UNIVERSITE DE GENEVE Section de Mathématiques
Faculté des sciences

Spectral Geometry - SS 2022
Sheet 3 - Discussed on 16.03.2022

Obligatory exercises are marked with a $(*)$. We ask you to solve these and be ready to present your solutions during the exercise session. Let me know one day in advance which ones you were able to solve and I will randomly assign some of you to present your solutions. You will get points for the exercises you announced as solved with possible deductions if your presentation is lacking.

Leftovers from last week:

Exercise 1. Let $\lambda > 0$. Show that any solution φ to $\Delta^2\varphi - \lambda^2\varphi = 0$ can be written as $\varphi = \varphi_+ + \varphi_-$ where

$$\Delta\varphi_+ + \lambda\varphi_+ = 0 \quad \text{and} \quad \Delta\varphi_- - \lambda\varphi_- = 0$$

Exercise 2. Let's remember that $\Delta_{\mathbb{R}^n} = \partial_r^2 + \frac{n-1}{r}\partial_r + \frac{1}{r^2}\Delta_{S^{n-1}}$. Describe all harmonic functions in $\mathbb{R}^n$ that are rotationally invariant.

New exercises:

Exercise 3 (*). Recall the definition of the Sobolev space $W^{1,2}(\Omega) = \{f \in L^2(\Omega) \text{ with } \nabla f \in L^2(\Omega)\}$ which is a Hilbert space with the inner product $\langle f, g\rangle_{W^{1,2}} = \langle f, g\rangle_2 + \langle \nabla f, \nabla g\rangle_2$. We denote by $W_0^{1,2}(\Omega)$ the closure of $C_0^\infty(\Omega)$ under $\|\cdot\|_{W^{1,2}}$.

1. Show that $f \equiv 1$ is not in $W_0^{1,2}(B_1)$.
2. Show that $1 - |x|$ is in $W_0^{1,2}(B_1)$.
3. If Ω is a bounded Lipschitz domain and $u \in C^1(\bar{\Omega})$ and $u = 0$ on $\partial\Omega$, then $u \in W_0^{1,2}(\Omega)$.

Exercise 4 (*). Let Ω be a bounded domain. Show that there exists some C such that for all $f \in C_0^\infty(\Omega)$ we have

$$\int_\Omega |f|^2 \le C \int_\Omega |\nabla f|^2.$$

Exercise 5. Like before we denote the k-th smallest eigenvalue of Δ on Ω under Dirichlet boundary conditions as λ_k. Show that

$$\lambda_k = \inf_{\substack{\dim(L)=k \\ L\subset W_0^{1,2}(\Omega)}} \sup_{f\in L} \frac{\int_\Omega |\nabla f|^2}{\int_\Omega |f|^2}$$

Exercise 6. Let u be a harmonic function in $\mathbb{R}^n \setminus \{0\}$ and $|u| \le 1$ in $B_1 \setminus \{0\}$. Show that u can be extended to a harmonic function on B_1.

Exercise 7 (*). For a function u in $\mathbb{R}^3$ we define the Kelvin transform as

$$u^*(x) = \frac{1}{|x|} u\left(\frac{x}{|x|^2}\right).$$

Let $f(x) := \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}\frac{1}{|x|}$. Show that f^* is a harmonic polynomial of degree $\alpha_1 + \alpha_2 + \alpha_3$.