bounded domain

Weyl's law: $\Omega \subset \mathbb{R}^d$

$$\lambda_k(\Omega) \underset{k\to\infty}{\sim} c_d \cdot \left(\frac{k}{|\Omega|}\right)^{\frac{2}{d}}$$

$$c_d = (2\pi)^2 \cdot |B_1|^{\frac{-2}{d}}$$

Polya's conjecture for Dirichlet eigenvalues in $\mathbb{R}^d$

$$\lambda_k(\Omega) \geq C_d \cdot \left(\frac{k}{|\Omega|}\right)^{2/d}$$

C_d — the same constant as in the Weyl law.

$u=0$ on $\partial\Omega$

Ω - smooth bounded domain in $\mathbb{R}^d$

Thm(Lieb).

$$\lambda_k \geq \tilde{C}_d \cdot \left(\frac{k}{|\Omega|} \right)^{2/d}$$

$$\tilde{C}_d << C_d$$

Thm(Li - Yau).

$$\lambda_k \geq \frac{d}{d+2} \cdot C_d \cdot \left(\frac{k}{|\Omega|} \right)^{2/d}$$

Thm (Li-Yau)

sharp

$$\sum_{i=1}^{k} \lambda_i \geq \frac{d}{d+2} C_d \cdot k^{\frac{d+2}{d}} \cdot |\Omega|^{-\frac{2}{d}}$$

$$k\lambda_k \geq \sum_{i=1}^{k} \lambda_i$$

$$\lambda_k \geq \frac{d}{d+2} C_d \cdot k^{\frac{2}{d}} \cdot |\Omega|^{-\frac{2}{d}}$$

likely non-sharp.

The proof is bare handed:

Basic properties of eigenfunctions + Fourier transform.

We follow the article: „On the Schrodinger equation and the eigenvalue problem" by P. Li & S.-T. Yau)

For the sake of simplicity, suppose Ω is (C^∞) smooth. So that eigenfunctions are smooth up to the boundary

$$\Delta \varphi_k + \lambda_k \varphi_k = 0 \qquad \text{in } \Omega$$

$$\varphi_k = 0 \qquad \text{in } \partial\Omega$$

Orthogonality of gradients.

$$\int \nabla \varphi_i \cdot \nabla \varphi_j = \delta_{i,j} \ \lambda_i$$

In particular,

$$\sum_{i=1}^{k} \int |\nabla \phi_i|^2 = \sum_{i=1}^{k} \lambda_i$$

Reproducing kernel:

$$K(x,y) = \sum_{i=1}^{k} \varphi_i(x) \cdot \varphi_i(y)$$

Integral operator T:

$$f \in L^2(\Omega) \longrightarrow Tf$$

$$Tf(y) := \int_{\Omega} K(x,y) f(x) dx$$

$$K(x,y) = \sum_{i=1}^{k} \varphi_i(x) \cdot \varphi_i(y)$$

$$Tf(y) := \int_{\Omega} K(x,y) f(x)\, dx$$

Exercise.

① If $f \in \mathrm{Span}(\varphi_1, \dots, \varphi_k)$, then $Tf = f$.

② If $f \perp \varphi_1, \dots, \varphi_k$, then $Tf = 0$

Corollary. $Tf(y) = \int K(x,y) f(x)\, dx$ is an orthonormal projection from $L^2(\mathbb{R}^d)$ onto $\operatorname{span}(\varphi_1, \dots, \varphi_k)$.

$$\|Tf\|_{L^2(\Omega)} \le \|f\|_{L^2(\Omega)}$$

Informal point of view.

If we fix y, then

$$K_n^y(x) = \sum_{i=1}^{n} \varphi_i(x)\varphi_i(y)$$

"converges" to δ_y as $n \to \infty$.

We would like to extend $\varphi_i(x)$ by 0 outside of $\Omega \subset \mathbb{R}^d$

$$\varphi_i \in W_0^{1,2}(\Omega) \subset W_0^{1,2}(\mathbb{R}^d)$$

Fourier tranform.

$$\hat{\varphi}(\xi) = (2\pi)^{-d/2} \cdot \int \varphi(x) \cdot e^{ix\xi}$$

$$\int |\hat{\varphi}|^2 = \int |\varphi|^2$$

$$\widehat{\partial_{x_i} \varphi} = (-i\xi_i) \cdot \widehat{\varphi}$$

$$|\xi|^2 = \sum |\xi_i|^2$$

$$\int |\nabla \varphi|^2 = \int |\xi|^2 \cdot |\hat{\varphi}|^2$$

$$\varphi \in W_0^{1,2}(\Omega)$$

$$\|\varphi\|^2_{W_0^{1,2}} = \int |\varphi|^2 + \int |\nabla \varphi|^2 =$$

$$= \int (1+|\xi|)^2 |\hat{\varphi}|^2$$

Fourier transform of the reproducing kernel

$$K(x,y) = \sum_{i=1}^{k} \varphi_i(x)\cdot\varphi_i(y)$$

$$K(x,y) \in W_0^{1,2}(\Omega\times\Omega) \subset W_0^{1,2}(\mathbb{R}^d\times\mathbb{R}^d)$$

extension by 0.

