Working Note: Time-Domain System Level Synthesis and MPC

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1 A time-domain derivation in terms of truncated Toeplitz operators

Consider the system

$$x_{k+1} = Ax_k + Bu_k + w_k. \tag{1.1}$$

In what follows, we will use $z(t)$ to denote the signal z over time, and z_k to denote its instantaneous value at $t = k$, i.e., $z(t) = (z_0, z_1, \dots)$. For a matrix X, we use X^j to denote its jth column; we also sometimes overload notation and use a superscript as a label for a vector, i.e., we will sometimes write $x^j(t)$ to denote a specific instantiation of a generic signal x.

Our goal is to characterize the closed loop maps from $w(t) \to x(t)$ and $w(t) \to u(t)$ under the assumption that $u(t) = (K \star x)(t)$ for some LTI controller $K(t)$. Here $(g \star h)(t)$ denotes the convolution of signals $g(t)$ and $h(t)$. Specifically, we seek filters $R(t)$ and $M(t)$ such that

$$\begin{array}{rcl} x(t) & = & (R \star w)(t) \\ u(t) & = & (M \star w)(t), \end{array} \tag{1.2}$$

under the assumption that $u(t) = (K \star x)(t)$.

First note that for any disturbance signal $w(t)$ we can write $w(t) = \sum_k \sum_j w_{jk} e_j \delta_{k-t}$, and hence by linearity of our dynamics (1.1), it suffices to characterize the impulse responses of the system due to $w^j(t) := e_j \delta_k$. To that end, we write $x^j(t)$ and $u^j(t)$ to denote the state and control responses due to the jth impulse disturbance $w^j(t)$; the dynamics (1.1) then simplify to

$$R^j_{k+1} = AR^j_k + BM^j_k,\ R^j_1 = e_j, \tag{1.3}$$

where we have used the fact that the resulting state and control responses satisfy, by definition,

$$\begin{array}{rcccccl} x^j_k & = & \sum_{t=1}^k R_t w^j_{k-t} & = & \sum_{t=1}^k R_t e_j \delta_{k-t} & = & R^j_k \\ u^j_k & = & \sum_{t=1}^k M_t w^j_{k-t} & = & \sum_{t=1}^k M_t e_j \delta_{k-t} & = & M^j_k. \end{array}$$

Therefore, concatenating the respective responses, we have that the responses $R(t)$ and $M(t)$ must satsify

$$R_{k+1} = AR_k + BM_k,\ R_1 = I. \tag{1.4}$$

Our next claim is that since $R(t)$ and $M(t)$ satisfy $R_0 = 0$ and $M_0 = 0$, the controller can be implemented directly as $u(t) = (M \star w)(t)$ by noting that u_k only depends on $w_1, \dots, w_{k-1}$, and that at time k, one can compute $w_{k-1} = x_k - (Ax_{k-1} + Bu_{k-1})$. It therefore follows from the constraints (1.4) that the desired state response is also achieved. To make this explicit, notice that for $w(t) = w^j(t) = e_j\delta_k$ we have from the dynamics (1.1) that

$$x_1^j = e_j = R_1^j.$$

Similarly, we have that

$$x_2^j = Ax_1^j + Bu_1^j = AR_1^j + BM_1^j = R_2^j$$

where the final equality follows from the constraints (1.4). This argument is easily extended to any $k \geq 2$ by induction.

Note that this argument is valid over finite or infinite horizons – note that in the infinite horizon case, additional stability constraints must be imposed on $R(t)$ and $M(t)$ to ensure that the resulting closed loop system is itself stable.

2 Finite-horizon LQR

2.1 Zero IC, Gaussian Noise

Here we consider the finite-horizon LQR problem with zero initial condition, driven by Gaussian noise:

$$\begin{array}{rl} \min_{x(t),u(t)} & \sum_{k=1}^{T} \mathbb{E}\left[x_k^\mathsf{T} Q x_k + u_k^\mathsf{T} S u_k\right] + \mathbb{E}\left[x_{T+1}^\mathsf{T} Q_F x_{T+1}\right] \\ \text{subject to} & x_{k+1} = Ax_k + Bu_k + w_k,\ x_0 = 0, \end{array} \tag{2.1}$$

for $w_k \overset{\text{i.i.d}}{\sim} \mathcal{N}(0, I)$ (note that this is wlog assuming a non-degenerate distribution for the process noise). We now show how this problem can be recast in terms of the system responses $R(t)$ and $M(t)$. First note that given equations (1.2), it follows that

$$\begin{array}{rcl} x_k & = & \sum_{t=1}^{k} R_t w_{k-t} \\ u_k & = & \sum_{t=1}^{k} M_t w_{k-t}. \end{array} \tag{2.2}$$

