EMF simulation

Finite-Difference Time-Domain (FDTD) Method – Part I

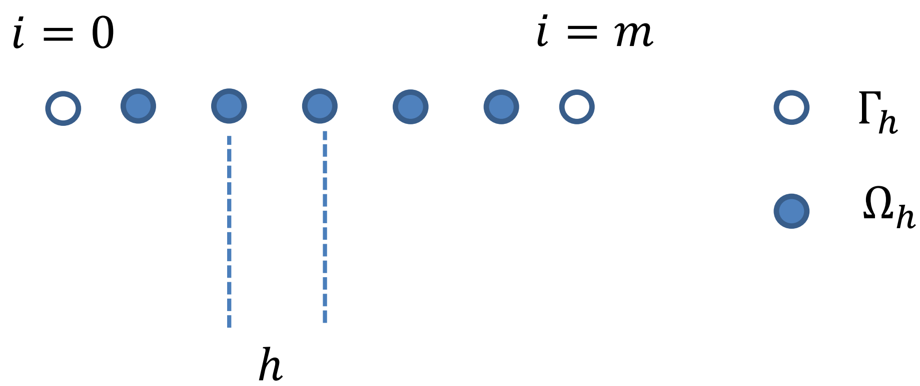
Finite differences

$$\Omega = (0, L)$$

$$\Omega_h = \{ih \mid i = 1, \dots, m-1\}$$

$$\overline{\Omega}_h = \{ih \mid i = 0, \dots, m\}$$

$$\Gamma_h := \overline{\Omega}_h \setminus \Omega_h$$

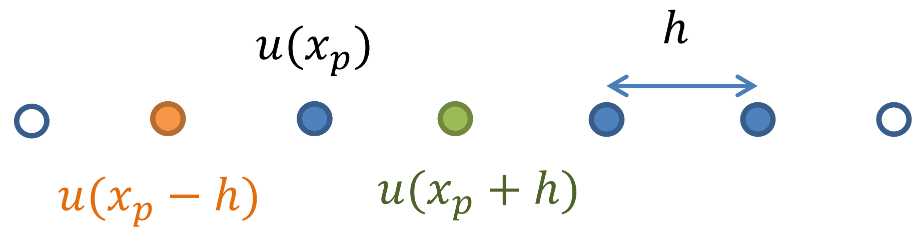


$$h = \frac{L}{m}$$

Finite differences

Consider a 1D function $u(x)$. We would like to discretize it on the interval $[0, L]$ with mesh size h .

$$h = \frac{L}{m}$$



Where m is an integer.

$$D_+u(x) = \frac{u(x+h) - u(x)}{h}$$

Forward difference

$$D_-u(x) = \frac{u(x) - u(x-h)}{h}$$

Backward difference

$$D_0u(x) = \frac{u(x+h) - u(x-h)}{2h}$$

Central difference

Taylor series

Error analysis:

Discrete representation of an analytical function introduces error.

Let‘s introduce Taylor expansion of a function:

$$f(x) = \sum_{x=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

Which is used for the error analysis of a discretization scheme.

FD error

Taylor expansions of $u(x)$:

$$u(x+h) = u(x) + hu'(x) + \frac{1}{2}h^2u''(x) + \frac{1}{6}h^3u'''(x) + \cdots$$

$$u(x-h) = u(x) - hu'(x) + \frac{1}{2}h^2u''(x) - \frac{1}{6}h^3u'''(x) + \cdots$$

$$D_+u(x) = \frac{u(x+h)-u(x)}{h} = u'(x) + \frac{1}{2}hu''(x) + \ldots$$

$\Rightarrow$

$$D_+u(x) = u'(x) + O(h)$$

$$D_-u(x) = u'(x) - O(h)$$

$$D_0u(x) = u'(x) + O(?)$$

Second order FD I

FD discretization of a second derivative:

$$D_0^2 u(x) = D_+ D_- u(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}$$

$$= \; u''(x) + \frac{1}{12} h^2 u^{(4)}(x)$$

$$D_0^2 u(x) = u''(x) + O(?\,)$$

Second order FD II

Consider this simple ODE (ordinary differential equation):

$$u''(x) = f(x) \qquad for \quad 0 < x < 1$$

$$\begin{cases} u(0) = a \\ u(1) = b \end{cases}$$ Boundary conditions

Discretize using FD:

