1)

a) We have $x' = f(t, x)$ where $f(t,x) = \frac{1}{8}(5 - x - 5025e^{-8t})$

$x(0) = 100 \quad t \in [0, 10]$

The Euler method is

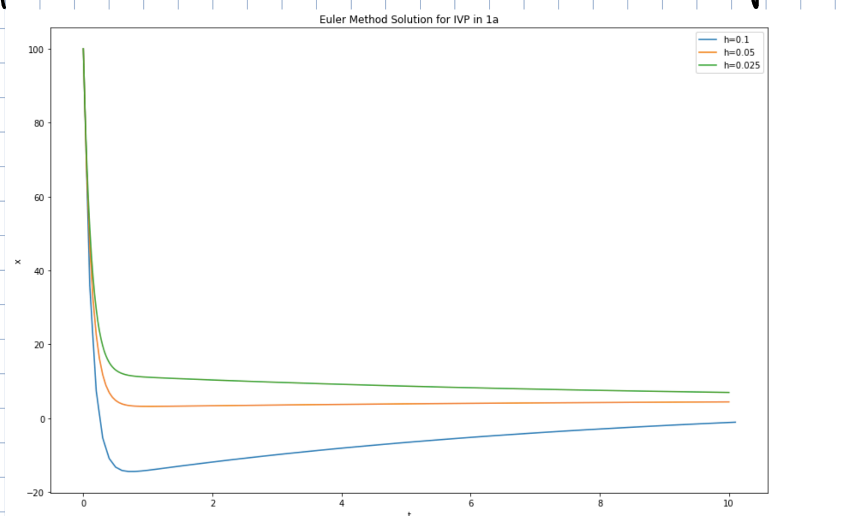
$$x_{n+1} = x_n + h f_n \quad \text{where } f_n = \frac{1}{8}(5 - x_n - 5025e^{-8t_n})$$

can also be written

$$x_{n+1} = x_n + \frac{h}{8}(5 - x_n - 5025e^{-8t_n}) \qquad x_0 = 100,\ t_0 = 0$$

$$t_{n+1} = t_n + h$$

b) We compute the solution and visualize for each h.



We notice that as $h \downarrow 0$ the solutions tend to get flatter after $t=1$.

c) To compute global error we first need to solve the IVP analytically.

$$\frac{dx}{dt} + \frac{x}{8} = \frac{1}{8}(-5025e^{-8t} + 5)$$

Let $\mu(t) = e^{\int \frac{1}{8} dt} = e^{\frac{t}{8}}$

Multiply both sides by $\mu(t)$

$$e^{\frac{t}{8}} \frac{dx}{dt} + \frac{1}{8} e^{\frac{t}{8}} \cdot x = -\frac{1}{8} e^{\frac{t}{8}} (5025e^{-8t} - 5)$$

$$e^{\frac{t}{8}}\frac{dx}{dt} + \frac{d}{dt}\left(e^{\frac{t}{8}}\right)x = -\frac{1}{8}e^{\frac{t}{8}}\left(5025e^{-8t} - 5\right)$$

Apply the reverse product rule to LHS

$$\frac{d}{dt}\left(e^{\frac{t}{8}}\cdot x\right) = -\frac{1}{8}e^{\frac{t}{8}}\left(5025e^{-8t} - 5\right)$$

$$\int \frac{d}{dt} e^{\frac{t}{8}}\cdot x \, dt = \int -\frac{1}{8}e^{\frac{t}{8}}\left(5025e^{-8t} - 5\right) dt$$

$$e^{\frac{t}{8}}\cdot x = \frac{5}{8}\left(8e^{t/8} + \frac{2680}{21}e^{\frac{-63t}{8}}\right) + C$$

