Robust Performance Guarantees for System Level Synthesis

Nikolai Matni

University of Pennsylvania

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Abstract

We generalize the system level parameterization of achievable closed loop responses to systems defined by bounded causal linear operators, and use this parameterization to make connections between robust system level synthesis and classical results from the robust control literature. In particular, by leveraging results from $\mathcal{L}_1$ robust control, we show that necessary and sufficient conditions for robust performance with respect to causal bounded linear uncertainty in the system dynamics can be translated into convex constraints on the system responses. We exploit this connection to show that these conditions naturally allow the incorporation of delay, sparsity, and locality constraints on the system responses and resulting controller implementation, allowing these methods to be applied to large-scale distributed control problems – to the best of our knowledge, these are the first robust performance guarantees for distributed control systems. We also show that for suitably small model uncertainty, sub-optimality bounds on robust performance, as measured with respect to performance achieved by an optimal controller for the true model, can be obtained as a function of the size of the model uncertainty, thus providing a natural bridge between tools from robust control and finite-data guarantees for system identification.

1 Introduction

TODO

2 Notation

We slightly adapt the notation used in [5]. We use Latin letters to denote vectors and matrices, e.g., $Ax = b$, and bold-face Latin letters to denote signals and operators, e.g., $\boldsymbol{x} = (x_t)_{t=0}^{\infty}$, and $\boldsymbol{y} = \boldsymbol{G}\boldsymbol{u}$. We let ℓ_∞ denote the space of all bounded sequences of real numbers, i.e., $\boldsymbol{x} = (x_t)_{t=0}^{\infty}$ if and only if $\sup_t |x_t| < \infty$, in which case we define $\|\boldsymbol{x}\|_{\ell_\infty} = \sup_t |x_t|$. Similarly, we let ℓ_∞^q denote the space of all q-tuples of elements of ℓ_∞: if $\boldsymbol{x} = (\boldsymbol{x}^1, \ldots, \boldsymbol{x}^q) \in \ell_\infty^q$, then $\|\boldsymbol{x}\|_{\ell_\infty} = \max_{i=1,\ldots,q} \left\|\boldsymbol{x}^i\right\|_{\ell_\infty}$. We also define the extended space $\ell_\infty^{q,e}$ which is equal to the space of all q-tuples of sequences of real numbers. We let $\boldsymbol{S}_+$ denote the right shift operator such tht if $\boldsymbol{x} = (x_t)_{t=0}^{\infty}$, then $\boldsymbol{S}_+\boldsymbol{x} = (0, x_0, x_1, \ldots)$. Similarly, we let $\boldsymbol{S}_-$ denote the left shift operator such that $\boldsymbol{S}_-\boldsymbol{x} = (x_1, x_2, \ldots)$. Hence $S_-S_+ = I$, but in general $S_+S_- \neq I$. We let $\mathcal{L}_{\text{TV}}^{p,q}$ be the space of all bounded linear causal operators mapping $\ell_\infty^q \to \ell_\infty^p$, and broadly refer to all such operators as ℓ_∞-stable. If $\boldsymbol{R} \in \mathcal{L}_{\text{TV}}^{p,q}$, then $\|\boldsymbol{R}\|_{\ell_\infty \to \ell_\infty} := \sup_{\|x\|_{\ell_\infty} \leq 1} \|\boldsymbol{R}\boldsymbol{x}\|_{\ell_\infty}$, which is the induced operator norm. Note that each $\boldsymbol{R} \in \mathcal{L}_{\text{TV}}^{p,q}$

can be completely characterized by its block lower-triangular pulse response matrix. We denote by $\mathcal{L}_{\text{TV}}^{p,q}$ the subspace of $\mathcal{L}_{\text{TV}}^{p,q}$ consisting of time-invariant operators.

