CPSC-354 Report

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Abstract

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2.1 Week 1

2.1.1 Notes and Exploration

In Week 1, we began with the MIU (or MU) puzzle from Hofstadter's *Gödel, Escher, Bach*. This puzzle introduces the idea of formal systems and rules of inference. It is a good starting point to think about what it means to derive a string from an axiom under a fixed set of rules.

The MU Puzzle

RULES:

1. (Append-`U`) If a string ends with `I`, you may append `U`:
   $x\texttt{I} \to x\texttt{IU}$.

2. (Double) From $\texttt{M}x$ you may produce $\texttt{M}xx$:
   $\texttt{M}x \to \texttt{M}xx$.
3. (III$\to$U) Replace any occurrence of `III` with `U`:
   $x\texttt{III}y \to x\texttt{U}y$.
4. (Delete `UU`) Delete any occurrence of `UU`:
   $x\texttt{UU}y \to xy$.

2.1.2 Homework

Problem: Can you derive `MU` from `MI`?

Solution: No, it is impossible to derive `MU` from `MI`. At first, playing around with the rules, I noticed that the goal was to create the correct number of `I`'s so that they could be converted to a single `U`. This means we would need $N_I \bmod 3 = 0$, where N_I counts the `I`'s.

Now, consider how each rule affects N_I:

- **(Append-`U`)** Appends a `U`, leaves N_I unchanged.
- **(Double)** $\texttt{M}x \to \texttt{M}xx$ doubles N_I; in modular arithmetic, $N_I \mapsto 2N_I \pmod 3$.
- **(III$\to$U)** Removes three `I`'s, leaving $N_I \bmod 3$ unchanged.
- **(Delete `UU`)** Only touches `U`'s, leaves N_I unchanged.

Starting from `MI`, we have $N_I = 1 \equiv 1 \pmod 3$. Doubling cycles between 1 and 2 modulo 3, never producing 0. Thus, it is impossible to reach $N_I \equiv 0 \pmod 3$.

Since `MU` has $N_I = 0$, it cannot be derived from `MI`.

2.1.3 Questions

What is the reasoning behind being able to convert `MIII` into `MU` (using the rule $\texttt{III} \to \texttt{U}$) but not being able to go the other way (from `MU` to `MIII`)?

2.2 Week 2

2.2.1 Homework

Problem: Consider the following ARSs. Draw a picture for each one. Are the ARSs terminating? Are they confluent? Do they have unique normal forms?

1. $A = \{\}, \quad R = \{\}$
2. $A = \{a\}, \quad R = \{\}$
3. $A = \{a\}, \quad R = \{(a,a)\}$
4. $A = \{a,b,c\}, \quad R = \{(a,b),(a,c)\}$
5. $A = \{a,b\}, \quad R = \{(a,a),(a,b)\}$
6. $A = \{a,b,c\}, \quad R = \{(a,b),(b,b),(a,c)\}$
7. $A = \{a,b,c\}, \quad R = \{(a,b),(b,b),(a,c),(c,c)\}$

Solution: For each ARS, I drew a graph (see figures) and analyzed:

- **Termination:** whether there are infinite chains. - **Confluence:** whether every divergence can rejoin. - **Unique normal forms:** whether each element has a unique NF.

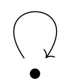
I will include one diagram for each ARS along with a short explanation of my analysis.

ARS 1: $A = \{\},\ R = \{\}$ There is no infinite chain, no diverging paths exist as there is no path at all, and nothing exists to violate uniqueness. Therefore ✓ terminating, ✓ confluent, and ✓ has unique normal form.

ARS 2: $A = \{a\},\ R = \{\}$

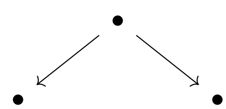
No rewrite steps: no infinite chain, no diverging paths exist or no path at all so it is vacuously satisfied and a is the only reachable normal form from a. Therefore ✓ terminating, ✓ confluent, and ✓ has unique normal form.

ARS 3: $A = \{a\},\ R = \{(a, a)\}$



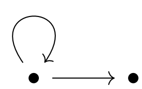
There is an infinite chain, $a \to^* a$. Since $a \to^* y$ and $a \to^* z$, therefore $y = z = a$, joining at a, and there are no normal forms as there is only one term, a, which has infinite outgoing rewrite steps. Therefore × terminating, ✓ confluent, and × unique normal form.