$$D^2 u_{hi} = \frac{u_{h(i+1)} - 2u_{hi} + u_{h(i-1)}}{h^2} = f(x_i)$$

$$h = 1/(m+1)$$

$$x_i = ih \qquad i = 1,2, \dots, m$$

$$\begin{cases} u_{h0} = a \\ u_{h(m+1)} = b \end{cases}$$ Discrete BC's

$$u_{h0}, u_{h1}, u_{h2}, u_{h3}, \dots, u_{hm}, u_{hm+1}$$

Which yields a linear system of m equations: $AU_h = F$

Second order FD III

$u''(x) = f(x) \quad for \;\; 0 < x < 1$

in discrete form:

$$U = \begin{bmatrix} U_{h1} \\ U_{h2} \\ \vdots \\ U_{h(m-1)} \\ U_{hm} \end{bmatrix}$$

$AU_h = F$ Linear system of equations

$$A = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & 0 & \dots & \dots & 0 \\ 1 & -2 & 1 & \dots & \dots & 0 \\ 0 & 1 & -2 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 1 & -2 & 1 \\ 0 & \dots & \dots & 0 & 1 & -2 \end{bmatrix}$$

$$F = \begin{bmatrix} f(x_1) - a/h^2 \\ f(x_2) \\ f(x_3) \\ \vdots \\ f(x_{m-1}) \\ f(x_m) - b/h^2 \end{bmatrix}$$

FD error I

U_j approximates $u_j(x)$, which is the exact solution.

$$\hat{u}_{ex} = \begin{bmatrix} u(x_1) \\ u(x_2) \\ \vdots \\ u(x_{m-1}) \\ u(x_m) \end{bmatrix}$$

Discretization error vector $\qquad E_h = U_h - \hat{u}_{ex}$

each element of E_h is the error between the numerical and exact value of the function at each grid point.

FD error II

Measuring magnitude of the error vector:

L_∞ norm $$\|E_h\|_\infty = \max_{1\le j\le m} |E_j| = \max_{1\le j\le m} |U_j - u(x_j)|$$

Measuring grid errors:

L_1 norm $$\|E_h\|_1 = h\sum_{j=1}^{m} |E_j|$$

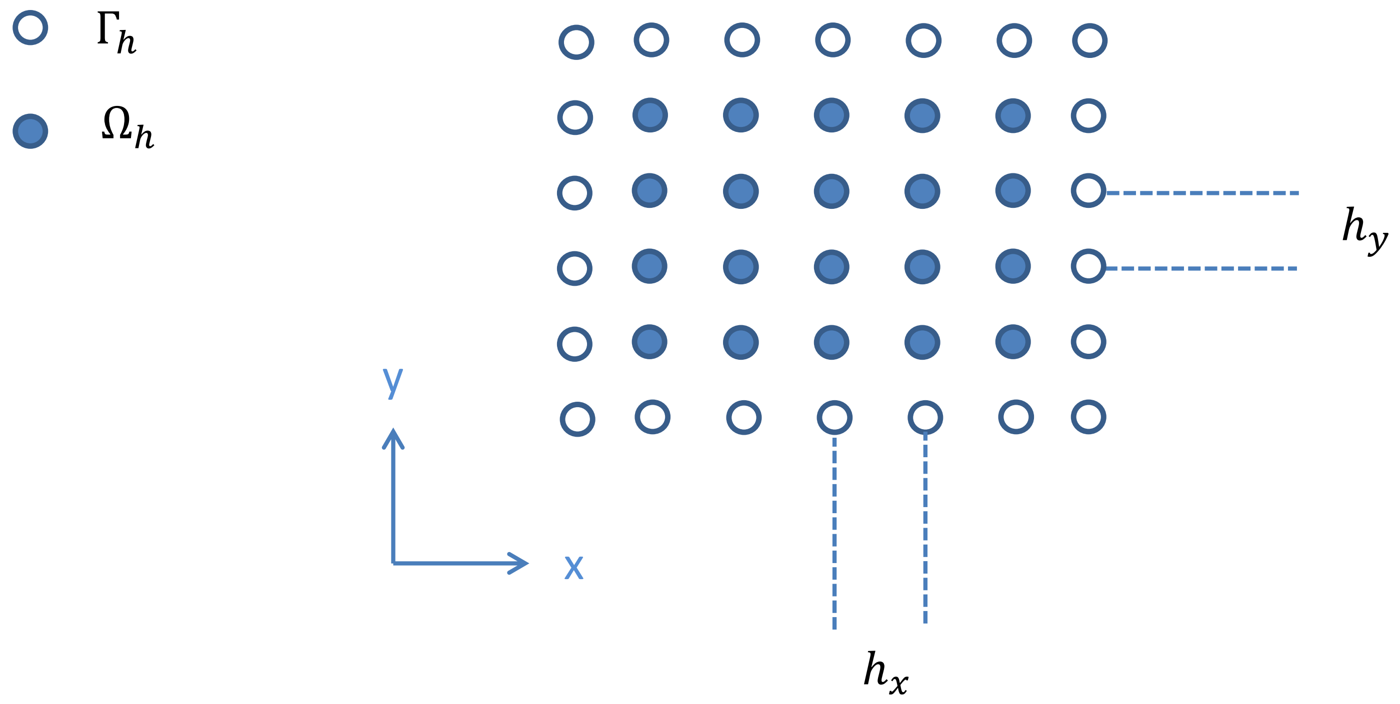
L_2 norm $$\|E_h\|_2 = (\,h\sum_{j=1}^{m} |E_j|^2)^{1/2}$$