3 Operator System Level Parameterization

Let $\boldsymbol{A} \in \mathcal{L}_{\text{TV}}^{n,n}$ and $\boldsymbol{B} \in \mathcal{L}_{\text{TV}}^{n,p}$, and consider the dynamical system mapping $\ell_\infty^{n,e} \times \ell_\infty^{p,e} \to \ell_\infty^{n,e}$ defined by

$$\boldsymbol{x} = \boldsymbol{S}_+\boldsymbol{A}\boldsymbol{x} + \boldsymbol{S}_+\boldsymbol{B}\boldsymbol{u} + \boldsymbol{S}_+\boldsymbol{w}. \tag{3.1}$$

As $\boldsymbol{S}_+\boldsymbol{A}$ is strictly causal, the dynamics (3.1) are well posed in the sense that $(I - \boldsymbol{S}_+\boldsymbol{A})^{-1}$ exists as an operator from $\ell_\infty^{n,e} \times \ell_\infty^{p,e} \to \ell_\infty^{n,e}$ [2]. We emphasize that although we impose that $\boldsymbol{A}$ be ℓ_∞-stable, this does not imply that the dynamics (3.1) are themselves open-loop stable. Rather, open-loop stability of the system is determined by the ℓ_∞ stability of the operator $(I - \boldsymbol{S}_+\boldsymbol{A})^{-1}$.

Note that if $\boldsymbol{A}$ and $\boldsymbol{B}$ are memoryless and time invariant, i.e., if their matrix representations are block-diagonal $\boldsymbol{A} = \text{blkdiag}(A, A, \dots)$, $\boldsymbol{B} = \text{blkdiag}(B, B, \dots)$, then the system (3.1) reduce to the familiar finite-dimensional linear time-invariant system

$$x_{t+1} = A_t x_t + Bu_t + w_t,\ x_0 = 0, \tag{3.2}$$

where once again stability of the closed loop system is determined by the stability of the operator $(zI - A)^{-1}$ as opposed to the boundedness of the matrix A.

For LTI systems (3.2), the System Level Synthesis (SLS) framework [10, 1] provides an appealing parameterization of all closed loop responses from $\boldsymbol{w} \to (\boldsymbol{x}, \boldsymbol{u})$ achievable by a causal state-feedback LTI control law $\boldsymbol{K}$ such that $\boldsymbol{u} = \boldsymbol{K}\boldsymbol{x}$, as summarized in the following theorem.

Theorem 3.1 (Theorem 1, [10])**.** *For the LTI system* (3.2) *with causal state-feedback LTI control law $\boldsymbol{u} = \boldsymbol{K}\boldsymbol{x}$, the follwing are true:*

1. *The affine subspace defined by*

$$\begin{bmatrix} zI - A & -B \end{bmatrix} \begin{bmatrix} \boldsymbol{\Phi}_x \\ \boldsymbol{\Phi}_u \end{bmatrix} = I \ \ \boldsymbol{\Phi}_x, \boldsymbol{\Phi}_u \in \frac{1}{z}\mathcal{RH}_\infty \tag{3.3}$$

   parameterizes all system responses

$$\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{u} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Phi}_x \\ \boldsymbol{\Phi}_u \end{bmatrix} \boldsymbol{w} \tag{3.4}$$

   achievable by an internally stabilizing state-feedback controller $\boldsymbol{K}$.

2. *For any transfer matrices $\{\boldsymbol{\Phi}_x, \boldsymbol{\Phi}_u\}$ satisfying the constraints* (3.3)*, the control signal computed via*[1]

$$\begin{aligned} \boldsymbol{u} &= z\boldsymbol{\Phi}_u\hat{\boldsymbol{w}} \\ \hat{\boldsymbol{w}} &= \boldsymbol{x} - \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{x}} &= (z\boldsymbol{\Phi}_x - I)\hat{\boldsymbol{w}} \end{aligned} \tag{3.5}$$

   is internally stabilizing and achieves the desired response (3.4)*.*

[1]We note that due to the affine constraints (3.3), $z\boldsymbol{\Phi}_x - I$ is strictly causal, and thus feedback loop between $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{w}}$ is well posed.

Thus, in the case of LTI systems (3.2), the search for an optimal controller $\boldsymbol{K}$ can be converted to a search over system responses $\{\boldsymbol{\Phi}_x, \boldsymbol{\Phi}_u\}$ constrained to lie in the affine space defined by equation (3.3). This fact, combined with the transparent mapping between the system responses and the corresponding controller implementation (3.5), has been successfully exploited for the synthesis of distributed optimal controllers for large-scale systems by introducing additional structural constraints on the system responses, and corresponding controller implementation (3.5), through additional subspace constraints – we refer the reader to [8, 7, 9] for more details.

