Thesis notes

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So far, we have several correspondent objects. Initially, we look at meanders. These are identified by intersections between a nonself-crossing closed curve and a straight line, unique up to deformation of the plane. The foremost question to be asked about these objects is, "how many of them are there?"

For some n, M_n is the number of meanders with n pairs of intersection points. M_n can be shown to be the

Story so far:

We began by introducing meandric systems [explain the basics]. Now, one thing that could be done with these meandric systems is to look at the Gram matrix of systems of size n.

$$U_m^{DF}(q) = U_m^{standard}(q/2)$$

Think about the last two pages of the proof, think about streamlining with regards to the open arch diagrams (are they necessary?)

1 Introduction

2 Important Equivalences

Below, we will introduce a series of important equivalences which we will proceed to interchange between somewhat freely depending on which construct best suits our needs.

2.1 Arch Configurations

To begin with, we consider $A_{2n} = \{\text{set of arch configurations of length 2n}\}$. These can be viewed as a series of non-crossing connections between $2n$ points

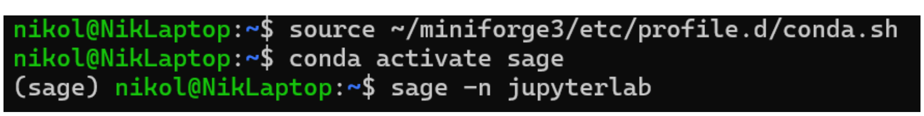


Figure 1: Commands to launch sage jupyter lab, start from ubuntu

on a line, where all the connections are above the line. Naturally, one wonders the number $|A_{2n}|$ of arch configurations of length $2n$.

Consider a connection from point 1 to point $2j$ for $1 \leq j \leq n$ (note that there must be an even number of points under any arch). This divides the line into sections $\{1, 2j\}$ and $\{2j+1, 2n\}$, which themselves can be viewed as $A_{2(j-1)}$ and $A_{2(n-j)}$. Thus, with $|A_0| = 1$, we have

$$|A_{2n}| = \sum_{1 \leq j \leq n} |A_{2(j-1)}||A_{2(n-j)}| = \frac{(2n)!}{(n+1)!n!} = c_n \tag{1}$$

where c_n is the nth Catalan number.

2.2 Closed Walks

For our first equivalent construction, let us consider W_{2n}, closed walk diagrams of length $2n$. These are formed by $2n$ steps either going up and to the right or down and to the left. Denoting these u and d respectively, for a walk diagram to be closed, we must have that $\#u = \#d$.

A bijection between A_{2n} and W_{2n} can be seen by taking any element $A \in A_{2n}$ and constructing a walk w with a u positioned wherever an arch opens and a d wherever an arch closes. One can also see that there is a unique solution to the inverse series of operations when constrained to noncrossing arches. This establishes the bijection, and so we also have that $|A_{2n}| = |W_{2n}|$.

2.3 Temperley-Lieb Algebra

Another equivalence can be made with reduced elements of the Temperley-Lieb Algebra $TL_n(q)$. The algebra can best be thought of as a series of n strings connecting $2n$ points, with the first n listed on a left line from top to bottom and the latter n on a right line listed from bottom to top. Each point is connecte to exactly one other point, and the strings cannot cross. Formally, $TL_n(q)$ is defined by the n generators $1, e_1, e_2, \ldots, e_{n-1}$ and a set of relations. The identity element 1 is portrayed by a diagram connecting each point on the left to it's corresponding point on the right with a straight string, and e_i is the same except with point i connected to point $i+1$ and point $n-i$ connected to point $n-i+1$. Combined with these generators, the following relations completely describe the algebra:

(i) $e_i^2 = qe_i$

(ii) $e_ie_j = e_je_i$ if $|i-j| > 1$

(iii) $e_ie_{i\pm1}e_i = e_i$

Relation *ii* simply says that if two strings are far apart, they commute. Relation *i* says that if a closed circle is made in the product of elements, it can be removed and accounted for by a factor of q. Relation *iii* says that unnecessary turns in the string can be removed by "pulling" a string straight.