FD convergence

The FD method in convergent, if:

$$\|E^h\| \to 0 \quad as \quad h \to 0$$

2D FD I



2D FD II

$$\Omega = (0, L)^2$$

A 2D domain

$$\Omega_h = \{(ih, jh) | i, j = 1, \dots, m-1\}$$

Inner grid

$$\overline{\Omega}_h = \{(ih, jh) | i, j = 0, \dots, m\}$$

Total grid= inner + boundaries

$$\Gamma_h \coloneqq \overline{\Omega}_h \backslash \Omega_h$$

$$h = \frac{L}{m}$$

$$\hat{e}_x = (1,0)$$

$$\hat{e}_y = (0,1)$$

$$\frac{\partial^+ u}{\partial x}(x, y) \approx \frac{u(x + h\hat{e}_x) - u(x)}{h}$$

$$\frac{\partial^- u}{\partial x}(x, y) \approx \frac{u(x) - u(x - h\hat{e}_x)}{h}$$

The same manner for y-direction!

EM wave

Maxwell's curl equations in Cartesian coordinates

$$\frac{\partial E_x}{\partial t} = \frac{1}{\varepsilon}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - \sigma E_x\right)$$

$$\frac{\partial E_y}{\partial t} = \frac{1}{\varepsilon}\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} - \sigma E_y\right)$$

$$\frac{\partial E_z}{\partial t} = \frac{1}{\varepsilon}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - \sigma E_z\right)$$

$$\frac{\partial H_x}{\partial t} = \frac{1}{\mu}\left(\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} - \sigma^* H_x\right)$$

$$\frac{\partial H_y}{\partial t} = \frac{1}{\mu}\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} - \sigma^* H_y\right)$$

$$\frac{\partial H_z}{\partial t} = \frac{1}{\mu}\left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} - \sigma^* H_z\right)$$

E_x, E_y, E_z electrical field components, v/m

H_x, H_y, H_z magnetic field components, v/m

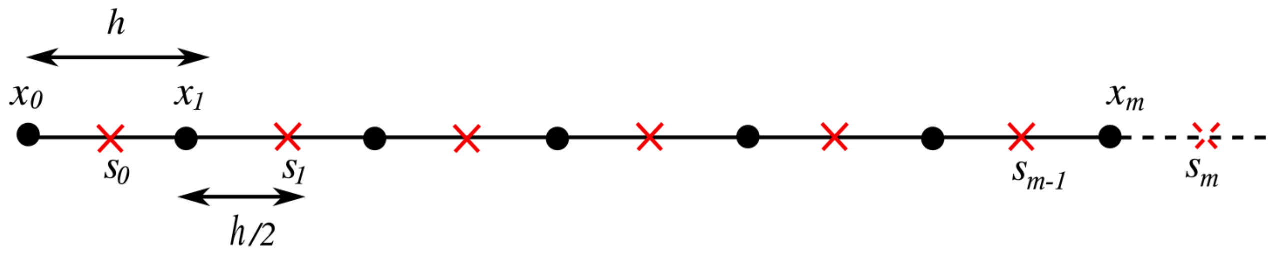
ε electric permittivity, F/m

σ electric conductivity, S/m

μ magnetic permeability, H/m

σ^* equivalent magnetic loss, Ω/m

Staggered grid – 1D



set of corner points in a grid: $\Omega_c := \{x_0, x_1, \ldots, x_m\}$

set of staggered points in the dual grid: $\Omega_s := \{s_0, s_1, \ldots, s_{m-1}\}$

where:

$$s_i = \frac{1}{2}(x_i + x_{i+1}) \quad for \quad i = 0, \ldots, m-1$$

Staggered grid – 3D

$\Omega = [0, L_x] \times [0, L_y] \times [0, L_z]$ A regular domain in 3D

$\Omega_c^x \coloneqq \{x_0, x_1, \dots, x_m\}$ set of corner points in x-direction

