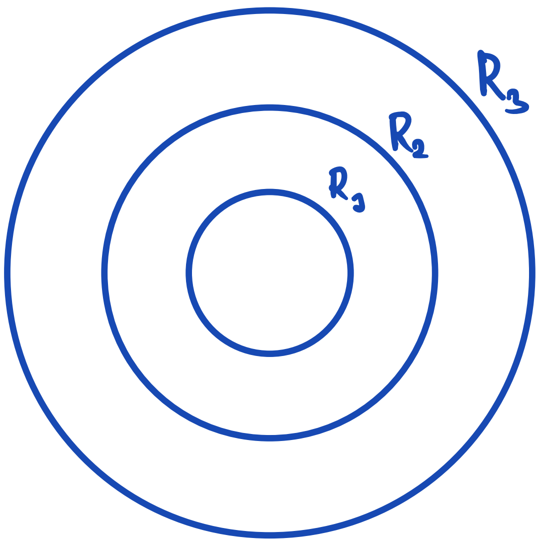
Hadamard's 3 circles thm.

$f \in Hol(\mathbb{C})$, then

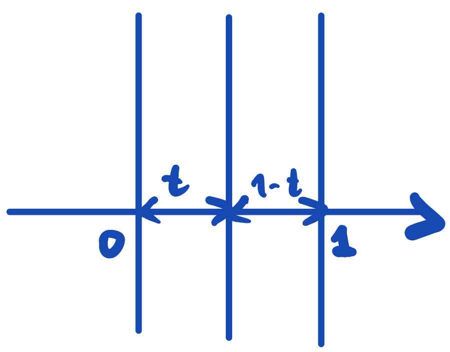
$M(r) := \max_{|z|=r} |f|$ satisfies



$$M(R_2) \le M^{\alpha}(R_1) \cdot M^{1-\alpha}(R_3)$$

where $R_2 = R_1^{\alpha} \cdot R_3^{1-\alpha}$

Three line thm.



f is a bounded holomorphic function in a strip $\mathrm{Re}\, z \in [0,1]$

$M_0 = \max_{\mathrm{Re}\, z=0} |f|$ $\quad M_1 = \max_{\mathrm{Re}\, z=1} |f|$

$t \in (0,1)$

$M_t = \max_{\mathrm{Re}\, z=t} |f|$

$$\boxed{M_t \le M_0^{1-t} \cdot M_1^{t}}$$

Proof is using a <u>fundamental fact</u>

$$\boxed{\log|f| \text{ is subharmonic}}$$

Subharmonic functions.

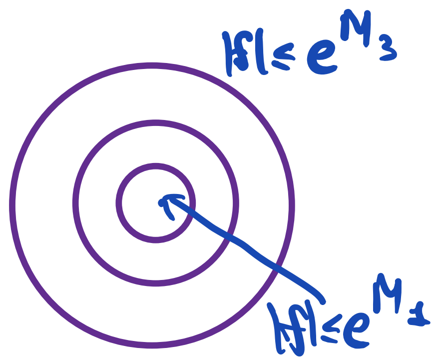
1) $u(x) \leq \fint_{B_r(x)} u \leq \fint_{\partial B_r(x)} u$

for any ball

2) $\Delta u \geq 0$ (in Generalized sense)

Example. Convex functions are subharmonic.

Proof of Hadamard's theorem.



$u = \log|f|$

$M_1 = \max\limits_{\partial B_{R_1}} u \qquad M_3 = \max\limits_{\partial B_{R_3}} u$

Define $h(x) = h(|x|) = (M_3 - M_1)\dfrac{\log|x| - \log R_1}{\log R_3 - \log R_1} + M_1$

h is harmonic in $B_{R_3} \setminus B_{R_1}$

$h(R_1) = M_1 \qquad h(R_3) = M_3$

$$\begin{matrix} u \le h \\ \text{on } \partial(B_{R_3}\setminus B_{R_1}) \end{matrix} \overset{\text{subharmonicity}}{\Longrightarrow} \begin{matrix} u \le h \\ \text{in } B_{R_3}\setminus B_{R_1} \end{matrix}$$

h is a linear function of $\log|x|$

$\log R_2 = \alpha \cdot \log R_1 + (1-\alpha)\cdot \log R_3$

Then $h(R_2) = \alpha \cdot h(R_1) + (1-\alpha) h(R_3)$

$$\sup_{\partial B_{R_2}} u \leq h(R_2) = \alpha \cdot \log M_1 + (1-\alpha) \log M_3$$

and therefore

$$|f(z)| = e^u \leq M_1^{\alpha} \cdot M_3^{1-\alpha}$$

for $|z| = R_2$.

