Beam Propagation

BPM, Gauss Modes,

Transfer Matrix Method

Paraxial Approximation

Motivation: In several applications, one has to simulate light propagating in one direction (z-direction):

- laser simulation
- light in fibers

Ansatz: E-field behaves like

$$E(x, y, z) = \exp(-ik_0 z)\Psi(x, y, z)$$

Here,

- $\Psi(x, y, z)$ is the amplitude of the wave and
- k_0 an average value of the wave number.

Paraxial Approximation

Inserting $E(x,y,z) = \exp(-ik_0 z)\Psi(x,y,z)$

in the scalar Helmholtz equation

$$\left(\Delta + k^2\right)E(x,y,z) = 0$$

yields $-\Delta\Psi + 2ik_0\frac{\partial\Psi}{\partial z} + \left(k_0^2 - k^2\right)\Psi = 0$

In paraxial approximation one neglects $\frac{\partial^2\Psi}{\partial z^2}$.
This leads to:

$$\boxed{2ik_0\frac{\partial\Psi}{\partial z} = \frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} - \left(k_0^2 - k^2\right)\Psi}$$

Beam Propagation Method

Discretization of $$2ik_0 \frac{\partial \Psi}{\partial z} = \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} - \left(k_0^2 - k^2\right)\Psi = 0$$

leads to $$\boxed{2ik_0 \frac{\Psi^{l+1} - \Psi^{l}}{\Delta z} = \frac{1}{2}\left(L_h \Psi^{l+1} + L_h \Psi^{l}\right)}$$

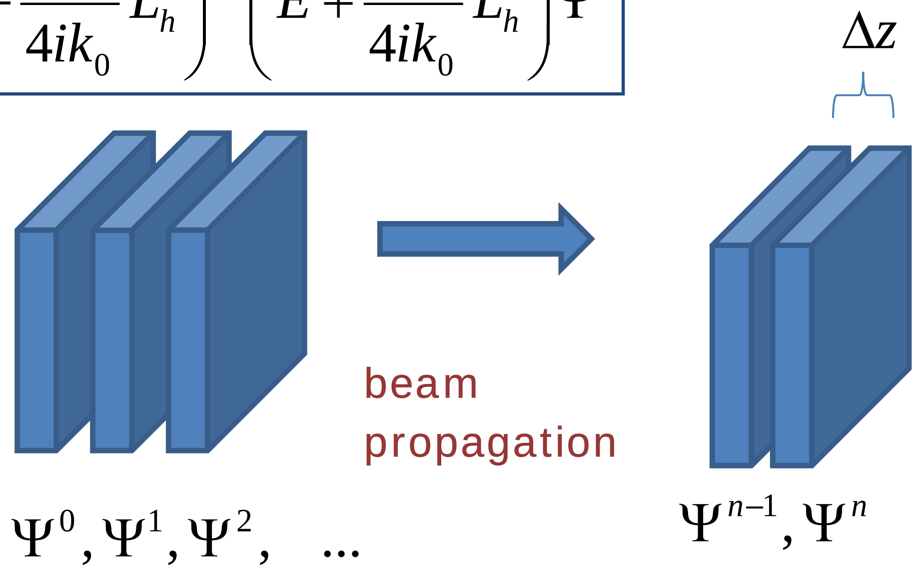
where $L_h \Psi$ is a spatial discretization of:

$$L(\Psi) := \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} - \left(k_0^2 - k^2\right)\Psi$$

Beam Propagation Method

The BPM methods performs the iteration:

$$\boxed{\Psi^{l+1} = \left(E - \frac{\Delta z}{4ik_0} L_h\right)^{-1} \left(E + \frac{\Delta z}{4ik_0} L_h\right) \Psi^l}$$



Gauss Modes

Assume $k = k_0$. Then, paraxial approximation leads to:

$$\boxed{2ik_0 \frac{\partial \Psi}{\partial z} = \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2}}$$

One solution of this equation is the TEM00 Gauss mode:

$$\boxed{E(x,y,z) = \frac{1}{q_0 + z} \exp\left(-ik\left(z + \frac{x^2 + y^2}{2(q_0 + z)} \right) \right)}$$

where q_0 is a complex parameter.