This equivalence can be seen visually as a sort of "opening up" of the Temperley-Lieb element, connecting the bottoms of the two lines of points and keeping the the connecting strings as they are, except they can now be thought of as arches.

A further equivalence can be made to an ideal $\mathcal{I}_n(q)$ of $TL_{2n}(q)$ generated by the element $u_n = e_1 e_3 \dots e_{2n-1}$. Pictorially, we make this equivalence simpler by turning diagrams of elements on their side, so that the top is made of n strings connecting the upper n pairs of points, and the strings connecting the points on the bottom are a "wiggly" arc diagram of order $2n$.

2.4 A New Tool: Boxes

Now we will introduce a map ρ from W_{2n}, the set of walks of length $2n$, to reduced elements of the ideal $\mathcal{I}_n(q)$ of $TL_{2n}(q)$. To begin with, let us consider the fundamental walk, denoted a_n. This consists of an up step and a down step repeated $n/2$ times, such that for any $0 \leq i \leq n/2 - 1$,

$$h(2i) = 0 \text{ and } h(2i+1) = 1$$

where h is a height function. ρ maps this walk to the reduced element

$$u_n = \rho(a_n) = e_1 e_3 ... e_{2n-1}$$

From this, we can build any reduced element by defining a box addition on the fundamental walk by

$$\rho(a + \beta_i) = e_i \rho(a)$$

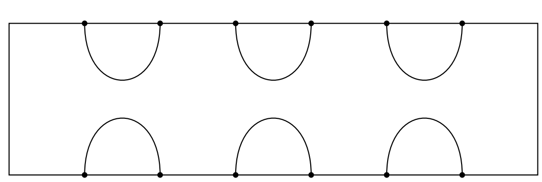
Diagrammatically, a box addition β_i takes a minimum at i in a to a maximum, taking $h(i) \to h(i) + 2$, increasing the height by 2.

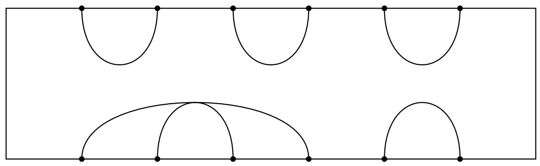
Therefore, we write a general element $e \in \mathcal{I}_n(q)$ as a product over all e_is corresponding to the necessary box additions to build from a_n to a such that $\rho(a) = e$. Further, we can define the length of a walk diagram $a \in W_{2n}$ to be the number of box additions needed to construct a from a_n.

2.5 Vector Space of $\mathcal{I}_n(q)$

From these reduced elements of $\mathcal{I}_N(q)$ we can construct a basis for the corresponding vector space over the complex numbers. Further, we know that $|W_{2n}| = c_n = \dim(\mathcal{I}_n(q))$. As an example, we then have that basis 1 for $\mathcal{I}_3(q)$ is given by the following $c_3 = 5$ elements

$$\{e_1e_3e_5, e_2e_1e_3e_5, e_4e_1e_3e_5, e_2e_4e_1e_3e_5, e_3e_2e_4e_1e_3e_5\} \tag{2}$$





To translate back to the Temperley-Lieb algebra from these box addition walks, simply let the corresponding arch diagram be the bottom half of an element $e \in \mathcal{I}_n(q)$, and let the top half be the strings connecting the n pairs of upper points as described above.

