Spectral Geometry.

Wednesday 8:15 - 11:55

Instructors:

Aleksandr Logunov
aleksandr.logunov@unige.ch

Nikolai Bobenko
nikolai.bobenko@unige.ch

Announcements via moodle

Please register on moodle

Office hours (floating): Monday 1 pm-2 pm
(office to be confirmed)

Grade : exercises 50%
mid-term 20%
final exam 30%

Functional analysis, measure theory are needed to comprehend the course

Eigenfunctions of Δ.

$$\Delta u + \lambda u = 0$$

Laplace operator in $\mathbb{R}^n$

$$\Delta u = u_{x_1 x_1} + \dots + u_{x_n x_n}$$

$\sin(kx)$ and $\cos(kx)$ examples of Laplace eigenfunctions on $\mathbb{T} = [0, 2\pi)$

$\mathbb{R}$ $\quad \Delta u = u_{xx}$ $\qquad u_{xx} + \lambda u = 0$

$\cos(kx + x_0)$ $\qquad \lambda = k^2$

$u = e^x$ $\qquad u_{xx} - u = 0$

Fourier's idea of series to solve the heat equation.

$$\Delta u - u_t = 0$$

- The case of a circular wire

$u_{xx} - u_t = 0$ has special solutions

$$\sin(kx)\cdot e^{-k^2 t}, \quad \cos(kx)\cdot e^{-k^2 t}, \quad \text{const}$$

Idea. A general solution can be written as a sum of simple solutions.

$$u(x,t) = \sum a_k \sin(kx)\cdot e^{-k^2 t} + \sum b_k \cos(kx)\cdot e^{-k^2 t} + b_0$$

$$u(x,0) = \sum a_k \sin(kx) + \sum b_k \cos(kx) + b_0$$

$$b_0 = \fint_{\pi} u(x,0) \quad \text{– average temperature}$$

Separation of variables.

$$\Delta u - u_t = 0 \quad (\text{in } \mathbb{R}^n)$$

$$u(x,t) = \varphi(x) \cdot w(t) \qquad \Delta_x \varphi \cdot w - \varphi \cdot w'(t) = 0$$

$\Rightarrow \Delta u - u_t = 0$

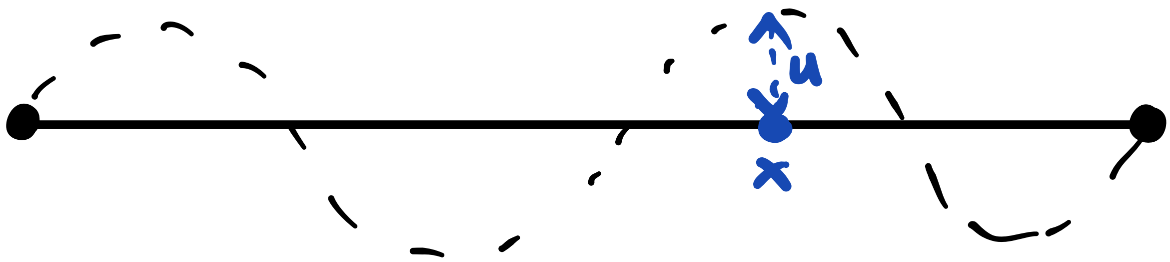
$$\boxed{\frac{\Delta \varphi(x)}{\varphi(x)} = \frac{w'(t)}{w(t)}} = -\lambda$$

– a scalar independent of t and x

$$w(t) = e^{-\lambda t}$$

$$\Delta \varphi + \lambda \varphi = 0$$

Vibrating string.



The vertical displacement $u(x,t)$ can be approximately described (for small oscilations) by the <u>wave equation</u>.

$$c \cdot u_{xx} - u_{tt} = 0$$

c depends on the physical properties of the string

The ends of the string are fixed

Boundary conditions

$u(0,t) = u(L,t) = 0$ for all t

L — the length of the string

Pure tones.

