Robust Performance Guarantees for System Level Synthesis

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Abstract

We generalize the system level synthesis framework to systems defined by bounded causal linear operators, and use this parameterization to make connections between robust system level synthesis and classical results from the robust control literature. In particular, by leveraging results from $\mathcal{L}_1$ robust control, we show that necessary and sufficient conditions for robust performance with respect to causal bounded linear uncertainty in the system dynamics can be translated into convex constraints on the system responses. We exploit this connection to show that these conditions naturally allow for the incorporation of delay, sparsity, and locality constraints on the system responses and resulting controller implementation, allowing these methods to be applied to large-scale distributed control problems – to the best of our knowledge, these are the first such robust performance guarantees for distributed control systems.

1 Introduction

TODO

2 Notation

We slightly adapt the notation used in [6]. We use Latin letters to denote vectors and matrices, e.g., $Ax = b$, and bold-face Latin letters to denote signals and operators, e.g., $\boldsymbol{x} = (x_t)_{t=0}^{\infty}$, and $\boldsymbol{y} = \boldsymbol{G}\boldsymbol{u}$. We let ℓ_∞ denote the space of all bounded sequences of real numbers, i.e., $\boldsymbol{x} = (x_t)_{t=0}^{\infty}$ if and only if $\sup_t |x_t| < \infty$, in which case we define $\|\boldsymbol{x}\|_{\ell_\infty} = \sup_t |x_t|$. Similarly, we let ℓ_∞^q denote the space of all q-tuples of elements of ℓ_∞: if $\boldsymbol{x} = (\boldsymbol{x}^1, \dots, \boldsymbol{x}^q) \in \ell_\infty^q$, then $\|\boldsymbol{x}\|_{\ell_\infty} = \max_{i=1,\dots,q} \left\|\boldsymbol{x}^i\right\|_{\ell_\infty}$. We also define the extended space $\ell_{\infty,e}^q$ which is equal to the space of all q-tuples of sequences of real numbers. We let $\boldsymbol{S}_+$ denote the right shift operator such tht if $\boldsymbol{x} = (x_t)_{t=0}^{\infty}$, then $\boldsymbol{S}_+\boldsymbol{x} = (0, x_0, x_1, \dots)$. Similarly, we let $\boldsymbol{S}_-$ denote the left shift operator such that $\boldsymbol{S}_-\boldsymbol{x} = (x_1, x_2, \dots)$. Hence $S_-S_+ = I$, but in general $S_+S_- \neq I$. We let $\mathcal{L}_{\text{TV}}^{p,q}$ be the space of all bounded linear causal operators mapping $\ell_\infty^q \to \ell_\infty^p$, and broadly refer to all such operators as ℓ_∞-stable. If $\boldsymbol{R} \in \mathcal{L}_{\text{TV}}^{p,q}$, then $\|\boldsymbol{R}\|_{\ell_\infty \to \ell_\infty} := \sup_{\|\boldsymbol{x}\|_{\ell_\infty} \leq 1} \|\boldsymbol{R}\boldsymbol{x}\|_{\ell_\infty}$, which is the induced operator norm. Note that each $\boldsymbol{R} \in \mathcal{L}_{\text{TV}}^{p,q}$ can be completely characterized by its block lower-triangular pulse response matrix. We denote by $\mathcal{L}_{\text{TI}}^{p,q}$ the subspace of $\mathcal{L}_{\text{TV}}^{p,q}$ consisting of time-invariant operators, and further denote by $\mathcal{RH}_\infty^{p,q}$ the subspace of $\mathcal{L}_{\text{TI}}^{p,q}$ consisting of all stable finite-dimensional systems. When the dimensions can be inferred from context, they will be omitted. Finally, we use z to denote the discrete-time z-transform variable: it follows that the restriction of $\boldsymbol{S}_+$ to $\mathcal{RH}_\infty$ is given by $\frac{1}{z}$, and similarly the restriction of $\boldsymbol{S}_-$ to $\mathcal{RH}_\infty$ is given by z.

3 Operator System Level Parameterization

Let $\boldsymbol{A} \in \mathcal{L}_{\text{TV}}^{n,n}$ and $\boldsymbol{B} \in \mathcal{L}_{\text{TV}}^{n,p}$, and consider the dynamical system mapping $\ell_{\infty,e}^{n} \times \ell_{\infty,e}^{p} \rightarrow \ell_{\infty,e}^{n}$ defined by

$$\boldsymbol{x} = \boldsymbol{S}_{+}\boldsymbol{A}\boldsymbol{x} + \boldsymbol{S}_{+}\boldsymbol{B}\boldsymbol{u} + \boldsymbol{S}_{+}\boldsymbol{w}. \tag{3.1}$$

As $\boldsymbol{S}_{+}\boldsymbol{A}$ is strictly causal, the dynamics (3.1) are well posed in the sense that $(I - \boldsymbol{S}_{+}\boldsymbol{A})^{-1}$ exists as an operator from $\ell_{\infty,e}^{n} \times \ell_{\infty,e}^{p} \rightarrow \ell_{\infty,e}^{n}$ [2]. We emphasize that although we impose that $\boldsymbol{A}$ be ℓ_{∞}-stable, this does not imply that the dynamics (3.1) are themselves open-loop stable. Rather, open-loop stability of the system is determined by the ℓ_{∞} stability of the operator $(I - \boldsymbol{S}_{+}\boldsymbol{A})^{-1}$.

