Spectral Geometry 2022

Midterm exam.

Instructions. This is a take home exam. Be ready to present the solutions on April 28, 9 AM - 10 AM using a blackboard and a piece of chalk.

The anticipated time for solving the problems is 3 hours.

You may use lecture notes, exercises, wikipedia

Problem 1. Let u be any solution to $\Delta u + u = 0$ on $\mathbb{R}^3$. Show that there is a numerical constant $R > 0$ independent of u such that

$$\int_{B_R} u = 0.$$

Problem 2. Let u be a solution to $\Delta u + u = 0$ on the plane $\mathbb{R}^2$. Assume that u is real and the zero set of u is a union of smooth non-intersecting curves. Show that

a) Z_u has infinite length.

b) $Z_u \cap B_r \geq c \cdot r^2$

for r sufficiently large.

Problem 3. Show that there is a sequence of eigenfunctions on $\mathbb{T}^2$ such that

$$\|\varphi_{\lambda_k}\|_{L^\infty} / \|\varphi_{\lambda_k}\|_{L^2} \to +\infty$$

Hint. Show that if $f_1, f_2, \dots, f_n$ are orthogonal functions on $L^2(\mathbb{T})$:

$$\int_{\mathbb{T}} f_i \cdot f_j = \delta_{ij}, \quad \text{then}$$

there is $f \in \operatorname{Span}(f_1, \dots, f_n)$ such that

$$\|f\|_\infty > c \cdot \sqrt{n}\, \|f\|_2$$

Problem 4.

Let Ω be a bounded domain in $\mathbb{R}^n$.

Let φ_{λ_k} be a sequence of Dirichlet eigenfunctions of the Laplace operator.

Show that for some $C_\Omega, C_d > 0$

$$\|\varphi_\lambda\|_{L^\infty} \leq C_\Omega \cdot \lambda^{C_d} \cdot \|\varphi_\lambda\|_{L^2}$$

for all of the eigenfunctions.