Working Note: Time-Domain System Level Synthesis

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1 A time-domain derivation

Consider the system

$$x_{k+1} = Ax_k + Bu_k + w_k. \tag{1.1}$$

In what follows, we will use $z(t)$ to denote the signal z over time, and z_k to denote its instantaneous value at $t = k$, i.e., $z(t) = (z_0, z_1, \dots)$. For a matrix X, we use X^j to denote its jth column; we also sometimes overload notation and use a superscript as a label for a vector, i.e., we will sometimes write $x^j(t)$ to denote a specific instantiation of a generic signal x.

Our goal is to characterize the closed loop maps from $w(t) \to x(t)$ and $w(t) \to u(t)$ under the assumption that $u(t) = (K \star x)(t)$ for some LTI controller $K(t)$. Here $(g \star h)(t)$ denotes the convolution of signals $g(t)$ and $h(t)$. Specifically, we seek filters $R(t)$ and $M(t)$ such that

$$\begin{array}{rcl} x(t) & = & (R \star w)(t) \\ u(t) & = & (M \star w)(t), \end{array} \tag{1.2}$$

under the assumption that $u(t) = (K \star x)(t)$.

First note that for any disturbance signal $w(t)$ we can write $w(t) = \sum_k \sum_j w_{jk} e_j \delta_{k-t}$, and hence by linearity of our dynamics (1.1), it suffices to characterize the impulse responses of the system due to $w^j(t) := e_j \delta_k$. To that end, we write $x^j(t)$ and $u^j(t)$ to denote the state and control responses due to the jth impulse disturbance $w^j(t)$; the dynamics (1.1) then simplify to

$$R^j_{k+1} = AR^j_k + BM^j_k,\ R^j_1 = e_j, \tag{1.3}$$

where we have used the fact that the resulting state and control responses satisfy, by definition,

$$\begin{array}{rcccccl} x^j_k & = & \sum_{t=1}^k R_t w^j_{k-t} & = & \sum_{t=1}^k R_t e_j \delta_{k-t} & = & R^j_k \\ u^j_k & = & \sum_{t=1}^k M_t w^j_{k-t} & = & \sum_{t=1}^k M_t e_j \delta_{k-t} & = & M^j_k. \end{array}$$

Therefore, concatenating the respective responses, we have that the responses $R(t)$ and $M(t)$ must satsify

$$R_{k+1} = AR_k + BM_k,\ R_1 = I. \tag{1.4}$$

Our next claim is that since $R(t)$ and $M(t)$ satisfy $R_0 = 0$ and $M_0 = 0$, the controller can be implemented directly as $u(t) = (M \star w)(t)$ by noting that u_k only depends on $w_1, \dots, w_{k-1}$, and that at time k, one can compute $w_{k-1} = x_k - (Ax_{k-1} + Bu_{k-1})$. It therefore follows from the constraints (1.4) that the desired state response is also achieved. To make this explicit, notice that for $w(t) = w^j(t) = e_j\delta_k$ we have from the dynamics (1.1) that

$$x_1^j = e_j = R_1^j.$$

Similarly, we have that

$$x_2^j = Ax_1^j + Bu_1^j = AR_1^j + BM_1^j = R_2^j$$

where the final equality follows from the constraints (1.4). This argument is easily extended to any $k \geq 2$ by induction.

Note that this argument is valid over finite or infinite horizons – note that in the infinite horizon case, additional stability constraints must be imposed on $R(t)$ and $M(t)$ to ensure that the resulting closed loop system is itself stable.

2 Finite-horizon LQR

2.1 Zero IC, Gaussian Noise

Here we consider the finite-horizon LQR problem with zero initial condition, driven by Gaussian noise:

$$\begin{array}{ll} \min_{x(t),u(t)} & \sum_{k=1}^{T} \mathbb{E}\left[x_k^\top Q x_k + u_k^\top S u_k\right] + \mathbb{E}\left[x_{T+1}^\top Q_F x_{T+1}\right] \\ \text{subject to} & x_{k+1} = Ax_k + Bu_k + w_k,\ x_0 = 0, \end{array} \tag{2.1}$$

for $w_k \overset{\text{i.i.d}}{\sim} \mathcal{N}(0, I)$ (note that this is wlog assuming a non-degenerate distribution for the process noise). We now show how this problem can be recast in terms of the system responses $R(t)$ and $M(t)$. First note that given equations (1.2), it follows that

$$\begin{array}{lcl} x_k & = & \sum_{t=1}^{k} R_t w_{k-t} \\ u_k & = & \sum_{t=1}^{k} M_t w_{k-t}. \end{array} \tag{2.2}$$