Gauss Modes

The spot size at point z is defined to be the radius r such that E-field decreases by e^{-1}:

$$\boxed{e^{-1} = \frac{|E(r,z)|}{|E(0,z)|}, \qquad \text{where} \quad r = \sqrt{x^2 + y^2}}$$

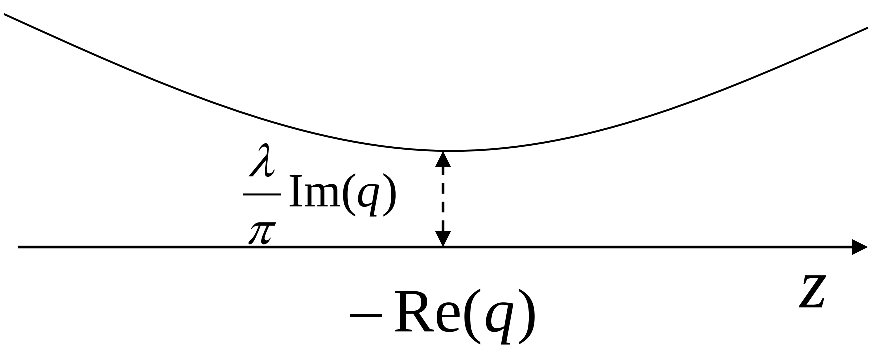
A Gauss mode with parameter q has beam radius

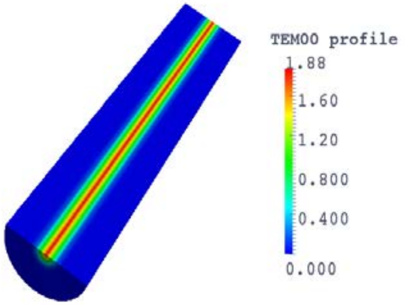
$$w(z)^2 = \frac{\lambda}{\pi}\left(\mathrm{Im}(q_0) + \frac{\left(\mathrm{Re}(q_0) + z\right)^2}{\mathrm{Im}(q_0)}\right)$$

Gaussian Modes

Gaussian mode TEM00:

$$E(x,y,z) = A\frac{1}{q_0+z}\exp\left(-ik\left(z+\frac{x^2+y^2}{2(q_0+z)}\right)\right)$$

