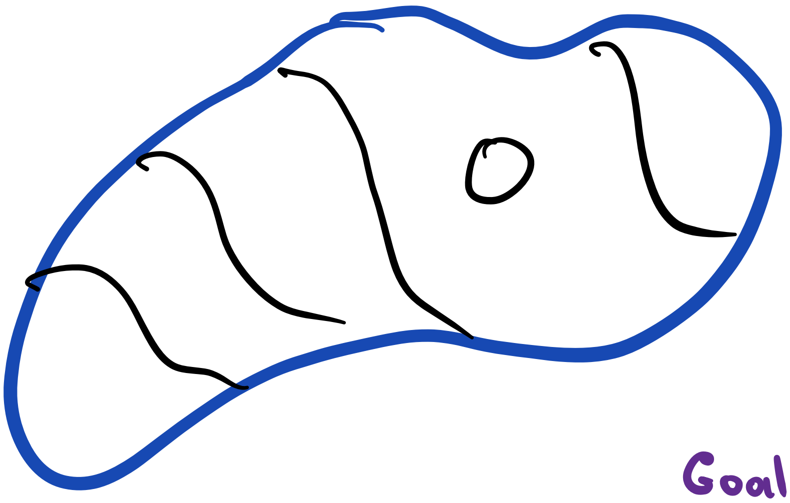
On the proof of Plejel's bound

Weyl's law

$$\lambda_k(\Omega) \underset{k\to\infty}{\sim} c_d \cdot \left(\frac{k}{|\Omega|} \right)^{\frac{2}{d}}$$

Estimate of the volume of nodal domain

$\Delta \varphi + \lambda \varphi = 0$ $\quad \Omega \subset \mathbb{R}^d$



$$\Omega = \bigcup \Omega_i \cup Z_\varphi$$

nodal domains

nodal set of φ

Goal. Estimate # nodal domains in terms of λ when $\lambda \to \infty$.

Reminder $\quad \lambda_1(\Omega_i) = \lambda$.

Faber-Krahn's inequality provides a lower bound for the volume of each nodal domain.

Faber-Krahn's inequality.

Among all domains with the same first eigenvalue, the ball has the smallest volume.

If $|\Omega| \leq |B|$, then

$$\lambda_1(\Omega) \geq \lambda_1(B)$$

Reminder. $\lambda_1(B_R) = \dfrac{\lambda_1(B_1)}{R^2}$

$$\lambda_1(B) = \lambda \iff |B| = \tilde{c}_d \cdot \left(\frac{1}{\sqrt{\lambda}}\right)^d$$

Plejel's bound. By Faber-Krahn's inequality the volume of each nodal domain $|\Omega_i| \geq \tilde{c}_d \cdot \left(\frac{1}{\sqrt{\lambda}}\right)^d$

$$|\Omega| = \sum |\Omega_i| \geq \#\text{ nodal domains} \cdot \tilde{c}_d \cdot \left(\frac{1}{\sqrt{\lambda}}\right)^d$$

Weyl's law: $\underbrace{\#\{\lambda_i \leq \lambda\}}_{=k} \sim c_d \cdot (\sqrt{\lambda})^d \cdot |\Omega|$

$\#$ nodal domains of $\varphi_k \lesssim p_d \cdot k$, $p_d < 1$

$\lesssim$ as $k \to +\infty$

Bourgain's improvement in 2D.

$\#$ nodal domains of $\varphi_k \leq \left(1 - \frac{1}{10^8}\right) \cdot k$
for large k

Idea: One cannot tile a plane by discs.

Improvement to Faber-Krahn's inequality for domains that are not discs.

Faber–Krahn's inequality.

B - is a ball in $\mathbb{R}^d$

If $|\Omega| \leq |B|$, then

$$\lambda_1(\Omega) \geq \lambda_1(B)$$

Assume Ω is smooth

The proof is based on symmetrization argument.

Tools:

1) Isoperimetric inequality
2) Coarea formula.

Isoperimetric inequality.