$\Omega_c^y \coloneqq \{y_0, y_1, \dots, y_m\}$ set of corner points in y-direction

$\Omega_c^z \coloneqq \{z_0, z_1, \dots, z_m\}$ set of corner points in z-direction

$\Omega_{t_x, t_y, t_z} \coloneqq \Omega_{t_x}^x \times \Omega_{t_y}^y \times \Omega_{t_z}^z$ set of 3D grids

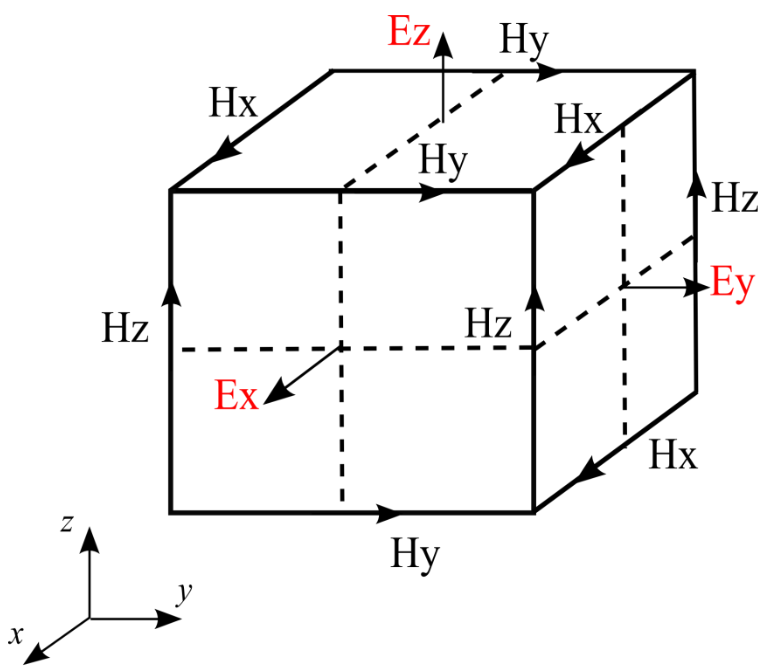
$(t_x, t_y, t_z) = \{c, s\}^3$

c: corner points

s: staggered points

$h_x = h_y = h_z = h = \frac{L}{m}$ for a cubic space lattice

Staggered grid – vector fields



$$\vec{E} = (E_x, E_y, E_z)$$

$$\vec{H} = (H_x, H_y, H_z)$$

$E_x \in \mathbb{R}^{|\Omega_{css}|}$ $\quad E_y \in \mathbb{R}^{|\Omega_{scs}|}$ $\quad E_z \in \mathbb{R}^{|\Omega_{ssc}|}$

$H_x \in \mathbb{R}^{|\Omega_{scc}|}$ $\quad H_y \in \mathbb{R}^{|\Omega_{csc}|}$ $\quad H_z \in \mathbb{R}^{|\Omega_{ccs}|}$

- ❑ Yee arrangement represents Ampere's law and Farady's law.
- ❑ Each $\vec{E}$ component, surrounded by four circulating $\vec{H}$ components, which are each surrounded by four circulating $\vec{E}$ components.
- ❑ The continuity of tangential electric and magnetic field components are preserved at an interface of different media, where the interface is parallel to a grid line.

