Thm (Zygmund): φ_λ: $\Delta\varphi+\lambda\varphi=0$ on $\mathbb{T}^2$

$$\|\varphi_\lambda\|_{L^4} \le C \cdot \|\varphi_\lambda\|_{L^2}$$

uniform in λ.

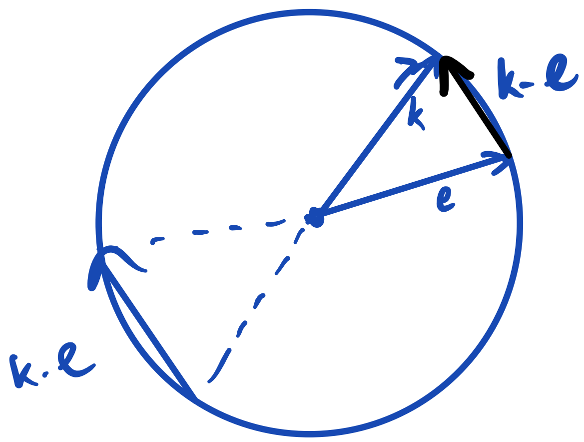
Proof. Nice exercise

Proof. $\varphi_\lambda(x) = \sum_{|k|^2=\lambda} a_k e^{ikx}$

$$|\varphi_\lambda(x)|^2 = \sum_{|k|^2=\lambda} a_k e^{ikx} \cdot \overline{\sum_{|l|^2=\lambda} a_l \cdot e^{ilx}} =$$

$$= \sum_{m=k-l} e^{imx} \cdot b_m \quad \text{, where } b_m = \sum_{\substack{k-l=m \\ |k|=|l|=\lambda^{1/2}}} a_k \bar{a}_l$$

Geometrical observation



k-l connects two integer points on the circle.

For $m \neq 0$, there are at most 2 ways to write m as $k-l$, $|k|=|l|$

$$\int |\varphi_\lambda|^4 = \|\varphi_\lambda^2\|_2^2 = \sum |b_m|^2$$

$$|b_0|^2 = \left|\sum |a_k|^2\right|^2$$

$$\sum_{m \neq 0} |b_m|^2 \leq 2 \cdot \sum_{k-l=m} |a_k|^2 |a_l|^2 \leq 2 \cdot \sum |a_k|^2 \cdot \sum |a_l|^2$$

$$|b_m|^2 = |a_k \bar{a}_l + a_{\tilde{k}} \bar{a}_{\tilde{l}}|^2 \leq 2|a_k a_l|^2 + 2|a_{\tilde{k}} a_{\tilde{l}}|^2$$

$$\int |\varphi_\lambda|^4 \leq C\left(\int |\varphi_\lambda|^2\right)^2$$

L^p bounds on $\mathbb{T}^2$

$$\Delta \varphi + \lambda \varphi = 0$$

$$\|\varphi_\lambda\|_{L^\infty} \leq C_\varepsilon \cdot \lambda^\varepsilon \cdot \|\varphi_\lambda\|_{L^2}$$

$$\|\varphi_\lambda\|_{L^4} \leq C \cdot \|\varphi_\lambda\|_{L^2}$$

Open question (1993 Bourgain)

$$\|\varphi_\lambda\|_{L^p} \leq C_p \cdot \|\varphi_\lambda\|_{L^2} \qquad p < +\infty$$

Conjecture (Bourgain - Zeev)

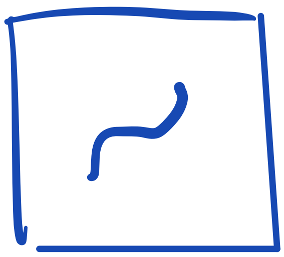
$\Delta \varphi + \lambda \varphi = 0 \qquad \mathbb{T}^d \qquad p = 2 + \varepsilon_d$ (suff. small)

$$\| \varphi \|_{L^p} \le C_p \cdot \| \varphi \|_{L^2}$$

Restriction problem.

smooth curve of finite length

$$\Gamma \subset \mathbb{T}^2$$



Γ can be a segment

$$\int_\Gamma |\varphi_\lambda|^2 \overset{?}{\le} C_\Gamma \cdot \int_{\mathbb{T}^d} |\varphi_\lambda|^2$$

Thm (Bourgain - Zeev)

If Γ is (strictly) convex

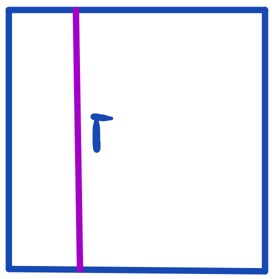
$$\int_\Gamma |\varphi_\lambda|^2 \le C_\Gamma \cdot \int_{\mathbb{T}^2} |\varphi_\lambda|^2$$

$\Delta \varphi + \lambda \varphi = 0$ $\mathbb{T}^2$

C_Γ independent of λ

The proof will take a while and is very related to integer points on the circle.