Note that if $\boldsymbol{A}$ and $\boldsymbol{B}$ are memoryless and time invariant, i.e., if their matrix representations are block-diagonal $\boldsymbol{A} = \text{blkdiag}(A, A, \dots)$, $\boldsymbol{B} = \text{blkdiag}(B, B, \dots)$, then the system (3.1) reduce to the familiar finite-dimensional linear time-invariant system

$$x_{t+1} = Ax_t + Bu_t + w_t,\ x_0 = 0, \tag{3.2}$$

where once again stability of the closed loop system is determined by the stability of the operator $(zI - A)^{-1}$ as opposed to the boundedness of the matrix A.

For LTI systems (3.2), the System Level Synthesis (SLS) framework [11, 1] provides an appealing parameterization of all closed loop responses from $\boldsymbol{w} \rightarrow (\boldsymbol{x}, \boldsymbol{u})$ achievable by a causal state-feedback LTI control law $\boldsymbol{K}$ such that $\boldsymbol{u} = \boldsymbol{K}\boldsymbol{x}$, as summarized in the following theorem.

Theorem 3.1 (Theorem 1, [11])**.** *For the LTI system* (3.2) *with causal state-feedback LTI control law $\boldsymbol{u} = \boldsymbol{K}\boldsymbol{x}$, the follwing are true:*

1. *The affine subspace defined by*

$$\begin{bmatrix} zI - A & -B \end{bmatrix} \begin{bmatrix} \boldsymbol{\Phi}_x \\ \boldsymbol{\Phi}_u \end{bmatrix} = I \ \ \boldsymbol{\Phi}_x, \boldsymbol{\Phi}_u \in \frac{1}{z}\mathcal{RH}_{\infty} \tag{3.3}$$

parameterizes all system responses

$$\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{u} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Phi}_x \\ \boldsymbol{\Phi}_u \end{bmatrix} \boldsymbol{w} \tag{3.4}$$

achievable by an internally stabilizing state-feedback controller $\boldsymbol{K}$.

2. *For any transfer matrices $\{\boldsymbol{\Phi}_x, \boldsymbol{\Phi}_u\}$ satisfying the constraints* (3.3)*, the control signal computed via*[1]

$$\begin{aligned} \boldsymbol{u} &= z\boldsymbol{\Phi}_u\hat{\boldsymbol{w}} \\ \hat{\boldsymbol{w}} &= \boldsymbol{x} - \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{x}} &= (z\boldsymbol{\Phi}_x - I)\hat{\boldsymbol{w}} \end{aligned} \tag{3.5}$$

defines the control law $\boldsymbol{u} = \boldsymbol{\Phi}_u\boldsymbol{\Phi}_x^{-1}\boldsymbol{x}$, is internally stabilizing, and achieves the desired response (3.4)*.*

[1]We note that due to the affine constraints (3.3), $z\boldsymbol{\Phi}_x - I$ is strictly causal, and thus feedback loop between $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{w}}$ is well posed.

Thus, in the case of LTI systems (3.2), the search for an optimal controller $\boldsymbol{K}$ can be converted to a search over system responses $\{\boldsymbol{\Phi}_x, \boldsymbol{\Phi}_u\}$ constrained to lie in the affine space defined by equation (3.3). This fact, combined with the transparent mapping between the system responses and the corresponding controller implementation (3.5), has been successfully exploited for the synthesis of distributed optimal controllers for large-scale systems by introducing additional structural constraints on the system responses, and corresponding controller implementation (3.5), through additional subspace constraints – we refer the reader to [9, 8, 10] for more details.

Another favorable feature of the parameterization defined in Theorem 3.1 is that it is provably stable under deviations from the subspace (3.3), as summarized in the following theorem from [7].

