UNIVERSITE DE GENEVE
Faculté des sciences

Section de Mathématiques

Topics In Probability And Analysis
Exercise Sheet 4 - Discussed on 15.10.2020

Exercise 1. (Entropy function) Show that the entropy function $h_2(p) = p\log_2(\frac{1}{p}) + (1-p)\log_2(\frac{1}{1-p})$ is smooth, has zeros at 0 and 1 and reaches the maximum of 1 at $p = \frac{1}{2}$.

Exercise 2. Let's generalize the dyadic measure from class to the b-adic context. Let $\mathbf{p} = (p_0, \ldots, p_{b-1})$ with $\sum_{i=0}^{b-1} p_i = 1$. We define the measure $\mu_{\mathbf{p}}(I_n(x)) = \prod_{i=1}^{n} p_{x_i}$ where $\{x_i\}$ is the b-ary expansion of x and $I_n(x)$ is the level n b-ary interval containing x. Show that

$$\dim(\mu_{\mathbf{p}}) = h_b(\mathbf{p}) := \sum_{i=0}^{b-1} p_i \log_b(\frac{1}{p_i}).$$

We define the graph of a function $f : [0,1] \to \mathbb{R}$ as $G_f := \{(x, f(x)) : x \in [0,1]\}$.

Exercise 3. Show that if f is smooth, then $\dim(G_f) = 1$. Furthermore construct a function f such that $\dim(G_f) > 1$.

Exercise 4. Show that for a set A with $\dim(A) < 1$ we have $A_x = \emptyset$ for almost every x.