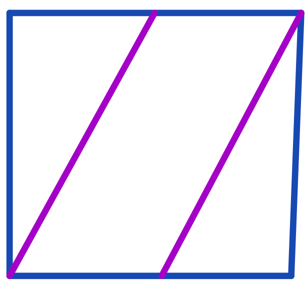
Exercise. a) Let Γ be a vertical segment on $\mathbb{T}^2$.



Show that

$$\int_\Gamma |\varphi_\lambda|^2 \le C \cdot \int_{\mathbb{T}^2} \varphi_\lambda^2$$

b)



Show that the same inequality is true when Γ is a closed geodesic.

Conjecture (Bourgain–Rudnick).
Upper bound should be true for any curves

Open even for geodesic segments and is related to open problems on integer points

Thm(Bourgain - Rudnick)

Γ - convex smooth curve on $\mathbb{T}^2$

If λ is sufficiently large, then

$\lambda > \lambda_0$

$$\int_\Gamma |\varphi_\lambda|^2 \geq c_\Gamma \cdot \int_{\mathbb{T}^2} |\varphi_\lambda|^2$$

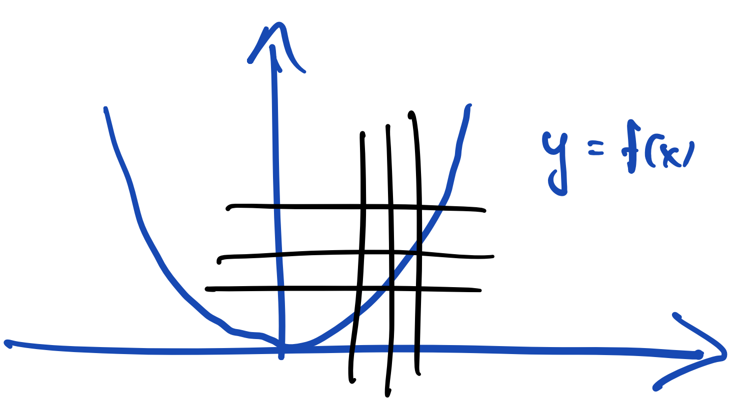
$c_\Gamma > 0$

Integer points on convex curves.

Thm (Jarnik, 1926) Every strictly convex arc $y = f(x)$ of length ℓ contains at most

$$3(4\pi)^{-\frac{1}{3}} \cdot \ell^{\frac{2}{3}} + O(\ell^{\frac{1}{3}})$$

integral lattice points



„The number of integral points on arcs and ovals"