Exploiting the i.i.d. nature of the disturbance, we can then write

$$\begin{array}{rcl} \mathbb{E}\left[x_k^\mathsf{T} Q x_k\right] & = & \sum_{t=1}^{k} \mathbf{Tr}\, R_t^\mathsf{T} Q R_t \\ \mathbb{E}\left[u_k^\mathsf{T} S u_k\right] & = & \sum_{t=1}^{k} \mathbf{Tr}\, M_t^\mathsf{T} S M_t \\ \mathbb{E}\left[x_{T+1}^\mathsf{T} Q_F x_{T+1}\right] & = & \sum_{t=1}^{T+1} \mathbf{Tr}\, R_t^\mathsf{T} Q_F R_t. \end{array} \tag{2.3}$$

Plugging equations (2.3) into the objective function of optimization problem (2.1), collecting like terms and optimizing over system responses as opposed to state and input trajectories, we may rewrite the standard LQR problem as

$$\begin{array}{rl} \min_{R(t),M(t)} & \sum_{\tau=1}^{T} (T+1-\tau)\left[\mathbf{Tr}\, R_\tau^\mathsf{T} Q R_\tau + \mathbf{Tr}\, M_\tau^\mathsf{T} S M_\tau\right] + \sum_{s=1}^{T+1} \mathbf{Tr}\, R_s^\mathsf{T} Q_F R_s \\ \text{subject to} & R_{k+1} = AR_k + BM_k,\ R_1 = I. \end{array} \tag{2.4}$$

2.2 Nonzero IC, no noise

Here we consider the finite-horizon LQR problem with nonzero initial condition and no noise:

$$\begin{array}{rl} \min_{x(t),u(t)} & \sum_{k=1}^{T} \left[x_k^\mathsf{T} Q x_k + u_k^\mathsf{T} S u_k\right] + x_{T+1}^\mathsf{T} Q_F x_{T+1} \\ \text{subject to} & x_{k+1} = Ax_k + Bu_k,\ x_0 = \xi. \end{array} \tag{2.5}$$

In this case, we can define the disturbance signal driving the system to be

$$w(t) := \xi\delta_k = (\xi, 0, 0, \dots). \tag{2.6}$$

It then follows that

$$\begin{array}{rclclcl} x_k &=& \sum_{t=1}^k R_t w_{k-t} &=& \sum_{t=1}^k R_t \xi \delta_{k-t} &=& R_k \xi \\ u_k &=& \sum_{t=1}^k M_t w_{k-t} &=& \sum_{t=1}^k M_t \xi \delta_{k-t} &=& M_k \xi, \end{array} \tag{2.7}$$

meaning that the the LQR problem (2.5) can be rewritten as

$$\begin{array}{rl} \min_{R(t),M(t)} & \sum_{k=1}^T \left[\mathbf{Tr}\, R_k^\top Q R_k \Xi + \mathbf{Tr}\, M_k^\top S M_k \Xi\right] + \mathbf{Tr}\, R_{T+1}^\top Q_F R_{T+1} \Xi \\ \text{subject to} & R_{k+1} = AR_k + BM_k,\ R_1 = I, \end{array} \tag{2.8}$$

where $\Xi := \xi\xi^\top$.

2.3 Putting them together

We now consider the standard LQR problem with nonzero initial condition and driving process noise

$$\begin{array}{rl} \min_{x(t),u(t)} & \sum_{k=1}^T \mathbb{E}\left[x_k^\top Q x_k + u_k^\top S u_k\right] + \mathbb{E}\left[x_{T+1}^\top Q_F x_{T+1}\right] \\ \text{subject to} & x_{k+1} = Ax_k + Bu_k + w_k,\ x_0 = \xi, \end{array} \tag{2.9}$$

for $w_k \overset{\text{i.i.d}}{\sim} \mathcal{N}(0, I)$. Exploiting linearity and the fact that $\mathbb{E}\left[w_k \xi^\top\right] = 0$ for all k we can simply combine the results of the previous subsections, allowing us to rewrite the standard LQR problem (2.9) as

