Latex Exercise

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An approximation for the exponential function

This should solve part A and B, as C is optional and don't give any points I honestly don't care

Table 1 shows a possible numerical implementation of the Exponential function in the c#

This function does two things, firstly it re-adjust the input x to be between 0 and 1/8, and secondly it calculates a reasonable approximation.

Setting aside the re-adjustment for now, we know that the taylor-expansion of $\exp(x)$ around $x = 0$ is

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{1}$$

For $x \approx 0$ it is a reasonable approximation to only use the first few terms, in this case we use the first 11 terms up until $n = 10$.

$$\exp(x) \approx \sum_{n=0}^{10} \frac{x^n}{n!}. \tag{2}$$

The error made here is on the order $O(x^{11})$, if only the user asked us for small x this would be very reasonable. In practice we want to reduce the number of multiplications used, and we can do that since:

```
static double ex(double x)
{
    if(x<0)           //Adjust the approximation to be positive
        return 1/ex(-x);
    if(x>1.0/8)
        return Pow(ex(x/2),2);//and "small", note I am using 2
           and not 2.0
    //Now use the approximation
    return 1+x*(1+x/2*(1+x/3*(1+x/4*(1+x/5*(1+x/6*(1+x/7*(1+x
       /8*(1+x/9*(1+x/10)))))))));
}
```

Table 1: A simple approximation of the exponential function, using only multiplication and division (the mono c# compiler do interpret `System.Math.Pow(double,int)` as a list of mulitiplications).

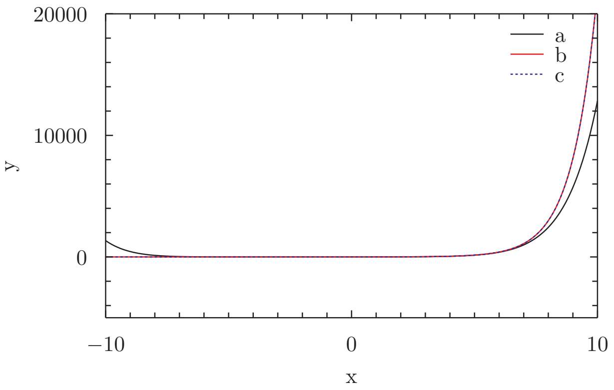

Figure 1: (a) the approximation without correcting the range, (b) the true exponential function and, (c) the approximation correcting x to be within a reasonable range.

$$\begin{aligned}\exp(x) &\approx \sum_{n=0}^{10} \frac{x^n}{n!}, && (3)\\ &\approx 1 + \sum_{n=1}^{10} \frac{x^n}{n!}, && (4)\\ &\approx 1 + \frac{x}{1}\sum_{n=0}^{9} \frac{x^n}{(n+1)!}. && (5)\end{aligned}$$

We can repeat the same trick 9 times on the sum, the result is:

$$\exp(x) \approx 1 + \frac{x}{1}\left(1 + \frac{x}{2}\left(1 + \frac{x}{3}\left(\ldots\left(1 + \frac{x}{9}\right)\right)\right)\right). \tag{6}$$

Now we just have 10 additions 9 divisions and 9 multiplications, which is considerably better, and this is the approximation used in Table 1.

In Figure 1 (a), this approximation is illustrated alongside the exponential function (b). Of course, we still need to have x relatively close to 0, as is evident on the figure the approximation diverges for x not close to 0. This can be fixed by using the relation $\exp(x/2)^2 = \exp(x)$ which is applied (recursively) whenever $x > 1/8$ to get within this range, if $x < 0$ we do know that $\exp(x) = 1/\exp(-x)$, where $-x$ is positive; making sure the argument is positive is necessary in order for the other check ($x > 1/8$) to be applied. So the two first if-statements simply make sure that we are within the range [0,0.125]. This is illustrated in Figure 1 (c). This ensures that the approximation always is applied when x is within where it should remain valid.