If $\quad \mathrm{Vol}_{d-1}(\partial\Omega) \leq \mathrm{Vol}_{d-1}(\partial B)$,

then $\quad \mathrm{Vol}(\Omega) \leq \mathrm{Vol}(B)$.

Coarea formula.

Consider a domain Ω in $\mathbb{R}^d$ and $F \in C^\infty(\Omega)$ with $F(\Omega) \subset [a,b]$. Let h be any measurable function such that $h|\nabla F|$ is integrable in Ω.

Then
$$\int_\Omega h|\nabla F| = \int_a^b \int_{\{x: F(x)=t\}} h(x)\, d\lambda_{d-1}(x)\, dt$$

$$\boxed{\int_\Omega h = \int \int_{\{F=t\}} h/|\nabla F|\, d\lambda_{d-1}\, dt}$$

$\int_{\{F(x)=t\}} h(x) \cdot d\lambda_{d-1}(x)$ denotes the integral over hyper surface $\{F(x)=t\}$ with respect to $(d-1)$-dimensional Lebesgue measure.

Sard's lemma $\Rightarrow$ for a.e. t $\{F(x)=t\}$ is a smooth $(d-1)$ dimensional surface.

Exercise. $$\int_{B_R(0)} h = \int_0^R \int_{\partial B_t(0)} h(x)\, d\lambda_{d-1}\, dt$$

$$F(x) = |x| \qquad |\nabla F| = 1$$

$$\int_{B_R(0)} h = \int_0^R \int_{\{F=t\}} \frac{h}{|\nabla F|}\, d\lambda_{d-1}\, dt$$

Symmetrization argument.

Let B be a ball in $\mathbb{R}^d$

with $|B|=|\Omega|$ Goal: $\lambda_1(\Omega)\ge\lambda_1(B)$

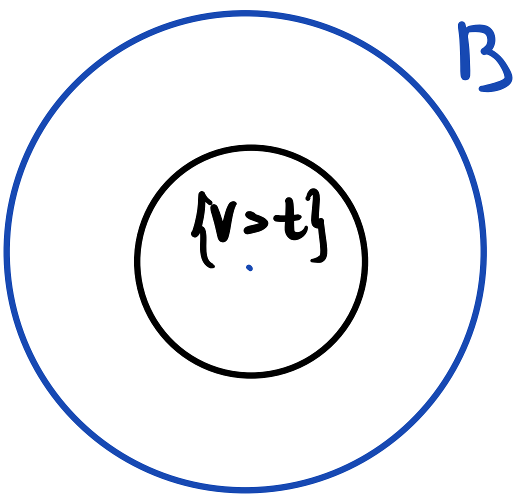
Given $u\in C^\infty(\Omega)$,

the <u>symmetrization</u> of u

is the <u>radial</u> function v in B

defined by

$\{x\in\Omega: u(x)>t\}$



$|\{v>t\}|=|\{u>t\}|$

The set $\{v>t\}$ is a ball centered at the center of B.

Formal contruction. $F(t)=|\{u>t\}|$

$r(t)$: $|B_{r(t)}|=|\{u>t\}|$

v-radial: $|\{v>t\}|=|B_{r(t)}|$

Claim 1 $\int_{\Omega} u^2 = \int_B v^2$ $\quad m(t) = |\{v > t\}|$

$= \int t^2 \cdot dm(t)$

$$|\{a < u < b\}| = |\{a < v < b\}|$$

Claim 2 For a.e. $t \in \mathbb{R}$

$$\int_{\{u=t\}} \frac{1}{|\nabla u|} = \int_{\{v=t\}} \frac{1}{|\nabla v|}$$

Justification:

$$\int_{\{a \le u \le b\}} 1 \cdot d\lambda_d = \int_{\{a \le v \le b\}} 1 \cdot d\lambda_d$$

$$\| $$

$$\int_{t \in (a,b)} \left(\int_{\{u=t\}} \frac{1}{|\nabla u|} \, d\lambda_{d-1} \right) dt = \int_{t \in (a,b)} \left(\int_{\{v=t\}} \frac{1}{|\nabla v|} \, d\lambda_{d-1} \right) dt$$