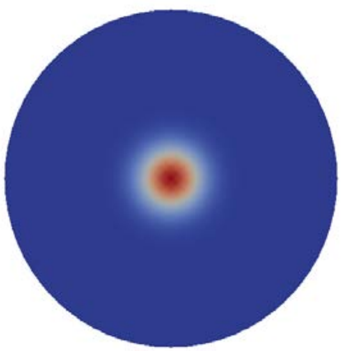




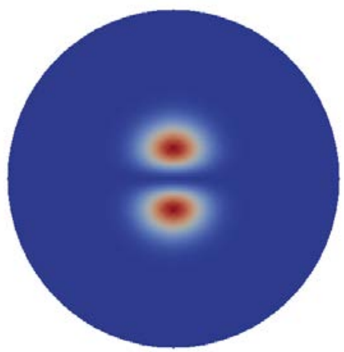
Gaussian Modes

<u>High order Gaussian modes</u>

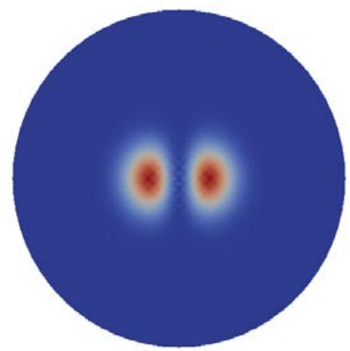
$u_{TEM\,00}$



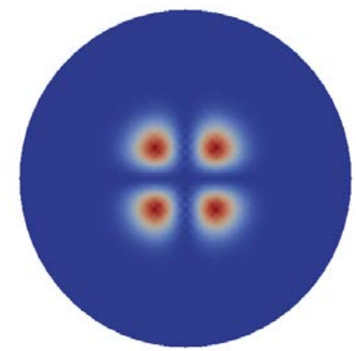
$u_{TEM\,01}$



$u_{TEM\,10}$



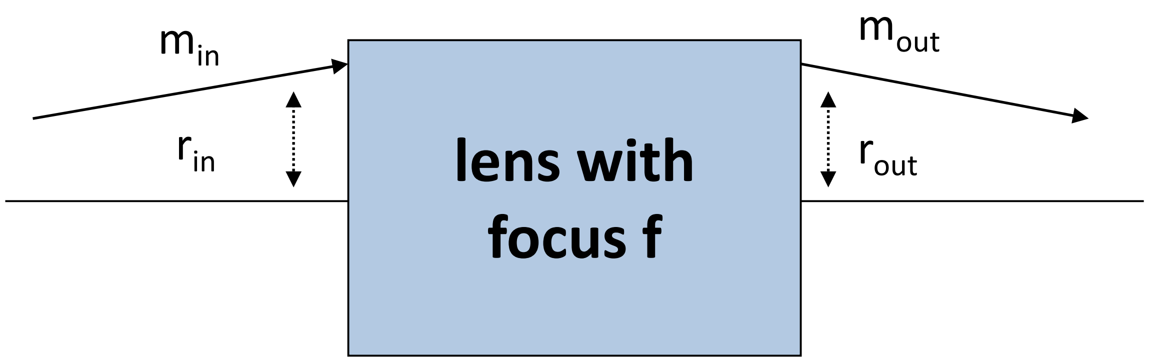
$u_{TEM\,11}$



$$r_m = r_{TEM\,00}\sqrt{m+1}$$

ABCD Matrix Calculation

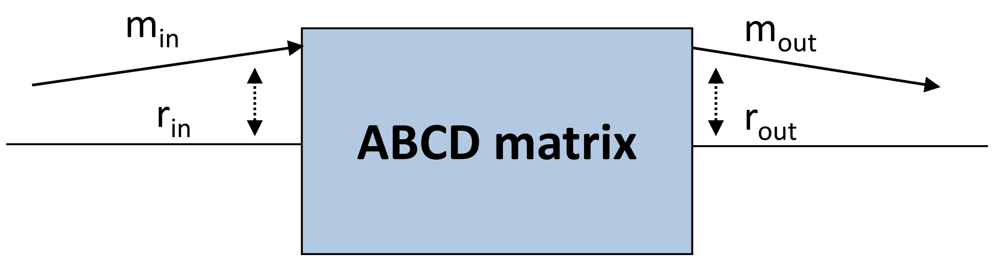
Propagation of ray through a lens:



$$\begin{pmatrix} r_{out} \\ m_{out} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\dfrac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} r_{in} \\ m_{in} \end{pmatrix}$$

ABCD Matrix Calculation

Lenses, mirrors, and other optical elements are described by ABCD matrices.

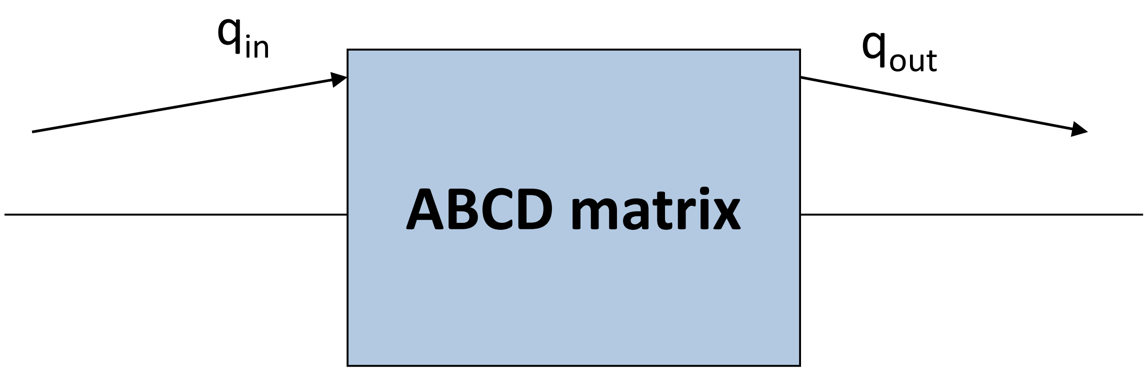


$$\begin{pmatrix} r_{out} \\ m_{out} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} r_{in} \\ m_{in} \end{pmatrix}$$

- Ray matrices: Propagation of diode pump light.
- Wave optics: Gaussian mode analysis of resonator.

ABCD Matrix Calculation

Propagation of Gaussian modes through lenses, mirrors, and other optical elements, described by ABCD matrices:



$$q_{out} = \frac{Aq_{in} + B}{Cq_{in} + D}$$

Gaussian mode TEM00:

$$E(x, y, z) = A \frac{1}{q+z} \exp\left(-ik \left(z + \frac{x^2 + y^2}{2(q+z)} \right) \right)$$

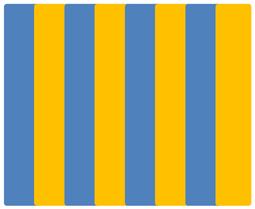
Transfer Matrix Method (TMM)

Consider the 1D scalar Helmholtz equation

$$\frac{\partial^2 E}{\partial x^2} + k(x)^2 E = 0$$

on a layers with piecewise constant wave number k_i.

<u>Example:</u> Periodic change of refraction index k_1, k_2:



Transfer Matrix Method (TMM)

On each layer the solution of scalar Helmholtz equation is:

$$E(x) = c_r \exp(-ikx) + c_l \exp(ikx)$$

On each layer this leads to the solutions:

$$E(x) = c_r^i \exp(-ik_i(x - l_{i-1})) + c_l^i \exp(ik_i(x - l_{i-1}))$$

where l_{i-1} is the location of the i-th layer

Transfer Matrix Method (TMM)

The transmission matrix of the i-th layer is:

$$\begin{pmatrix} c_r^{i+1} \\ c_l^{i+1} \end{pmatrix} = M_i \begin{pmatrix} c_r^{i} \\ c_l^{i} \end{pmatrix}$$