Another favorable feature of the parameterization defined in Theorem 3.1 is that it is provably stable under deviations from the subspace (3.3), as summarized in the following theorem from [6].

Theorem 3.2 (Theorem 2, [6])**.** *Let* $(\hat{\boldsymbol{\Phi}}_x, \hat{\boldsymbol{\Phi}}_u, \boldsymbol{\Delta})$ *be a solution to*

$$\begin{bmatrix} zI - A & -B \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{\Phi}}_x \\ \hat{\boldsymbol{\Phi}}_u \end{bmatrix} = I - \boldsymbol{\Delta} \;\; \hat{\boldsymbol{\Phi}}_x, \hat{\boldsymbol{\Phi}}_u \in \frac{1}{z}\mathcal{RH}_\infty. \tag{3.6}$$

If $(I - \boldsymbol{\Delta})^{-1}$ *exists as on operator in* $\mathcal{RL}_\infty$*, then the controller implementation* (3.5) *defined in terms of the transfer matrices* $\left\{\hat{\boldsymbol{\Phi}}_x, \hat{\boldsymbol{\Phi}}_u\right\}$ *achieves the closed loop responses*

$$\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{u} \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{\Phi}}_x \\ \hat{\boldsymbol{\Phi}}_u \end{bmatrix} (I - \boldsymbol{\Delta})^{-1}\boldsymbol{w}, \tag{3.7}$$

on the LTI system (3.2)*, and is internally stabilizing if and only if* $(I - \boldsymbol{\Delta})^{-1} \in \mathcal{RH}_\infty$*.*

This parameterization has proved crucial in providing tractable approximations to non-convex distributed control problems [6], and in providing sub-optimality bounds for robust controllers as applied to learned systems [3, 4]. However, in these past works, very crude approximations based solely on small gain bounds and triangle inequalities were used to control the effects of the uncertain map $(I - \boldsymbol{\Delta})^{-1}$ on system stability and performance. In this work we show that Theorems 3.1 and 3.2 can be extended to a more general setting that subsequently allows us to make connections to well developed theory in the robust control literature [5, 2]. Although we focus on $\mathcal{L}_1$ optimal control in this paper due to its favorable separability structure (cf. §**??**), we expect analogous results to carry over naturally in the $\mathcal{H}_\infty$ setting.

3.1 Necessary Conditions

Here we characterize a set of affine constraints that the closed loop system responses of system (3.1) must satisfy if they are induced by a linear and causal controller $\boldsymbol{K} : \ell_\infty^{n,e} \to \ell_\infty^{p,e}$ via the control law $\boldsymbol{u} = \boldsymbol{K}\boldsymbol{x}$.

Proposition 3.3. *Let* $\boldsymbol{K} : \ell_\infty^{n,e} \to \ell_\infty^{p,e}$ *be a linear and causal map such that* $(I - \boldsymbol{S}_+(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}))^{-1}\boldsymbol{S}_+ \in \mathcal{L}_{\mathrm{TV}}^{n,n}$ *and* $\boldsymbol{K}(I - \boldsymbol{S}_+(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}))^{-1}\boldsymbol{S}_+ \in \mathcal{L}_{\mathrm{TV}}^{n,p}$*, i.e., let* $\boldsymbol{K}$ *be a linear causal and stabilizing controller such that closed loop maps from* $\boldsymbol{w} \to (\boldsymbol{x}, \boldsymbol{u})$ *are* ℓ_∞*-stable. Then all maps taking* $\boldsymbol{w} \to (\boldsymbol{x}, \boldsymbol{u})$ *achievable by such a* $\boldsymbol{K}$ *satisfy the constraints*

$$\begin{bmatrix} I - \boldsymbol{S}_+\boldsymbol{A} & -\boldsymbol{S}_+\boldsymbol{B} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Phi}_x \\ \boldsymbol{\Phi}_u \end{bmatrix} = \boldsymbol{S}_+, \;\; \boldsymbol{\Phi}_x, \boldsymbol{\Phi}_u \textit{ strictly causal, linear, and } \ell_\infty\textit{-stable.} \tag{3.8}$$

