Exercise

Show that $e^{ikx} \cdot e^{imy}$, $k, m \in \mathbb{Z}$

form orth. L^2- basis on $\mathbb{T}^2$ (use that e^{ikx} form orth basis on $\mathbb{T}^1$)

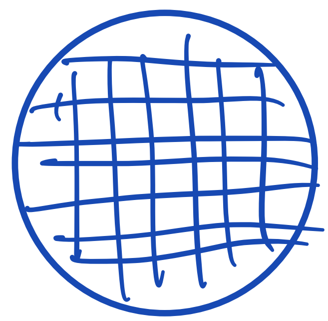
In this lecture it is more convenient to work with complex-valued eigenfunctions, but for the rest of the course we stick to real-valued eigenfunctions).

Eigenfunctions on $\mathbb{T}^d$.

$$f(x) = \sum_{n=(n_1,\dots,n_d)\in\mathbb{Z}^d} e^{inx}\cdot\hat{f}(n)$$

$$x\in\mathbb{T}^d = [0,2\pi)^d$$

$$\Delta f+\lambda f=0 \qquad \lambda = n_1^2+\dots+n_d^2$$



$$N(\lambda) = \{\#\lambda_k(\Omega)<\lambda\} =$$

$= \#$ integer points in $B_{\sqrt{\lambda}}(0)$

Gauss circle proplem

$N(r) =$ The number of integer points in a 2D disc of radius r.

$$N(r) \sim \pi r^2 \qquad r \to \infty$$

Thm (Gauss) $N(r) - \pi r^2 = O(r)$

Gauss' Conjecture. $N(r) - \pi r^2 = O(r^{\frac{1}{2}+\varepsilon})$ for all $\varepsilon > 0$.

A lot of results are fighting for improving ε. Current best $\varepsilon = 0.12\ldots$

Multiplicity of eigenvalues on $\mathbb{T}^d$.

$$\lambda = \sum n_1^2 + \ldots + n_d^2$$

Multiplicity of λ = # integer points on $\partial B_{\frac{1}{\sqrt{\lambda}}}(0)$

How many integer points on a sphere of radius λ?

Number of integer points on a circle.

Theorem (upper bound)

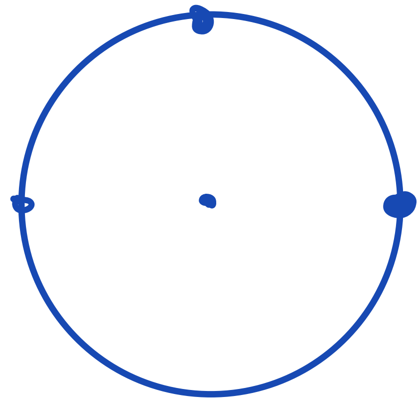
$$\#\{(a,b)\in\mathbb{Z}^2 : a^2+b^2=\lambda\} = o(\lambda^{\varepsilon})$$

for every $\varepsilon>0$.

Remark.

prime

$\lambda = p^2$ $\qquad p = 4k+3$



only 4 point on the circle of radius p^2

There is only one way to represent

$\lambda = a^2 + b^2$ $\qquad p^2 = p^2 + 0 \qquad p^2 = 0 + p^2$

multiplicity of $\lambda \asymp$

representations $\lambda = a^2 + b^2$

Gaussian numbers

$a+bi \qquad a, b \in \mathbb{Z}$

$a+bi$ is called <u>Gaussian prime</u> if

$$(a+bi) \neq (c+di)(e+f\cdot i)$$

$$|c+di|>1 \qquad |e+fi|>1$$

$\pm 1, \pm i$

Unique factorization

(unique up to multiplying primes by $\pm 1, \pm i$)

Gaussian

$$A = c\,(1+i)^{\alpha_0} \cdot P_1^{\alpha_1} P_2^{\alpha_2} \cdots P_n^{\alpha_n}$$

unimodular

$c \in \{\pm 1, \pm i\}$

P_k - Gaussian primes

$$\alpha_k > 0$$

$$|P_k| > |1+i| = \sqrt{2}$$

$$2 = (1+i)(1-i) = 1^2 + 1^2 = \frac{(1+i)^2}{i}$$

$$N = a^2 + b^2 = (a+bi)(a-bi)$$

Description of Gaussian primes

⓪ $(1 \pm i)$ are G-primes

① $p = 4k+3$ prime is G-prime

$p = (a+bi)(a-bi)$

$$p = a^2 + b^2 \mod 4$$

$a^2 + b^2$ can take only values $\{0, 1, 2\}$ mod 4

② Fermats thm (first proof due Euler)

If $p = 4k+1$ is a prime, then
$p = a^2 + b^2 \qquad a^2, b^2 > 1$

$p = (a+bi)(a-bi) \qquad a+bi, a-bi$ are Gaussian primes

Proof uses number theory.