Staggered grid – lattices

Position of finite-difference lattices with respect to a given point

$$E = M + \hat{e}_x \frac{h_x}{2} \qquad W = M - \hat{e}_x \frac{h_x}{2}$$

$$S = M + \hat{e}_y \frac{h_y}{2} \qquad N = M - \hat{e}_y \frac{h_y}{2}$$

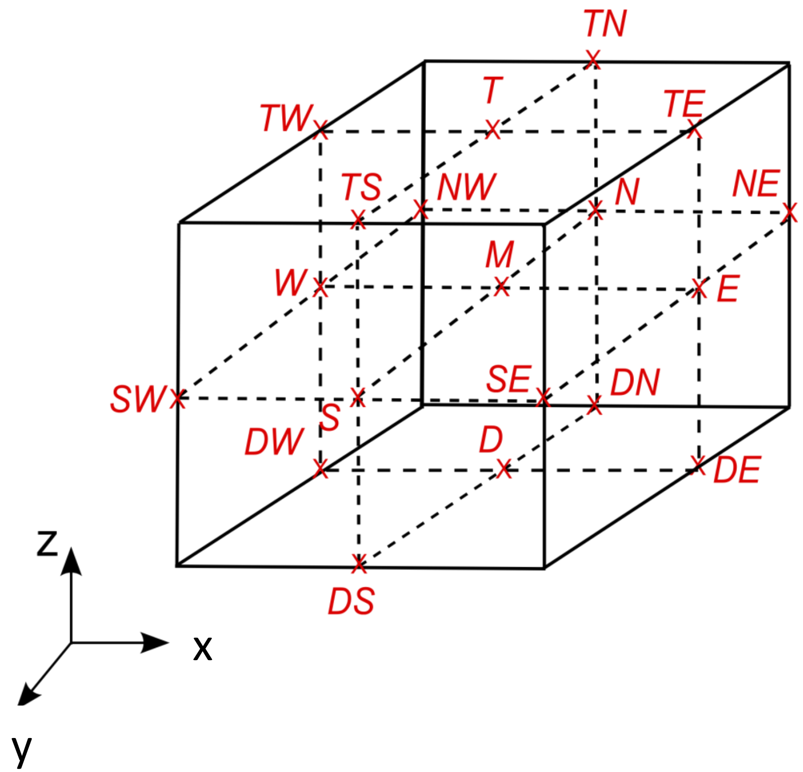
$$T = M + \hat{e}_z \frac{h_z}{2} \qquad D = M - \hat{e}_z \frac{h_z}{2}$$

$\hat{e}_x, \hat{e}_y, \hat{e}_z$: unit vectors in a Cartesian coordinate system

W, E : shift of a grid point, located in M, to the west and east neighboring lattices

N, S : shifts to north and south directions

T, D : shift to top and down directions.



FDTD discretization

$$(i, j, k) = (i\Delta x, j\Delta y, k\Delta z)$$

$$F^n_{i,j,k} = F(i\Delta x, j\Delta y, k\Delta z, n\Delta t)$$

$$\frac{\partial F^n(i\Delta x, j\Delta y, k\Delta z, n\Delta t)}{\partial x} = \frac{F^n_{i+\frac{1}{2},j,k} - F^n_{i-\frac{1}{2},j,k}}{\Delta x} + O[(\Delta x^2)]$$

The $\pm\frac{1}{2}$ increment in the i subscript, denotes a space and time finite difference over $\pm\frac{1}{2}\Delta x$.