Theorem 3.2 (Theorem 2, [7])**.** *Let* $(\hat{\boldsymbol{\Phi}}_x, \hat{\boldsymbol{\Phi}}_u, \boldsymbol{\Delta})$ *be a solution to*

$$\begin{bmatrix} zI - A & -B \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{\Phi}}_x \\ \hat{\boldsymbol{\Phi}}_u \end{bmatrix} = I - \boldsymbol{\Delta} \;\; \hat{\boldsymbol{\Phi}}_x, \hat{\boldsymbol{\Phi}}_u \in \frac{1}{z}\mathcal{RH}_\infty. \tag{3.6}$$

If $(I - \boldsymbol{\Delta})^{-1}$ *exists as an operator from* $\ell^n_{\infty,e} \to \ell^n_{\infty,e}$*, then the controller implementation* (3.5) *defined in terms of the transfer matrices* $\left\{\hat{\boldsymbol{\Phi}}_x, \hat{\boldsymbol{\Phi}}_u\right\}$ *achieves the closed loop responses*

$$\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{u} \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{\Phi}}_x \\ \hat{\boldsymbol{\Phi}}_u \end{bmatrix} (I - \boldsymbol{\Delta})^{-1} \boldsymbol{w}, \tag{3.7}$$

on the LTI system (3.2)*, and is internally stabilizing if and only if* $(I - \boldsymbol{\Delta})^{-1} \in \mathcal{RH}_\infty$*.*

This parameterization has proved crucial in providing tractable approximations to non-convex distributed control problems [7], and in providing sub-optimality bounds for robust controllers as applied to learned systems [3, 5]. However, in these past works, very crude approximations based solely on small gain bounds and triangle inequalities were used to control the effects of the uncertain map $(I - \boldsymbol{\Delta})^{-1}$ on system stability and performance. In this work we show that Theorems 3.1 and 3.2 can be extended to a more general setting that subsequently allows us to make connections to well developed theory in the robust control literature [6, 2]. Although we focus on $\mathcal{L}_1$ optimal control in this paper due to its favorable separability structure (cf. §4.1), we expect these results to carry over naturally to the $\mathcal{H}_\infty$ setting.

3.1 Necessary Conditions

Here we characterize a set of affine constraints that the closed loop system responses of system (3.1) must satisfy if they are induced by a linear and causal controller $\boldsymbol{K} : \ell^n_{\infty,e} \to \ell^p_{\infty,e}$ via the control law $\boldsymbol{u} = \boldsymbol{K}\boldsymbol{x}$.

Proposition 3.3. *Let* $\boldsymbol{K} : \ell^n_{\infty,e} \to \ell^p_{\infty,e}$ *be a linear and causal map such that* $(I - \boldsymbol{S}_+(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}))^{-1}\boldsymbol{S}_+ \in \mathcal{L}^{n,n}_{\mathrm{TV}}$ *and* $\boldsymbol{K}(I - \boldsymbol{S}_+(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}))^{-1}\boldsymbol{S}_+ \in \mathcal{L}^{n,p}_{\mathrm{TV}}$*, i.e., let* $\boldsymbol{K}$ *be a linear causal and stabilizing controller such that closed loop maps from* $\boldsymbol{w} \to (\boldsymbol{x}, \boldsymbol{u})$ *are* ℓ_∞*-stable. Then all maps taking* $\boldsymbol{w} \to (\boldsymbol{x}, \boldsymbol{u})$ *achievable by such a* $\boldsymbol{K}$ *satisfy the constraints*

$$\begin{bmatrix} I - \boldsymbol{S}_+\boldsymbol{A} & -\boldsymbol{S}_+\boldsymbol{B} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Phi}_x \\ \boldsymbol{\Phi}_u \end{bmatrix} = \boldsymbol{S}_+, \;\; \boldsymbol{\Phi}_x, \boldsymbol{\Phi}_u \textit{ strictly causal, linear, and } \ell_\infty\textit{-stable.} \tag{3.8}$$

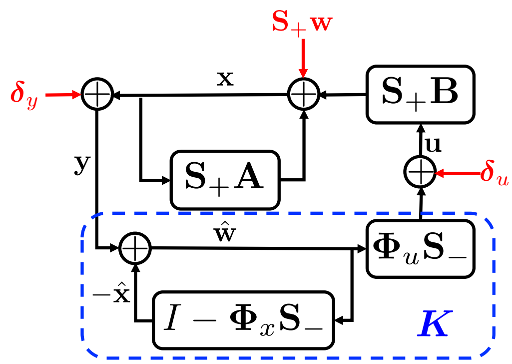

Figure 1: The proposed state-feedback controller structure defined by equations (3.10).

Proof. As $\boldsymbol{S}_+(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K})$ is strictly causal, the feedback interconnection is well posed in the sense that $(I-\boldsymbol{S}_+(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}))^{-1}$ exists as a map from $\ell_{\infty,e}^n \to \ell_{\infty,e}^n$. By assumption, both $(I-\boldsymbol{S}_+(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}))^{-1}$ and $\boldsymbol{K}(I-\boldsymbol{S}_+(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}))^{-1}$ are ℓ_∞-stable. Defining

$$\begin{bmatrix}\boldsymbol{\Phi}_x\\ \boldsymbol{\Phi}_u\end{bmatrix} = \begin{bmatrix} I \\ \boldsymbol{K}\end{bmatrix}(I-\boldsymbol{S}_+(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}))^{-1},$$

it is easily verified that $\{\boldsymbol{\Phi}_x, \boldsymbol{\Phi}_u\}$ satisfy constraint (3.8). $\square$

Remark 1. *If $\boldsymbol{A}$ and $\boldsymbol{B}$ are memoryless and LTI, and $\boldsymbol{K}$ is LTI, then constraint* (3.8) *simplifies to*

$$\begin{bmatrix} I-\frac{1}{z}A & -\frac{1}{z}B\end{bmatrix}\begin{bmatrix}\boldsymbol{\Phi}_x\\ \boldsymbol{\Phi}_u\end{bmatrix} = \frac{1}{z}I \quad \boldsymbol{\Phi}_x, \boldsymbol{\Phi}_u \in \frac{1}{z}\mathcal{RH}_\infty, \tag{3.9}$$

which is clearly equivalent to (3.3).

