UNIVERSITE DE GENEVE Section de Mathématiques
Faculté des sciences

**Topics In Probability And Analysis**
**Exercise Sheet 1 - Discussed on 24.09.2020**

Recall that we defined the Minkowski content of a set $K$ as $M_\epsilon^\alpha(K) = \epsilon^\alpha N(K,\epsilon)$ where $N(K,\epsilon)$ is the minimal amount of $\frac{\epsilon}{2}$-balls needed to cover $K$. In the following we assume that $K$ is a bounded set in $\mathbb{R}^d$.

**Exercise 1** (Equivalence of Minkowski contents)**.** Show that for any $\epsilon = \frac{1}{2^n}$ for $n \in \mathbb{N}$ we have

$$M_\epsilon^\alpha(K) \asymp \inf\{\sum_j |Q_j|^\alpha : \, Q_j \text{ are dyadic cubes}, K \subset \bigcup_j Q_j, |Q_j| = \epsilon\}.$$

**Exercise 2** (Equivalence of Hausdorff dimension definitions)**.** Show that

$$\begin{aligned}\dim(K) =& \inf\{\alpha : \mathscr{H}^\alpha(K) = 0\} = \inf\{\alpha : \mathscr{H}^\alpha(K) < \infty\} \\ =& \sup\{\alpha : \mathscr{H}^\alpha(K) = \infty\} = \sup\{\alpha : \mathscr{H}^\alpha(K) > 0\}\end{aligned}$$

**Exercise 3.** Show that

$$\dim(K) = 0$$

holds for any countable set $K$.

**Exercise 4.** Show that $\dim(K) \leq \underline{\dim}_M(K) \leq \overline{\dim}_M(K)$.

**Exercise 5.** Construct a set $K$ such that $\dim(K) < \underline{\dim}_M(K) < \overline{\dim}_M(K)$ holds.