E. Bombieri & J. Pila

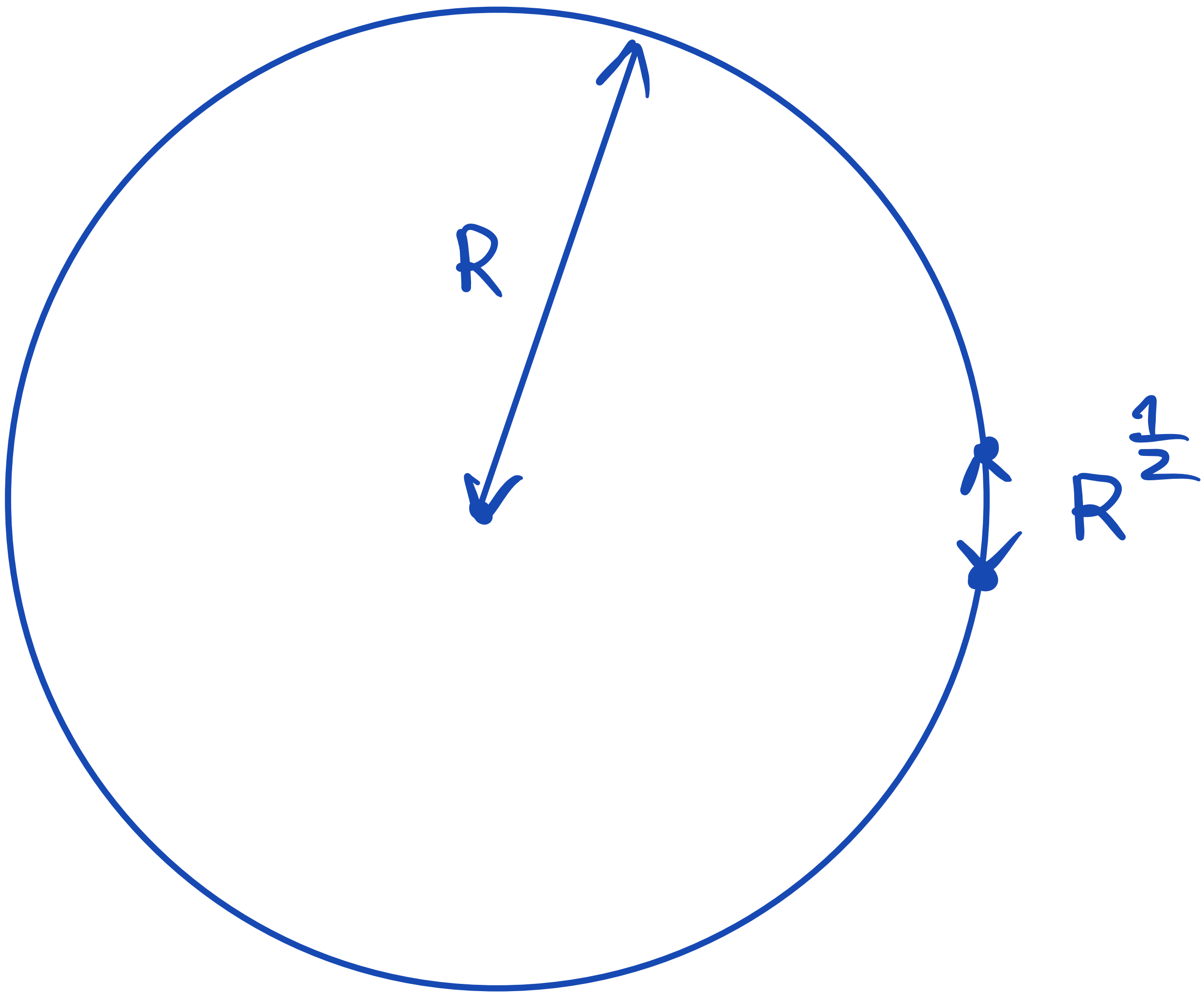
Thm* (Bombieri & Pila)

If $f \in C^\infty([0,1])$ is strictly convex and $\Gamma = \{(f(x), x), x \in [0,1]\}$, then

$$|t\Gamma \cap \mathbb{Z}^2| \leq c_{f,\varepsilon} \cdot t^{\frac{1}{2}+\varepsilon}$$

$$|\Gamma \cap \frac{1}{t} \cdot \mathbb{Z}^2| \leq c_{f,\varepsilon} \cdot t^{\frac{1}{2}+\varepsilon}$$

Open question.

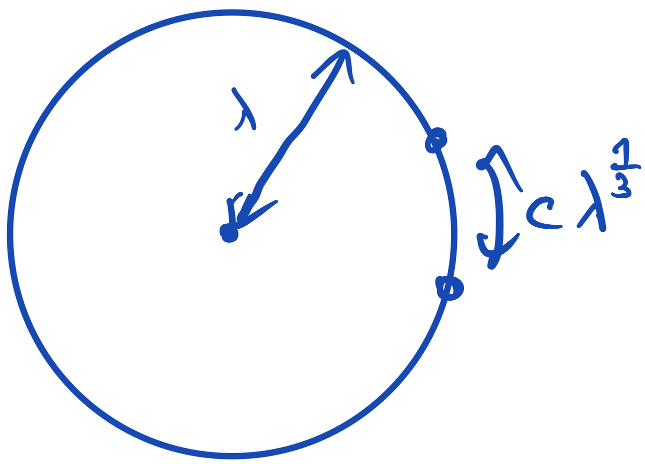


The number of lattice points on an arc of length $R^{\frac{1}{2}}$ admits a uniform upper bound

Update

It is also conjectured that there is a uniform bound C_ε for the number of integer points on an arc of length $R^{1-\varepsilon}$ of the circle of radius R.