Agmon's thm.

Let $\Delta u = 0$ in $\mathbb{R}^n$.

$$H(r) = ⨍_{\partial B_r} u^2 = \int_{\partial B_r} u^2 / |\partial B_r|$$

Then $H(R_2) \le H(R_1)^{\alpha} \cdot H(R_3)^{1-\alpha}$

$$R_2 = R_1^{\alpha} \cdot R_3^{1-\alpha}$$

Proof. Decomposition as a sum of harmonic polynomials.

$u(x) = \sum a_k p_k(x)$ - converge in some neighborhood of 0 as u is real analytic.

$p_k(x) = |x|^k \cdot \varphi_k\left(\frac{x}{|x|}\right)$

homogeneous polynomial

Exercise. p_k are harmonic

Exercise. $\int_{\partial B_1} \varphi_k \cdot \varphi_l = 0 \quad k \neq l$

$$⨍_{\partial B_r} u^2 = \sum r^{2k} a_k$$, where $a_k > 0$

$$a_k = ⨍_{\partial B_1} \varphi_k^2$$

The functions of the form

$$\sum r^k \cdot a_k = f(r) \qquad a_k > 0$$

are called <u>absolutely monotone</u>

<u>Exercise</u>. Show that any absolutely monotone function $f(r)$ satisfy $\log f(e^t)$ is a convex function of t.

As corollary we get Agmon's thm.

$$H(r) = \fint_{\partial B_r} u^2 \quad \text{satisfies} \quad (\log H(e^t))'' \geq 0$$

$F(r) = \dfrac{r \cdot H'(r)}{H(r)}$ is called <u>frequency</u>

and F is a monotonically increasing function. $\Leftrightarrow (\log H(e^t))'' \geq 0$

$(\log H(e^t))' = \dfrac{e^t \cdot H'(e^t)}{H(e^t)} = F(e^t)$

$(F(e^t))' \geq 0 \Leftrightarrow F$ is monotone

<u>Exercise.</u> Show that monotonicity of frequency implies

$$\left(\frac{R}{r}\right)^{F(r)} \leq \frac{H(R)}{H(r)} \leq \left(\frac{R}{r}\right)^{F(R)}$$

and three balls for L^2-averages holds

Remark. Case of polynomials.

$u = p_k(x)$ $\qquad \Delta p_k = 0$ $\qquad p_k\left(\frac{x}{|x|}\right) \cdot |x|^k = p_k(x)$

$$H(r) = \fint_{\partial B_r} u^2 = c \cdot r^{2k}$$

$$F(r) = \frac{r \cdot H'}{H} = 2k$$

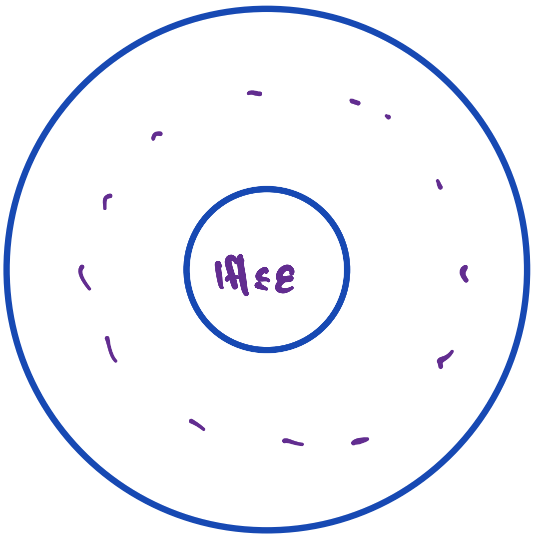
$F = 2 \cdot \deg u$

Exercise. ① Show that if $u = \sum_{k=0}^{n} p_k(x)$, then (hom. harmonic)

$F(r) \le 2n$

② $u = \sum_{k \ge n} p_k(x)$, then $F(r) \ge 2 \cdot n$

Propagation of smallness.