ARS 4: $A = \{a, b, c\},\ R = \{(a, b), (a, c)\}$



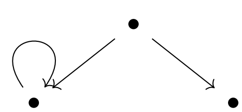
There is no infinite chain, two diverging paths that do not get joined, and two different normal forms from a. Therefore ✓ terminating, × confluent, and × unique normal form.

ARS 5: $A = \{a, b\},\ R = \{(a, a), (a, b)\}$



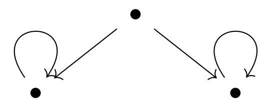
Even with one infinite chain it does not terminate, $a \to a$ and $a \to b$ and they join at b, b is the only normal form since a has an infinite chain therefore × terminating, ✓ confluent, has a ✓ unique normal form.

ARS 6: $A = \{a, b, c\},\ R = \{(a, b), (b, b), (a, c)\}$



Infinite chain at b (since $b \to b \to b \cdots$). From a the system diverges to b and c with no connecting path to join them. The only normal form is c, but not every element reduces to a unique normal form, so the system is × terminating, × confluent, and × has unique normal form.

ARS 7: $A = \{a, b, c\},\ R = \{(a, b), (a, c), (b, b), (c, c)\}$



There are two infinite chains ($b \to b \to \cdots$ and $c \to c \to \cdots$). From a the system diverges to b and c, but since b only reaches b and c only reaches c, the peak at a does not join. Every element has outgoing rewrite steps, so there are no normal forms. Therefore $\times$ terminating, $\times$ confluent, and $\times$ has unique normal form.

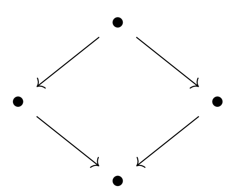
All Eight Combinations

The homework also asked to find examples of ARSs for each of the eight possible combinations of confluence, termination, and unique normal forms. The table below summarizes the results.

Confluent	**Terminating**	**Unique Normal Forms**	**Example** (A, R)
True	True	True	$A = \{a, b, c, d\},\ R = \{(a, b), (a, c), (b, d), (c, d)\}$
True	True	False	N/A
True	False	True	$A = \{a, b, c, d\},\ R = \{(a, a), (a, b), (a, c), (b, d), (c, d)\}$
True	False	False	$A = \{a, b, c, d\},\ R = \{(a, b), (a, c), (b, d), (c, d), (d, d)\}$
False	True	True	N/A
False	True	False	$A = \{a, b, c, d\},\ R = \{(a, b), (a, c), (b, d)\}$
False	False	True	$A = \{a, b, c\},\ R = \{(a, b), (a, c), (b, d), (c, d), (d, d)\}$
False	False	False	$A = \{a, b, c\},\ R = \{(a, a), (a, b), (a, c)\}$

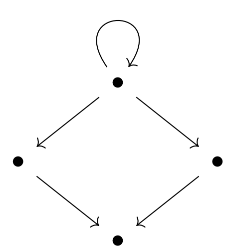
Example Graphs

Graph 1

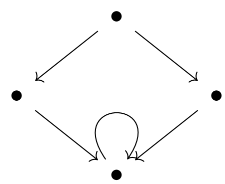


Graph 2 This comination is impossible as termination gives a normal form for each element AND confluence guarantees any two reduction paths from the same element to join therefore all reductions end at some normal form.

Graph 3

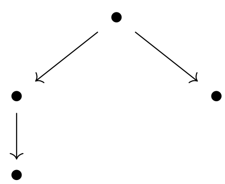


Graph 4

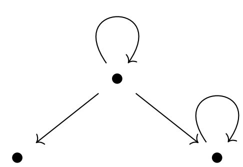


Graph 5 This case is impossible. Every system that is terminating and where every element has a unique normal form must also be confluent. Indeed, if $a \to^* y$ and $a \to^* z$, then both y and z reduce to some normal forms. Since the normal form is unique, y and z must reduce to the same normal form, which means the system is confluent.

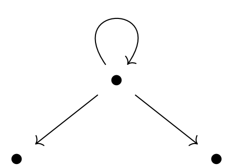
Graph 6



Graph 7



Graph 8



2.2.2 Questions

How can thinking about ARSs help us better understand the way programming languages define and control the process of evaluating programs?

3 Essay

4 Evidence of Participation

5 Conclusion

References

[BLA] Author, Title, Publisher, Year.