On solutions to $\mathbf{K} = 3\mathbf{K}^2 - 2\mathbf{K}^3$

Nikolaj B. Jensen

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We are interested in proving whether there exists any **non-idempotent** matrices $\mathbf{K}$ that are solutions to $\mathbf{K} = 3\mathbf{K}^2 - 2\mathbf{K}^3$. The problem is easy to solve if $\mathbf{K}$ is decomposable through diagonalization, for instance. In general, we cannot assume $\mathbf{K}$ to be diagonalizable, but we can still construct a nearly diagonal matrix (a Jordan Normal Form).

1 Jordan Normal Forms

The Jordan normal form of a matrix $\mathbf{K}$ expresses $\mathbf{K}$ in a form that is nearly diagonal. Specifically, for a square matrix $\mathbf{K}$, there exists an invertible matrix $\mathbf{P}$ such that:

$$\mathbf{K} = \mathbf{P}\mathbf{J}\mathbf{P}^{-1}$$

where $\mathbf{J}$ is a Jordan normal form of $\mathbf{K}$. The Jordan normal form of a $\mathbf{K} \in \mathbb{R}^{m\times n}$ may be a $\mathbf{J} \in \mathbb{C}^{m\times n}$. The block-diagonal matrix $\mathbf{J}$ is composed of Jordan blocks, each associated with an eigenvalue of $\mathbf{K}$. A Jordan block $\mathbf{J}_\lambda$ has size $(d \times d)$, where d is the *algebraic* multiplicity of λ, $d = \alpha(\lambda)$. We further decompose each $\mathbf{J}_\lambda$ into square Jordan boxes along the diagonal of each $\mathbf{J}_\lambda$. There are n Jordan boxes in each Jordan block, where n is the *geometric* multiplicity of λ, $n = \gamma(\lambda)$. There are 1's along the superdiagonal in each Jordan box, and 0's everywhere else in $\mathbf{J}$.

For instance, a $\mathbf{J}_\lambda$ with $\alpha(\lambda) = 3$ can have three possible geometric multiplicities of λ, giving rise to one of three different Jordan blocks:

- When $\gamma(\lambda) = 3$:

$$\mathbf{J}_\lambda = \begin{pmatrix} \boxed{\lambda} & 0 & 0 \\ 0 & \boxed{\lambda} & 0 \\ 0 & 0 & \boxed{\lambda} \end{pmatrix}$$

- When $\gamma(\lambda) = 2$:

$$\mathbf{J}_\lambda = \begin{pmatrix} \boxed{\begin{matrix} \lambda & 1 \\ 0 & \lambda \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \boxed{\lambda} \end{pmatrix}$$

- When $\gamma(\lambda) = 1$:

$$\mathbf{J}_\lambda = \begin{pmatrix} \boxed{\begin{matrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{matrix}} \end{pmatrix}$$

In some situations, the algebraic and geometric multiplicities are insufficient to fully determine $\mathbf{J}_\lambda$, but these cases are irrelevant here. Note that when $\alpha(\lambda) = \gamma(\lambda)$, the matrix $\mathbf{J}_\lambda$ is diagonal. When this holds for all λ then $\mathbf{J}$ is also diagonal which means $\mathbf{K}$ is diagonalizable.

2 Solving the equation

Given the equation:

$$\mathbf{K} = 3\mathbf{K}^2 - 2\mathbf{K}^3 \tag{1}$$

We can substitute $\mathbf{K} = \mathbf{PJP}^{-1}$:

$$\mathbf{PJP}^{-1} = 3(\mathbf{PJP}^{-1})^2 - 2(\mathbf{PJP}^{-1})^3$$

Which can be simplified as:

$$\mathbf{PJP}^{-1} = 3\mathbf{PJ}^2\mathbf{P}^{-1} - 2\mathbf{PJ}^3\mathbf{P}^{-1}$$

Since $\mathbf{P}$ is invertible, this is equivalent to:

$$\mathbf{J} = 3\mathbf{J}^2 - 2\mathbf{J}^3 \tag{2}$$

Thus for an arbitrary $\mathbf{K}$, solving Eq. (1) equates to solving Eq. (2).