FDTD time discretization

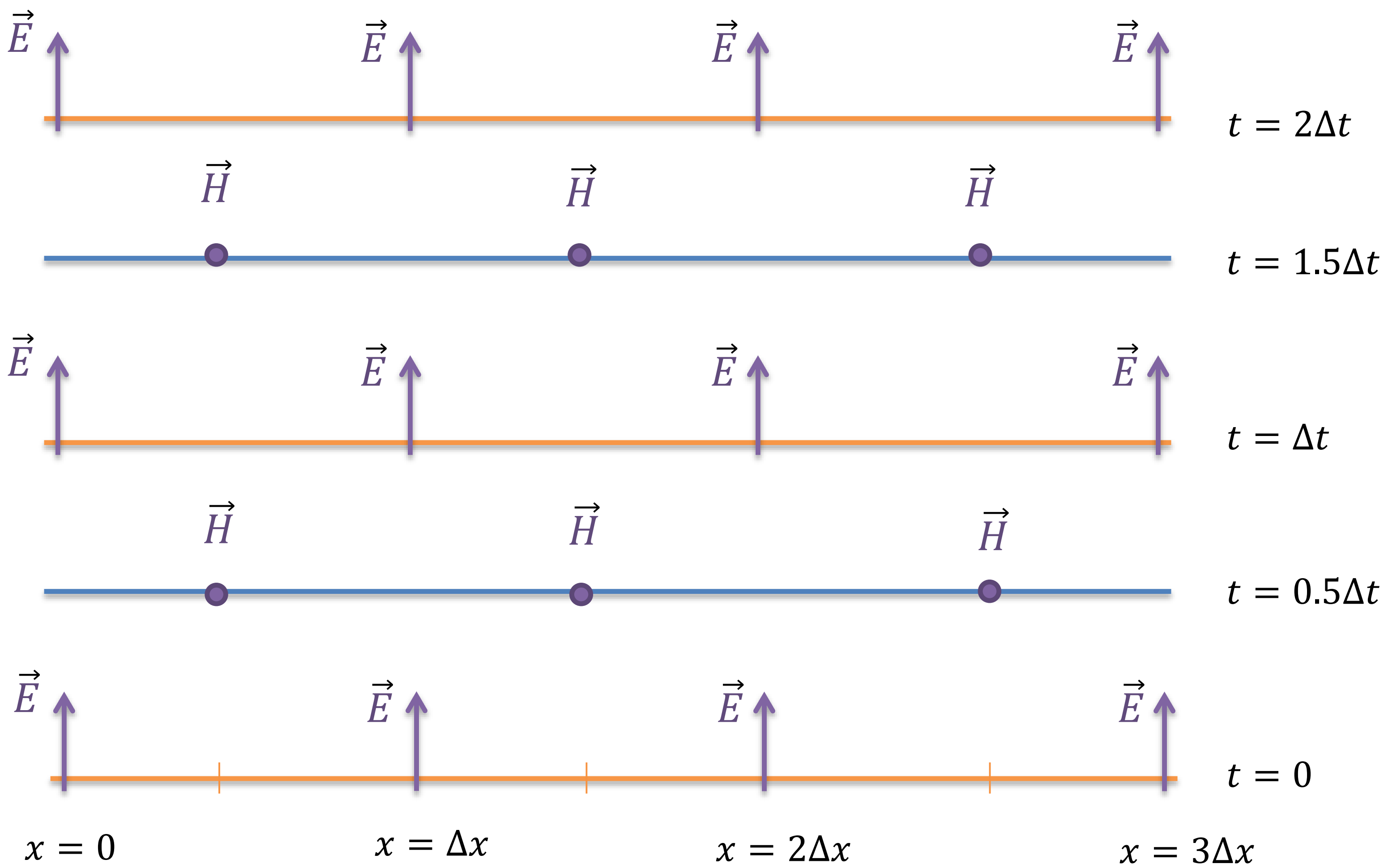
- Time derivatives of Maxwell's equations are discretized using central finite-difference expressions.

- Chronological values of EM-field components at each location are obtained in a temporal leapfrog manner.

- Similar to spatial arrangements, electric and magnetic vector components are interleaved in time with $\pm\frac{1}{2}\Delta t$.

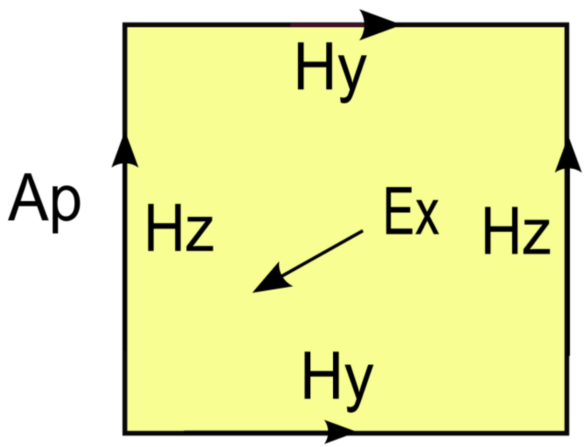
$$\frac{\partial F^{n}(i\Delta x, j\Delta y, k\Delta z, n\Delta t)}{\partial t} = \frac{F_{i,j,k}^{n+\frac{1}{2}} - F_{i,j,k}^{n-\frac{1}{2}}}{\Delta t} + O[(\Delta t^{2})]$$

FDTD discretization: leapfrog scheme



FDTD discretization: Ex-field

Update equation for E_x component



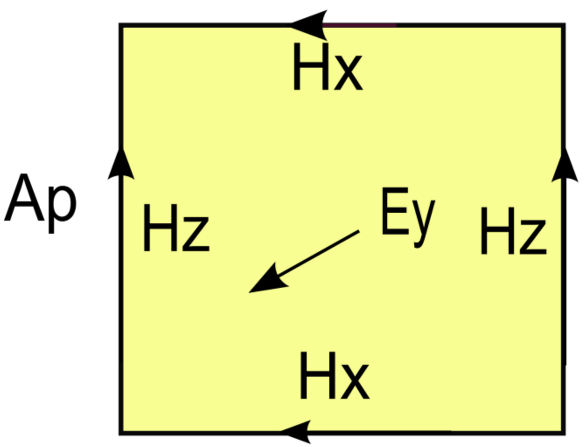
$$\frac{\partial E_x}{\partial t} = \frac{1}{\varepsilon}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - \sigma E_x\right)$$

$$\varepsilon_p \frac{E_x|_M^{n+1} - E_x|_M^{n}}{\tau}$$

$$= \frac{H_z|_S^{n+\frac{1}{2}} - H_z|_N^{n+\frac{1}{2}}}{\Delta y} - \frac{H_y|_T^{n+\frac{1}{2}} - H_y|_D^{n+\frac{1}{2}}}{\Delta z} - \sigma_p E_x|_M^{n+1}$$