There is $a \in \mathbb{Z}$: $a^2 \underset{p}{\equiv} -1 \qquad p = 4k+1$

Thm (Euler) If each of two integer numbers is a sum of two squares, then the product is also a sum of two squares

$$N = a^2+b^2 = (a+bi)(a-bi)$$
$$M = c^2+d^2 = (c+di)(c-di)$$

$$NM = (a+bi)(c+di)\cdot(a-bi)(c-di)$$

$$(a+bi)(c+di) = ac-bd+i(bc+ad)$$

$$NM = (ac-bd)^2+(bc+ad)^2$$

$$(a+bi)(c-di) = (ac+bd)+i(bc-ad)$$

$$NM = (ac+bd)^2+(bc-ad)^2$$

Multiplicity of eigenvalues on $\mathbb{T}^2$

$$\varphi_\lambda = \sum_{\substack{n=(n_1,n_2) \\ n_1^2+n_2^2=\lambda}} e^{inx} \cdot c_n \qquad \Delta\varphi + \lambda\varphi = 0$$

$$\text{Multiplicity of } \lambda \leq C_\varepsilon \cdot \lambda^\varepsilon$$

$$\#\{\text{representations } \lambda = a^2+b^2\} \leq C_\varepsilon \cdot \lambda^\varepsilon$$

Proof. $\lambda = a^2 + b^2$

$$\lambda = 2^k \cdot \prod_{p_k - 3 \vdots 4} p_k^{\alpha_k} \cdot \prod_{p_e - 1 \vdots 4} p_e^{\beta_e}$$

$$p_e - 1 \vdots 4 \overset{\text{Fermat}}{\Rightarrow} p_e = (a_e + b_e \cdot i)(a_e - b_e \cdot i)$$

Gaussian primes

$$\lambda = \underset{|c|=1}{c} \cdot (1+i)^{2k} \cdot \prod_{p_k - 3 \vdots 4} p_k^{\alpha_k} \prod (a_e + b_e \cdot i)^{\beta_e} \cdot \prod (a_e - b_e \cdot i)^{\beta_e}$$

Claim. If $\alpha_k \not\vdots 2$, then there is no way to write λ as a sum of 2 squares.

If $\lambda = (a+bi)(a-bi)$

b : prime $p \in \mathbb{R}$

$a+bi \vdots p^k$; $(a+bi) = p \cdot q$

$\Downarrow$

$a-bi \vdots p^k$ $(a-bi) = \bar{p} \cdot \bar{q}$

$(a+bi)(a-bi) = p^{2k}, \ldots$

$p^\alpha \quad p^\beta \quad p^\gamma$

$\gamma = \alpha + \beta$

The end of lecture 7

Gaussian primes

$$|c|=1$$

$$\lambda = c\cdot(1+i)^{2k}\cdot \prod_{p_k \equiv 3 \,(4)} p_k^{\alpha_k} \prod (a_e+b_e\cdot i)^{\beta_e}\cdot \prod (a_e-b_e\cdot i)^{\beta_e}$$

$$\alpha_k \, \vdots \, 2$$

$$\lambda = (a+bi)(a-bi)$$

$$[a+bi]_p = \frac{\alpha_k}{2}$$

$$(a+bi) = (1+i)^k \prod p_k^{\frac{\alpha_k}{2}} \cdot (\tilde{a}+\tilde{b}i)$$

$$(a-bi) = (1-i)^k\cdot \prod p_k^{\frac{\alpha_k}{2}}\cdot (\tilde{a}-\tilde{b}i)$$