Since $\mathbf{J}$ is block-diagonal, the equation in (2) can be broken down into smaller equations for each block:

$$\mathbf{J}_\lambda = 3\mathbf{J}_\lambda^2 - 2\mathbf{J}_\lambda^3$$

Which we can write as a system of equations:

$$\begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ 0 & 0 & \lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda \end{pmatrix} = 3 \begin{pmatrix} \lambda^2 & 2\lambda & 1 & \dots & 0 \\ 0 & \lambda^2 & 2\lambda & \dots & 0 \\ 0 & 0 & \lambda^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda^2 \end{pmatrix} - 2 \begin{pmatrix} \lambda^3 & 3\lambda^2 & 3\lambda & 1 & \dots & 0 \\ 0 & \lambda^3 & 3\lambda^2 & 3\lambda & \dots & 0 \\ 0 & 0 & \lambda^3 & 3\lambda^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \lambda^3 \end{pmatrix}$$

Thus, for a $(d \times d)$ Jordan block we have up to four equations per eigenvalue:

$$\lambda = 3\lambda^2 - 2\lambda^3 \tag{3}$$

$$1 = 3(2\lambda) - 2(3\lambda^2) = 6\lambda - 6\lambda^2 \qquad \text{Only when } d \geq 2. \tag{4}$$

$$0 = 3(1) - 2(3\lambda) = 3 - 6\lambda \qquad \text{Only when } d \geq 3. \tag{5}$$

$$0 = 3(0) - 2(1) = 0 - 2 \qquad \text{Only when } d \geq 4. \tag{6}$$

There are never more equations than this, since all other entries in a Jordan block must be 0.

Since Eq. (6) is a contradiction, we can have no solution which solves all Eqs. (3), (4), (5), and (6). Note also that there exists no solutions satisfying Eqs. (3), (4), and (5), nor do any solutions exist for both Eqs. (3) and (4). The following are solutions which satisfy only Eq. (3):

$$\lambda = \{0, 0.5, 1\}$$

The only situation where Eqs. (4), (5), and (6) do not arise is when $\alpha(\lambda) = \gamma(\lambda)$, which is precisely the case when $\mathbf{K}$ is diagonalizable. Therefore, any $\mathbf{K}$ which is a solution to (1) must have a Jordan Normal Form which is a diagonal matrix:

$$\mathbf{J} = \begin{pmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_3 \end{pmatrix}$$

We can therefore arrive at the following conclusions:

- $\mathbf{K}$ must be diagonalizable.
- $\mathbf{K}$ cannot have complex eigenvalues (they must be from the set $\{0, 0.5, 1\}$).
- All idempotent matrices are possible choices for $\mathbf{K}$. A matrix $\mathbf{A}$ is idempotent if and only if all its eigenvalues are either 0 or 1.
- There are non-idempotent matrices which are choices for $\mathbf{K}$. Any matrix with 0.5 as an eigenvalue (and possibly any other eigenvalues) will be a fixed point, but it will not be idempotent. This is an example:

$$\mathbf{A} = \begin{pmatrix} 0.5 & 0 \\ 0 & 1 \end{pmatrix}$$

3 Stability analysis

The previous section tells us which $\mathbf{K}$ are fixed points of $f(\mathbf{K}) = 3\mathbf{K}^2 - 2\mathbf{K}^3$, but in this section we show that not all such fixed points will be found in practice when repeatedly applying f.

Stability analysis tells us that the first-order derivative $f'(x)$ at a fixed point x_0 indicates whether that point is stable (attracting), unstable (repelling), or neutral:

- If $|f'(x_0)| < 1$, x_0 is a stable (attracting) fixed point.
- If $|f'(x_0)| > 1$, x_0 is an unstable (repelling) fixed point.
- If $|f'(x_0)| = 1$, the first-order derivative alone cannot determine stability, and higher-order derivatives must be used.