Proof. As $\boldsymbol{S}_+(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K})$ is strictly causal, the feedback interconnection is well posed in the sense that $(I-\boldsymbol{S}_+(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}))^{-1}$ exists as a map from $\ell_\infty^{n,e}\to\ell_\infty^{n,e}$. By assumption, both $(I-\boldsymbol{S}_+(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}))^{-1}$ and $\boldsymbol{K}(I-\boldsymbol{S}_+(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}))^{-1}$ are ℓ_∞-stable. Defining

$$\begin{bmatrix}\boldsymbol{\Phi}_x\\ \boldsymbol{\Phi}_u\end{bmatrix}=\begin{bmatrix}I\\ \boldsymbol{K}\end{bmatrix}(I-\boldsymbol{S}_+(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}))^{-1},$$

it is easily verified that $\{\boldsymbol{\Phi}_x,\boldsymbol{\Phi}_u\}$ satisfy constraint (3.8). $\square$

Remark 1. *If $\boldsymbol{A}$ and $\boldsymbol{B}$ are memoryless and LTI, and $\boldsymbol{K}$ is LTI, then constraint* (3.8) *simplifies to*

$$\begin{bmatrix}I-\frac{1}{z}A & -\frac{1}{z}B\end{bmatrix}\begin{bmatrix}\boldsymbol{\Phi}_x\\ \boldsymbol{\Phi}_u\end{bmatrix}=\frac{1}{z}I \;\; \boldsymbol{\Phi}_x,\boldsymbol{\Phi}_u\in\frac{1}{z}\mathcal{RH}_\infty, \tag{3.9}$$

which is clearly equivalent to (3.3).

3.2 Controller Implementation

We now show how to construct an internally stabilizing controller from any operators $\{\boldsymbol{\Phi}_x,\boldsymbol{\Phi}_u\}$ satisfying constraint (3.8) that achieves the desired response from $\boldsymbol{w}\to(\boldsymbol{x},\boldsymbol{u})$.

Proposition 3.4. *Let $\{\boldsymbol{\Phi}_x,\boldsymbol{\Phi}_u\}$ satisfy constraint* (3.8)*. Then the controller implementation shown in Fig.* **??**, *described by the equations*

$$\begin{array}{rcl}\boldsymbol{u} & = & \boldsymbol{\Phi}_x\boldsymbol{S}_-\hat{\boldsymbol{w}}\\ \hat{\boldsymbol{w}} & = & x-\hat{\boldsymbol{x}}\\ \hat{\boldsymbol{x}} & = & (\boldsymbol{\Phi}_x\boldsymbol{S}_- - I)\hat{\boldsymbol{w}},\end{array} \tag{3.10}$$

is well-posed and ℓ_∞ stable as a map from $(\boldsymbol{w},\boldsymbol{\delta}_y,\boldsymbol{\delta}_u)\to(\boldsymbol{x},\boldsymbol{u},\hat{\boldsymbol{w}})$ and achieves the desired closed loop response

$$\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{u}\end{bmatrix}=\begin{bmatrix}\boldsymbol{\Phi}_x\\ \boldsymbol{\Phi}_u\end{bmatrix}\boldsymbol{w}. \tag{3.11}$$

Proof. From equations (3.1) and (3.10) (alternatively, from Fig. **??**), we have that

$$\begin{array}{rcl}\boldsymbol{x} & = & \boldsymbol{S}_+\boldsymbol{A}\boldsymbol{x}+\boldsymbol{S}_+\boldsymbol{B}\boldsymbol{u}+\boldsymbol{S}_+\boldsymbol{u}\\ \boldsymbol{u} & = & \boldsymbol{\Phi}_u\boldsymbol{S}_-\hat{\boldsymbol{w}}+\boldsymbol{\delta}_u\\ \hat{\boldsymbol{w}} & = & \boldsymbol{x}+\boldsymbol{\delta}_y+(I-\boldsymbol{\Phi}_x\boldsymbol{S}_-)\hat{\boldsymbol{w}},\end{array} \tag{3.12}$$