Exploiting the i.i.d. nature of the disturbance, we can then write

$$\begin{array}{rcl} \mathbb{E}\left[x_k^\top Q x_k\right] & = & \sum_{t=1}^{k} \mathbf{Tr}\, R_t^\top Q R_t \\ \mathbb{E}\left[u_k^\top S u_k\right] & = & \sum_{t=1}^{k} \mathbf{Tr}\, M_t^\top S M_t \\ \mathbb{E}\left[x_{T+1}^\top Q_F x_{T+1}\right] & = & \sum_{t=1}^{T+1} \mathbf{Tr}\, R_t^\top Q_F R_t. \end{array} \tag{2.3}$$

Plugging equations (2.3) into the objective function of optimization problem (2.1), collecting like terms and optimizing over system responses as opposed to state and input trajectories, we may rewrite the standard LQR problem as

$$\begin{array}{ll} \min_{R(t),M(t)} & \sum_{\tau=1}^{T} (T+1-\tau)\left[\mathbf{Tr}\, R_\tau^\top Q R_\tau + \mathbf{Tr}\, M_\tau^\top S M_\tau\right] + \sum_{s=1}^{T+1} \mathbf{Tr}\, R_s^\top Q_F R_s \\ \text{subject to} & R_{k+1} = AR_k + BM_k,\ R_1 = I. \end{array} \tag{2.4}$$

2.2 Nonzero IC, no noise

Here we consider the finite-horizon LQR problem with nonzero initial condition and no noise:

$$\begin{array}{ll} \min_{x(t),u(t)} & \sum_{k=1}^{T} \left[x_k^\top Q x_k + u_k^\top S u_k\right] + x_{T+1}^\top Q_F x_{T+1} \\ \text{subject to} & x_{k+1} = Ax_k + Bu_k,\ x_0 = \xi. \end{array} \tag{2.5}$$

In this case, we can define the disturbance signal driving the system to be

$$w(t) := \xi\delta_k = (\xi, 0, 0, \dots). \tag{2.6}$$

It then follows that

$$\begin{array}{rclclcl} x_k & = & \sum_{t=1}^k R_t w_{k-t} & = & \sum_{t=1}^k R_t \xi \delta_{k-t} & = & R_k \xi \\ u_k & = & \sum_{t=1}^k M_t w_{k-t} & = & \sum_{t=1}^k M_t \xi \delta_{k-t} & = & M_k \xi, \end{array} \tag{2.7}$$

meaning that the the LQR problem (2.5) can be rewritten as

$$\begin{array}{rl} \min_{R(t),M(t)} & \sum_{k=1}^T \left[\mathbf{Tr}\, R_k^\mathsf{T} Q R_k \Xi + \mathbf{Tr}\, M_k^\mathsf{T} S M_k \Xi\right] + \mathbf{Tr}\, R_{T+1}^\mathsf{T} Q_F R_{T+1} \Xi \\ \text{subject to} & R_{k+1} = AR_k + BM_k,\ R_1 = I, \end{array} \tag{2.8}$$

where $\Xi := \xi\xi^\mathsf{T}$.

2.3 Putting them together

We now consider the standard LQR problem with nonzero initial condition and driving process noise

$$\begin{array}{rl} \min_{x(t),u(t)} & \sum_{k=1}^T \mathbb{E}\left[x_k^\mathsf{T} Q x_k + u_k^\mathsf{T} S u_k\right] + \mathbb{E}\left[x_{T+1}^\mathsf{T} Q_F x_{T+1}\right] \\ \text{subject to} & x_{k+1} = Ax_k + Bu_k + w_k,\ x_0 = \xi, \end{array} \tag{2.9}$$

for $w_k \overset{\text{i.i.d}}{\sim} \mathcal{N}(0, I)$. Exploiting linearity and the fact that $\mathbb{E}\left[w_k \xi^\mathsf{T}\right] = 0$ for all k we can simply combine the results of the previous subsections, allowing us to rewrite the standard LQR problem (2.9) as