$$\begin{array}{rl} \min_{R(t),M(t)} & J_{\text{stoch}}(R(t), M(t)) + J_{\text{det}}(R(t), M(t), \Xi) \\ \text{subject to} & R_{k+1} = AR_k + BM_k,\ R_1 = I, \end{array} \tag{2.10}$$

where

$$\begin{array}{rcl} J_{\text{stoch}}(R(t), M(t)) &:=& \sum_{\tau=1}^T (T+1-\tau)\left[\mathbf{Tr}\, R_\tau^\top Q R_\tau + \mathbf{Tr}\, M_\tau^\top S M_\tau\right] + \sum_{s=1}^{T+1} \mathbf{Tr}\, R_s^\top Q_F R_s \\ J_{\text{det}}(R(t), M(t), \Xi) &:=& \sum_{k=1}^T \left[\mathbf{Tr}\, R_k^\top Q R_k \Xi + \mathbf{Tr}\, M_k^\top S M_k \Xi\right] + \mathbf{Tr}\, R_{T+1}^\top Q_F R_{T+1} \Xi. \end{array} \tag{2.11}$$

This will be the form we will want to use for studying the MPC problem (modulo the addition of a Δ term to capture model error) – below I show how to more compactly write the objective function in terms of block-Toeplitz operators that are more amenable to analysis, but not necessarily computation.

3 A redo of the above using block-Toeplitz operators

In the following we work with the signals and block-Toeplitz lower-triangular operators defined below:

$$\mathbf{x} = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_T \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_T \end{bmatrix} \quad \mathbf{w} = \begin{bmatrix} w_{-1} \\ w_0 \\ \vdots \\ w_{T-1} \end{bmatrix} \quad \mathbf{R} = \begin{bmatrix} R_1 & & & \\ R_2 & R_1 & & \\ \vdots & \ddots & \ddots & \\ R_T & \cdots & R_2 & R_1 \end{bmatrix} \quad \mathbf{M} = \begin{bmatrix} M_1 & & & \\ M_2 & M_1 & & \\ \vdots & \ddots & \ddots & \\ M_T & \cdots & M_2 & M_1 \end{bmatrix}. \tag{3.1}$$

Note that the disturbance signal $\mathbf{w}$ begins at $k = -1$, whereas the remaining signals begin at $k = 0$ – this allows us to write the strictly proper operators $\mathbf{R}$ and $\mathbf{M}$ as lower-block-triangular, as opposed to strictly lower-block-triangular, matrices. This will be more convenient. Further, if x_0 is known to be equal to some value ξ, this can be encoded in the above framework by making w_{-1} have mean ξ and covariance 0.

Now let Z be the block-downshift operator, i.e., a matrix with identity matrices along its first block sub-diagonal and zeros elsewhere. We can then equivalently write equation (1.2) as

$$\begin{array}{rcl} \mathbf{x} & = & \mathbf{R}\mathbf{w} \\ \mathbf{u} & = & \mathbf{M}\mathbf{w}. \end{array} \tag{3.2}$$

Following the same reasoning as above, it is straightforward to show that the responses $\{\mathbf{R}, \mathbf{M}\}$ must satisfy the equation

$$\begin{bmatrix} I - Z\mathcal{A} & -Z\mathcal{B} \end{bmatrix} \begin{bmatrix} \mathbf{R} \\ \mathbf{M} \end{bmatrix} = I, \tag{3.3}$$

where

$$\mathcal{A} := \text{blkdiag}\left(A_1, A_2, \ldots, A_T\right), \ \mathcal{B} := \text{blkdiag}\left(B_1, B_2, \ldots, B_T\right), \tag{3.4}$$

Remark 3.1. *Notice that time-varying dynamics are now trivial to incorporate into this framework.*

Theorem 3.2. *For the dynamics (1.1) with block-Toeplitz lower-triangular state feedback law $\mathbf{K}$ defining the control action as $\mathbf{u} = \mathbf{K}\mathbf{x}$, the following are true*

1. *the affine subspace defined by (3.3) parameterizes all possible responses (3.2) of the system.*
2. *for any block-Toeplitz matrices $\{\mathbf{R}, \mathbf{M}\}$ satisfying (3.3), the controller $\mathbf{K} = \mathbf{M}\mathbf{R}^{-1}$ achieves the desired response.*

Proof. First note that we can equivalently write the dynamics (1.1) as

$$\mathbf{x} = Z\mathcal{A}\mathbf{x} + Z\mathcal{B}\mathbf{u} + \mathbf{w} \tag{3.5}$$