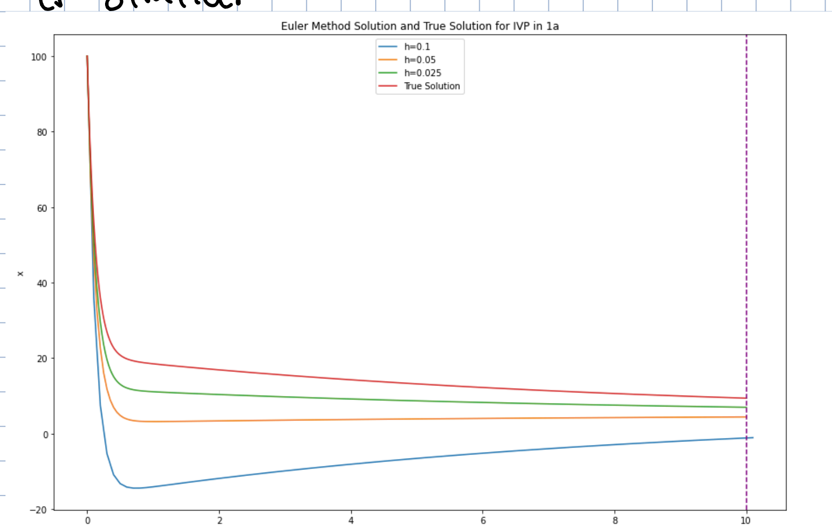
$$\Rightarrow \quad x = \frac{1675e^{-8t}}{21} + Ce^{-\frac{t}{8}} + 5$$

Plug in initial conditions:

$$C + \frac{1780}{21} = 100 \quad \Rightarrow \quad C = \frac{320}{21}$$

$$\Rightarrow \quad x = \frac{5}{21}\left(64e^{-\frac{t}{8}} + 335e^{-8t} + 21\right)$$

We visualize the true solution to verify the Euler method solution is similar.



We notice that the solutions converge to the true solution as $h \downarrow 0$ and the difference between solutions is quite clear since with Euler we have convergence of $O(h)$.

We can also compute the global error $|e_N| = |x(t_N) - x_n|$ at $t=10$ for the given h.

h	|e_N|
0.1	10.461190676644826
0.05	4.966152138155016
0.025	2.4066008156698206

We can see that as h gets smaller $|e_N|$ also decreases. This is a good verification of the implementation since we know the Euler method is convergent.

2)

a) In order to apply TS3 we convert the ODE to a first order system.

$$\begin{aligned} x &= u \\ x' &= v \end{aligned} \Rightarrow \begin{aligned} u' &= v \\ v' &= t^2 - 2u - 3v \end{aligned} \qquad \begin{aligned} x(0) &= 1 \\ x'(0) &= 0 \end{aligned} \Rightarrow \begin{aligned} u(0) &= 1 \\ v(0) &= 0 \end{aligned}$$