3.2 Controller Implementation

We now show how to construct an internally stabilizing controller from any operators $\{\boldsymbol{\Phi}_x, \boldsymbol{\Phi}_u\}$ satisfying constraint (3.8) that achieves the desired response from $\boldsymbol{w} \to (\boldsymbol{x}, \boldsymbol{u})$.

Proposition 3.4. *Let $\{\boldsymbol{\Phi}_x, \boldsymbol{\Phi}_u\}$ satisfy constraint* (3.8). *Then the controller implementation shown in Fig. 1, described by the equations*

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{\Phi}_x\boldsymbol{S}_-\hat{\boldsymbol{w}} \\ \hat{\boldsymbol{w}} &= \boldsymbol{x}-\hat{\boldsymbol{x}},\ \hat{w}_0 = 0, \\ \hat{\boldsymbol{x}} &= (\boldsymbol{\Phi}_x\boldsymbol{S}_- - I)\hat{\boldsymbol{w}}, \end{aligned} \tag{3.10}$$

is well-posed, and the resulting closed loop system is ℓ_∞-stable as a map from $(\boldsymbol{w}, \boldsymbol{\delta}_y, \boldsymbol{\delta}_u) \to (\boldsymbol{x}, \boldsymbol{u}, \hat{\boldsymbol{w}})$ and achieves the desired closed loop response

$$\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{u}\end{bmatrix} = \begin{bmatrix}\boldsymbol{\Phi}_x\\ \boldsymbol{\Phi}_u\end{bmatrix}\boldsymbol{w}. \tag{3.11}$$

Proof. From equations (3.1) and (3.10) (alternatively, from Fig. 1), we have that

$$\begin{aligned} \boldsymbol{x} &= \boldsymbol{S}_+\boldsymbol{A}\boldsymbol{x} + \boldsymbol{S}_+\boldsymbol{B}\boldsymbol{u} + \boldsymbol{S}_+\boldsymbol{u} \\ \boldsymbol{u} &= \boldsymbol{\Phi}_u\boldsymbol{S}_-\hat{\boldsymbol{w}} + \boldsymbol{\delta}_u \\ \hat{\boldsymbol{w}} &= \boldsymbol{x} + \boldsymbol{\delta}_y + (I-\boldsymbol{\Phi}_x\boldsymbol{S}_-)\hat{\boldsymbol{w}}, \end{aligned} \tag{3.12}$$

with $\hat{w}_0 = 0$, i.e., $\hat{\boldsymbol{w}}$ is a strictly causal signal. We first observe that the restriction of $(I - \boldsymbol{\Phi}_x \boldsymbol{S}_-)$ to strictly causal signals is itself strictly causal, as the constraint (3.8) enforces that the block lower triangular matrix representation of $\boldsymbol{\Phi}_x$ has identify matrices along its first block sub-diagonal, i.e., for $\boldsymbol{\Phi}_x = (\Phi_x(i,j))_{i,j=0}^{\infty}$ the block lower triangular matrix representation of $\boldsymbol{\Phi}_x$, we have that $\Phi_x(i, i-1) = I$ for all $i \geq 1$. It therefore follows that the feedback loop between $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{w}}$ is well posed. By rote calculation, tt follows from equation (3.12) that the closed loop maps from $(\boldsymbol{w}, \boldsymbol{\delta}_y, \boldsymbol{\delta}_u) \to (\boldsymbol{x}, \boldsymbol{u}, \hat{\boldsymbol{w}})$ are given by

$$\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{u} \\ \hat{\boldsymbol{w}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Phi}_x & \boldsymbol{\Phi}_x(\boldsymbol{S}_- - \boldsymbol{A}) & \boldsymbol{\Phi}_x \boldsymbol{B} \\ \boldsymbol{\Phi}_u & \boldsymbol{\Phi}_u(\boldsymbol{S}_- - \boldsymbol{A}) & I + \boldsymbol{\Phi}_u \boldsymbol{B} \\ \boldsymbol{S}_+ & I - \boldsymbol{S}_+ \boldsymbol{A} & \boldsymbol{S}_+ \boldsymbol{B} \end{bmatrix} \begin{bmatrix} \boldsymbol{w} \\ \boldsymbol{\delta}_y \\ \boldsymbol{\delta}_u \end{bmatrix}. \tag{3.13}$$