Fourier transform of $K(x,y)$:

with respect to x-variable

y

$$\hat{K}(\xi, y) = (2\pi)^{-d/2} \int_{x\in\mathbb{R}^d} K(x,y)\, e^{ix\cdot\xi}\, dx$$

Exercise.

Show that $\int |K(x,y)|^2 = k$

L^2-estimate of slices:

$$f(\xi) = \int |\hat{K}(\xi, y)|^2 dy =$$

$$= \int_{\Omega} (2\pi)^{-d} \left(\int_{\mathbb{R}^d} K(x,y) \cdot e^{ix\xi} dx \right)^2 dy$$

Claim 1. $\int_{\mathbb{R}^d} f(\xi) d\xi = k$

Claim 2. $f(\xi) \le (2\pi)^{-d} \cdot |\Omega|$

Claim 3. $\int f(\xi) \cdot |\xi|^2 d\xi = \lambda_1 + \ldots + \lambda_k$

$$f(\xi) = \int |\hat{K}(\xi, y)|^2 dy =$$

$$= \int_\Omega (2\pi)^{-d} \left(\int_\Omega K(x,y) \cdot e^{ix\xi} dx \right)^2 dy$$

$$= (2\pi)^{-d} \, \| T e^{ix\xi} \|^2_{L^2(\Omega)} \leq (2\pi)^{-d} \cdot \| e^{ix\xi} \|^2_{L^2(\Omega)}$$

$$= (2\pi)^{-d} \cdot |\Omega|$$

Claim 3. $\int f(\xi)\cdot|\xi|^2 d\xi = \lambda_1 + \dots + \lambda_k$

$$\iint |\xi|^2 |\hat{K}(\xi, y)|^2 dy\, d\xi =$$

$$= \iint |\widehat{\nabla_x K}|^2(\xi, y)\, dy\, d\xi$$

$$= \iint |\nabla_x K|^2(x, y)\, dx\, dy$$

$$\iint \sum \nabla \varphi_i(x)\cdot\varphi_i(y) \cdot \sum \nabla \varphi_i(x)\cdot\varphi_i(y) =$$

$$= \sum \int |\nabla \varphi_i|^2(x) \cdot \int |\varphi_i|^2(y) = \sum \lambda_i$$

All together:

① $\int f(\xi) = k$

② $0 \le f(\xi) \le (2\pi)^{-d} \cdot |\Omega|$

③ $\int f(\xi) \cdot |\xi|^2 = \lambda_1 + \dots + \lambda_k$.

① + ② + ③ implies

a bound for $\lambda_1 + \dots + \lambda_k$ in terms of k and $|\Omega|$

Exercise. If $\int f = k$, $0 \le f \le M$, then

$$\int f(\xi)|\xi|^2 \ge \tilde{c}_d \cdot k^{\frac{d+2}{d}} \cdot M^{-\frac{2}{d}}$$

calculate $\tilde{c}_d$

Exercise The LHS is minimal when f is the step function:

$$f = \chi_{B_R} \cdot M$$

where is chosen in such way that

$$\int \chi_{B_R} \cdot M = k \; :$$

In other words, $|B_1| \cdot R^d \cdot M = k$

Exercise Use coarea formula to show $|\partial B_1| = d \cdot |B_1|$

$0 \le f \le M \qquad \int f = k \qquad \min \int_{\mathbb{R}^d} f \cdot |\xi|^2$

$f = \chi_{B_R} \cdot M \qquad R = ? \qquad |B_R| \cdot M = \int f = k$

$$|B_1| \cdot R^d \cdot M = k$$

$$M \int_{B_R} |\xi|^2 = M \cdot \int_0^R d\rho \int_{\partial B_\rho} \rho^2 = M \cdot \int_0^R \rho^2 \cdot \overbrace{|\partial B_\rho|}^{\rho^{d-1}\cdot|\partial B_1|} \, d\rho =$$

$$= M \cdot |\partial B_1| \cdot \int_0^R \rho^{d+1} = \frac{M \cdot |\partial B_1|}{d+2} \cdot R^{d+2} =$$

$$= \frac{d}{d+2} M |B_1| \cdot R^{d+2} = \frac{d}{d+2} R^2 \cdot k =$$

$$= \frac{d}{d+2} \cdot k \cdot \left(\frac{k}{|B_1| \cdot M} \right)^{\frac{2}{d}}$$