Now we can write the TS3 method for the IVP

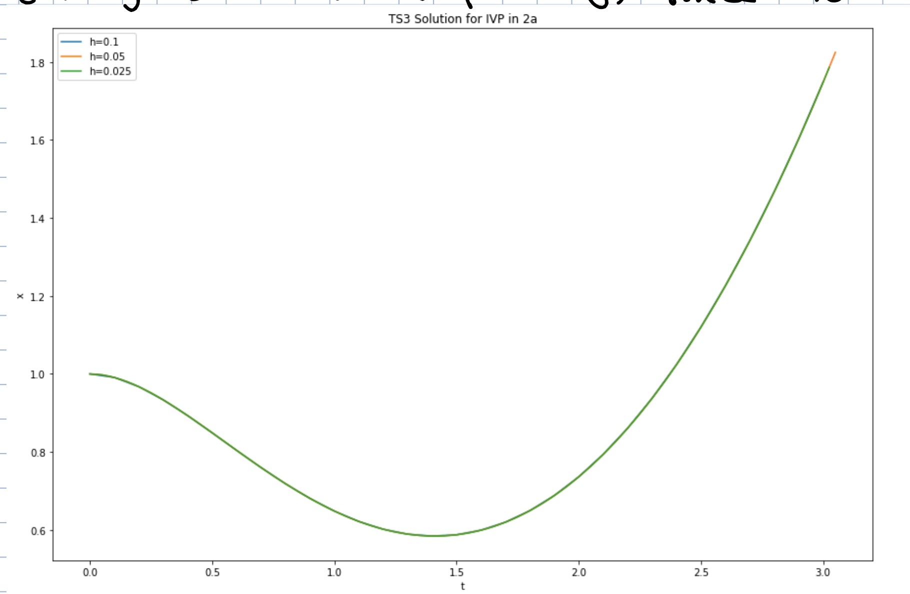
$$y_{n+1} = y_n + h f_n + \frac{h^2}{2!} f_n' + \frac{h^3}{3!} f_n'' \quad \text{where } y_n = \begin{pmatrix} u_n \\ v_n \end{pmatrix}, \; y_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$f_n = \begin{pmatrix} v_n \\ t_n^2 - 2u_n - 3v_n \end{pmatrix} \quad f_n' = \begin{pmatrix} v_n' \\ 2t_n - 2u_n' - 3v_n' \end{pmatrix} \quad f_n'' = \begin{pmatrix} v_n'' \\ 2 - 2u_n'' - 3v_n'' \end{pmatrix}$$

where $v_n'' = 2t_n - 2v_n - 3v_n'$ $\quad u_n'' = t_n^2 - 2u_n - 3v_n$

$v_n' = t_n^2 - 2u_n - 3v_n$ $\quad u_n' = v_n$

b) We compute the solution and visualize for each h, by just taking the u values from y, since $x = u$.



The solutions overlap heavily. This is since with TS3 we have order 3 convergence hence we can afford to have less time steps (and so h a little larger than Euler) for good convergence.

c) First we must solve the ODE analytically:

Complementary solution:

Assume solution $\propto e^{\lambda t}$

$$\frac{d^2}{dt}\left(e^{\lambda t}\right) + 3\frac{d}{dt}\left(e^{\lambda t}\right) + 2e^{\lambda t} = 0$$

$$(\lambda^2 + 3\lambda + 2)e^{\lambda t} = 0 \Rightarrow \lambda = -2, -1$$

$$x = c_1 e^{-2t} + c_2 e^{-t}$$

Particular solution:

Assume $x_p = a_1 + a_2 t + a_3 t^2$

Substitute in $\Rightarrow 2a_3 + 3(a_2 + 2a_3 t) + 2(a_1 + a_2 t + a_3 t^2) = t^2$

equating coefficients $\Rightarrow a_1 = \frac{7}{4},\ a_2 = -\frac{3}{2},\ a_3 = \frac{1}{2}$

$$\Rightarrow x = \frac{t^2}{2} - \frac{3t}{2} + c_1 e^{-2t} + c_2 e^{-t} + \frac{7}{4}$$

$$x' = t - 2c_1 e^{-2t} - c_2 e^{-t} - 3/2$$

Plug in initial conditions

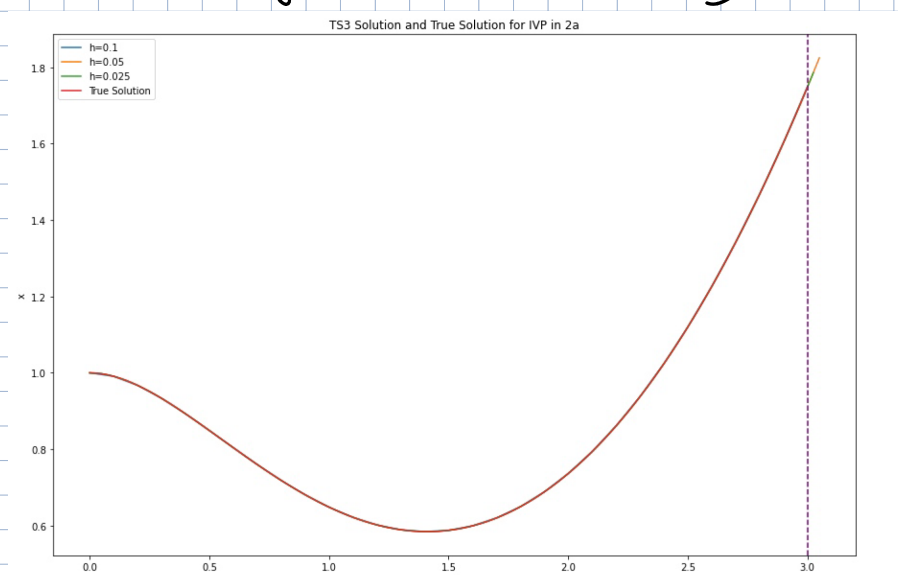
$$\Rightarrow \quad c_1 + c_2 + \frac{7}{4} = 1, \quad -2c_1 - c_2 - \frac{3}{2} = 0$$