By assumption, $\boldsymbol{\Phi}_x$, $\boldsymbol{\Phi}_u$, $\boldsymbol{A}$, and $\boldsymbol{B}$ are all ℓ_∞-stable, and hence the interconnection illustrated in Fig. 1, and described by equations (3.1) and (3.12) is well-posed and ℓ_∞-stable. ☐

Remark 2. *If $\boldsymbol{A}$ and $\boldsymbol{B}$ are memoryless and LTI, and $\boldsymbol{K}$ is LTI, then so are the system responses $\{\boldsymbol{\Phi}_x, \boldsymbol{\Phi}_u\}$, and consequently the right and left shift operators $\boldsymbol{S}_+$ and $\boldsymbol{S}_-$ become $\frac{1}{z}$ and z, respectively, recovering the controller implementation* (3.5)*.*

3.3 Robust Operator System Level Synthesis

Thus Propositions 3.3 and 3.4 show that the parameterization of Theorem 3.1 can be extended to a class of dynamics described by bounded and causal linear operators in feedback with a causal linear controller. We now show that this extension enjoys similar stability properties with respect to perturbations from the subspace (3.8).

Theorem 3.5. *Let $\boldsymbol{A} \in \mathcal{L}_{\mathrm{TV}}^{n,n}$ and $\boldsymbol{B} \in \mathcal{L}_{\mathrm{TV}}^{n,p}$, and suppose that $\left\{\hat{\boldsymbol{\Phi}}_x, \hat{\boldsymbol{\Phi}}_u\right\}$ satisfy*

$$\begin{bmatrix} I - \boldsymbol{S}_+ \boldsymbol{A} & -\boldsymbol{S}_+ \boldsymbol{B} \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{\Phi}}_x \\ \hat{\boldsymbol{\Phi}}_u \end{bmatrix} = \boldsymbol{S}_+(I - \boldsymbol{\Delta}),\ \boldsymbol{\Phi}_x, \boldsymbol{\Phi}_u \text{ strictly causal, linear, and } \ell_\infty\text{-stable,} \tag{3.14}$$

for $\boldsymbol{\Delta}$ a strictly causal linear operator from $\ell_{\infty,e}^n \to \ell_{\infty,e}^n$. Then the controller implementation (3.10) *defined in terms of the operators $\left\{\hat{\boldsymbol{\Phi}}_x, \hat{\boldsymbol{\Phi}}_u\right\}$ is well posed and achieves the following response*

$$\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{u} \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{\Phi}}_x \\ \hat{\boldsymbol{\Phi}}_u \end{bmatrix} (I - \boldsymbol{\Delta})^{-1} \boldsymbol{u}. \tag{3.15}$$

Further, this interconnection is ℓ_∞-stable if and only if $(I - \boldsymbol{\Delta})^{-1}$ is ℓ_∞-stable.

Proof. As $\boldsymbol{\Delta}$ is strictly causal by assumption, $(I - \boldsymbol{\Delta})^{-1}$ eists as a map from $\ell_{\infty,e}^n \to \ell_{\infty,e}^n$. Going through a similar argument as that in the proof of Proposition 3.4, we observe that

$$\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{u} \\ \hat{\boldsymbol{w}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Phi}_x I_{\boldsymbol{\Delta}} & \boldsymbol{\Phi}_x I_{\boldsymbol{\Delta}}(\boldsymbol{S}_- - \boldsymbol{A}) & \boldsymbol{\Phi}_x I_{\boldsymbol{\Delta}} \boldsymbol{B} \\ \boldsymbol{\Phi}_u I_{\boldsymbol{\Delta}} & \boldsymbol{\Phi}_u I_{\boldsymbol{\Delta}}(\boldsymbol{S}_- - \boldsymbol{A}) & I + \boldsymbol{\Phi}_u I_{\boldsymbol{\Delta}} \boldsymbol{B} \\ \boldsymbol{S}_+ I_{\boldsymbol{\Delta}} & I_{\boldsymbol{\Delta}}(I - \boldsymbol{S}_+ \boldsymbol{A}) & I_{\boldsymbol{\Delta}} \boldsymbol{S}_+ \boldsymbol{B} \end{bmatrix} \begin{bmatrix} \boldsymbol{w} \\ \boldsymbol{\delta}_y \\ \boldsymbol{\delta}_u \end{bmatrix}, \tag{3.16}$$

where we let $I_{\boldsymbol{\Delta}} := (I - \boldsymbol{\Delta})^{-1}$. Thus we see that the desired map (3.15) from $\boldsymbol{u} \to (\boldsymbol{x}, \boldsymbol{u})$ is achived. Further, as $\boldsymbol{\Phi}_x$, $\boldsymbol{\Phi}_u$, $\boldsymbol{A}$, $\boldsymbol{B}$ are all ℓ_∞-stable by assumption, it follows that the ℓ_∞-stability of the map from $(\boldsymbol{w}, \boldsymbol{\delta}_y, \boldsymbol{\delta}_u) \to (\boldsymbol{x}, \boldsymbol{u}, \hat{\boldsymbol{w}})$ is determined by the ℓ_∞-stability of $I_{\boldsymbol{\Delta}}$, from which the result follows. ☐