3 Meander Determinant

The Temperley-Lieb Algebra is equipped with a trace operation

$$\mathrm{Tr}(e) = q^{\kappa(e)} \tag{3}$$

where $\kappa(e)$ is the number of connected components after adding extraneous strings connecting opposite points i and $n-i+1$. Now, noting that $e_i^t = e_i$ and $(ef)^t = f^t e^t$, we introduce the bilinear form

$$(e, f) = \mathrm{Tr}(ef^t) \tag{4}$$

Using this, we can consider the Gram matrix of basis 1 of $I_n(q)$ formed by running over the walk diagrams $(a)_1, (b)_1 \in W_{2n}$, where

$$[\Gamma_{2n}(q)]_{a,b} = ((a)_1, (b)_1) = \mathrm{Tr}((a)_1(b)_1^t) \tag{5}$$

The entries are then given by $q^{n+\kappa(a|b)}$, where the n is present because of the linearity of trace, and comes from the n connections of the factors of $u_n = e_1e_3 \dots e_{2n-1}$ in the $I_n(q)$ representations of both a and b, forming n closed circles (see relation 1 of the Temperley Lieb algebra). The $\kappa(a|b)$ is the standard exponent given by the trace, connecting ends of strings on the top and bottom of the I_nq diagram. In this case, $q^{\kappa(a|b)}$ represents the number of connected components of the meandric system corresponding to $(a|b) = (a)_1(b)_1^t = ef^t$.

Now, recalling the definition of the meander matrix $\mathcal{G}_{2n}(q)$ where each entry is the number of connected components of the meandric system, we have that

$$[\Gamma_{2n}(q)]_{a,b} = q^{n+\kappa(a|b)} = q^n[\mathcal{G}_{2n}(q)]_{a,b}$$

Thus, calculating the Gram determinant of basis 1 will also yield the Meander determinant.

4 Creating a new orthonormal basis

To help us compute the Gram determinant of basis 1, we will perform an explicit Gram-Schmidt orthonormalization of basis in terms of the above bilinear form.

This orthonormalization is a change of basis from basis 1 to a new basis, basis 2. Basis 2 satisfies the following properties

1. Basis 2 elements are still indexed in the same way as basis 1 elements, denoted as $(a)_2$ for $a \in W_{2n}$.

2. Basis 2 elements are orthogonal, meaning that $((a)_2, (b)_2) = 0$ when $a \neq b$.

3. Basis 2 elements all have norm 1, meaning that $((a)_2, (a)_2) = 1$ for $a \in W_{2n}$.

To begin the construction of basis 2 elements, we start with the fundamental element $(a_n)_2$, similar to basis 1 but with a new normalization factor introduced.

$$(a_n)_2 = q^{-n} e_1 e_3 \ldots e_{2n-1} \tag{6}$$

This normalization factor is necessary for property 3, such that

$$((a_n)_2, (a_n)_2) = \mathrm{Tr}((a_n)_2 (a_n)_2^t) = \mathrm{Tr}((a_n)_2) = q^{-n} q^n = 1$$

as n internal loops create a q^n to cancel out one of the two q^{-n}s, and n external loops from the trace cancel out the other q^{-n}.

Now, to continue the constrution of basis 2 elements, we return to box additions, this time with a new definition. A box addition takes place at index i and height ℓ (recall that the base walk consists of heights $0, 1, 0, 1, \ldots, 0, 1, 0$, and that a box addition occurs at a minimum, increasing the height by 2). Then, we have that

$$(a + \beta_{i,\ell}) = \sqrt{\frac{\mu_{\ell+1}}{\mu_\ell}} (e_i - \mu_\ell)(a)_2$$

where we denote μ_ℓ as

$$\mu_\ell = \frac{U_{\ell-1}(q)}{U_\ell(q)} \quad \text{for } \ell \geq 1$$

where U_m is a Chebyshev polynomial of the second kind. Together with $(a_n)_2$, the above box addition rule completely defines basis 2. The recursion relation

$$U_{m+1}(q) = qU_m(q) - U_{m-1}(q)$$

now turns into

$$\frac{1}{\mu_1} - \mu_m = \frac{1}{\mu_{m+1}}$$

This new box addition for basis 2 can be viewed as analogous to the box additions for basis 1, but with some new baggage in order to ensure the basis elements remain orthonormal. For instance, the $\sqrt{\mu_{\ell+1}/\mu_\ell}$ factor ensures the basis elements remain normal, though this is certainly not obvious. However, the algebra simply works out, as will soon be seen.