with $\hat{w}_0=0$, i.e., $\hat{\boldsymbol{w}}$ a strictly causal signal. We first observe that the restriction of $(I-\boldsymbol{\Phi}_x\boldsymbol{S}_-)$ to strictly causal signals is itself strictly causal, as the constraint (3.8) enforces that the block lower triangular matrix representation of $\boldsymbol{\Phi}_x$ has identify matrices along its first block sub-diagonal, i.e., for $\boldsymbol{\Phi}_x=(\Phi_x(i,j))_{i,j=0}^\infty$ the block lower triangular matrix representation of $\boldsymbol{\Phi}_x$, we have that $\Phi_x(i,i-1)=I$ for all $i\geq 1$. It therefore follows that the feedback loop between $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{w}}$ is well posed. By rote calculation, tt follows from equation (3.12) that the closed loop maps from $(\boldsymbol{w},\boldsymbol{\delta}_y,\boldsymbol{\delta}_u)\to(\boldsymbol{x},\boldsymbol{u},\hat{\boldsymbol{w}})$ are given by

$$\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{u}\\ \hat{\boldsymbol{w}}\end{bmatrix}=\begin{bmatrix}\boldsymbol{\Phi}_x & \boldsymbol{\Phi}_x(\boldsymbol{S}_- - \boldsymbol{A}) & \boldsymbol{\Phi}_x\boldsymbol{B}\\ \boldsymbol{\Phi}_u & \boldsymbol{\Phi}_u(\boldsymbol{S}_- - \boldsymbol{A}) & I+\boldsymbol{\Phi}_u\boldsymbol{B}\\ \boldsymbol{S}_+ & I-\boldsymbol{S}_+\boldsymbol{A} & \boldsymbol{S}_+\boldsymbol{B}\end{bmatrix}\begin{bmatrix}\boldsymbol{w}\\ \boldsymbol{\delta}_y\\ \boldsymbol{\delta}_u\end{bmatrix}. \tag{3.13}$$

By assumption, $\boldsymbol{\Phi}_x$, $\boldsymbol{\Phi}_u$, $\boldsymbol{A}$, and $\boldsymbol{B}$ are all ℓ_∞-stable, and hence the interconnection illustrated in Fig. **??**, and described by equations (3.1) and (3.12) is well-posed and ℓ_∞-stable. $\square$

Remark 2. *If $\boldsymbol{A}$ and $\boldsymbol{B}$ are memoryless and LTI, and $\boldsymbol{K}$ is LTI, then so are the system responses $\{\boldsymbol{\Phi}_x, \boldsymbol{\Phi}_u\}$, and consequently the right and left shift operators $\boldsymbol{S}_+$ and $\boldsymbol{S}_-$ become $\frac{1}{z}$ and z, respectively, recovering the controller implementation* (3.5)*.*

3.3 Robust Operator System Level Synthesis

Thus Propositions 3.3 and 3.4 show that the parameterization of Theorem 3.1 can be extended to a class of dynamics described by bounded and causal linear operators in feedback with a causal linear controller. We now show that this extension enjoys similar stability properties with respect to perturbations from the subspace (3.8).

Theorem 3.5. *Let $\boldsymbol{A} \in \mathcal{L}_{\mathrm{TV}}^{n,n}$ and $\boldsymbol{B} \in \mathcal{L}_{\mathrm{TV}}^{n,p}$, and suppose that $\left\{\hat{\boldsymbol{\Phi}}_x, \hat{\boldsymbol{\Phi}}_u\right\}$ satisfy*

$$\begin{bmatrix} I - \boldsymbol{S}_+\boldsymbol{A} & -\boldsymbol{S}_+\boldsymbol{B} \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{\Phi}}_x \\ \hat{\boldsymbol{\Phi}}_u \end{bmatrix} = \boldsymbol{S}_+(I - \boldsymbol{\Delta}),\ \boldsymbol{\Phi}_x, \boldsymbol{\Phi}_u \text{ strictly causal, linear, and } \ell_\infty\text{-stable,} \tag{3.14}$$

for $\boldsymbol{\Delta}$ a strictly causal linear operator from $\ell_\infty^{n,e} \to \ell_\infty^{n,e}$. Then the controller implementation (3.10) *defined in terms of the operators $\left\{\hat{\boldsymbol{\Phi}}_x, \hat{\boldsymbol{\Phi}}_u\right\}$ is well posed and achieves the following response*

$$\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{u} \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{\Phi}}_x \\ \hat{\boldsymbol{\Phi}}_u \end{bmatrix} (I - \boldsymbol{\Delta})^{-1}\boldsymbol{u}. \tag{3.15}$$

Further, this interconnection is ℓ_∞-stable if and only if $(I - \boldsymbol{\Delta})^{-1}$ is ℓ_∞-stable.