4 Robust Performance under Model Uncertainty

We now use the tools developed in the previous section to identify necessary and sufficient conditions for the robust stability and robust performance of a system (3.1) subject to bounded perturbations in its $\boldsymbol{A}$ and $\boldsymbol{B}$ operators. In particular consider the system

$$\boldsymbol{x} = \boldsymbol{S}_+(\boldsymbol{A}_0 + \boldsymbol{\Delta_A})\boldsymbol{x} + \boldsymbol{S}_+(\boldsymbol{B}_0 + \boldsymbol{\Delta_B})\boldsymbol{u} + \boldsymbol{S}_+\boldsymbol{u}, \tag{4.1}$$

where $\boldsymbol{A}_0 = \text{blkdiag}(\hat{A}, \hat{A}, \dots)$ and $\boldsymbol{B}_0 = \text{blkdiag}(\hat{B}, \hat{B}, \dots)$ are memoryless LTI operators defining a nominal LTI system $x_{t+1} = \hat{A}x_t + \hat{B}u_t + w_t$, and $\boldsymbol{\Delta_A}$ and $\boldsymbol{\Delta_B}$ are ℓ_∞-stable and satisfy

$$\|[\boldsymbol{\Delta_A},\, \boldsymbol{\Delta_B}]\|_{\ell_\infty \to \ell_\infty} \leq \varepsilon. \tag{4.2}$$

We first identify SLS based necessary and sufficient conditions for robust stability, and then build upon those to formulate a robust performance problem.

We consider the following robust control problem: find a LTI controller $\boldsymbol{K} : \ell^n_{\infty,e} \to \ell^p_{\infty,e}$, using only the nominal dynamics $(\hat{A}, \hat{B})$ and uncertainty bound ε, such that the dynamics (4.1) in closed loop with the control law $\boldsymbol{u} = \boldsymbol{K}\boldsymbol{x}$ is ℓ_∞-stable for all admissible uncertainty realizations (4.2). To do so, we recognize that for any LTI $\{\hat{\boldsymbol{\Phi}}_x, \hat{\boldsymbol{\Phi}}_u\}$ satisfying the LTI achievability constraints (3.3), we then have that

$$\begin{aligned}
&\begin{bmatrix} I - \boldsymbol{S}_+\boldsymbol{A}_0 - \boldsymbol{S}_+\boldsymbol{\Delta_A} & -\boldsymbol{S}_+\boldsymbol{B}_0 - \boldsymbol{S}_+\boldsymbol{\Delta_B} \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{\Phi}}_x \\ \hat{\boldsymbol{\Phi}}_u \end{bmatrix} \\
&\qquad = \begin{bmatrix} I - \boldsymbol{S}_+\boldsymbol{A}_0 & -\boldsymbol{S}_+\boldsymbol{B}_0 \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{\Phi}}_x \\ \hat{\boldsymbol{\Phi}}_u \end{bmatrix} - \begin{bmatrix} \boldsymbol{S}_+\boldsymbol{\Delta_A} & \boldsymbol{S}_+\boldsymbol{\Delta_B} \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{\Phi}}_x \\ \hat{\boldsymbol{\Phi}}_u \end{bmatrix} \\
&\qquad = \boldsymbol{S}_+ \left(I - \boldsymbol{S}_+ \begin{bmatrix} \boldsymbol{\Delta_A} & \boldsymbol{\Delta_B} \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{\Phi}}_x \\ \hat{\boldsymbol{\Phi}}_u \end{bmatrix} \right)
\end{aligned} \tag{4.3}$$

where the final inequality follows from the assumption that $\{\hat{\boldsymbol{\Phi}}_x, \hat{\boldsymbol{\Phi}}_u\}$ satisfy the LTI achievability constraints (3.3), or equivalently (3.8), defined in terms of the dynamics $(\hat{A}, \hat{B})$. Noting that

$$\hat{\boldsymbol{\Delta}} := \boldsymbol{S}_+ \begin{bmatrix} \boldsymbol{\Delta_A} & \boldsymbol{\Delta_B} \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{\Phi}}_x \\ \hat{\boldsymbol{\Phi}}_u \end{bmatrix}, \tag{4.4}$$

is a strictly causal ℓ_∞-stable operator, we conclude by the robust parameterization of Theorem 3.5 that the controller implementation (3.10) defined in terms of the LTI operators $\{\hat{\boldsymbol{\Phi}}_x, \hat{\boldsymbol{\Phi}}_u\}$ achieves the following closed loop behavior when applied to the uncertain dynamics (4.1):

$$\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{u} \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{\Phi}}_x \\ \hat{\boldsymbol{\Phi}}_u \end{bmatrix} (I + \hat{\boldsymbol{\Delta}})^{-1}. \tag{4.5}$$