$$\begin{array}{rl} \min_{R(t),M(t)} & J_{\text{stoch}}(R(t), M(t)) + J_{\text{det}}(R(t), M(t), \Xi) \\ \text{subject to} & R_{k+1} = AR_k + BM_k,\ R_1 = I, \end{array} \tag{2.10}$$

where

$$\begin{array}{rcl} J_{\text{stoch}}(R(t), M(t)) & := & \sum_{\tau=1}^T (T+1-\tau)\left[\mathbf{Tr}\, R_\tau^\mathsf{T} Q R_\tau + \mathbf{Tr}\, M_\tau^\mathsf{T} S M_\tau\right] + \sum_{s=1}^{T+1} \mathbf{Tr}\, R_s^\mathsf{T} Q_F R_s \\ J_{\text{det}}(R(t), M(t), \Xi) & := & \sum_{k=1}^T \left[\mathbf{Tr}\, R_k^\mathsf{T} Q R_k \Xi + \mathbf{Tr}\, M_k^\mathsf{T} S M_k \Xi\right] + \mathbf{Tr}\, R_{T+1}^\mathsf{T} Q_F R_{T+1} \Xi. \end{array} \tag{2.11}$$

3 A redo of the above using block-Toeplitz operators

In the following we work with the signals and block-Toeplitz lower-triangular operators, and assume $x_0 = 0$, but allow $w_0 \neq 0$[1]:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{T+1} \end{bmatrix} \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{T+1} \end{bmatrix} \mathbf{w} = \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_T \end{bmatrix} \mathbf{R} = \begin{bmatrix} R_1 & & & \\ R_2 & R_1 & & \\ \vdots & \ddots & \ddots & \\ R_{T+1} & \cdots & R_2 & R_1 \end{bmatrix} \mathbf{M} = \begin{bmatrix} M_1 & & & \\ M_2 & M_1 & & \\ \vdots & \ddots & \ddots & \\ M_{T+1} & \cdots & M_2 & M_1 \end{bmatrix}. \tag{3.1}$$

[1]This is wlog, as $x_1 = w_0$ can be taken as the nonzero initial condition.

Further let Z be the block-downshift operator, i.e., a matrix with identity matrices along its first block sub-diagonal and zeros elsewhere. We can then equivalently write equation (1.2) as

$$\begin{array}{rcl} \mathbf{x} & = & \mathbf{R}\mathbf{w} \\ \mathbf{u} & = & \mathbf{M}\mathbf{w}. \end{array} \tag{3.2}$$

Following the same reasoning as above, it is straightforward to show that the responses $\{\mathbf{R}, \mathbf{M}\}$ must satisfy the equation

$$\begin{bmatrix} I - Z\mathcal{A} & -Z\mathcal{B} \end{bmatrix} \begin{bmatrix} \mathbf{R} \\ \mathbf{M} \end{bmatrix} = I, \tag{3.3}$$

where

$$\mathcal{A} := \text{blkdiag}\left(A_1, A_2, \ldots, A_T\right), \ \mathcal{B} := \text{blkdiag}\left(B_1, B_2, \ldots, B_T\right), \tag{3.4}$$

Theorem 3.1. *For the dynamics (1.1) with block-Toeplitz lower-triangular state feedback law* $\mathbf{K}$ *defining the control action as* $\mathbf{u} = \mathbf{K}\mathbf{x}$*, the following are true*

1. *the affine subspace defined by (3.3) parameterizes all possible responses (3.2) of the system.*
2. *for any block-Toeplitz matrices* $\{\mathbf{R}, \mathbf{M}\}$ *satisfying (3.3), the controller* $\mathbf{K} = \mathbf{M}\mathbf{R}^{-1}$ *achieves the desired response.*

Proof. First note that we can equivalently write the dynamics (1.1) as

$$\mathbf{x} = Z\mathcal{A}\mathbf{x} + Z\mathcal{B}\mathbf{u} + \mathbf{w} \tag{3.5}$$

Proof of 1.: Let $\mathbf{K}$ be any block-Toeplitz lower-triangular operator, and $\mathbf{u} = \mathbf{K}\mathbf{x}$. Then

$$\mathbf{x} = Z(\mathcal{A} + \mathcal{B}\mathbf{K})\mathbf{x} + \mathbf{w}, \tag{3.6}$$

which implies that

$$\begin{array}{rcl} \mathbf{x} & = & (I - Z(\mathcal{A} + \mathcal{B}\mathbf{K}))^{-1}\mathbf{w} \\ \mathbf{u} & = & \mathbf{K}(I - Z(\mathcal{A} + \mathcal{B}\mathbf{K}))^{-1}\mathbf{w}. \end{array} \tag{3.7}$$