FDTD discretization: Ey-field

Update equation for E_y component

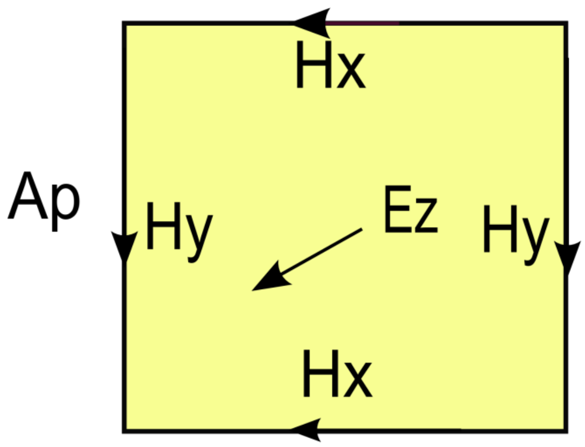


$$\frac{\partial E_y}{\partial t} = \frac{1}{\varepsilon}\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} - \sigma E_y\right)$$

$$\varepsilon_p \frac{E_y|_M^{n+1} - E_y|_M^{n}}{\tau} = \frac{H_x|_T^{n+\frac{1}{2}} - H_x|_D^{n+\frac{1}{2}}}{\Delta z} - \frac{H_z|_E^{n+\frac{1}{2}} - H_z|_W^{n+\frac{1}{2}}}{\Delta x} - \sigma_p E_y|_M^{n+1}$$

Finite-difference discretization: Ez-field

Update equation for E_z component



$$\frac{\partial E_z}{\partial t} = \frac{1}{\varepsilon}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - \sigma E_z\right)$$

$$\varepsilon_p \frac{E_z|_M^{n+1} - E_z|_M^{n}}{\tau}$$

$$= \frac{H_y|_E^{n+\frac{1}{2}} - H_y|_W^{n+\frac{1}{2}}}{\Delta x} - \frac{H_x|_S^{n+\frac{1}{2}} - H_x|_N^{n+\frac{1}{2}}}{\Delta y} - \sigma_p E_z|_M^{n+1}$$

E-field update

Set of three equations for updating $\vec{E}$ components:

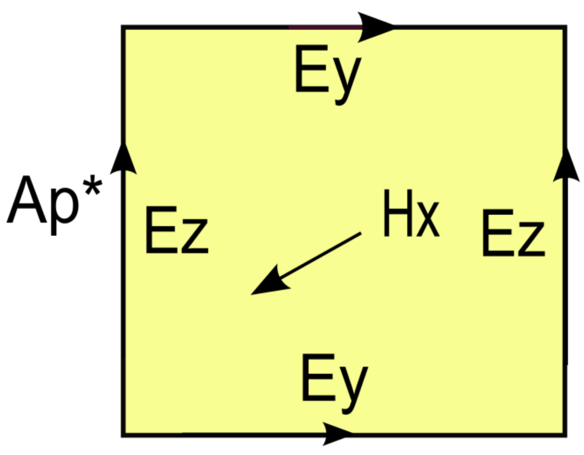
$$E_x\Big|_M^{n+1} = \frac{1}{1+\tau\frac{\sigma_p}{\varepsilon_p}} E_x\Big|_M^{n} + \frac{\frac{\tau}{\varepsilon_p}}{1+\tau\frac{\sigma_p}{\varepsilon_p}} \left(\frac{H_z\Big|_S^{n+\frac{1}{2}} - H_z\Big|_N^{n+\frac{1}{2}}}{\Delta y} - \frac{H_y\Big|_T^{n+\frac{1}{2}} - H_y\Big|_D^{n+\frac{1}{2}}}{\Delta z}\right)$$

