Scientific Computation Project 4

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1.

In "Compressive Sensing", R. Baraniuk presents the concept of compressive sensing, a framework for signal acquisition and processing that is suitable for signals that are compressible in some domain. The paper argues that traditional methods of signal acquisition, which uniformly sample a signal in time or space, are inefficient for compressible signals because most of the samples are redundant and do not provide new information. The author proposes a new approach where the signal is acquired using a smaller number of non-uniform measurements.

The paper introduces the concept of sparsity as a measure of how many non-zero coefficients a signal has in a given basis. The author then introduces the concept of measurement vectors, which are linear projections of the signal onto a set of vectors forming the measurement matrix used to acquire the compressive measurements of the signal. The measurement process is not adaptive and the measurement matrix is fixed and does not depend on the signal.

We came across sparsity during our study of principal component analysis and recommender systems. We saw that PCA works by finding the principal components of the data, which are the directions of maximum variance in the data. These principal components can then be used as a new set of coordinates for the data, allowing for the original high-dimensional data to be represented using a smaller number of dimensions, similar to the compressive sensing measurement vectors.

To implement the measurement process, the measurement matrix is designed to satisfy the restricted isometry property (RIP) and the incoherence property. These properties are sufficient for stable solution for both K-sparse and compressible signals. However, checking all possibilities for the measurement matrix and verifying each one to have these properties is computationally expensive. Therefore, a random Gaussian matrix with mean 0 and variance $\frac{1}{N}$ is used for the measurement matrix, where N is the signal length. This formulation gives the measurement matrix to have incoherence with the identity basis of delta spikes with high probability, and hence the measurement process can recover K-sparse and compressible signals of length N from only $M \geq cKlog(\frac{N}{K})$ random Gaussian measurements, where M is the chosen number of measurements.

Finally, the signal is reconstructed from the measurement vector using an L1 norm optimization problem. The geometry of L2 and L0 norms is not suitable for reconstructing the signal. The L2 norm will not find the sparse solution, while the L0 norm is numerically unstable and NP-complete. However, the L1 norm allows the exact recovery of the signal with high probability if $M \geq cKlog(\frac{N}{K})$.

With PCA reconstruction we learned that the approximation will not be exact, but the amount of information lost can be controlled by adjusting the number of principal components used in the compression. This is different to the compressive sensing reconstruction in which we have no control over how much of the variance is explained of the original, rather having a high probability of exact reconstruction.

2.(a)

Let

$$s_i^+ = \begin{cases} 0 & s_i \leq 0 \\ |s_i| & s_i > 0 \end{cases}, s_i^- = \begin{cases} |s_i| & s_i \leq 0 \\ 0 & s_i > 0 \end{cases}$$

So we have $s = s^+ - s^-$. Say that we have a solution to optimisation problem (2) in the project document. Let $v = s^+ + s^-, u = s^+ - s^- \iff s^+ = \frac{v+u}{2}, s^- = \frac{v-u}{2}$. We see that the constraints match up from problem (2) to (1). $v_j - u_j \geq 0 \quad \forall j \iff s^- \geq 0 \quad \forall j$ and $v_j + u_j \geq 0 \quad \forall j \iff s^+ \geq 0 \quad \forall j$. Further, since $u = s$ we have $\Theta u = y \iff \Theta s = y$. Now we see why the minimization of $\sum v_i$ is equivalent to the minimization of $||s||_1$. $\sum_i v_i = \sum_i (s_i^+ + s_i^-) = \sum_i |s_i| = ||s||_1$. The second inequality here is true since s^+ and s^- partition s into positive and negative components and so will never both have a non zero value at the same index. So we have shown that we can solve (1) using the solution to (2).

3.(b)

The resulting plot shows that the error decreases as the number of measurements M increases. This is expected, as the more measurements we take, the better our approximation of the original image will be. We observe that after a certain M value the reconstruction error reduces to $< 10^{-6}$. This result is consistent with the explanation in the paper. Since we use the Gaussian measurement matrix, from the paper, we know Φ is incoherent with the basis $\Psi = I$ of delta spikes with high probability. Additionally, from the paper, we know that the M $\times$ N i.i.d. Gaussian matrix $\Theta = \Phi I = \Phi$ has the RIP with high probability if $M \geq cKlog(\frac{N}{K})$, with c a small constant. The RIP is a sufficient condition for a stable solution for both K-sparse and compressible signals. Therefore, Ksparse and compressible signals of length N can be recovered from only $M \geq cKlog(\frac{N}{K})$ random Gaussian measurements. We are able to find $K = 187$ in our specific data signal. Then we are able to estimate $c = \frac{M^*}{Klog(\frac{N}{K})}$ where M^* is the first M value that results a reconstruction error of $< 10^{-6}$. For seed $= 1$ we are able to find that $c = 1.234$ which is indeed a small constant as we expected. In the code for part3analyze() we allow an input of number of seeds $0, ..., \text{num\_seed}$ for which we output all the c values for each seed as well as the mean and standard deviation of the seeds. For the first 10 seeds we find the average $c = 1.221$ with standard deviation 0.0123. This gives mean $M^* = 415.6$. So we can expect a stable solution to the compressive sampling problem with only a little over 400 samples rather than $N = 1155$ which is a significant improvement.

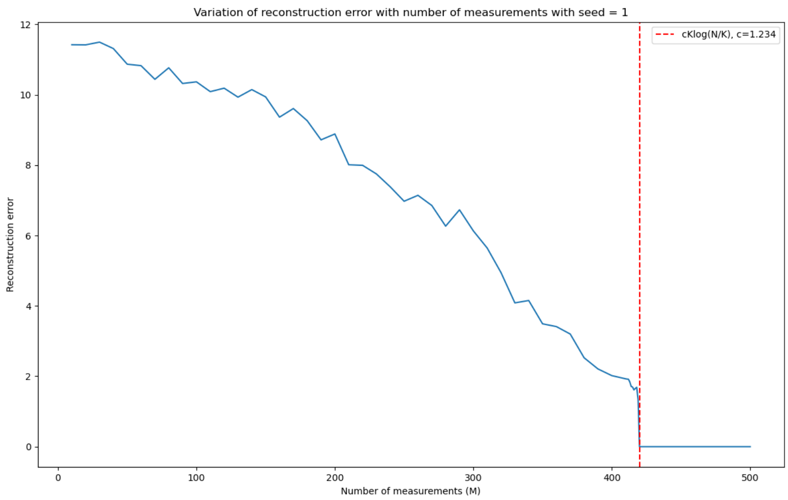

Figure 1: Variation of reconstruction error with number of measurements with seed = 1