Method of stationary phase.

(we follow lecture notes of T.Tao)

Oscillatory integral.

$$I(\lambda) = \int_{\Omega} a(x) \cdot e^{i\lambda \varphi(x)} dx \qquad \Omega \subset \mathbb{R}^d$$

$a(x) \in C_0^{\infty}(\Omega)$ or $C^{\infty}(\Omega)$

$a \equiv 1$

Phase : $\varphi(x)$ – real function

Question. How does $I(\lambda)$ behave as $\lambda \to +\infty$?

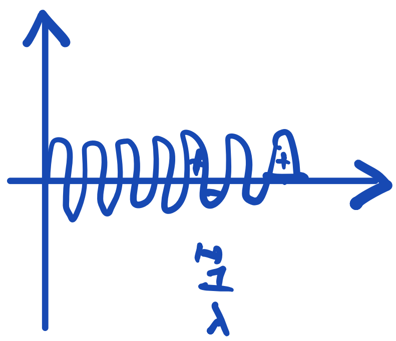
First thoughts

① Certainly, $|I(\lambda)| \le \int |a(x)|$ is bounded

② $\mathrm{Im}\, e^{i\lambda \varphi(x)}$ and $\mathrm{Re}\, e^{i\lambda \varphi(x)}$ rapidly oscillate if $|\nabla \varphi| \neq 0$.

<u>Example.</u> $\int_{-1}^{1} e^{i\lambda x} = \frac{1}{i\lambda}\left(e^{i\lambda} - e^{-i\lambda}\right)$

<u>Intuition</u> $\mathrm{Re}\, e^{i\lambda\varphi} =$



The integral of a bounded rapidly osilating function is small.

Method of non-stationary phase.

1-dimensional case.

$$I(\lambda) = \int_{[0,1]} a(x)\, e^{i\lambda \varphi(x)} dx \qquad a \in C_0^1[0,1]$$

$$|a|, |a'| \leq C$$

Assume $\varphi' \geq \delta > 0$ $\qquad |\varphi''| \leq C$

Claim 1 $\quad |I(\lambda)| \lesssim \frac{1}{\delta^n \lambda}$

$$I(\lambda) = \int a(x) \underbrace{\frac{1}{i\lambda \varphi'}}\, d e^{i\lambda\varphi} = \frac{-1}{i\lambda} \cdot \int \left(\frac{a}{\varphi'}\right)' \cdot e^{i\lambda\varphi} dx$$

$$\left| \frac{a' \cdot \varphi' - a \cdot \varphi''}{(\varphi')^2} \right| \leq \frac{C}{\delta^2}$$

Proof.

$$I(\lambda) = \int_0^1 a \frac{1}{i\lambda \varphi'} \cdot \partial_x e^{i\lambda\varphi} dx \underset{\text{by parts}}{\overset{\text{integrating}}{=\!=\!=}}$$

$$= \frac{-1}{\lambda} \int_0^1 \partial_x \left(a \cdot \frac{1}{i\varphi'}\right) \cdot e^{i\lambda\varphi} + \frac{1}{\lambda} \Big|_0^1 \frac{a}{i\varphi'} \cdot e^{i\lambda\varphi}$$

Van der Corput lemma (monotone first derivative)

$|\varphi'| \geq \delta \qquad \varphi'$ monotone on $(0,1)$

Then $\left| \int_0^1 e^{i\lambda\varphi} \right| \lesssim \frac{1}{\delta\lambda}$

Proof.

$$I(\lambda) = \int_0^1 a \frac{1}{i\lambda\varphi'} \cdot \partial_x e^{i\lambda\varphi} dx \underline{\underline{\text{integrating by parts}}}$$

$$= \frac{-1}{\lambda} \int_0^1 \partial_x \left(a \cdot \frac{1}{i\varphi'} \right) \cdot e^{i\lambda\varphi} dx + \frac{1}{\lambda} \Big|_0^1 \frac{a}{i\varphi'} \cdot e^{i\lambda\varphi}$$

$$\int \partial_x \left(a \cdot \frac{1}{\varphi'} \right) \cdot e^{i\lambda\varphi} dx =$$

$$= \int a' \cdot \frac{1}{\varphi'} \cdot e^{i\lambda\varphi} dx +$$

$$+ \int a \cdot e^{i\lambda\varphi} \cdot \left(\frac{1}{\varphi'} \right)' dx = I + II$$

φ' and $\frac{1}{\varphi'}$ are monotone

$$|\mathrm{I}| \underset{?}{\leq} C \cdot \frac{1}{\delta}$$

$$|\mathrm{II}| \leq C \cdot \frac{1}{\delta}$$

$$\varphi' \geq \delta$$

$$\int \left| \left(\frac{1}{\varphi'} \right)' \right| dx \leq \max \frac{1}{\varphi'} - \min \frac{1}{\varphi'} \leq \frac{1}{\delta}$$

$$\int a \cdot e^{i\lambda\varphi} \left(\frac{1}{\varphi'} \right)' dx \leq \max|a| \cdot \left| \int \left| \frac{1}{\varphi'} \right|' dx \right| \leq \frac{c}{\delta}$$

$$I(\lambda) \leq \frac{C}{\delta\lambda}$$

Van der Corput lemma.