Note since all of these elements are built on the factor $e_1 e_3 \dots e_{2n-1}$, they all belong to the ideal $\mathcal{I}_n(q)$. Therefore, basis 2 elements can be expressed in terms of basis 1 with μ_ℓ coefficients as follows:

$$(a)_2 = \sum_{b \subset a} P_{b,a(b_1)} \tag{7}$$

Here, each entry $P_{b,a}$ is the μ-coefficients of the front of the expanded product created by all the basis 2 box additions needed to create a from b. More specifically, $P_{b,a}$ are the coefficients in front of the basis 1 representation of b. For example, the change of basis matrix for $c_3 = 5$ looks like

$$P_3 = \begin{pmatrix} \mu_1^3 & -\mu_1^{7/2}\mu_2^{1/2} & -\mu_1^{7/2}\mu_2^{1/2} & \mu_1^4\mu_2^1 & -\mu_1^4\mu_2^{3/2}\mu_3^{1/2} \\ 0 & \mu_1^{5/2}\mu_2^{1/2} & 0 & -\mu_1^3\mu_2^1 & \mu_1^3\mu_2^{3/2}\mu_3^{1/2} \\ 0 & 0 & \mu_1^{5/2}\mu_2^{1/2} & -\mu_1^3\mu_2^1 & \mu_1^3\mu_2^{3/2}\mu_3^{1/2} \\ 0 & 0 & 0 & \mu_1^2\mu_2 & -\mu_1^2\mu_2^{3/2}\mu_3^{1/2} \\ 0 & 0 & 0 & 0 & \mu_1^2\mu_2^{1/2}\mu_3^{1/2} \end{pmatrix} \tag{8}$$

In Dyck path and box diagram notation, the basis 2 elements of $\mathcal{I}_3(q)$ look like

$$\begin{aligned}
&\mu_1^3 e_1 e_3 e_5 \\
&\mu_1^{5/2}\mu_2^{1/2}(e_2 - \mu_1)e_1 e_3 e_5 \\
&\mu_1^{5/2}\mu_2^{1/2}(e_4 - \mu_1)e_4 e_1 e_3 e_5 \\
&\mu_1^2\mu_2(e_2 - \mu_1)(e_4 - \mu_1)e_2 e_4 e_1 e_3 e_5 \\
&\mu_1^2\mu_2^{1/2}\mu_3^{1/2}(e_3 - \mu_2)(e_2 - \mu_1)(e_4 - \mu_1)e_3 e_2 e_4 e_1 e_3 e_5
\end{aligned}$$

The expression resulting from multiplying out each of these expressions corresponds to a column of P.

5 Proposition 1.

The basis 2 is orthonormal w.r.t. the bilinear form, namely

$$((a)_2, (b)_2) = \delta_{a,b} \qquad \text{for all } a, b \in W_{2n}$$

First, and here we are very faithful to Di Francesco's work, we will recurse on box additions to prove the following

5.1 Lemma 1.

$$(a)_2^t (b)_2 = 0 \qquad \text{for all } a, b \in W_{2n} \text{ such that } |a| \leq |b| \text{ and } a \neq b$$