The total transmission matrix is: $M = \prod_{i=1}^{n} M_i$

$$\boxed{\begin{pmatrix} c_r^{n+1} \\ c_l^{n+1} \end{pmatrix} = M \begin{pmatrix} c_r^{1} \\ c_l^{1} \end{pmatrix}}$$

transmission matrix

$$\boxed{\begin{pmatrix} c_r^{n+1} \\ c_l^{1} \end{pmatrix} = S \begin{pmatrix} c_r^{1} \\ c_l^{n+1} \end{pmatrix}}$$

scattering matrix

Transfer Matrix Method (TMM)

$$\boxed{\begin{pmatrix} c_r^{n+1} \\ c_l^{n+1} \end{pmatrix} = M \begin{pmatrix} c_r^{1} \\ c_l^{1} \end{pmatrix}}$$

transmission matrix

$$\boxed{\begin{pmatrix} c_r^{n+1} \\ c_l^{1} \end{pmatrix} = S \begin{pmatrix} c_r^{1} \\ c_l^{n+1} \end{pmatrix}}$$

scattering matrix

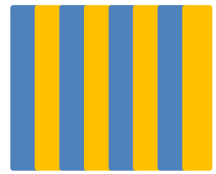
Transfer Matrix Method (TMM)

Using the continuity of E(x) and E'(x) one obtains:

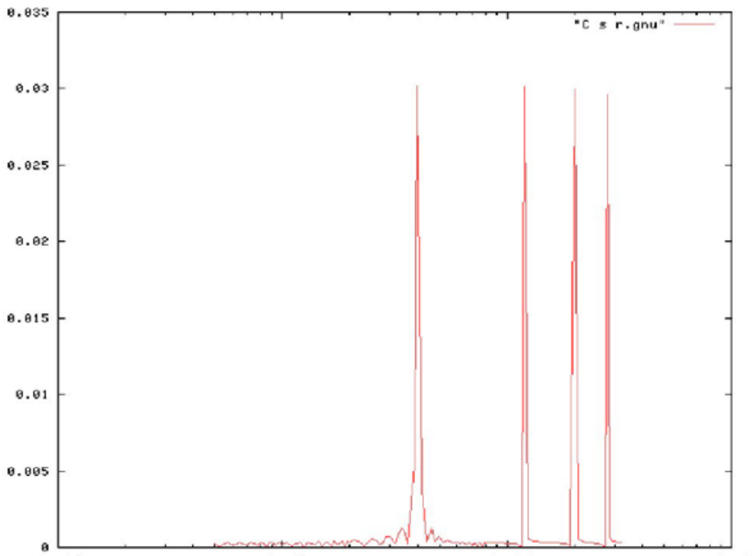
$$\begin{pmatrix} 1 & 1 \\ -k_{i+1} & k_{i+1} \end{pmatrix} \begin{pmatrix} c_r^{i+1} \\ c_l^{i+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -k_i & k_i \end{pmatrix} \begin{pmatrix} \exp(-ik_i h_i) & 0 \\ 0 & \exp(ik_i h_i) \end{pmatrix} \begin{pmatrix} c_r^i \\ c_l^i \end{pmatrix}$$

Transfer Matrix Method (TMM)

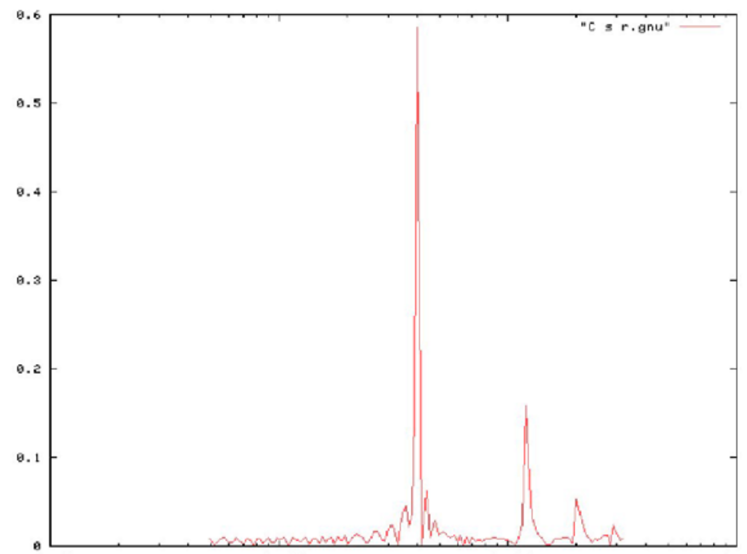
<u>Example:</u> Periodic change of refraction index k_1, k_2 on layers of size $\lambda/4$:



<u>Transmission:</u>



$n_1 = 3.275$



$n_1 = 3.220$

$n_0 = 3.277$