UNIVERSITE DE GENEVE
Faculté des sciences

Section de Mathématiques

Spectral Geometry - SS 2022
Sheet 10 - Discussed on 18.05.2022

Obligatory exercises are marked with a $(*)$. We ask you to solve these and be ready to present your solutions during the exercise session. Let me know one day in advance which ones you were able to solve and I will randomly assign some of you to present your solutions. You will get points for the exercises you announced as solved with possible deductions if your presentation is lacking.

Exercise 1. Let f be a holomorphic function in the ball B_5 and N be the number of zeroes of f in B_1. Show that

$$\frac{\max_{B_5}|f|}{\max_{B_1}|f|} \geq e^{cN}.$$

Exercise 2. Let $\Delta u = 0$ in $\mathbb{R}^n$ and $u(x) = \sum a_k p_k(x)$ its series expansion into homogeneous polynomials $p_k(x) = |x|^k p_k(\frac{x}{|x|})$. This expansion converges locally around 0 since u is real analytic.

1. Show that the p_k are harmonic.
2. Show $\int_{\partial B_1} p_k p_l = 0$.

Exercise 3. Show that any absolutely monotone function $f(r)$ satisfies that $\log f(e^t)$ is a convex function of t.

Exercise 4. We define $H(r) := \fint_{\partial B_r} u^2$. Then we call $F(r) = \frac{rH'(r)}{H(r)}$ the frequency. Show that monotonicity of frequency implies

$$\left(\frac{R}{r}\right)^{F(r)} \leq \frac{H(R)}{H(r)} \leq \left(\frac{R}{r}\right)^{F(R)}$$

and that the three balls theorem holds for L^2 averages.

Exercise 5. Show that if $u = \sum_{k=0}^n p_k(x)$ is a sum of harmonic k-homogeneous polynomials then $F(r) \leq 2n$.

Exercise 6. Show that if $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_3$ with Ω_1 open and Ω_3 bounded then there exists some $\alpha \in (0,1)$ such that for any harmonic function u we have that $|u| \leq \epsilon$ on Ω_1 and $|u| \leq 1$ on Ω_3 implies $u \leq \epsilon^\alpha$ on Ω_2.