Proof of 1.: Let $\mathbf{K}$ be any block-Toeplitz lower-triangular operator, and $\mathbf{u} = \mathbf{K}\mathbf{x}$. Then

$$\mathbf{x} = Z(\mathcal{A} + \mathcal{B}\mathbf{K})\mathbf{x} + \mathbf{w}, \tag{3.6}$$

which implies that

$$\begin{array}{rcl} \mathbf{x} & = & (I - Z(\mathcal{A} + \mathcal{B}\mathbf{K}))^{-1}\mathbf{w} \\ \mathbf{u} & = & \mathbf{K}(I - Z(\mathcal{A} + \mathcal{B}\mathbf{K}))^{-1}\mathbf{w}. \end{array} \tag{3.7}$$

It is then easily seen that

$$\begin{bmatrix} I - Z\mathcal{A} & -Z\mathcal{B} \end{bmatrix} \begin{bmatrix} (I - Z(\mathcal{A} + \mathcal{B}\mathbf{K}))^{-1} \\ \mathbf{K}(I - Z(\mathcal{A} + \mathcal{B}\mathbf{K}))^{-1} \end{bmatrix} = (I - Z\mathcal{A} - Z\mathcal{B}\mathbf{K})(I - Z(\mathcal{A} + \mathcal{B}\mathbf{K}))^{-1} = I. \tag{3.8}$$

Proof of 2.: Let $\mathbf{K} = \mathbf{M}\mathbf{R}^{-1}$. Then

$$\mathbf{x} = (I - Z(\mathcal{A} + \mathcal{B}\mathbf{M}\mathbf{R}^{-1}))^{-1}\mathbf{w}. \tag{3.9}$$

But we then have that

$$(I - Z(\mathcal{A} + \mathcal{B}\mathbf{M}\mathbf{R}^{-1}))^{-1} = ((I - Z\mathcal{A})\mathbf{R} - Z\mathcal{B}\mathbf{M})\mathbf{R}^{-1})^{-1} = \mathbf{R}((I - Z\mathcal{A})\mathbf{R} - Z\mathcal{B}\mathbf{M})^{-1} = \mathbf{R} \tag{3.10}$$

where the last equality follows form the fact that $\{\mathbf{R}, \mathbf{M}\}$ satisfy (3.3). Similarly we have that

$$\mathbf{u} = \mathbf{M}\mathbf{R}^{-1}\mathbf{x} = \mathbf{M}\mathbf{R}^{-1}\mathbf{R}\mathbf{w} = \mathbf{M}\mathbf{w}, \tag{3.11}$$

where the second equality follows from the fact that $\mathbf{x} = \mathbf{R}\mathbf{w}$. ∎

Next we consider a robust variant of the above theorem. Before stating the result, we let

$$\mathbf{\Delta} = \begin{bmatrix} \Delta_1 & & & \\ \Delta_2 & \Delta_1 & & \\ \vdots & \ddots & \ddots & \\ \Delta_{T+1} & \cdots & \Delta_2 & \Delta_1 \end{bmatrix} \tag{3.12}$$

be an arbitrary block-Toeplitz lower-triangular matrix.

Theorem 3.3. *Let $\mathbf{\Delta}$ be defined as in equation (3.12), and suppose that $\{\mathbf{R}, \mathbf{M}\}$ satisfy*

$$\begin{bmatrix} I - Z\mathcal{A} & -Z\mathcal{B} \end{bmatrix} \begin{bmatrix} \mathbf{R} \\ \mathbf{M} \end{bmatrix} = I + \mathbf{\Delta}. \tag{3.13}$$

If $(I + \Delta_1)^{-1}$ exists, then the controller $\mathbf{K} = \mathbf{M}\mathbf{R}^{-1}$ achieves the system response

$$\begin{bmatrix} \mathbf{R} \\ \mathbf{M} \end{bmatrix} (I + \mathbf{\Delta})^{-1} \tag{3.14}$$

Proof. If $(I+\Delta_1)^{-1}$ exists, then so does $(I+\mathbf{\Delta})^{-1}$, and therefore the constraint (3.13) is equivalent to

$$\begin{bmatrix} I - Z\mathcal{A} & -Z\mathcal{B} \end{bmatrix} \begin{bmatrix} \mathbf{R} \\ \mathbf{M} \end{bmatrix} (I + \mathbf{\Delta})^{-1} = I. \tag{3.15}$$