$$\Rightarrow \quad c_1 = -\frac{3}{4} \qquad c_2 = 0$$

Hence the general solution is:

$$x = \frac{1}{4}\left(-3e^{-2t} + 2t^2 - 6t + 7\right).$$

We plot the true solution with the TS3 solution for the given h to verify TS3 solutions.



We can see that $h=0.1$ was already enough to have a near identical solution to the true one, again due to higher order (3) convergence of this method.

To find the number of time steps needed for $|e_N| < 10^{-3}$ at $t=3$ we try for $n = 1, \dots, 30$ and use the relation $h = 3/n$ at $t = 3$.

We obtain that for the following values of n that $|e_N| < 10^{-3}$.

N	h	|e_N|
10	0.3333333333333333	0.009459484208271585
13	0.25	0.004937837225500896
14	0.23076923076923078	0.004124079097300948
15	0.21428571428571427	0.003492448864515163
16	0.2	0.0029931336954396315
17	0.1875	0.002592072402723966
20	0.15789473684210525	0.001771310901792944
24	0.13043478260869565	0.0011648955745104939
25	0.125	0.0010617144985360927
26	0.12	0.0009715450402125736
27	0.11538461538461539	0.0008923029696439322
28	0.1111111111111111	0.0008223028525846665
29	0.10714285714285714	0.0007601703205200039
30	0.10344827586206896	0.0007047759619065452

We see that 10 time steps is sufficient enough for $|e_N| < 10^{-3}$ at $t = 3$. We would take the smallest n that satisfies the bound since this minimizes computational work. Of course, as n gets larger than all n will provide the bound, we see that after 24 steps all number of step methods give the bound. Overall 10 steps is relatively low which is due to higher order (3) convergence resulting in less computational work which is desirable.

4)

a) Consider $x' = f(t,x) \quad x(t_0) = \alpha, \quad t \in [t_0, t_N]$

We can integrate over $[t_n, t_{n+1}]$ giving

$$\int_{t_n}^{t_{n+1}} x' \, dt = \int_{t_n}^{t_{n+1}} f(t,x) \, dt$$

$$\Rightarrow x(t_{n+1}) = x(t_n) + \int_{t_n}^{t_{n+1}} f(t,x) \, dt$$

To proceed with the numerical solution we approximate $f(t,x)$ with the Lagrange polynomial through t_n, t_{n-1}, t_{n-2} (in order to have 3 steps)

$$\ell(t) = \frac{(t-t_n)(t-t_{n-1})}{(t_{n-2}-t_n)(t_{n-2}-t_{n-1})} f_{n-2} + \frac{(t-t_{n-2})(t-t_n)}{(t_{n-1}-t_{n-2})(t_{n-1}-t_n)} f_{n-1} + \frac{(t-t_{n-2})(t-t_{n-1})}{(t_n-t_{n-2})(t_n-t_{n-1})} f_n$$

$$\int_{t_n}^{t_{n+1}} f(t,x)\,dt \approx \int_{t_n}^{t_{n+1}} \alpha(t)\,dt$$

$$= \int_{t_n}^{t_{n+1}} \frac{(t-t_n)(t-t_{n-1})}{(t_{n-2}-t_n)(t_{n-2}-t_{n-1})} f_{n-2} + \frac{(t-t_{n-2})(t-t_n)}{(t_{n-1}-t_{n-2})(t_{n-1}-t_n)} f_{n-1} + \frac{(t-t_{n-2})(t-t_{n-1})}{(t_n-t_{n-2})(t_n-t_{n-1})} f_n \,dt$$