$u = f(x) \cdot \sin(\omega t)$

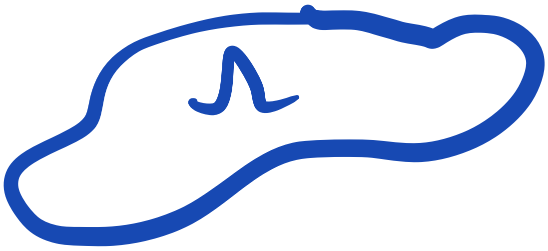
$$k \cdot \frac{\pi}{L} = \omega_k$$

$$u_k(x,t) = \sin(\omega_k x) \sin(\omega_k t)$$

$\omega_1 = \dfrac{\pi}{L}$ - the lowest frequency

Pure tones of a vibrating membrane

$\Omega \subset \mathbb{R}^d$



the shape of a drum

Vertical displacement.

$u(x,t) = \varphi(x) \cdot \sin(\sqrt{\lambda}\, t)$ solves $\Delta u - u_{tt} = 0$

Dirichlet eigenfunctions

$$\begin{cases} \Delta \varphi + \lambda \varphi = 0 & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega \end{cases}$$

Dirichlet boundary conditions

Spectral theorem (black box)

Given $\Omega \subset \mathbb{R}^d$ - bounded domain with smooth boundary,

there is a sequence of functions φ_k and numbers λ_k
(Laplace eigenfunctions), (Laplace eigenvalues of Ω)

- $\Delta \varphi_k + \lambda_k \varphi_k = 0$ in Ω
- $\varphi_k = 0$ on $\partial\Omega$ $\left(\varphi_k(x) \to 0 \text{ as } x \to \partial\Omega\right)$
- $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \ldots \nearrow +\infty$
- φ_k form an orthonormal basis in $L^2(\Omega)$

Exercise (solving the heat equation on $\mathbb{T}$)

Assume that $u \in C^2(\mathbb{T} \times \mathbb{R}^+)$

solves $u_{xx} - u_t = 0$,

then the convolution $u * e^{ikx}$

$$v_k(x,t) = \int_0^{2\pi} u(x-y,t) \cdot e^{iky} \, dy$$

a) solves the heat equation

b) $v_k(x,t) = c_k \cdot e^{iky} \cdot e^{-k^2 t}$

c) Assuming C^2-smoothness show that

$$\frac{d}{dt}\int_0^{2\pi} u^2(x,t)\,dx \le 0$$

Hint. $$\int_0^{2\pi} u_{xx}\cdot u = -\int_0^{\pi} u_x^2$$

Corollary. Uniqueness of the solution.

d) Show that as $t \to +\infty$.

$$u(x,t) \overset{\text{uniformly}}{\rightrightarrows} \underbrace{\underset{\pi}{⨍} u(x,0)\,dx}_{\text{average temperature at time 0.}}$$

(Real solution)

Application to PDE.

① $\Delta u - u_t = 0$ in $\Omega \subset \mathbb{R}^d$

with zero boundary conditions

$f(x) = u(x,0)$ - temperature at time zero

$\int_\Omega f \cdot \varphi_k = c_k$

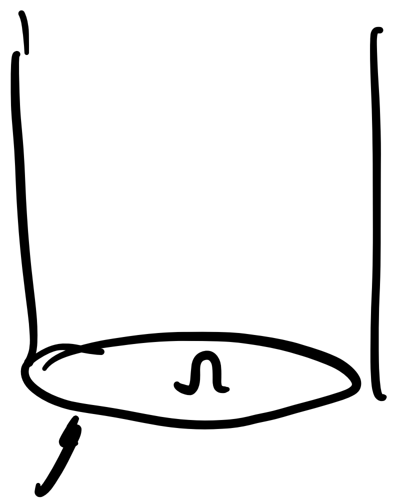
$f(x) = \sum c_k \varphi_k$ $\quad c_k = \langle f, \varphi_k \rangle$

$$u(x,t) = \sum c_k \varphi_k(x) \cdot e^{-\lambda_k t}$$

$$\int_\Omega u^2(x,t)\,dx = \sum |c_k|^2 \cdot e^{-2\lambda_k t}$$

$e^{-2\lambda_1 t}$ - the rate of decay.