$$E_y\Big|_M^{n+1} = \frac{1}{1+\tau\frac{\sigma_p}{\varepsilon_p}} E_y\Big|_M^{n} + \frac{\frac{\tau}{\varepsilon_p}}{1+\tau\frac{\sigma_p}{\varepsilon_p}} \left(\frac{H_x\Big|_T^{n+\frac{1}{2}} - H_x\Big|_D^{n+\frac{1}{2}}}{\Delta z} - \frac{H_z\Big|_E^{n+\frac{1}{2}} - H_z\Big|_W^{n+\frac{1}{2}}}{\Delta x}\right)$$

$$E_z\Big|_M^{n+1} = \frac{1}{1+\tau\frac{\sigma_p}{\varepsilon_p}} E_z\Big|_M^{n} + \frac{\frac{\tau}{\varepsilon_p}}{1+\tau\frac{\sigma_p}{\varepsilon_p}} \left(\frac{H_x\Big|_N^{n+\frac{1}{2}} - H_x\Big|_S^{n+\frac{1}{2}}}{\Delta \mathrm{y}} - \frac{H_y\Big|_W^{n+\frac{1}{2}} - H_y\Big|_E^{n+\frac{1}{2}}}{\Delta x}\right)$$

Finite-difference discretization: Hx-field

Update equation for H_x component



$$\frac{\partial H_x}{\partial t} = \frac{1}{\mu}\left(\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} - \sigma^* H_x\right)$$

$$\mu \frac{H_x|_M^{n+\frac{1}{2}} - H_x|_M^{n-\frac{1}{2}}}{\tau}$$

$$= \frac{E_y|_T^n - E_y|_D^n}{\Delta z} - \frac{E_z|_S^n - E_z|_N^n}{\Delta y} - \sigma_p^* H_x|_M^{n+\frac{1}{2}}$$

H-field update

Set of three equations for updating $\vec{H}$ components:

$$H_x\Big|_M^{n+\frac{1}{2}} = \frac{1}{1+\tau\frac{\sigma_p^*}{\mu_0}} H_x\Big|_M^{n-\frac{1}{2}} + \frac{\frac{\tau}{\sigma_p^*}}{1+\tau\frac{\sigma_p^*}{\mu_0}} \left(\frac{E_y|_T^n - E_y|_D^n}{\Delta \mathrm{z}} - \frac{E_z|_S^n - E_z|_N^n}{\Delta \mathrm{y}}\right)$$

$$H_y\Big|_M^{n+\frac{1}{2}} = \frac{1}{1+\tau\frac{\sigma_p^*}{\mu_0}} H_y\Big|_M^{n-\frac{1}{2}} + \frac{\frac{\tau}{\sigma_p^*}}{1+\tau\frac{\sigma_p^*}{\mu_0}} \left(\frac{E_z|_E^n - E_z|_W^n}{\Delta \mathrm{x}} - \frac{E_x|_T^n - E_x|_D^n}{\Delta \mathrm{z}}\right)$$

$$H_z\Big|_M^{n+\frac{1}{2}} = \frac{1}{1+\tau\frac{\sigma_p^*}{\mu_0}} H_z\Big|_M^{n-\frac{1}{2}} + \frac{\frac{\tau}{\sigma_p^*}}{1+\tau\frac{\sigma_p^*}{\mu_0}} \left(\frac{E_x|_S^n - E_x|_N^n}{\Delta \mathrm{y}} - \frac{E_y|_E^n - E_y|_W^n}{\Delta \mathrm{x}}\right)$$

Stability condition

CFL condition: Assume that $\sigma \geq 0$ and $\varepsilon > 0$. Let:

$$\xi = \frac{1}{\sqrt{|\mu\varepsilon|}} \tau \sqrt{\frac{1}{h_x^2} + \frac{1}{h_y^2} + \frac{1}{h_z^2}}$$

The finite difference time domain method is stable if:

$$\xi < 1$$

For a cubic domain, the CFL condition is:

$$\tau < \frac{\sqrt{3}}{c} h$$

FDTD advantages

A fully explicite method:

All quantities on the RHS of each FD expression are known from the previous time-step.

No need to store EM-fields at all previous time steps.

$$H_y\big|_D^{n+\frac{1}{2}} \quad H_y\big|_T^{n+\frac{1}{2}} \quad H_z\big|_S^{n+\frac{1}{2}} \quad H_z\big|_N^{n+\frac{1}{2}} \quad E_x\big|_M^{n} \Rightarrow E_x\big|_M^{n+1}$$

Parallel computing

Spatially, the computations are dependent on nearby field components.

Facilitating implementation of the method on parallel machines.