Let $u = \frac{t-t_n}{h}$ $\quad 0 \le u \le 1$ $\quad$ s.t $t = t_n + hu$ $\quad dt = h\,du$

$t-t_{n+1} = -h+hu$, $t-t_n = hu$, $t-t_{n-1} = h+hu$, $t-t_{n-2} = 2h+hu$

Substituting new variable $\Rightarrow$

$$\int_{t_n}^{t_{n+1}} \frac{(t-t_n)(t-t_{n-1})}{(t_{n-2}-t_n)(t_{n-2}-t_{n-1})} f_{n-2} + \frac{(t-t_{n-2})(t-t_n)}{(t_{n-1}-t_{n-2})(t_{n-1}-t_n)} f_{n-1} + \frac{(t-t_{n-2})(t-t_{n-1})}{(t_n-t_{n-2})(t_n-t_{n-1})} f_n \,dt$$

$$= f_{n-2}\cdot h\int_0^1 \tfrac{1}{2}u(u+1)\,du + f_{n-1}\cdot h\int_0^1 -u(u+2)\,du + f_n\cdot h\int_0^1 \tfrac{1}{2}(u+1)(u+2)\,du$$

$$= \frac{5}{12}hf_{n-2} - \frac{4}{3}hf_{n-1} + \frac{23}{12}hf_n$$

So we obtain the AB(3) method

$$x_{n+1} = x_n + \frac{h}{12}(5f_{n-2} - 16f_{n-1} + 23f_n)$$

b) To compute the LTE we first rewrite AB(3) as

$$x(t_n+h) = x(t_n) + \frac{h}{12}\left(23x'(t_n) - 16x'(t_n-h) + 5x'(t_n-2h)\right)$$

now we find the two taylor expansions in the above

$$x'(t_n-h) = x'(t_n) - hx''(t_n) + \frac{h^2}{2}x'''(t_n) - \frac{h^3}{6}x''''(t_n) + O(h^4)$$

$$x'(t_n-2h) = x'(t_n) - 2hx''(t_n) + 2h^2x'''(t_n) - \frac{4}{3}h^3x''''(t_n) + O(h^4)$$

Now we plug this into the AB(3) to give

$$x(t_n+h) = x(t_n) + \frac{h}{12}\Big(23x'(t_n) - 16\big(x'(t_n) - hx''(t_n) + \frac{h^2}{2}x'''(t_n) - \frac{h^3}{6}x''''(t_n)\big)$$
$$+ 5\big(x'(t_n) - 2hx''(t_n) + 2h^2x'''(t_n) - \frac{4}{3}h^3x''''(t_n)\big)\Big) + O(h^5)$$

$= x(t_n) + \frac{h}{12}\left(12x'(t_n) + 6hx''(t_n) + 2h^2x'''(t_n) - 4h^3x''''(t_n)\right) + O(h^5)$

$= x(t_n) + hx'(t_n) + \frac{h^2}{2}x''(t_n) + \frac{h^3}{3!}x'''(t_n) - \frac{h^4}{3}x''''(t_n) + O(h^5)$

$x(t+h) = x(t) + hx'(t) + \frac{h^2}{2}x''(t) + \frac{h^3}{3!}x'''(t) + \frac{h^4}{4!}x''''(t_n) + O(h^5)$

Hence $x(t+h) - x(t_n+h) = \frac{3h^4}{8}x''''(t_n) + O(h^5) = O(h^4)$

where we use the localizing assumption

$\Rightarrow$ LTE is of order 4.