$$\int_\Omega u(x,t)\,dx = \sum c_k \int \varphi_k(x) \cdot e^{-\lambda_k t}$$



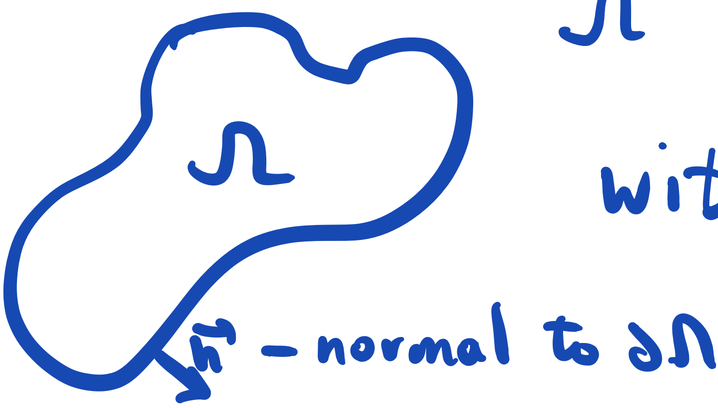
$$\Delta u - u_t = 0$$

Boundary conditions.

$$u(x,0) = f$$

$$u(x,t) = g(x,t) \quad x \in \partial\Omega$$

Neumann boundary conditions.



Ω is a room in $\mathbb{R}^d$ with insulating walls

If the insulation is perfect, i.e. no heat comes in or gets out, then $\frac{\partial u}{\partial n} = 0$ at $\partial\Omega$

Under the smoothness assumptions for Ω a similar spectral theorem holds for Neumann boundary conditions, but eigenvalues start with $\lambda_0 = 0$
$\varphi_0 = \text{const}$

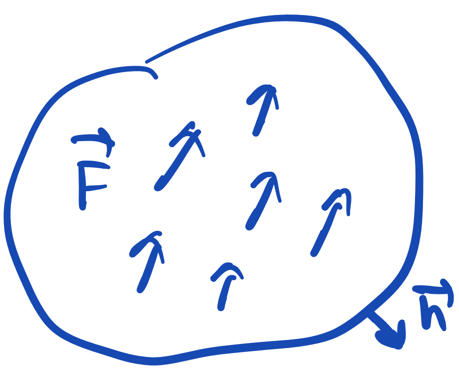
Exercise. Green's furmulas.

1) $$\int_{\Omega} \Delta u \cdot v + \int_{\Omega} \nabla u \cdot \nabla v = \int_{\partial\Omega} \partial_n u \cdot v$$

2) $$\int_{\Omega} \Delta u \cdot v - \int_{\Omega} u \Delta v = \int_{\partial\Omega} \partial_n u \cdot v - \int_{\partial\Omega} u \, \partial_n v$$

(Under C^2 assumptions, but the formulas usually hold in greater generality that we don't discuss yet)

Prove 1) and 2) using



$$\int_{\Omega} \operatorname{div} \vec{F} = \int_{\partial\Omega} \vec{F} \cdot \vec{n}$$

$$\vec{F} = (F_1, F_2, F_3)$$

$$\operatorname{div} \vec{F} = \partial_{x_1} F_1 + \partial_{x_2} F_2 + \partial_{x_3} F_3$$

$$\operatorname{div}(\overset{\text{vector field}}{\vec{F}} \cdot \overset{\text{scalar}}{f}) = \operatorname{div}(\vec{F}) \cdot f + \vec{F} \cdot \nabla f$$

3) Show that Δ eigenfunctions with different eigenvalues are orthogonal.

$$\Delta\varphi_1 + \lambda_1 \varphi_2 = 0 \qquad \Delta\varphi_2 + \lambda_2 \varphi_2 = 0$$

$$\varphi_1 = 0 \quad \partial\Omega \qquad \varphi_2 = 0 \quad \partial\Omega$$

$$\lambda_1 \neq \lambda_2 \implies \int_\Omega \varphi_1 \varphi_2 = 0$$

Gravitational potential..