Further this control law is internally stabilizing if and only if $(I + \hat{\boldsymbol{\Delta}})^{-1}$ is ℓ_∞-stable. Defining the controlled output signal as

$$\boldsymbol{z} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u}, \tag{4.6}$$

for $\boldsymbol{C} = \text{blkdiag}(C, C, \dots)$ and $\boldsymbol{D} = \text{blkdiag}(D, D, \dots)$, user specified cost matrices, and consider the goal of minimizing the $\ell_\infty \to \ell_\infty$ induced gain from $\boldsymbol{w} \to \boldsymbol{z}$ of the uncertain system (4.1). We can then pose the robust performance problem for a specified performance level $\gamma \geq 0$ as finding LTI operators $\{\hat{\mathbf{\Phi}}_x, \hat{\mathbf{\Phi}}_u\}$ that satisfy

$$\begin{aligned}
&\left\| \begin{bmatrix} \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{\Phi}}_x \\ \hat{\mathbf{\Phi}}_u \end{bmatrix} \left(I - \boldsymbol{S}_+ \begin{bmatrix} \mathbf{\Delta}_{\boldsymbol{A}} & \mathbf{\Delta}_{\boldsymbol{B}} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{\Phi}}_x \\ \hat{\mathbf{\Phi}}_u \end{bmatrix} \right)^{-1} \right\|_{\ell_\infty \to \ell_\infty} \leq \gamma \\
&\begin{bmatrix} zI - \hat{A} - \hat{B} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{\Phi}}_x \\ \hat{\mathbf{\Phi}}_u \end{bmatrix} = I, \ \hat{\mathbf{\Phi}}_x, \hat{\mathbf{\Phi}}_u \in \frac{1}{z}\mathcal{RH}_\infty, \\
&\left(I - \boldsymbol{S}_+ \begin{bmatrix} \mathbf{\Delta}_{\boldsymbol{A}} & \mathbf{\Delta}_{\boldsymbol{B}} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{\Phi}}_x \\ \hat{\mathbf{\Phi}}_u \end{bmatrix} \right)^{-1} \text{ is } \ell_\infty\text{-stable}
\end{aligned} \tag{4.7}$$

for all $(\mathbf{\Delta}_{\boldsymbol{A}}, \mathbf{\Delta}_{\boldsymbol{B}})$ satisfying bound (4.2).

To lighten notation going forward, we let

$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}, \ \hat{\mathbf{\Phi}} = \begin{bmatrix} \hat{\mathbf{\Phi}}_x \\ \hat{\mathbf{\Phi}}_u \end{bmatrix}, \ \mathbf{\Delta} = \frac{1}{\varepsilon}\boldsymbol{S}_+ \begin{bmatrix} \mathbf{\Delta}_{\boldsymbol{A}} & \mathbf{\Delta}_{\boldsymbol{B}} \end{bmatrix}, \ Z_{AB} = \begin{bmatrix} zI - \hat{A} & -\hat{B} \end{bmatrix}. \tag{4.8}$$

With this notation, the robust performance problem (4.7) is equivalent to finding an LTI operator $\hat{\mathbf{\Phi}}$ satisfying

$$\begin{aligned}
&\left\| \frac{1}{\gamma}\boldsymbol{Q}\hat{\mathbf{\Phi}} + \frac{1}{\gamma}\boldsymbol{Q}\hat{\mathbf{\Phi}}\mathbf{\Delta}(I - (\varepsilon\hat{\mathbf{\Phi}})\mathbf{\Delta})^{-1}(\varepsilon\hat{\mathbf{\Phi}}) \right\|_{\ell_\infty \to \ell_\infty} \leq 1 \\
&Z_{AB}\hat{\mathbf{\Phi}} = I, \ \hat{\mathbf{\Phi}} \in \frac{1}{z}\mathcal{RH}_\infty, \ (I - \hat{\mathbf{\Phi}}\mathbf{\Delta})^{-1} \text{ is } \ell_\infty\text{-stable}
\end{aligned} \tag{4.9}$$

for all $\mathbf{\Delta}$ satisfying $\|\mathbf{\Delta}\|_{\ell_\infty \to \ell_\infty} \leq 1$, where we have used that $(I - \Delta\hat{\mathbf{\Phi}})^{-1} = I + \Delta(I - \hat{\mathbf{\Phi}}\Delta)^{-1}\hat{\mathbf{\Phi}}$ and that $(I - GH)^{-1}$ is ℓ_∞-stable if and only if $(I - HG)^{-1}$ is ℓ_∞-stable [6] to recast the expression (4.7) in a form that matches the linear-fractional-transform (LFT) structure studied in classical robust control.

