UNIVERSITE DE GENEVE Section de Mathématiques
Faculté des sciences

Spectral Geometry - SS 2022
Sheet 6 - Discussed on 06.04.2022

Obligatory exercises are marked with a (∗). We ask you to solve these and be ready to present your solutions during the exercise session. Let me know one day in advance which ones you were able to solve and I will randomly assign some of you to present your solutions. You will get points for the exercises you announced as solved with possible deductions if your presentation is lacking.

Exercise 1. Let u be given by $u(x,t) = \int_\Omega f(y) p_\Omega(x,y,t)dy$ where p_Ω is the heat kernel associated to the domain Ω. Show that

$$\|u(\cdot,t) - f\|_{L^2(\Omega)} \xrightarrow{t\to 0} 0.$$

Exercise 2. Let's work through the Wiener-Tauberian theorem.

1. Let $f \in L^1(\mathbb{R})$, $h \in L^\infty(\mathbb{R})$ and $\lim_{x\to\infty} h(x) = A$. Show that

$$\lim_{x\to\infty} (f * h)(x) = A \int f.$$

2. (hard): Let $f \in L^1(\mathbb{R})$. If $\hat{f}$ has no zeroes, then linear combinations of translations of f are dense in $L^1(\mathbb{R})$.

3. Let $f \in L^1(\mathbb{R})$ with $\hat{f}$ having no zeroes and $h \in L^\infty(\mathbb{R}$ such that

$$(f * h)(x) \xrightarrow{x\to\infty} A \int f.$$

   Then for any $g \in L^1(\mathbb{R})$ we also have

$$(g * h)(x) \xrightarrow{x\to\infty} A \int g.$$

Exercise 3. Let A be a finite and graph-connected subset of $\mathbb{Z}^d$ with discrete boundary $\partial A \subset \mathbb{Z}^d$. We define the linear operators $Q, \mathcal{L}$ by

$$QF(x) = \frac{1}{2d} \sum_{|x-y|=1} F(y)$$
$$\mathcal{L}F(x) = (Q - I)F(x).$$

Show that there exists a unique function $p_n(x)$ that solves the Dirichlet heat equation:

$$\begin{aligned} p_{n+1}(x) - p_n(x) &= \mathcal{L}p_n(x), \quad x \in A \\ p_0(x) &= f(x), \quad x \in A \\ p_n(x) &= 0, \quad x \in \partial A \end{aligned}$$

for any given initial function $f : A \to \mathbb{R}$. What is the probabilistic interpretation of this p_n? How can we compute p_n?