Distribution of integer points on $\partial B_\lambda(0)$



Thm (Jarnik or earlier)

Every arc of Length $c \cdot \lambda^{\frac{1}{3}}$ on circle of radius λ contains at most two integer points.

Proof.

Distractive question.

Given a curve on $\mathbb{T}^2$,

can u_x vanish on it?

$\sin(\alpha x+\beta y \underset{\vee}{} {}^{+c})$ vanishes

on a geodesic curve.

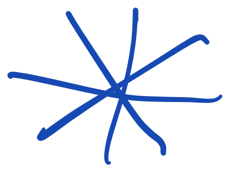
Exercise

Thm of Bourgain and Rudnick implies that only finitely many of eigenfunctions on $\mathbb{T}^2$ can vanish on a curve, which is not a closed geodesic.

For the exercise it will be helpful to know.

Zero sets of solutions to $\Delta u+\lambda u=0$ on $\mathbb{T}^2$ are real-analytic curves.

Thm (Bers). The intersiction of several nodal curves is equiangular.



$\mathbb{R}e(z^k)$

Remark.

$x^3-y^2=0$

$\leftarrow$ cannot happen

Demonstration.

$\Delta u + \lambda u = 0$

Taylor's expansion

$$u(z) = P_k(z) + P_{k+1}(z) + \dots$$

$P_e(z)$ – homogeneous polynomial of order e

<u>Exercise.</u> P_k is harmonic.

$$0 = \Delta u + \lambda u = \frac{\Delta P_k(x) + O(|x|^{k-1})}{|x|^{k-2}} \to \Delta P_k\left(\frac{x}{|x|}\right)$$

Homogeneous harmonic polynomials of 2-variables

$$u(z) = u\left(\frac{z}{|z|}\right) \cdot |z|^k .$$

$$z = r \cdot e^{i\varphi}$$

$$u(z) = r^k \cdot \sin(k(\varphi + \varphi_0))$$

$$u(z) \in \operatorname{span} \mathrm{Re}(z^k), \; \mathrm{Im}(z^k)$$

<u>Exercise</u> There are no others.

Exercise. Let $u \in C^\infty(\mathbb{R}^2)$

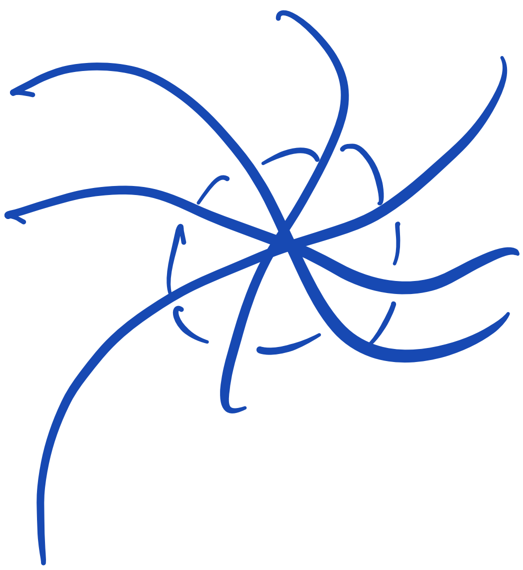
If $u = \mathrm{Re}(z^k) + O(z^{k+1})$,

$\nabla u = \nabla \mathrm{Re}(z^k) + O(|z|^k)$

then the zero set of u

looks like the zero

set of $\mathrm{Re}(z^k)$

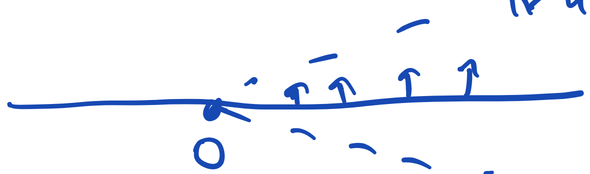


$u = \mathrm{Im}(z^k) + \text{Error}$

$|\nabla u| = |k \cdot z^{k-1}|$

$|\nabla \text{Error}| =$

$= O(z^k)$



Remark. It is also true but non-trivial that solutions to

$$\operatorname{div}(A\nabla u) + b\cdot\nabla u + c\cdot u = 0 \quad (*)$$

in $\Omega \subset \mathbb{R}^d$

$0 \in \Omega$.

$A(0) = Id$, A - Lipschitz

b, c - bounded

can be approximated by harmonic polynomials.