We can therefore leverage the equivalence between robust stability and performance [6, 2] to conclude that $\hat{\mathbf{\Phi}}$ satisfies the robust performance conditions (4.9) for all $\|\mathbf{\Delta}\|_{\ell_\infty \to \ell_\infty} \leq 1$ if and only if the augmented LTI system

$$\boldsymbol{M} = \begin{bmatrix} \frac{1}{\gamma}\boldsymbol{Q}\hat{\mathbf{\Phi}} & \frac{1}{\gamma}\boldsymbol{Q}\hat{\mathbf{\Phi}} \\ \varepsilon\hat{\mathbf{\Phi}} & \varepsilon\hat{\mathbf{\Phi}} \end{bmatrix} \tag{4.10}$$

is robustly stable for all structured perturbations $\tilde{\mathbf{\Delta}}$ satisfying

$$\tilde{\mathbf{\Delta}} = \operatorname{blkdiag}(\mathbf{\Delta}_1, \mathbf{\Delta}_2), \ \mathbf{\Delta}_1, \mathbf{\Delta}_2 \in \mathcal{L}_{\text{TV}}, \ \|\mathbf{\Delta}_1\|_{\ell_\infty \to \ell_\infty} \leq 1, \|\mathbf{\Delta}_2\|_{\ell_\infty \to \ell_\infty} \leq 1. \tag{4.11}$$

The necessary and sufficient conditions for robust stability of the resulting two-block problem can be derived as a special case of Theorem 6.3 of [6]. The particular case of an augmented LTI system $\boldsymbol{M}$ satisfying $\boldsymbol{M}_{11} = \boldsymbol{M}_{12}$ and $\boldsymbol{M}_{21} = \boldsymbol{M}_{22}$, as is the case for our problem (4.10), is addressed in Ch 8.3 of [6]. The necessary and sufficient conditions specified in Theorem 6.3 of [6] reduce to the following *convex* constraints on the system response $\hat{\mathbf{\Phi}}$

$$\begin{aligned}
&Z_{AB}\hat{\mathbf{\Phi}} = I, \ \hat{\mathbf{\Phi}} \in \frac{1}{z}\mathcal{RH}_\infty, \\
&\left\| \boldsymbol{Q}\hat{\mathbf{\Phi}} \right\|_{\ell_\infty \to \ell_\infty} + \gamma \left\| \varepsilon\hat{\mathbf{\Phi}} \right\|_{\ell_\infty \to \ell_\infty} < \gamma.
\end{aligned} \tag{4.12}$$

Finally, we remark that although the constraints (4.12) are in general infinite-dimensional due to the transfer matrix $\hat{\mathbf{\Phi}}$, principled finite-dimensional approximations, some of which enjoy provable sub-optimality guarantees, are available [7, 1, 3, 4]. Further, for the $\mathcal{L}_1$ problem considered here, the resulting optimization problem can be posed as a linear program, thus enjoying favorable computational complexity properties. It then follows that by bisecting on γ, e.g., by using golden search, we can find a performance level γ, and corresponding system responses, that satisfies $\gamma \leq \gamma_\star + \epsilon$ in $O \log(1/\epsilon)$ iterations, for $\gamma_\star$ the smallest γ such that the set defined by (4.12) is non-empty.

4.1 Robust Performance Guarantees for Large-Scale Distributed Control

In previous work [9, 8, 10], the author and his colleagues showed that for LTI dynamical systems (3.2) defined by structured (i.e., sparse) matrices (A, B), imposing *locality constraints* on the system responses $\{\mathbf{\Phi}_x, \mathbf{\Phi}_u\}$, i.e., imposing that $\{\mathbf{\Phi}_x, \mathbf{\Phi}_u\} \in \mathcal{S}$, for $\mathcal{S}$ a suitably defined structure inducing subspace constraint, led to distributed controllers that enjoyed scalable synthesis and implementation complexity. Although formally defining these concepts is beyond the scope of this paper, we note that such conditions can be easily enforced on the solution of the robust performance conditions (4.12) by additionally imposing that $\{\hat{\mathbf{\Phi}}_x, \hat{\mathbf{\Phi}}_u\} \in \mathcal{S}$. The resulting problem satisfies a notion of *partial separability*, c.f. §IV of [10], which allows for the problem to be solved at scale using distributed optimization techniques. To the best of our knowledge, this is the first such robust performance guarantee that is applicable to large-scale distributed systems.

We emphasize that the conditions identified in (4.12) remain necessary and sufficient when additional structure is imposed on the system responses $\{\hat{\mathbf{\Phi}}_x, \hat{\mathbf{\Phi}}_u\}$ so long as the dynamic perturbations $(\mathbf{\Delta}_{\boldsymbol{A}}, \mathbf{\Delta}_{\boldsymbol{B}})$ remain unstructured. An exciting direction for future work will be to explore the interplay between locality in the closed loop responses and its consequences on necessary and sufficient conditions for robust performance when the perturbations $(\mathbf{\Delta}_{\boldsymbol{A}}, \mathbf{\Delta}_{\boldsymbol{B}})$ are constrained to respect the topology of the nominal system, as defined by the support of $(\hat{A}, \hat{B})$.

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