Working Note: Time-Domain System Level Synthesis

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1 A time-domain derivation

Consider the system

$$x_{k+1} = Ax_k + Bu_k + w_k. \tag{1.1}$$

In what follows, we will use $z(t)$ to denote the signal z over time, and z_k to denote its instantaneous value at $t = k$, i.e., $z(t) = (z_0, z_1, \dots)$. For a matrix X, we use X^j to denote its jth column; we also sometimes overload notation and use a superscript as a label for a vector, i.e., we will sometimes write $x^j(t)$ to denote a specific instantiation of a generic signal x.

Our goal is to characterize the closed loop maps from $w(t) \to x(t)$ and $w(t) \to u(t)$ under the assumption that $u(t) = (K \star x)(t)$ for some LTI controller $K(t)$. Here $(g \star h)(t)$ denotes the convolution of signals $g(t)$ and $h(t)$. Specifically, we seek filters $R(t)$ and $M(t)$ such that

$$\begin{array}{rcl} x(t) & = & (R \star w)(t) \\ u(t) & = & (M \star w)(t), \end{array} \tag{1.2}$$

under the assumption that $u(t) = (K \star x)(t)$.

First note that for any disturbance signal $w(t)$ we can write $w(t) = \sum_k \sum_j w_{jk} e_j \delta_{k-t}$, and hence by linearity of our dynamics (1.1), it suffices to characterize the impulse responses of the system due to $w^j(t) := e_j \delta_k$. To that end, we write $x^j(t)$ and $u^j(t)$ to denote the state and control responses due to the jth impulse disturbance $w^j(t)$; the dynamics (1.1) then simplify to

$$R^j_{k+1} = AR^j_k + BM^j_k, \; R^j_1 = e_j, \tag{1.3}$$

where we have used the fact that the resulting state and control responses satisfy, by definition,

$$\begin{array}{rcccccl} x^j_k & = & \sum_{t=1}^k R_t w^j_{k-t} & = & \sum_{t=1}^k R_t e_j \delta_{k-t} & = & R^j_k \\ u^j_k & = & \sum_{t=1}^k M_t w^j_{k-t} & = & \sum_{t=1}^k M_t e_j \delta_{k-t} & = & M^j_k. \end{array}$$

Therefore, concatenating the respective responses, we have that the responses $R(t)$ and $M(t)$ must satsify

$$R_{k+1} = AR_k + BM_k, \; R_1 = I. \tag{1.4}$$

Our next claim is that since $R(t)$ and $M(t)$ satisfy $R_0 = 0$ and $M_0 = 0$, the controller can be implemented directly as $u(t) = (M \star w)(t)$ by noting that u_k only depends on $w_1, \dots, w_{k-1}$, and that at time k, one can compute $w_{k-1} = x_k = (Ax_{k-1} + Bu_{k-1})$. It therefore follows from the constraints (1.4) that the desired state response is also achieved. To make this explicit, notice that for $w(t) = w^j(t) = e_j \delta_k$ we have from the dynamics (1.1) that

$$x_1^j = e_j = R_1^j.$$

Similarly, we have that

$$x_2^j = Ax_1^j + Bu_1^j = AR_1^j + BM_1^j = R_2^j$$

where the final equality follows from the constraints (1.4). This argument is easily extended to any $k \geq 2$ by induction.

Note that this argument is valid over finite or infinite horizons – note that in the infinite horizon case, additional stability constraints must be imposed on $R(t)$ and $M(t)$ to ensure that the resulting closed loop system is itself stable.

2 Finite-horizon LQR

2.1 Zero IC, Gaussian Noise

Here we consider the finite-horizon LQR problem with zero initial condition, driven by Gaussian noise:

$$\begin{array}{rl} \min_{x(t),u(t)} & \sum_{k=1}^{T} \mathbb{E}\left[x_k^\mathsf{T} R x_k + u_k^\mathsf{T} S u_k\right] + \mathbb{E}\left[x_{T+1}^\mathsf{T} Q_F x_{T+1}\right] \\ \text{subject to} & x_{k+1} = Ax_k + Bu_k + w_k,\ x_0 = 0, \end{array} \tag{2.1}$$

for $w_k \overset{\text{i.i.d}}{\sim} \mathcal{N}(0, I)$ (note that this is wlog assuming a non-degenerate distribution for the process noise). We now show how this problem can be recast in terms of the system responses $R(t)$ and $M(t)$. First note that given equations (1.2), it follows that

$$\begin{array}{rcl} x_k & = & \sum_{t=1}^{k} R_t w_{k-t} \\ u_k & = & \sum_{t=1}^{k} M_t w_{k-t}. \end{array} \tag{2.2}$$

