UNIVERSITE DE GENEVE
Faculté des sciences
Section de Mathématiques

Spectral Geometry - SS 2022
Sheet 7 - Discussed on 13.04.2022

Obligatory exercises are marked with a $(*)$. We ask you to solve these and be ready to present your solutions during the exercise session. Let me know one day in advance which ones you were able to solve and I will randomly assign some of you to present your solutions. You will get points for the exercises you announced as solved with possible deductions if your presentation is lacking.

Exercise 1. Let's proove the local Weyl's law using Karamata's Tauberian theorem. Let $x, y \in \Omega$ be fixed. To this end

1. Show that $\sum_{\lambda_k \leq \lambda} |\varphi_k(x)|^2 \sim C_d \lambda^{\frac{d}{2}}$.
2. Show that $\frac{1}{\#\{\lambda_k \leq \lambda\}} \sum_{\lambda_k \leq \lambda} \varphi_k(x)\varphi_k(y) \xrightarrow{\lambda \to \infty} 0$.

Exercise 2. Eigenfunctions on the torus $\mathbb{T}^2$.

1. Show that $\{e^{inx}e^{imy}\}_{n,m \in \mathbb{Z}}$ form and orthonormal basis of $L^2(\mathbb{T}^2)$.
2. Classify doubly periodic functions in $\mathbb{R}^2$ solving $\Delta u + \lambda u = 0$. For which λ does a solution exist?
3. Prove Weyl's law in the case of a torus.

Exercise 3. Show that if $p = 4k + 1$ is a prime then it can be written as the sum of two integer squares $p = a^2 + b^2$ where $a^2, b^2 > 1$.