It is then easily seen that

$$\begin{bmatrix} I - Z\mathcal{A} & -Z\mathcal{B} \end{bmatrix} \begin{bmatrix} (I - Z(\mathcal{A} + \mathcal{B}\mathbf{K}))^{-1} \\ \mathbf{K}(I - Z(\mathcal{A} + \mathcal{B}\mathbf{K}))^{-1} \end{bmatrix} = (I - Z\mathcal{A} - Z\mathcal{B}\mathbf{K})(I - Z(\mathcal{A} + \mathcal{B}\mathbf{K}))^{-1} = I. \tag{3.8}$$

Proof of 2.: Let $\mathbf{K} = \mathbf{M}\mathbf{R}^{-1}$. Then

$$\mathbf{x} = (I - Z(\mathcal{A} + \mathcal{B}\mathbf{M}\mathbf{R}^{-1}))^{-1}\mathbf{w}. \tag{3.9}$$

But we then have that

$$(I - Z(\mathcal{A} + \mathcal{B}\mathbf{M}\mathbf{R}^{-1}))^{-1} = ((I - Z\mathcal{A})\mathbf{R} - Z\mathcal{B}\mathbf{M})\mathbf{R}^{-1})^{-1} = \mathbf{R}((I - Z\mathcal{A})\mathbf{R} - Z\mathcal{B}\mathbf{M})^{-1} = \mathbf{R} \tag{3.10}$$

where the last equality follows form the fact that $\{\mathbf{R}, \mathbf{M}\}$ satisfy (3.3). Similarly we have that

$$\mathbf{u} = \mathbf{M}\mathbf{R}^{-1}\mathbf{x} = \mathbf{M}\mathbf{R}^{-1}\mathbf{R}\mathbf{w} = \mathbf{M}\mathbf{w}, \tag{3.11}$$

where the second equality follows from the fact that $\mathbf{x} = \mathbf{R}\mathbf{w}$. ∎

Next we consider a robust variant of the above theorem. Before stating the result, we let

$$\mathbf{\Delta} = \begin{bmatrix} \Delta_1 & & & \\ \Delta_2 & \Delta_1 & & \\ \vdots & \ddots & \ddots & \\ \Delta_{T+1} & \cdots & \Delta_2 & \Delta_1 \end{bmatrix} \tag{3.12}$$

be an arbitrary block-Toeplitz lower-triangular matrix.

Theorem 3.2. *Let $\mathbf{\Delta}$ be defined as in equation (3.12), and suppose that $\{\mathbf{R}, \mathbf{M}\}$ satisfy*

$$\begin{bmatrix} I - Z\mathcal{A} & -Z\mathcal{B} \end{bmatrix} \begin{bmatrix} \mathbf{R} \\ \mathbf{M} \end{bmatrix} = I + \mathbf{\Delta}. \tag{3.13}$$

If $(I + \Delta_1)^{-1}$ exists, then the controller $\mathbf{K} = \mathbf{M}\mathbf{R}^{-1}$ achieves the system response

$$\begin{bmatrix} \mathbf{R} \\ \mathbf{M} \end{bmatrix} (I + \mathbf{\Delta})^{-1} \tag{3.14}$$

Proof. If $(I + \Delta_1)^{-1}$ exists, then so does $(I + \mathbf{\Delta})^{-1}$ does too, and therefore the constraint (3.13) is equivalent to

$$\begin{bmatrix} I - Z\mathcal{A} & -Z\mathcal{B} \end{bmatrix} \begin{bmatrix} \mathbf{R} \\ \mathbf{M} \end{bmatrix} (I + \mathbf{\Delta})^{-1} = I. \tag{3.15}$$

Then noting that $\mathbf{K} = \mathbf{M}\mathbf{R}^{-1} = \mathbf{M}(I + \mathbf{\Delta})^{-1}(\mathbf{R}(I + \mathbf{\Delta})^{-1})^{-1}$, we then have, by Theorem 3.1, that the controller $\mathbf{K} = \mathbf{M}\mathbf{R}^{-1}$ achieves the system responses (3.14). ∎

