A recurrence relation for idempotence

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Basic idea

We work to some numerical accuracy ϵ, and we say that a matrix is *near-idempotent to order n* when it takes the form $k = p + d$, where p is idempotent, and where there exists some $n \in \mathbb{N}$ such that d^{n+1} has coefficients with absolute value below ϵ. The quantity d^{n+1} can therefore considered as negligible.

Suppose that k is near-idempotent to first order, so we have $k = p + d$ where d^2 is negligible. We may wonder whether k is in fact idempotent to first order, which we investigate with the following calculation:

$$k^2 - k = p^2 + pd + dp + dd - p - d = pd + dp - d$$

Perhaps unsurprisingly, we see that k is not idempotent to order 1.

We now define a new operator $k' = 2k^2 - k^4$. (Note that this can be obtained by concatenating the transformations $k \mapsto k^2$ and $k \mapsto 2k - k^2$.) Clearly, if k is idempotent, then $k = k'$; otherwise we may have $k' \neq k$. We compute the value of k' as follows, to first order in d:

$$\begin{aligned}
k' &= 2k^2 - k^4 \\
&= 2(p+d)^2 - (p+d)^4 \\
&= 2(p + pd + dp) - (pppp + pppd + ppdp + pdpp + dppp)) \\
&= 2p + 2pd + 2dp - p - pd - pdp - pdp - dp \\
&= p + pd + dp - 2pdp
\end{aligned}$$

We now ask once again the same question: is k' idempotent to first order in d? We calculate as follows:

$$\begin{aligned}
(k')^2 - k' &= (p + pd + dp - 2pdp)^2 - (p + pd + dp - 2pdp) \\
&= pp + ppd + pdp - 2ppdp + pdp + dpp - 2pdpp - p - pd - dp + 2pdp \\
&= p + \ pd + \ pdp - 2pdp + pdp + \ dp - 2pdp - p - pd - dp + 2pdp \\
&= 0
\end{aligned}$$

Perhaps surprisingly we see that k' is indeed an idempotent to first order in d, even though k was not. The recurrence relation $k' = 2k^2 - k^4$ has *corrected* the idempotent.

Higher order calculation

Above we considered a quartic formula for k'. We may consider a more general ansatz to arbitrary degree:

$$k' = \alpha_0 \mathbb{I} + \alpha_1 k + \alpha_2 k^2 + \alpha_3 k^3 + \alpha_4 k^4 + \alpha_5 k^5 + \cdots$$

This raises a number of questions, in particular:

- Above we showed that the quartic transformation $k' = 2k^2 - k^4$ was able to correct a first-order near-idempotent. Can this also be achieved with a cubic transformation, i.e. with $a_i = 0$ for $i > 3$?
- Can a transformation be found which can correct a second-order near-idempotent? What is the smallest polynomial degree for such a transformation?