Then noting that $\mathbf{K} = \mathbf{M}\mathbf{R}^{-1} = \mathbf{M}(I + \mathbf{\Delta})^{-1}(\mathbf{R}(I + \mathbf{\Delta})^{-1})^{-1}$, we then have, by Theorem 3.2, that the controller $\mathbf{K} = \mathbf{M}\mathbf{R}^{-1}$ achieves the system responses (3.14). ∎

4 LQR optimal control problems

Going forward, we assume that $\mathbf{w} \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, I)$.[1]

Without loss of generality, we assume that u_T is included in the cost functional, i.e., that the cost we seek to minimize is

$$\begin{array}{rl} \min_{x(t),u(t)} & \sum_{k=0}^{T} \mathbb{E}\left[x_k^\mathsf{T} Q_k x_k + u_k^\mathsf{T} S_k u_k\right] \\ \text{subject to} & x_{k+1} = Ax_k + Bu_k + w_k. \end{array} \tag{4.1}$$

[1]If $x_0 = \xi$ is known, then a similar decomposition technique as done above can be used to split the cost into a stochastic and deterministic component by exploiting independence between $x_0 = w_{-1}$ and the rest of the disturbance signal.

This problem can be rewritten as

$$\begin{aligned} \min_{\mathbf{R},\mathbf{M}} \quad & \left\| \begin{bmatrix} \mathcal{Q}^{\frac{1}{2}} & 0 \\ 0 & \mathcal{S}^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \mathbf{R} \\ \mathbf{M} \end{bmatrix} \right\|_F^2 \\ \text{subject to} \quad & \begin{bmatrix} I - Z\mathcal{A} & -Z\mathcal{B} \end{bmatrix} \begin{bmatrix} \mathbf{R} \\ \mathbf{M} \end{bmatrix} = I, \end{aligned} \tag{4.2}$$

where

$$\mathcal{Q} := \text{blkdiag}(Q_1, Q_2, \ldots, Q_T), \ \mathcal{S} := \text{blkdiag}(S_1, S_2, \ldots, S_T). \tag{4.3}$$

5 A Robust LQR result

The above conditions/results should look extremely similar to what we have seen in the infinite horizon setting. Let $\hat{\mathcal{A}} = \mathcal{A} + \mathcal{D}_{\mathcal{A}}$ and $\hat{\mathcal{B}} = \mathcal{B} + \mathcal{D}_{\mathcal{B}}$, with the assumption that $\|\mathcal{D}_{\mathcal{A}}\| \le \epsilon_A$ and $\|\mathcal{D}_{\mathcal{B}}\| \le \epsilon_B$. We then have the following robust performance result (note that since we are working in finite-time, there is no concern for stability):

Lemma 5.1. *Let $\hat{\mathcal{A}}, \hat{\mathcal{B}}, \mathcal{D}_{\mathcal{A}}, \mathcal{D}_{\mathcal{B}}$ be defined as above, and let*

$$\hat{\mathbf{\Delta}} := Z \begin{bmatrix} \mathcal{D}_{\mathcal{A}} & \mathcal{D}_{\mathcal{B}} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{R}} \\ \hat{\mathbf{M}} \end{bmatrix}. \tag{5.1}$$

Then if $\hat{\mathbf{R}}, \hat{\mathbf{M}}$ satisfy

$$\begin{bmatrix} I - Z\hat{\mathcal{A}} & -Z\hat{\mathcal{B}} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{R}} \\ \hat{\mathbf{M}} \end{bmatrix} = I, \tag{5.2}$$

the controller $\widehat{K} = \hat{\mathbf{M}}\hat{\mathbf{R}}^{-1}$ achieves the following performance on the true system $(\mathcal{A}, \mathcal{B})$:

$$\left\| \begin{bmatrix} \mathcal{Q}^{\frac{1}{2}} & 0 \\ 0 & \mathcal{S}^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{R}} \\ \hat{\mathbf{M}} \end{bmatrix} g(\hat{\mathbf{\Delta}}) \right\|_F^2, \tag{5.3}$$

where

$$g(\hat{\mathbf{\Delta}}) := (I + \hat{\mathbf{\Delta}})^{-1} = \sum_{t=0}^{T-1} \hat{\mathbf{\Delta}}^t. \tag{5.4}$$

Proof. This follows immediately from Theorem 3.2 and by noting that $\hat{\mathbf{\Delta}}$ is strictly block-lower-triangular, and hence nilpotent. ∎