We leave away the part of plugging in

$$M = (2\pi)^{-d} \cdot |\Omega| \; ; \; R = \left(\frac{k}{M \cdot |B_d|} \right)^{\frac{1}{d}}$$

Li-Yau's bound:

$$\lambda_1+\lambda_2+\dots+\lambda_k \geq \frac{d}{d+2} \underbrace{(2\pi)^2 \cdot |B_1|^{-\frac{2}{d}}}_{C_d} \cdot k^{\frac{d+2}{d}} \cdot |\Omega|^{-\frac{2}{d}}.$$

Weyl's law: $\lambda_k \sim c_d \cdot \left(\frac{k}{|\Omega|}\right)^{2/d}$

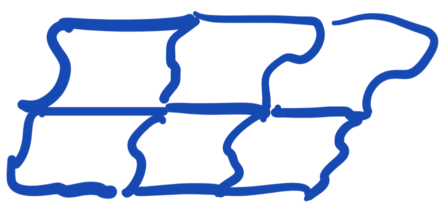
Polya's conjecture (for domains in $\mathbb{R}^d$)

$$\lambda_k \geq c_d \cdot \left(\frac{k}{|\Omega|}\right)^{2/d}$$

Li-Yau: $\lambda_k \geq \frac{d}{d+2} c_d \left(\frac{k}{|\Omega|}\right)^{2/d}$

Def. A domain $\Omega \subset \mathbb{R}^d$ is called tiling if one can tile $\mathbb{R}^d$ with translations of Ω.

Example. A square is a tiling domain, but a ball isn't.



Exercise. Prove Polya's conjecture for tiling domains.

Reminder.

$$\lambda_{k+1} = \sup_{\substack{L \subset W_0^{1,2}(\Omega) \\ \dim(L)=k}} \ \inf_{\substack{\varphi \in W_0^{1,2}(\Omega) \\ \varphi \perp L}} \frac{\int |\nabla \varphi|^2}{\int \varphi^2}$$

λ_k are monotone:

$$\Omega \subset \tilde{\Omega}$$

$$\lambda_k(\Omega) \geq \lambda_k(\tilde{\Omega})$$

Min-max principle

Neumann eigenvalues

$$\mu_k = \sup_{\substack{L \subset W^{1,2}(\Omega) \\ \dim(L)=k-1}} \inf_{\substack{\varphi \in W^{1,2}(\Omega) \\ \varphi \perp L}} \frac{\int |\nabla \varphi|^2}{\int \varphi^2}$$

Compare to Dirichlet eigenvalues.

$$\lambda_{k+1} = \sup_{\substack{L \subset W_0^{1,2}(\Omega) \\ \dim(L)=k}} \inf_{\substack{\varphi \in W_0^{1,2}(\Omega) \\ \varphi \perp L}} \frac{\int |\nabla \varphi|^2}{\int \varphi^2}$$

Weyl's law for Neumann's eigenvalues is the same as for Dirichlet's.

$$\lambda_k, \mu_k \sim C_d \cdot \left(\frac{k}{|\Omega|}\right)^{2/d}$$

Weyl's conjecture

Ω - smooth, bounded domain in $\mathbb{R}^d$.

The number of Dirichlet's eigenvalues λ_i in the interval $(0,\lambda)$ behaves as follows

$$(2\pi)^{-d}\cdot \lambda^{d/2} |B_1|\cdot|\Omega| - \frac{1}{4}(2\pi)^{1-d}\cdot|\partial B_1|\cdot\lambda^{\frac{d-1}{2}}\cdot|\partial\Omega| + o(\lambda^{\frac{d-1}{2}})$$

—— " —— for Neumann eigenvalues with +

$$(2\pi)^{-d}\cdot \lambda^{d/2} |B_1|\cdot|\Omega| + \frac{1}{4}(2\pi)^{1-d}\cdot|\partial B_1|\cdot\lambda^{\frac{d-1}{2}}\cdot|\partial\Omega| + o(\lambda^{\frac{d-1}{2}})$$

Polya's conjecture.

$$\mu_k \leq c_d \cdot \left(\frac{k}{|\Omega|}\right)^{\frac{2}{d}} \leq \lambda_k$$