If $\varphi^{(k)} \geq \delta > 0$, $k \geq 2$, then

$$\left| \int_0^1 e^{i\lambda\varphi} \right| \lesssim \frac{1}{(\delta\lambda)^{1/k}}$$

Proof. Induction on k. $\left|\int_0^a e^{i\lambda\varphi}\right| \leq \frac{C_k}{(\delta\lambda)^{1/k}}$

$a \in (0,1)$

At most one point x_0: $\varphi^{(k-1)}(x_0) = 0$

$I_\alpha = [x_0 - \alpha, x_0 + \alpha]$ α - parameter to be chosen later

$$\left|\int_{I_\alpha} e^{i\lambda\varphi}\right| \lesssim \alpha$$

$$\varphi^{(k-1)}(x) = \int_{x_0}^{x} \varphi^{(k)}$$

Outside I_α $|\varphi^{(k-1)}| \geq \delta' = \alpha \cdot \delta$

and monotone on each of the intervals

of $[0,1] \setminus I_\alpha$.

$$\int_{[0,1]\setminus I_\alpha} e^{i\lambda\varphi} \lesssim \frac{1}{(\delta'\lambda)^{1/k-1}} \qquad \int_{I_\alpha} e^{i\lambda\varphi} \leq 2\alpha$$

$$\int_{[0,1]} e^{i\lambda\varphi} \lesssim \alpha + \frac{1}{(\alpha\cdot\delta\cdot\lambda)^{1/k-1}}$$

Optimisation by choosing α.

$$\alpha + \underbrace{\frac{C}{\alpha^{1/k-1}} + \dots + \frac{C}{\alpha^{1/k-1}}}_{k-1} \geq k \cdot C^{\frac{k-1}{k}}$$

Equality is achieved when all terms are equal. $\alpha = C^{\frac{k}{k-1}}$

$$\int_{[0,1]} e^{i\lambda\varphi} \lesssim \alpha + \frac{1}{(\alpha \cdot \delta \cdot \lambda)^{1/k-1}} \overset{\text{optimisation}}{\lesssim} \left(\frac{1}{(\delta\lambda)^{1/k-1}}\right)^{\frac{k-1}{k}} = \frac{1}{(\delta\lambda)^{1/k}}$$

Claim

$$I(\lambda) = \int_0^1 a(x) \cdot e^{i\lambda \varphi(x)} dx$$

$$a(x) \in C_0^\infty(0,1)$$

$$|\varphi'| \geq c \qquad \varphi \in C^\infty(0,1)$$

Then $I(\lambda) = O(\lambda^{-n})$ for all $n \in \mathbb{N}$

depends on C^n norm of φ.

Proof.

$$I(\lambda) = \int_0^1 a \frac{1}{i\lambda\varphi'} \cdot \partial_x e^{i\lambda\varphi} dx \quad \underline{\underline{\text{integrating by parts}}}$$

$$= -\frac{1}{\lambda} \int_0^1 \partial_x \Big(a \cdot \underbrace{\frac{1}{i\varphi'}}_{\text{new } a}\Big) \cdot e^{i\lambda\varphi} + \frac{1}{\lambda} \underbrace{\Big| _0^1 \frac{a}{i\varphi'} \cdot e^{i\lambda\varphi}}_{= 0}$$

Bad about the argument: the bound depends on higher derivatives of φ

A clever thing to do

Van der Corput lemma.

If $\varphi^{(k)} \geq \delta > 0$, $k \geq 2$, $\boxed{\begin{matrix} a(0)=a(1)=0 \\ |a'| \leq C \end{matrix}}$

$$\left| \int_0^1 a(x)\, e^{i\lambda\varphi} \right| \lesssim \frac{C}{(\delta\lambda)^{1/k}}$$

Proof. We know that

$$F(t) = \int_0^t e^{i\lambda\varphi} \lesssim \frac{1}{(\delta\lambda)^{1/k}} \qquad t \in (0,1)$$

$$\left| \int_0^1 a(x) \cdot e^{i\lambda\varphi}\, dx \right| = \left| \int_0^1 dt\, F(t) \cdot a'(t) \right| \lesssim \frac{1}{(\delta\lambda)^{1/k}}$$

Higher-dimensional case

$\Omega \subset \mathbb{R}^n$ – smooth bounded

$$\int_\Omega a(x) e^{i\lambda \varphi(x)} dx = I(\lambda)$$

Non-stationary case: $|\nabla \varphi| > \delta > 0$

If $a \in C_0^\infty(\Omega)$, $\varphi \in C^\infty$, then

$$|I(\lambda)| \overset{a,\varphi,n}{\lesssim} \frac{1}{\lambda^n} \quad \text{for every } n.$$

If f has a singularity (deep zero), then the asymptotics of

$$\int_{\Omega} a\, e^{i\lambda f} = I(\lambda)$$

is not completely understood in higher dimensions.

$$I(\lambda) = O\left(\frac{1}{\lambda^s}\right)$$

for some $s > 0$.