$f \in Hol(D)$

$|f| \le \varepsilon$ on $\frac{1}{4}D$

$|f| \le 1$ on D

Then $f \le \varepsilon^{\frac{1}{2}}$ on $\frac{1}{2}D$

Unique continuation property (UCP)

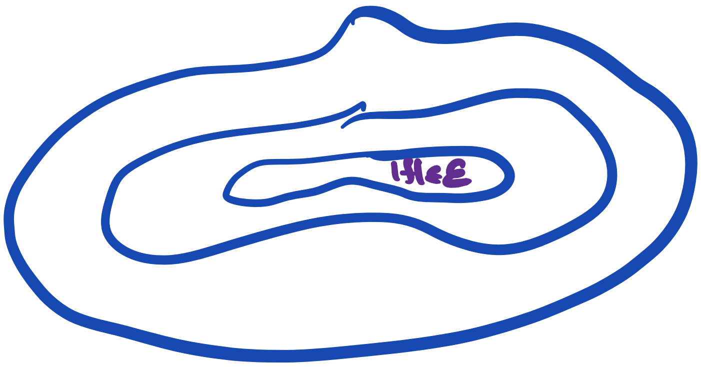
Def. A PDE $Lu=0$ is said to have UCP

if for any solution u

$u=0$ on an open set $\Rightarrow$ $u \equiv 0$.

open   bounded $\subset \mathbb{R}^d$

Exercise. Show that if $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_3$



then there is $\alpha \in (0,1)$ such that for any harmonic function u

$$\begin{cases} |u| \le \varepsilon \text{ on } \Omega_1 \\ |u| \le 1 \text{ on } \Omega_3 \end{cases} \Rightarrow u \le |\varepsilon|^{\alpha} \text{ on } \Omega_2$$

Hint. Start with the case
$\Omega_1 = \frac{1}{2} B$, $\Omega_2 = B$, $\Omega_3 = 2B$

Apply Harnack chain argument.

General fact about linear elliptic PDE with sufficiently smooth coefficients.

$$Lu = \sum_{|\alpha| \le n} a_\alpha D^\alpha u$$ - elliptic of order n

$\alpha = (\alpha_1, \dots, \alpha_d)$

ellipticity means a condition on the coefficients of the highest order.

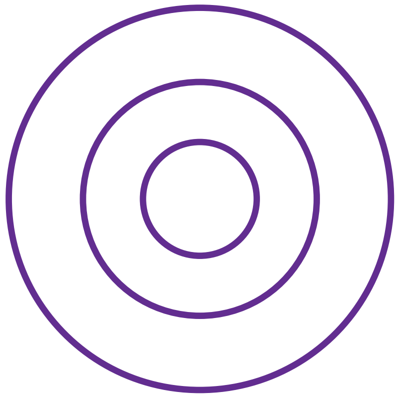
$\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d}$

$$c|\xi|^n \le \sum_{|\alpha|=n} a_\alpha \cdot \xi^\alpha \le C \cdot |\xi|^n$$

$c, C > 0$ - ellipticity constants

Under assumption of ellipticity if a_α are sufficiently smooth, then UCP holds.

3 balls thm for Linear elliptic PDE.



$Lu = 0$ $\quad$ L - as before

$$\max_{B_{R_2}} |u| \le C \cdot \max_{B_{R_1}} |u|^{\alpha} \cdot \max_{B_{R_3}} |u|^{1-\alpha}$$

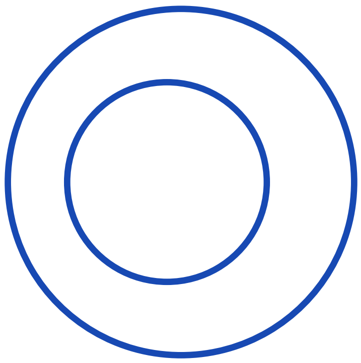
$C > 1$.

Methods for proving UCP

1) Carleman inequalities

2) Monotonicity formulas

Comparison of L^p norms.



$$\Delta u = 0$$

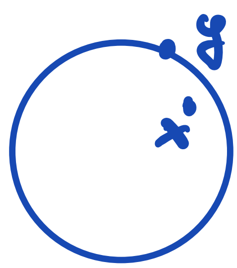
$$B_r \ll B_R$$

$$\|u\|_{L^\infty(B_r)} \le C \cdot \|u\|_{L^1(B_R)}$$

Demonstration. $$\max_{B_{\frac{1}{2}}} |u| \le C \cdot \int_{\partial B_1} |u|$$

Poisson formula

$$u(x) = \int_{\partial B_1} u(y)\, P_x(y)\, dy \qquad P_x(y) = C_d \frac{1-|x|^2}{|x-y|^d}$$



$$|x-y| \ge \frac{1}{2} \qquad u(x) \lesssim \int |u(y)|\, dy$$

Open problem for non-linear PDE.

p-harmonic functions

1) $\operatorname{div}(|\nabla u|^{p-2} \cdot \nabla u) = 0$

2) minimize $\int |\nabla u|^p$

Do p-harmonic functions have UCP?