Proof. As $\boldsymbol{\Delta}$ is strictly causal by assumption, $(I - \boldsymbol{\Delta})^{-1}$ eists as a map from $\ell_\infty^{n,e} \to \ell_\infty^{n,e}$. Going through a similar argument as that in the proof of Proposition 3.4, we observe that

$$\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{u} \\ \hat{\boldsymbol{w}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Phi}_x I_{\boldsymbol{\Delta}} & \boldsymbol{\Phi}_x I_{\boldsymbol{\Delta}}(\boldsymbol{S}_- - \boldsymbol{A}) & \boldsymbol{\Phi}_x I_{\boldsymbol{\Delta}} \boldsymbol{B} \\ \boldsymbol{\Phi}_u I_{\boldsymbol{\Delta}} & \boldsymbol{\Phi}_u I_{\boldsymbol{\Delta}}(\boldsymbol{S}_- - \boldsymbol{A}) & I + \boldsymbol{\Phi}_u I_{\boldsymbol{\Delta}} \boldsymbol{B} \\ \boldsymbol{S}_+ I_{\boldsymbol{\Delta}} & I_{\boldsymbol{\Delta}}(I - \boldsymbol{S}_+\boldsymbol{A}) & I_{\boldsymbol{\Delta}} \boldsymbol{S}_+ \boldsymbol{B} \end{bmatrix} \begin{bmatrix} \boldsymbol{w} \\ \boldsymbol{\delta}_y \\ \boldsymbol{\delta}_u \end{bmatrix}, \tag{3.16}$$

where we let $I_{\boldsymbol{\Delta}} := (I - \boldsymbol{\Delta})^{-1}$. Thus we see that the desired map (3.15) from $\boldsymbol{u} \to (\boldsymbol{x}, \boldsymbol{u})$ is achived. Further, as $\boldsymbol{\Phi}_x$, $\boldsymbol{\Phi}_u$, $\boldsymbol{A}$, $\boldsymbol{B}$ are all ℓ_∞-stable by assumption, it follows that the ℓ_∞-stability of the map from $(\boldsymbol{w}, \boldsymbol{\delta}_y, \boldsymbol{\delta}_u) \to (\boldsymbol{x}, \boldsymbol{u}, \hat{\boldsymbol{w}})$ is determined by the ℓ_∞-stability of $I_{\boldsymbol{\Delta}}$, from which the result follows. □

4 Robust Performance under Model Uncertainty

We now use the tools developed in the previous section to identify necessary and sufficient conditions for the robust stability and robust performance of a system (3.1) subject to bounded perturbations in its $\boldsymbol{A}$ and $\boldsymbol{B}$ operators. In particular consider the system

$$\boldsymbol{x} = \boldsymbol{S}_+(\boldsymbol{A}_0 + \boldsymbol{\Delta}_{\boldsymbol{A}})\boldsymbol{x} + \boldsymbol{S}_+(\boldsymbol{B}_0 + \boldsymbol{\Delta}_{\boldsymbol{B}})\boldsymbol{u} + \boldsymbol{S}_+ w, \tag{4.1}$$

where $\boldsymbol{A}_0 = \mathrm{blkdiag}(A, A, \dots)$ and $\boldsymbol{B}_0 = \mathrm{blkdiag}(B, B, \dots)$ are memoryless LTI operators defining a nominal LTI system $x_{t+1} = Ax_t + Bu_t + w_t$, and $\boldsymbol{\Delta}_{\boldsymbol{A}}$ and $\boldsymbol{\Delta}_{\boldsymbol{B}}$ are ℓ_∞-stable and satisfy $\|[\boldsymbol{\Delta}_{\boldsymbol{A}}, \boldsymbol{\Delta}_{\boldsymbol{B}}]\|_{\ell_\infty \to \ell_\infty} \le \varepsilon$.

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