Exploiting the i.i.d. nature of the disturbance, we can then write

$$\begin{array}{rcl} \mathbb{E}\left[x_k^\mathsf{T} Q x_k\right] & = & \sum_{t=1}^{k} \mathbf{Tr}\, R_t^\mathsf{T} Q R_t \\ \mathbb{E}\left[u_k^\mathsf{T} S u_k\right] & = & \sum_{t=1}^{k} \mathbf{Tr}\, M_t^\mathsf{T} S M_t \\ \mathbb{E}\left[x_{T+1}^\mathsf{T} Q_F x_{T+1}\right] & = & \sum_{t=1}^{T+1} \mathbf{Tr}\, R_t^\mathsf{T} Q_F R_t. \end{array} \tag{2.3}$$

Plugging equations (2.3) into the objective function of optimization problem (2.1), collecting like terms and optimizing over system responses as opposed to state and input trajectories, we may rewrite the standard LQR problem as

$$\begin{array}{rl} \min_{R(t),M(t)} & \sum_{\tau=1}^{T} (T+1-\tau)\left[\mathbf{Tr}\, R_\tau^\mathsf{T} Q R_\tau + \mathbf{Tr}\, M_\tau^\mathsf{T} S M_\tau\right] + \sum_{s=1}^{T+1} \mathbf{Tr}\, R_s^\mathsf{T} Q_F R_s \\ \text{subject to} & R_{k+1} = AR_k + BM_k,\ R_1 = I. \end{array} \tag{2.4}$$

2.2 Nonzero IC, no noise

Here we consider the finite-horizon LQR problem with nonzero initial condition and no noise:

$$\begin{array}{rl} \min_{x(t),u(t)} & \sum_{k=1}^{T} \left[x_k^\mathsf{T} R x_k + u_k^\mathsf{T} S u_k\right] + x_{T+1}^\mathsf{T} Q_F x_{T+1} \\ \text{subject to} & x_{k+1} = Ax_k + Bu_k,\ x_0 = \xi. \end{array} \tag{2.5}$$

In this case, we can define the disturbance signal driving the system to be

$$w(t) := \xi\delta_k = (\xi, 0, 0, \ldots). \tag{2.6}$$

It then follows that

$$\begin{array}{rclclcl} x_k & = & \sum_{t=1}^k R_t w_{k-t} & = & \sum_{t=1}^k R_t \xi \delta_{k-t} & = & R_k \xi \\ u_k & = & \sum_{t=1}^k M_t w_{k-t} & = & \sum_{t=1}^k M_t \xi \delta_{k-t} & = & M_k \xi, \end{array} \tag{2.7}$$

meaning that the the LQR problem (2.5) can be rewritten as

$$\begin{array}{rl} \min_{R(t),M(t)} & \sum_{k=1}^T \left[\mathbf{Tr}\, R_k^\mathsf{T} Q R_k \Xi + \mathbf{Tr}\, M_k^\mathsf{T} S M_k \Xi\right] + \mathbf{Tr}\, R_{T+1}^\mathsf{T} Q_F R_{T+1} \Xi \\ \text{subject to} & R_{k+1} = AR_k + BM_k,\ R_1 = I, \end{array} \tag{2.8}$$

where $\Xi := \xi\xi^\mathsf{T}$.

2.3 Putting them together

We now consider the standard LQR problem with nonzero initial condition and driving process noise

$$\begin{array}{rl} \min_{x(t),u(t)} & \sum_{k=1}^T \mathbb{E}\left[x_k^\mathsf{T} R x_k + u_k^\mathsf{T} S u_k\right] + \mathbb{E}\left[x_{T+1}^\mathsf{T} Q_F x_{T+1}\right] \\ \text{subject to} & x_{k+1} = Ax_k + Bu_k + w_k,\ x_0 = \xi, \end{array} \tag{2.9}$$

for $w_k \overset{\text{i.i.d}}{\sim} \mathcal{N}(0, I)$. Exploiting linearity and the fact that $\mathbb{E}\left[w_k \xi^\mathsf{T}\right] = 0$ for all k we can simply combine the results of the previous subsections, allowing us to rewrite the standard LQR problem (2.9) as

$$\begin{array}{rl} \min_{R(t),M(t)} & J_{\text{stoch}}(R(t), M(t)) + J_{\text{det}}(R(t), M(t), \Xi) \\ \text{subject to} & R_{k+1} = AR_k + BM_k,\ R_1 = I, \end{array} \tag{2.10}$$

where

$$\begin{array}{rcl} J_{\text{stoch}}(R(t), M(t)) & := & \sum_{\tau=1}^T (T + 1 - \tau) \left[\mathbf{Tr}\, R_\tau^\mathsf{T} Q R_\tau + \mathbf{Tr}\, M_\tau^\mathsf{T} S M_\tau\right] + \sum_{s=1}^{T+1} \mathbf{Tr}\, R_s^\mathsf{T} Q_F R_s \\ J_{\text{det}}(R(t), M(t), \Xi) & := & \sum_{k=1}^T \left[\mathbf{Tr}\, R_k^\mathsf{T} Q R_k \Xi + \mathbf{Tr}\, M_k^\mathsf{T} S M_k \Xi\right] + \mathbf{Tr}\, R_{T+1}^\mathsf{T} Q_F R_{T+1} \Xi. \end{array} \tag{2.11}$$