c) Let $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Then we have $X' = f(X)$ where

$f(X) = \begin{pmatrix} \sigma(y-x) \\ x(\rho-z)-y \\ xy-\beta z \end{pmatrix}$ $\quad X(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

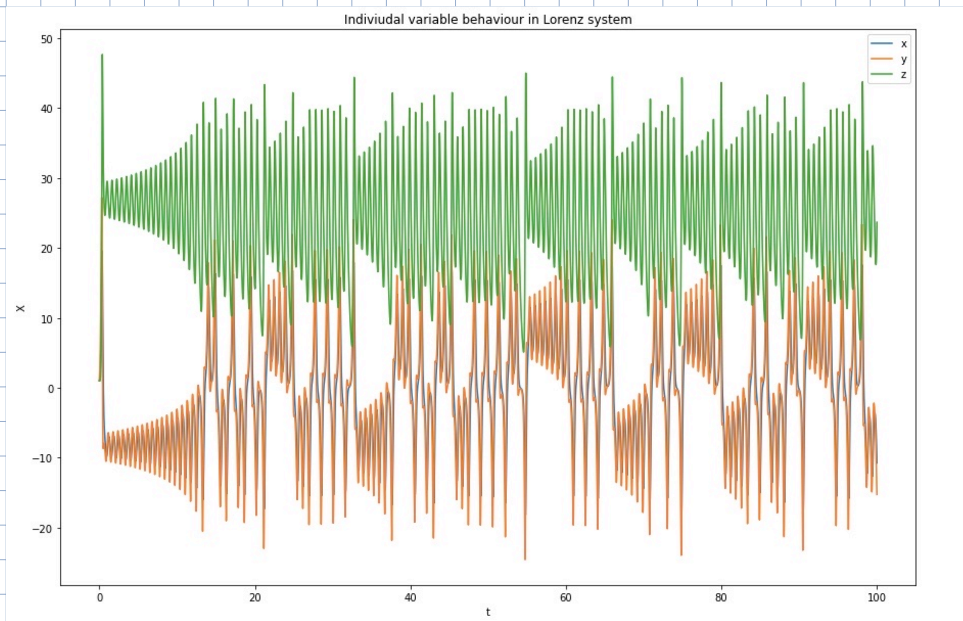
we apply AB(3) method to get

$X_{n+1} = X_n + \frac{h}{12}(23f_n - 16f_{n-1} + 5f_{n-2})$, $X_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $t_0 = 0$

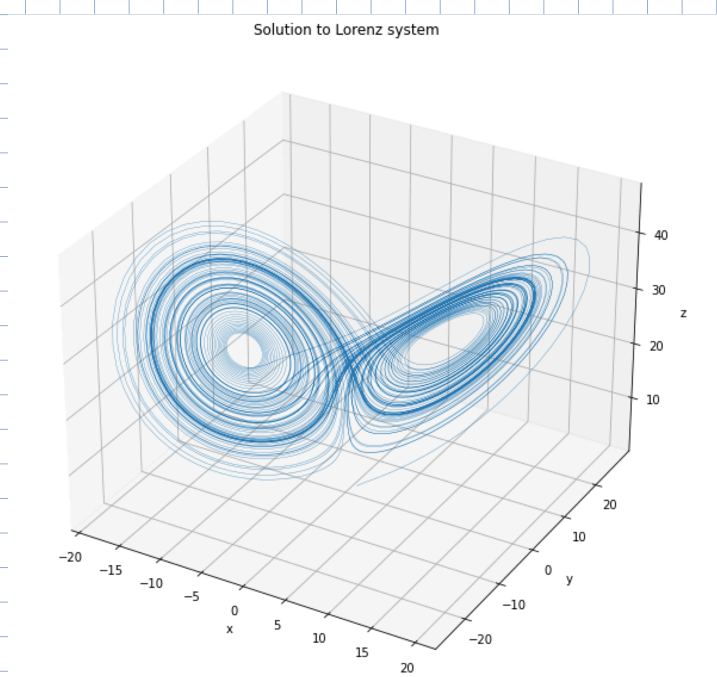
where $f_n = \begin{pmatrix} \sigma(y_n - x_n) \\ x_n(\rho - z_n) - y_n \\ x_n y_n - \beta z_n \end{pmatrix}$, $f_{n-1} = \begin{pmatrix} \sigma(y_{n-1} - x_{n-1}) \\ x_{n-1}(\rho - z_{n-1}) - y_{n-1} \\ x_{n-1} y_{n-1} - \beta z_{n-1} \end{pmatrix}$,