4 LQR optimal control problems

Going forward, we assume that $\mathbf{w} \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, I)$. For simplicity of notation, we assume that u_{T+1} is included in the cost functional, i.e., that the cost we seek to minimize is

$$\begin{array}{ll} \min_{x(t),u(t)} & \sum_{k=1}^{T+1} \mathbb{E}\left[x_k^\intercal Q_k x_k + u_k^\intercal S_k u_k\right] \\ \text{subject to} & x_{k+1} = Ax_k + Bu_k + w_k, \; x_0 = 0. \end{array} \tag{4.1}$$

This problem can be rewritten as

$$\begin{array}{ll} \min_{\mathbf{R},\mathbf{M}} & \left\| \begin{bmatrix} \mathcal{Q}^{\frac{1}{2}} & 0 \\ 0 & \mathcal{S}^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \mathbf{R} \\ \mathbf{M} \end{bmatrix} \right\|_F^2 \\ \text{subject to} & \begin{bmatrix} I - Z\mathcal{A} & -Z\mathcal{B} \end{bmatrix} \begin{bmatrix} \mathbf{R} \\ \mathbf{M} \end{bmatrix} = I, \end{array} \tag{4.2}$$

where

$$\mathcal{Q} := \text{blkdiag}(Q_1, Q_2, \ldots, Q_{T+1}), \; \mathcal{S} := \text{blkdiag}(S_1, S_2, \ldots, S_{T+1}). \tag{4.3}$$

5 A statement of the Learning-LQR result

The above conditions/results should look extremely similar to what we have seen in the infinite horizon setting. Letting $\hat{\mathcal{A}} = \mathcal{A} + \mathcal{D}_\mathcal{A}$ and $\hat{\mathcal{B}} = \mathcal{B} + \mathcal{D}_\mathcal{B}$, with the assumption that $\|\mathcal{D}_\mathcal{A}\| \leq \epsilon_A$ and $\|\mathcal{D}_\mathcal{B}\| \leq \epsilon_B$, we can step through the same proof procedure as that in the infinite horizon setting, replacing $\mathcal{H}_\infty$ norms with operator norms, and $\mathcal{H}_2$ norms with Frobenius norms, to obtain an analogous result.

We define the finite-time resolvent $\mathfrak{R}_\mathbf{M}$ of an operator $\mathbf{M}$ to be

$$\mathfrak{R}_\mathbf{M} := (I - Z\mathbf{M})^{-1}, \tag{5.1}$$

and let

$$\mathbf{\Delta} := -Z(\mathcal{D}_\mathcal{A} + \mathcal{D}_\mathcal{B}\mathbf{K}_\star)\mathfrak{R}_{\mathcal{A}+\mathcal{B}\mathbf{K}_\star}, \tag{5.2}$$

for $\mathbf{K}_\star$ the optimal LQR controller. Further let

$$\zeta := (\epsilon_A + \epsilon_B\|\mathbf{K}_\star\|)\mathfrak{R}_{\mathcal{A}+\mathcal{B}\mathbf{K}_\star}. \tag{5.3}$$

Then, if $\zeta \leq \frac{1}{5}$, the controller $\mathbf{K} = \mathbf{M}\mathbf{R}^{-1}$ computed by solving the optimization problem

$$\begin{array}{rl} \min_{\gamma\in[0,1)} \frac{1}{1-\gamma} \min_{\mathbf{R},\mathbf{M}} & \left\| \begin{bmatrix} \mathcal{Q}^{\frac{1}{2}} & 0 \\ 0 & \mathcal{S}^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \mathbf{R} \\ \mathbf{M} \end{bmatrix} \right\|_F \\ \text{subject to} & \begin{bmatrix} I - Z\hat{\mathcal{A}} & -Z\hat{\mathcal{B}} \end{bmatrix} \begin{bmatrix} \mathbf{R} \\ \mathbf{M} \end{bmatrix} = I, \ \sqrt{2}\left\| \begin{matrix} \epsilon_A\mathbf{R} \\ \epsilon_B\mathbf{M} \end{matrix} \right\| \leq \gamma \end{array} \tag{5.4}$$

achieves the relative performance bound

$$\frac{J(\mathcal{A}, \mathcal{B}, \mathbf{K}) - J_\star}{J_\star} \leq 5\zeta, \tag{5.5}$$

where $J_\star$ is the optimal performance achievable if $\mathcal{A}$ and $\mathcal{B}$ are known.