Given a measure μ in $\mathbb{R}^3$

Newton potential

$$u_\mu = \mu * \frac{1}{|x|}$$

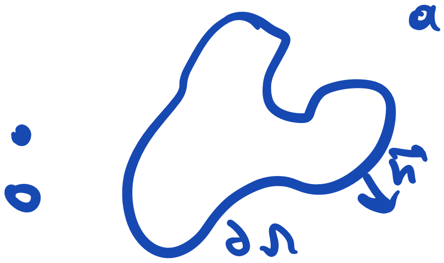
∇u_μ has a physical meaning of the gravitional force created by the mass μ.

Exercise. a) u_μ is harmonic outside supp μ.

b) For a point mass δ_0

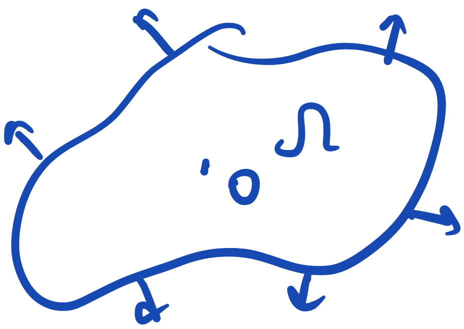
the gragient $\nabla u = \nabla \frac{1}{|x|}$ satisfies

a) the flux of ∇u across $\partial\Omega$



$$\int_{\Omega} \nabla u \cdot \vec{n} = 0 \quad \text{if } 0 \notin \bar{\Omega}$$

b)



Calculate $\int_{\Omega} \nabla u \cdot \vec{n}$

for the case $0 \in \Omega$.

c) The Fundamental solution for Δ (in $\mathbb{R}^d$)

$$G(x,y) = \frac{1}{|x-y|^{d-2}}, \ d>2 \quad \left(\log|x-y|, \ d=2 \right)$$

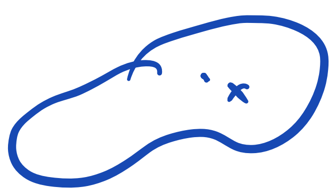
Given $f \in C_0^\infty(\mathbb{R}^d)$, (lower index 0 means that f has compact support,)

The equation $\Delta u = f$ has a solution

$$u(x) = C_d \cdot \int f(y) \cdot G(x,y)$$

d) There is a simple way to determine the value of a harmonic function inside of the domain if we know the Cauchy data on the boundary.

Cauchy data: $u, \partial u / \partial n$ on $\partial \Omega$



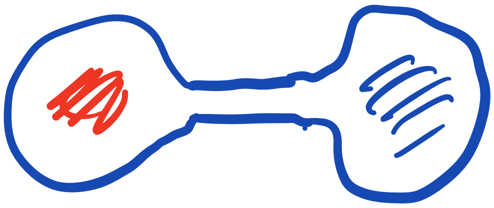
Lebesgue measure on $\partial\Omega$

$$\int_{\Omega} G(x,y) \cdot \frac{\partial u}{\partial n}(y) - \frac{\partial G(x,y)}{\partial n} u(y) \, d\sigma(y) = $$

$$= \begin{cases} c_d \cdot u(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \notin \Omega \end{cases}$$