In fact, this lemma is a stronger result than we need, but does imply the orthogonality required. In the following computations, we will take the length of a basis 2 element $(a)_2$, $|a|$, to be the number of basis 2 box additions needed to make $(a)_2$ from the fundamental walk $(a_n)_2$. We will now prove Lemma 1 by recursion on basis 2 box addition. First, suppose that the property described in Lemma 1 holds for some $(a)_2$. Now, we will show that it holds for any basis 2 box addition at some minimum of $(a)_2$, written as $(a)_2 + \beta_i$. Namely, we show that for $b \geq |a + \beta_i|$,

$$(a + \beta_{i,\ell})_2^t (b)_2 = 0 \tag{9}$$

The trick here is that since e_i commutes with e_j for $|i - j| > 1$ and basis 2 box additions are self adjoint, we can simply transfer the box addition from a to b, at which point $(a + \beta_{i,\ell})_2$ will again just become $(a)_2$ and the aforementioned property will apply. **IRON THIS PART OUT MORE, ASK KYLE WHAT'S HAPPENING**

However, when transferring the basis 2 box addition, there are three possible outcomes in b:

1. b has a minimum at i with $h(i) = m$.
2. b has a maximum at i with $h(i) = m$.
3. b has a slope at i with $h(i) = m$.

Also, note that this height m is liable to be different from the height ℓ at which the box addition would occur in a. With this in mind, our goal here is to, after transferring the box addition over, express $\beta_{i,\ell}(b)_2$ as a linear combination of box additions on box diagrams b', all of which have $|b'| \geq |b| - 1$ so that our inductive hypothesis can take effect on each term, reducing the whole thing to 0.

First, we deal with the most natural case, where b also has a minimum at i at height $m - 1$ and the box transfer takes place at height m. We can write the basis 2 box addition as

$$\sqrt{\frac{\mu_{\ell+1}}{\mu_\ell}}(e_i - \mu_\ell) = \sqrt{\frac{\mu_{\ell+1}\mu_m}{\mu_\ell\mu_{m+1}}} \times \sqrt{\frac{\mu_{m+1}}{\mu_m}}(e_i - \mu_m) + \sqrt{\frac{\mu_{\ell+1}}{\mu_\ell}}(\mu_m - \mu_\ell)$$

where in the first term we have introduced a box addition at point i and height m. Part of this is canceled out in the leading coefficient, and part of is canceled in the second term. With this, we can write the box addition on b as

$$\sqrt{\frac{\mu_{\ell+1}}{\mu_\ell}}(e_i - \mu_\ell)(b)_2 = \sqrt{\frac{\mu_{\ell+1}\mu_m}{\mu_\ell\mu_{m+1}}}(b + \beta_{i,m})_2 + \sqrt{\frac{\mu_{\ell+1}}{\mu_\ell}}(\mu_m - \mu_\ell)(b)_2$$

leaving us with a linear combination as desired.

Let us now deal with case 2, where b has a maximum at i. Here, b is the result of a previous box addition at i on the walk $b - \beta_{i,m}$, where previously there was a minimum at i with $h(i) = m - 1$. Therefore, we can write the basis

2 box addition of $(b)_2$ as

$$\begin{aligned}
\beta_{i,\ell}(b)_2 &= \sqrt{\frac{\mu_{\ell+1}}{\mu_\ell}}(e_i - \mu_\ell)(b)_2 \\
&= \sqrt{\frac{\mu_{\ell+1}\mu_{m+1}}{\mu_\ell \mu_m}}(e_i - \mu_\ell)(e_i\mu_m)(b - \beta_{i,m})_2 \\
&= \sqrt{\frac{\mu_{\ell+1}\mu_{m+1}}{\mu_\ell \mu_m}}\big[(e_1^{-1}\big]
\end{aligned}$$

6 TO DO

1. figure out how to share my stuff on github with kyle
2. Using stanley's product of power sum stuff, how can we, interpreting

$$n = \sum i\lambda_i$$

   as a partition of n, figure out how many meandric systems are represented by that partition. In other words, given a partition of n, how many meandric systems are represented by that in $\mathcal{H}_{2n}$?
3. more coding, figure out determinant stuff of both $\mathcal{G}_{2n}$ and $\mathcal{H}_{2n}$.
4. Work more on the first chapter.