Thm*. For every non-zero solution u to $*$ there is an integer $k > 0$ and a non-zero homogeneous harmonic polynomial P_k of degree k:

$$u(x) = P_k(x) + O(|x|)^{k+1}$$

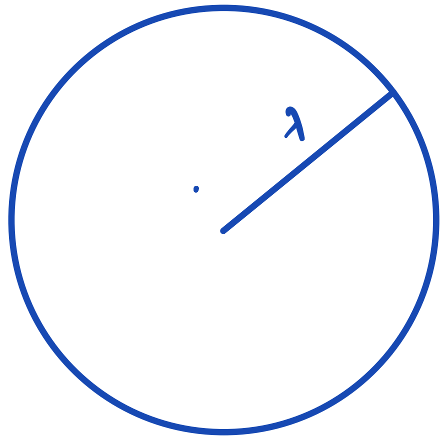
Back to nodal sets on $\mathbb{T}^2$.

Thm (Bourgain-Rudnick).

For every C^∞-smooth curve $\Sigma \subset \mathbb{T}^2$ with non-vanishing curvature there is $\Lambda = \Lambda(\Sigma)$ such that for $\lambda > \Lambda$ no eigenfunction φ : $\Delta\varphi + \lambda\varphi = 0$ on $\mathbb{T}^2$ vanishes on Λ.

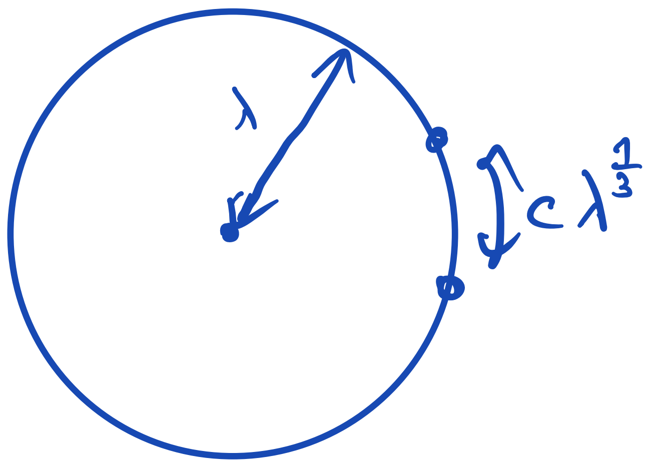
Moreover, $\lambda > \Lambda_\Sigma$

$$c_\Sigma \cdot \|\varphi_\lambda\|_2 \le \|\varphi_\lambda\|_{L^2(\Sigma)} \le C_\Sigma \|\varphi\|_{L^2}$$

Tools: ① 

$\varepsilon > 0$

integer points on $\{|z| = \lambda\}$ is $\leq C_\varepsilon \cdot \lambda^\varepsilon$

② 

$c = \frac{1}{10^{20}}$

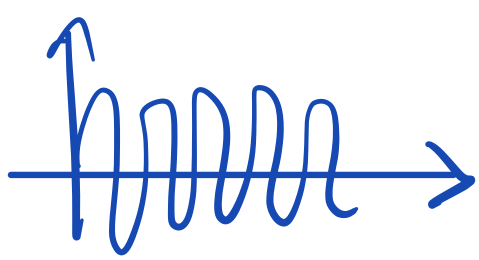
Every arc of Length $c \cdot \lambda^{\frac{1}{3}}$ on circle of radius λ contains at most two points.

③ Useful tool: oscilatory integrals

$$I(\lambda) = \int_{-1}^{1} e^{i\lambda\varphi(x)}\,dx \qquad \varphi \text{ real}$$

under some assumptions on φ

$$I(\lambda) \xrightarrow{\lambda\to+\infty} 0$$



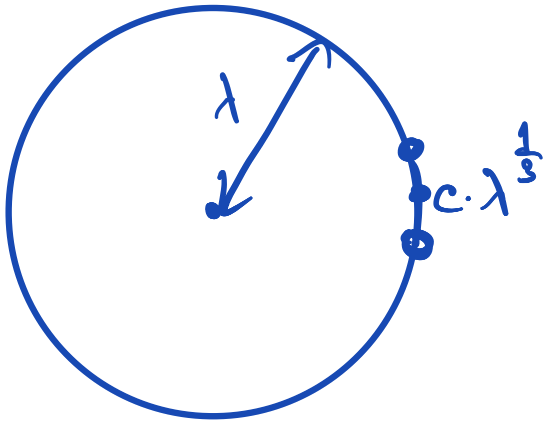
Proposition.