The physical meaning of Laplace eigenvalues

① $\partial_n u = 0$ $\quad \Delta u + \lambda u = 0$

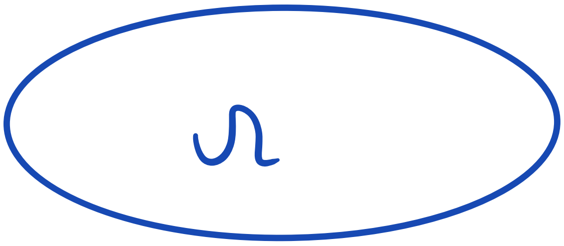


First Neumann eigenvalue has a physical meaning of how fast the temperature stabilizes:

$$u(x,t) - \fint u(x,0) = \sum c_k \varphi_k \cdot e^{-\lambda_k t}$$

the rate of decay $\lesssim e^{-\lambda_1 t}$

② Fundamental frequency of the drum



$$\begin{cases} \Delta u + \lambda u = 0 & \Omega \\ u = 0 & \partial\Omega \end{cases}$$

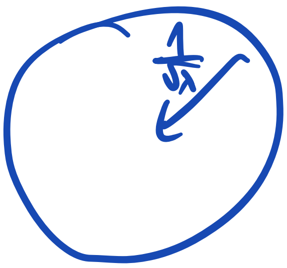
$$0 < \lambda_1 \le \lambda_2 \le \dots \nearrow +\infty$$

<u>Faber-Krahn's</u> inequality

$\lambda_1(\Omega) \geq \lambda_1(B)$, where B is a ball with the same volume as Ω.

$$\Delta u + \lambda u = 0$$

$$u(x) = v(kx)$$



$$\frac{1}{\sqrt{\tilde{\lambda}}} = k \cdot \frac{1}{\sqrt{\lambda}}$$

Exercise. By rescaling show that

homothetic copy of Ω

$$\lambda_1(k\cdot\Omega) = \lambda_1(\Omega)/k^2$$

Faber-Krahn's inequality implies that

$$\lambda_1(\Omega) \geq c_d\,|\Omega|^{-\frac{2}{d}}$$

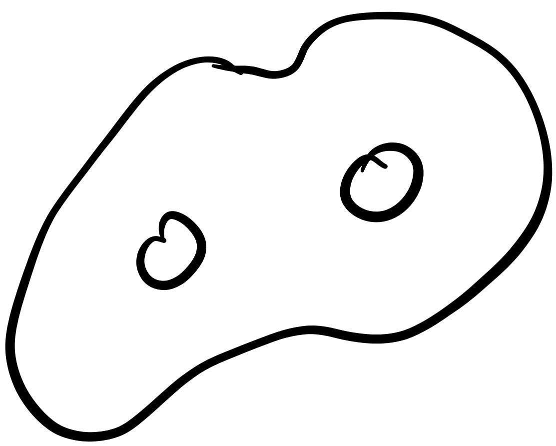
(the bigger the drum,
the better the bass
the lower the frequency)

Big question.

What kind of information about the domain is reflected in eigenvalues?

For planar domains $(\mathbb{R}^2)$ with nice boundary

$$\sum_{k=1}^{\infty} e^{-\lambda_k t} \sim \frac{1}{4\pi t} \cdot \left[\text{Area}(\Omega) - \sqrt{4\pi t} \cdot \text{length}(\Omega) + \frac{2\pi t}{3}(1-\gamma(\Omega)) \right]$$



where γ is the number of holes in Ω.

λ_k - Dirichlet eigenvalues

Weyl law

Ω - bounded domain in $\mathbb{R}^d$

$$\lambda_k(\Omega) \sim C_d \cdot \left(\frac{k}{|\Omega|}\right)^{2/d}$$

for Dirichlet eigenvalues ($\varphi_k = 0$ on $\partial\Omega$)

For Neumann eigenvalues minimal assumptions on the boundary are needed

$$C_d = \frac{(2\pi)^2}{|B_1|^{2/d}}$$