$f_{n-2} = \begin{pmatrix} \sigma(y_{n-2} - x_{n-2}) \\ x_{n-2}(\rho - z_{n-2}) - y_{n-2} \\ x_{n-2} y_{n-2} - \beta z_{n-2} \end{pmatrix}$, $\sigma = 10$, $\beta = \frac{8}{3}$, $\rho = 28$

d) We compute the AB(3) method solution with $h = 0.01$ and plot it for each individual variable.



We then plot in 3 dimensions and obtain the well known Lorenz attractor.



e) We start the method using the Euler method (used in the implementation in part d) as AB(3) is a 3 step method but we only have X_0 given ($X_0 = \begin{pmatrix}1\\1\\1\end{pmatrix}$). So we use Euler to obtain X_1 and X_2.

$$X_1 = X_0 + hf_0 = X_0 + h \cdot \begin{pmatrix} \sigma(y_0 - x_0) \\ z_0(\rho - z_0) - y_0 \\ x_0 y_0 - \beta z_0 \end{pmatrix}$$ where $\sigma = 10$, $\beta = \frac{8}{3}$, $\rho = 28$, $\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = X_0$

and then $$X_2 = X_1 + hf_1 = X_1 + h \cdot \begin{pmatrix} \sigma(y_1 - x_1) \\ z_1(\rho - z_1) - y_1 \\ x_1 y_1 - \beta z_1 \end{pmatrix}$$ where $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = X_1$

Then we are able to find

$$X_3 = X_2 + \frac{h}{12}(23f_2 - 16f_1 + 5f_0)$$

where $f_2 = \begin{pmatrix} \sigma(y_2 - x_2) \\ z_2(\rho - z_2) - y_2 \\ x_2 y_2 - \beta z_2 \end{pmatrix}$, $f_1 = \begin{pmatrix} \sigma(y_1 - x_1) \\ z_1(\rho - z_1) - y_1 \\ x_1 y_1 - \beta z_1 \end{pmatrix}$,

$f_0 = \begin{pmatrix} \sigma(y_0 - x_0) \\ z_0(\rho - z_0) - y_0 \\ x_0 y_0 - \beta z_0 \end{pmatrix}$ using the AB(3) method

and all X_n $n \geq 4$ we also find using AB(3)

$$X_{n+1} = X_n + \frac{h}{12}(23f_n - 16f_{n-1} + 5f_{n-2})$$

where $f_n = \begin{pmatrix} \sigma(y_n - x_n) \\ z_n(\rho - z_n) - y_n \\ x_n y_n - \beta z_n \end{pmatrix}$, $f_{n-1} = \begin{pmatrix} \sigma(y_{n-1} - x_{n-1}) \\ z_{n-1}(\rho - z_{n-1}) - y_{n-1} \\ x_{n-1} y_{n-1} - \beta z_{n-1} \end{pmatrix}$,

$f_{n-2} = \begin{pmatrix} \sigma(y_{n-2} - x_{n-2}) \\ z_{n-2}(\rho - z_{n-2}) - y_{n-2} \\ x_{n-2} y_{n-2} - \beta z_{n-2} \end{pmatrix}$, $\sigma = 10$, $\beta = \frac{8}{3}$, $\rho = 28$