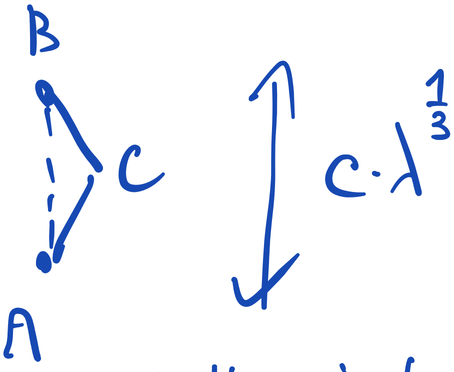
The number of lattice points on an arc of length $c \cdot \lambda^{\frac{1}{3}}$ of $\{|z|=\lambda\}$, where $c>0$ is small, is at most 2.

Plan. We will prove this proposition and will use it in the proof of Bourgain-Rudnick's bound.

Proof

Assume the contrary





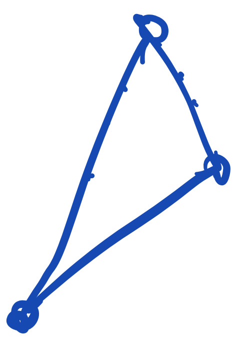
convexity of the circle

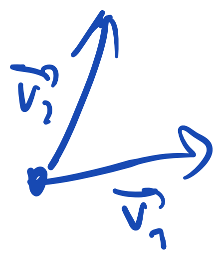
$\Downarrow$

Claim. a) $\text{Area}(\triangle ABC) > 0$

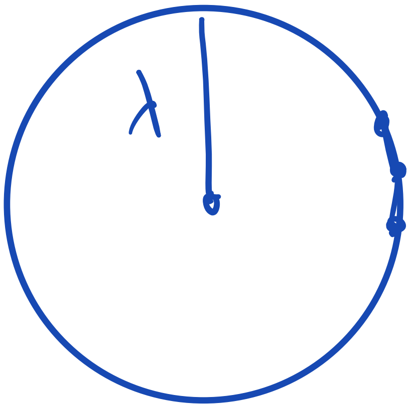
A, B, C – have integer coordinate

b) $\text{Area}(\triangle ABC) \geq \frac{1}{2}$





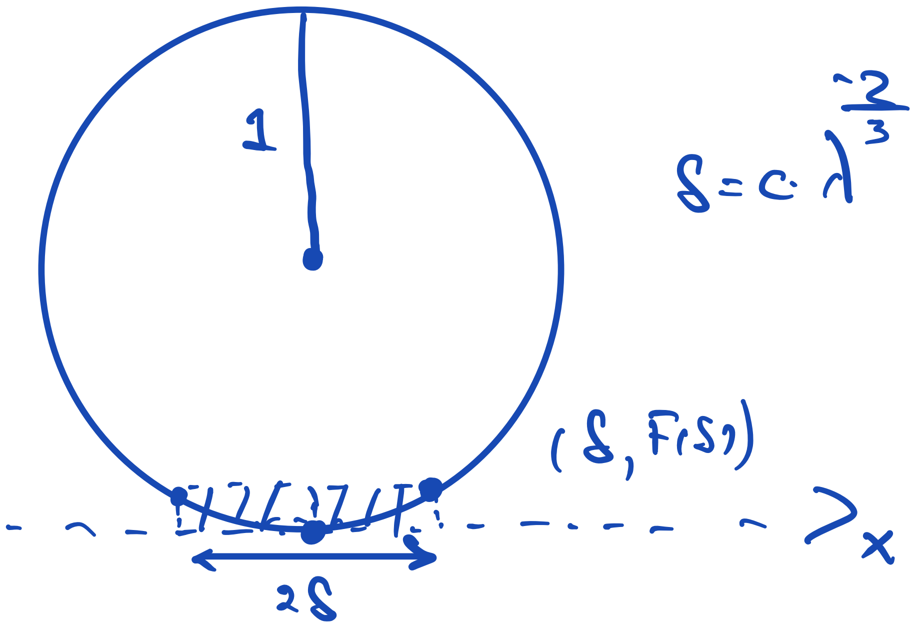
$\det(\vec{v}_1\ \vec{v}_2) = \pm\,\text{Area}($  $)$



$c\cdot\lambda^{\frac{1}{3}}$

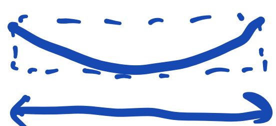
Λ

$\downarrow$ $\frac{1}{\lambda}$ rescaling



$\delta = c\cdot\lambda^{-\frac{2}{3}}$

$F(\delta) \asymp \delta^2$

Area( $\updownarrow c\cdot\delta^2$) $\leq C\cdot\delta^3 \asymp c\cdot\lambda^{-2}$

After rescaling Area(ABC) $\leq c\cdot\lambda^{-2}$

Before rescaling Area(ABC) $\leq c$.