$$\tilde{a}^2+\tilde{b}^2 = \prod (a_e+b_e\cdot i)^{\beta_e}\cdot \prod (a_e-b_e\cdot i)^{\beta_e}$$

$$(\tilde{a}+\tilde{b}\cdot i) \, \vdots \, (a_e+b_e\cdot i)^{\gamma_e}$$

$$\Downarrow !$$

$$(\tilde{a}-\tilde{b}\cdot i) \, \vdots \, (a_e-b_e\cdot i)^{\gamma_e}$$

$$\gamma_e + \delta_e = \beta$$

$$\tilde{a} + \tilde{b} \cdot i = \prod (a_e + b_e \cdot i)^{\gamma_e} \cdot \prod (a_e - b_e \cdot i)^{\delta_e}$$

$$\tilde{a} - \tilde{b} \cdot i = \prod (a_e + b_e \cdot i)^{\beta_e - \gamma_e} \cdot \prod (a_e - b_e \cdot i)^{\beta_e - \delta_e}$$

$$\beta_e - \delta_e = \gamma_e$$

γ_e can be $0, 1, \dots, \beta_e$

Conclusion

$$\#\{a + b \cdot i : a^2 + b^2 = \lambda\} = \prod (\beta_e + 1)$$

$$\lambda = 2^k \cdot \prod_{p_k \equiv 3 : 4} p_k^{\alpha_k} \cdot \prod_{p_e \equiv 1 : 4} p_e^{\beta_e}$$

Claim $\quad \prod (\beta_e + 1) \le C_\varepsilon \cdot \lambda^\varepsilon$

Fix $\varepsilon > 0$

Proof. $\prod p_e^{\beta_e} = \lambda$

$\beta_e \leq \log_2 \lambda$

$$\prod_{p_e \leq A} (\beta_e + 1) \leq (\log \lambda)^{C_A} \qquad C_A \lesssim \frac{A}{\ln A}$$

$$\wedge\!\!/$$

$$C_{\varepsilon, A} \cdot \lambda^{\varepsilon/2}$$

$$\prod_{p_e \geq A} (\beta_e + 1) \leq \prod p_e^{\beta_e \cdot \varepsilon/2} \leq \lambda^{\varepsilon/2}$$

$\beta_e + 1 \leq p_e^{\beta_e \cdot \varepsilon/2}$ if $p_e \geq A$ suff. large

Worst case scenario.

$$\lambda = \prod_{p_k - 1 \,\vdots\, 4} p_k$$

Good (calculation) proplem. Use asymptotics of prime numbers to specify what is the better bound for multiplicity of eigenvalues on $\mathbb{T}^2$

Multiplicity of λ on $\mathbb{T}^2 \leq C_\varepsilon \cdot \lambda^\varepsilon$

Corollary.

u_λ: $\Delta u + \lambda u = 0$ on $\mathbb{T}^2$

$$\| u_\lambda \|_{L^\infty} \leq C_\varepsilon \cdot \lambda^\varepsilon \cdot \| u \|_{L^2}$$

Proof. $\sum c_k e^{ikx} = u$ $\quad x \in \mathbb{T}^2$ $\quad k = a + bi$, $a^2 + b^2 = \lambda$

$$\sum c_k^2 = \int u^2$$

$$\left(\| u \|_\infty \right)^2 \leq \left(\sum \| c_k \cdot e^{ikx} \| \right)^2 \leq \left(\sum | c_k | \right)^2 \leq \int \sum 1 \cdot \int | c_k |^2$$

$$\leq \text{Multiplicity of } \lambda \cdot \| u \|_{L^2}$$

$$\| u \|_\infty \leq \sqrt{\text{Multiplicity of } \lambda} \cdot \| u \|_{L^2}$$

Exercise (Multiplicity vs L^∞-norm)

a) $f_1, \dots, f_k$ be orthonormal and continuous (on some finite measure space)

Show that there is $f \in \mathrm{Span}\{f_1, \dots, f_k\}$

$$\|f\|_{L^2} = 1 \quad \text{such that} \quad \|f\|_{L^\infty} \geq c\sqrt{k}$$

b) Show that there is a sequence of eigenfunctions f_{λ_k} on $\mathbb{T}^2$ such that $\|f_{\lambda_k}\|_\infty / \|f_{\lambda_k}\|_2 \to +\infty$