UNIVERSITE DE GENEVE
Faculté des sciences

Section de Mathématiques

Spectral Geometry - SS 2022
Sheet 1 - Discussed on 02.03.2022

Obligatory exercises are marked with a $(*)$. We ask you to solve these and be ready to present your solutions during the exercise session. Let me know one day in advance which ones you were able to solve and I will randomly assign some of you to present your solutions. You will get points for the exercises you announced as solved with possible deductions if your presentation is lacking.

Exercise 1 (*)**.** Assume that $u \in C^2(\Pi \times \mathbb{R}^+)$ solves $u_{xx} - u_t = 0$. Show that then for the convolution

$$v_k(x,t) := u * e^{iky}(x,t) = \int_0^{2\pi} u(x-y,t)e^{iky}dy$$

the following holds.

1. v_k solves the heat equation
2. $v_k(x,t) = c_k e^{ikx}e^{-k^2 t}$
3. Assuming C^2-smoothness show that

$$\frac{d}{dt}\int_0^{2\pi} u^2(x,t)dx \leq 0$$

4. Show that this implies uniqueness of the solution
5. Show that as $t \to \infty$ we have uniform convergence

$$u(x,t) \rightrightarrows \fint_0^{2\pi} u(x,0)dx$$

Exercise 2 (Green's formulas)**.** Using the divergence theorem $\int_\Omega \nabla \cdot F = \int_{\partial\Omega} F \cdot n$ where n is the outward pointing normal on $\partial\Omega$ and F is a vector field on the domain Ω show the following Green's identities.

1. $\int_\Omega \Delta u v + \int_\Omega \nabla u \cdot \nabla v = \int_{\partial\Omega} \partial_n u \cdot v$
2. $\int_\Omega \Delta u v - \int_\Omega u \Delta v = \int_{\partial\Omega} \partial_n u \cdot v - \int_{\partial\Omega} u \cdot \partial_n v$
3. Show that Δ eigenfunctions with different eigenvalues are orthogonal.

Exercise 3 (*)**.** Given a measure μ in $\mathbb{R}^3$ we define $u_\mu = \mu * \frac{1}{|x|}$. Show that

1. u_μ is harmonic outside the support of μ.
2. For a point mass δ_0 show that

$$\int_{\partial\Omega} \nabla u_{\delta_0} \cdot n = 0$$

   if $0 \notin \Omega$. Furthermore calculate $\int_{\partial\Omega} \nabla u_{\delta_0} \cdot n$ for $0 \in \Omega$.

3. Let $d > 2$. Show that for $f \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ the equation $\Delta u = f$ has a solution on $\mathbb{R}^d$ given by
$$u(x) = C_d \int_{\mathbb{R}^d} f(y) G(x,y) dy$$
where $G(x,y) = \frac{1}{|x-y|^{d-2}}$

4. Let u be harmonic in Ω. Show that then
$$\int_{\partial\Omega} G(x,y)\frac{\partial u}{\partial n} - \frac{\partial G(x,y)}{\partial n} u(y) dS(y) = \begin{cases} c_d u(x) & \text{if } x \in \Omega \\ 0 & \text{otherwise} \end{cases}$$
This provides a nice way to evaluate harmonic functions given its Cauchy data.

Exercise 4. Show the rescaling equality
$$\lambda_1(k\Omega) = \frac{\lambda_1(\Omega)}{k^2}$$
where $\lambda_1(\Omega)$ is the smallest eigenvalue of Δ on Ω under Dirichlet boundary conditions.