Proof (Bourgain - Zeev)

σ - normalized arc-length measure on Σ.

$$\sigma(\Sigma) = \hat{\sigma}(0) = 1$$

$$\int e^{ikx} \, d\sigma(\xi) = \hat{\sigma}(k)$$

Claim (easy).

$$|\hat{\sigma}(\xi)| < 1 \quad \text{for} \quad \xi \neq 0.$$

$$\int e^{i\xi x} d\sigma(x) =$$

$= \hat{\sigma}(\xi)$

Σ - has non-vanishing curvature.

Claim (method of stationary phase)

$$|\hat{\sigma}(\xi)| \le \frac{1}{\sqrt{|\xi|}}, \quad |\xi| >> 1.$$

Corollary. $|\hat{\sigma}(\xi)| \le 1 - \delta$, $\delta > 0$

for $\xi \in \mathbb{Z}^2 \setminus \{0\}$

Back to eigenfunctions.

$$f(x) = \sum_{|n|=R} \hat{f}(n) \cdot e^{2\pi i n x}$$

$$R \cdot S^1 = \bigcup I_\alpha$$

$$|I_\alpha| \asymp R^{\frac{1}{3}}$$

Each I_α contains one or two integer points.

$n \in I_\alpha$, $m \in I_\beta$ $\quad n, m \in \mathbb{Z}^2$

$$\Rightarrow |n-m| \gtrsim c \cdot R^{\frac{1}{3}}$$

$$\varphi^{\alpha}_{(x)} = \sum_{n \in I_\alpha} \hat{\varphi}(n) \cdot e^{2\pi i n x}$$

$$\varphi = \sum \varphi^{\alpha}$$

$$\int |\varphi|^2 \, d\sigma = \sum_{I_\alpha, I_\beta} \int \varphi^{\alpha} \cdot \overline{\varphi^{\beta}} \, d\sigma$$

$$\sum$$

$$\int \varphi^{\alpha} \cdot \overline{\varphi^{\beta}} \, d\sigma$$

Oscilatory integral

$\alpha \neq \beta$

$$\sum \int e^{2\pi i n_\alpha \cdot x} \cdot e^{-2\pi i n_\beta \cdot x} \, d\sigma \leq C \cdot R^{-\frac{1}{6}}$$

$$|n_\alpha - n_\beta| \geq c \cdot \lambda^{\frac{1}{3}}$$

$$\left| \int e^{i \xi x} \, d\sigma \right| \leq \frac{C}{\sqrt{|\xi|}}$$

$$\sum_{\text{non-diagonal}} \left| \int \varphi^{\alpha} \cdot \overline{\varphi^{\beta}} \, d\sigma \right| \le C \cdot R^{-\frac{1}{6}} \cdot R^{2\varepsilon}$$

$\#$ terms in the sum $\le C \cdot R^{2\varepsilon}$

$\#$ integer points on $\{|z| = R\}$ is $\le C \cdot R^{\varepsilon}$

$$\sum \left| \text{non-diagonal terms} \right| \le C \cdot R^{-\frac{1}{6} + 2\varepsilon}$$

$\varphi = \sum c_k \cdot e^{ikx}$

$$\sum \int |\varphi^{\alpha}|^2 \simeq \sum \text{diagonal terms} \simeq \sum c_k^2$$

$$\int |e^{inx}|^2 \, d\sigma = 1$$

$n_1 \ne n_2$

$$\int |e^{in_1 x} + e^{in_2 x}|^2 \, d\sigma$$

$$= 2 + \int e^{i(n_1 - n_2)x} \, d\sigma + \int e^{i(n_2 - n_1)x} \, d\sigma$$

$1-\delta$ $1-\delta$

$$\varphi = \sum c_k \cdot e^{ikx}$$

$$\int |\varphi|^2 \, d\sigma \asymp \sum c_k^2$$