Lemma 5.1 then immediately suggests the following robust synthesis procedure, which is convex for any fixed $\gamma > 0$:

$$\begin{aligned} \min_{\gamma>0} \min_{\mathbf{R},\mathbf{M},\tau>0} \quad & \gamma^2 \left\| \begin{bmatrix} \mathcal{Q}^{\frac{1}{2}} & 0 \\ 0 & \mathcal{S}^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \mathbf{R} \\ \mathbf{M} \end{bmatrix} \right\|_F^2 \\ \text{subject to} \quad & \begin{bmatrix} I - Z\hat{\mathcal{A}} & -Z\hat{\mathcal{B}} \end{bmatrix} \begin{bmatrix} \mathbf{R} \\ \mathbf{M} \end{bmatrix} = I, \ \sqrt{2} \left\| \begin{matrix} \epsilon_A \mathbf{R} \\ \epsilon_B \mathbf{M} \end{matrix} \right\| \le \tau, \ \textstyle\sum_{t=0}^{T} \tau^t \le \gamma \end{aligned} \tag{5.5}$$

where we have obtained an upper-bound to the cost (5.3) by applying the sub-multiplicative property, the triangle inequality and the trick to upper-bound the spectral-norm of $\hat{\mathbf{\Delta}}$.

With this in hand, we can now step through the same proof procedure as the learning LQR paper to get a relative performance bound for this controller in terms of the error sizes ϵ_A and ϵ_B. In particular, if we let

$$\zeta := (\epsilon_A + \epsilon_B \|\mathbf{K}_\star\|)\|\mathfrak{R}_{\mathcal{A}+\mathcal{B}\mathbf{K}_\star}\|, \tag{5.6}$$

where we define the finite-time resolvent $\mathfrak{R}_{\mathbf{M}}$ of an operator $\mathbf{M}$ to be

$$\mathfrak{R}_{\mathbf{M}} := (I - Z\mathbf{M})^{-1}, \tag{5.7}$$

then we can bound the relative performance achieved by[2]

$$\frac{J(\mathcal{A}, \mathcal{B}, \widehat{K}) - J_\star}{J_\star} \leq \left[\frac{(1-\gamma_0^T)(1-\zeta)}{(1-\gamma^0)(1-\zeta)}\right]^2, \tag{5.8}$$

where ζ is as defined above, and

$$\gamma_0 := \frac{\sqrt{2}\zeta(1-\zeta^T)}{1-\zeta}. \tag{5.9}$$

Note that we have used the relation

$$\sum_{t=0}^{T-1} r^t = \frac{1-r^T}{1-r} \tag{5.10}$$

repeatedly in the above.

6 Extending this to Analyzing MPC

So what we have above is essentially a sub-routine for MPC that could be called at each update. So here is what I would suggest considering as a first problem to analyze: pick the baseline to be a standard LQR MPC problem with a terminal state-cost specified by the solution to the Riccati equation. This ensures that MPC behaves exactly like an optimal infinite horizon LQR controller. This then opens up some interesting questions:

- How small do ϵ_A and ϵ_B have to be to ensure that an MPC scheme implemented using the above robust controller is stable?
- Can we quantify the end-to-end performance loss in the MPC scheme?
- Can we suitably pick a terminal cost (or equivalently add constraints to our robust synthesis problem) such that the end-to-end performance loss in the MPC scheme is the same as that in the infinite horizon setting we have already considered?

The last question is a bit of a flyer, but given that for known (A, B) we know how to pick a terminal cost to get optimal infinite horizon performance, I think it's an interesting question to investigate.

[2]Note the costs here are the squared Frobenius norms, so the actual LQR costs, not their square roots.

Once asymptotic stability is established, the next interesting question I think is considering robust satisfaction of state/input constraints over one iteration, i.e., take the robust problem I analyzed above, and ask that $\mathbf{x} \in \mathcal{X}$ and $\mathbf{u} \in \mathcal{U}$. What is the appropriate notion of a robust-invariant set in this setting where we have this funny $(I + \hat{\mathbf{\Delta}})^{-1}$ term modulating our disturbance signal. I have some ideas on this, but they are all quite primitive.

Once the robust feasibility of one iteration is guaranteed, the next task will be to prove recursive feasibility: I suspect that if we get these robust-invariant sets right in the previous step, then this should be pretty straightforward.

I think that with these three pieces, stability, feasibility and recursive feasibility, we have most (all?) of the tools needed to analyze Ugo's algorithm.