UNIVERSITE DE GENEVE
Faculté des sciences

Section de Mathématiques

Topics In Probability And Analysis
Exercise Sheet 3 - Discussed on 08.10.2020

Exercise 1. (Billingsley's Lemma with Vitali coverings) In class we showed Billingsley's Lemma for measures on the interval $[0,1]$. In the statement and proof we used the dyadic intervals $I_n(x)$. Show that we can generalize the statement by replacing this with a Vitali covering with the bounded subcover property.

Exercise 2. We defined the dimension of a measure μ as

$$\dim(\mu) = \inf\{\dim(A) : \ A \text{ Borel}, \mu(A^c) = 0\}.$$

Show that this is equivalent to

$$\dim(\mu) = \inf\{\alpha : \mu \perp \mathscr{H}^\alpha\}$$

where two measures μ, ν are called orthogonal if there exists a set A such that $\mu(A^c) = 0 = \nu(A)$.

Exercise 3. Let μ be the measure assigning a mass $\frac{1}{2^n}$ to each n-th level interval of the middle-third Cantor set $K \subset [0,1]$. Calculate $\dim(\mu)$.

Exercise 4. Construct a set A which has zero Hausdorff measure of its own dimension which is not 0. That is $\dim(A) = \alpha > 0$ and $\mathscr{H}^\alpha(A) = 0$.