Claim 3. $\displaystyle\int_{\{u=t\}} |\nabla u| \, d\lambda_{d-1} \geq \int_{\{v=t\}} |\nabla v| \, d\lambda_{d-1}$

Proof. Equivalent statement

$$\begin{array}{ccc}
\displaystyle\int_{\{u=t\}} |\nabla u| \cdot \int_{\{u=t\}} \frac{1}{|\nabla u|} & \overset{?}{\geq} & \displaystyle\int_{\{v=t\}} |\nabla v| \cdot \int_{\{v=t\}} \frac{1}{|\nabla v|} \\
\rotatebox{90}{$\leq$} & & \rotatebox{90}{$=$} \ \text{- } v \text{ is radial} \\
\displaystyle\left(\int_{\{u=t\}} 1 \right)^2 & & \displaystyle\left(\int_{\{v=t\}} 1 \right)^2 \\
\rotatebox{90}{$=$} & & \rotatebox{90}{$=$} \\
\lambda_{d-1}\left| \{u=t\} \right| & \geq & \lambda_{d-1}\left| \{v=t\} \right|
\end{array}$$

Isoperimetric inequality

$$\lambda_d(\{u>t\}) = \lambda_d(\{v>t\})$$

We showed $\displaystyle\int_{\{u=t\}} |\nabla u| \geq \int_{\{v=t\}} |\nabla v|$

Claim 4. $\displaystyle\int_{\Omega} |\nabla u|^2 \geq \int_{B} |\nabla v|^2$

$$\begin{array}{ccc} \displaystyle\int_{\Omega} |\nabla u|^2 & \geq & \displaystyle\int_{B} |\nabla v|^2 \\ \| & & \| \\ \displaystyle\int \int_{\{u=t\}} |\nabla u| \, d\lambda_{d-1} dt & \geq & \displaystyle\int \int_{\{v=t\}} |\nabla v| \, d\lambda_{d-1} dt \end{array}$$

Final step:

Comparing Rayleigh quotients.

Ω: $\lambda_1(\Omega) = \inf\limits_{\substack{u \in C_0^\infty(\Omega) \\ u \not\equiv 0 \; u \geq 0}} \dfrac{\int |\nabla u|^2}{\int u^2}$

for every $u \in C_0^\infty(\Omega)$

we can find the symmetrization

$$\frac{\int_B |\nabla v|^2}{\int_B |v|^2} \leq \frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2}$$

$\Rightarrow \lambda_1(B) \leq \inf \dfrac{\int |\nabla v|^2}{\int |v|^2} \leq \lambda_1(\Omega)$

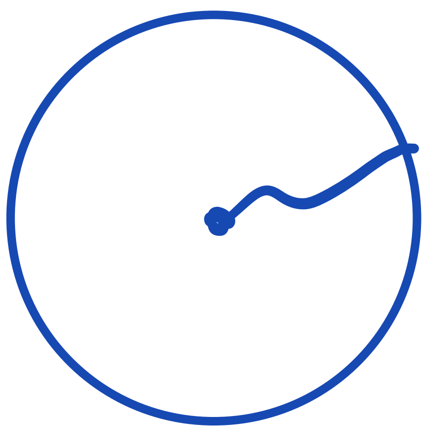
Take u - almost the first eigenf.

Exercise. Show that if $\Omega \subset \mathbb{R}^2$ is a simply connected domain with $\lambda_1(\Omega) = \lambda$, then one can inscribe a disc of radius $\frac{c}{\sqrt{\lambda}}$ in Ω $\left(c = \frac{1}{10^{90}}\right)$ and cannot inscribe a disc of radius $\frac{C}{\sqrt{\lambda}}$.

Hint 1. Monotonicity property.

$$\Omega_1 \subset \Omega_2 \Rightarrow \lambda_1(\Omega_1) \geq \lambda_1(\Omega_2)$$

Hint 2. Prove a version of Poincare's inequality



Γ is a curve connecting 0 and $\partial B_{\frac{1}{\sqrt{\lambda}}}(0)$

If $u=0$ on Γ, then

$$\int_B |\nabla u|^2 \geq c\lambda \int_B u^2$$