The matrix $\mathbf{K}$ is diagonalizable, meaning it can be written as $\mathbf{K} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$ where $\mathbf{\Lambda}$ is a diagonal matrix of eigenvalues and $\mathbf{V}$ is a matrix of eigenvectors. This implies that the behavior of $3\mathbf{K}^2 - 2\mathbf{K}^3$ is directly governed by the behavior of $3\mathbf{\Lambda}^2 - 2\mathbf{\Lambda}^3$. We can therefore analyze the stability of a potential solution $\mathbf{M}$ by analyzing the stability of its eigenvalues under the same transformation.

Consider the first-order derivative of $f(\lambda) = 3\lambda^2 - 2\lambda^3$:

$$f'(\lambda) = 6\lambda - 6\lambda^2$$

Recall from the previous section that eigenvalues of $\mathbf{K}$ must be drawn from the set

$$\lambda = \{0, 0.5, 1\}$$

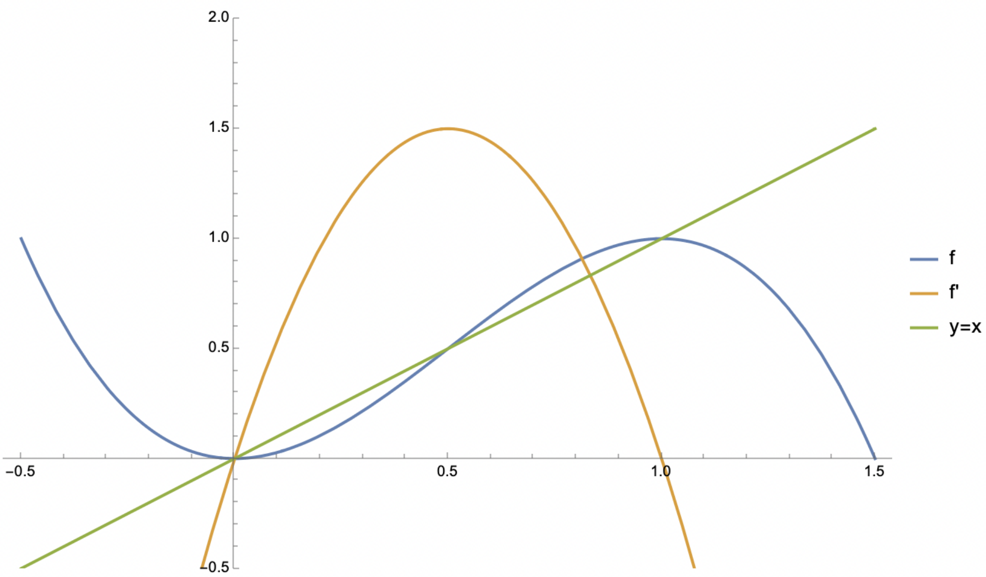
We then evaluate f' at these points:

$$f'(0) = 0, \quad f'(0.5) = 1.5, \quad f'(1) = 0$$

which shows that eigenvalues $\lambda = 0$ and $\lambda = 1$ are attracting points, while $\lambda = 0.5$ is a repelling point. We can therefore conclude that any $\mathbf{K}$ with only 0 and/or 1 as eigenvalue is a stable/attracting fixed point to Eq. (1), while any $\mathbf{K}$ with an eigenvalue of 0.5 must be an unstable/repellant point to Eq. (1).

We can therefore conclude that there exist some non-idempotent fixed point solutions to Eq. (1), but only those $\mathbf{K}$ which are idempotent are attracting solutions.

We may also plot f, its first order derivative f', and the line $y = x$ to gain some intuition:



If $f(x) > x$, then the point $(x, f(x))$ is above the line $y = x$, which means it is increasing when applied repeatedly. Similarly, if $f(x) < x$, then $(x, f(x))$ is below the line $y = x$ which means it is decreasing when applied repeatedly.

4 Some empirical results

Due to the above, it is unsurprising to see that the matrix

$$\mathbf{A} = \begin{pmatrix} 0.5 & 0 \\ 0 & 1 \end{pmatrix} \tag{7}$$

is a fixed point to the equation. This is easily seen in this picture (no noise is added):

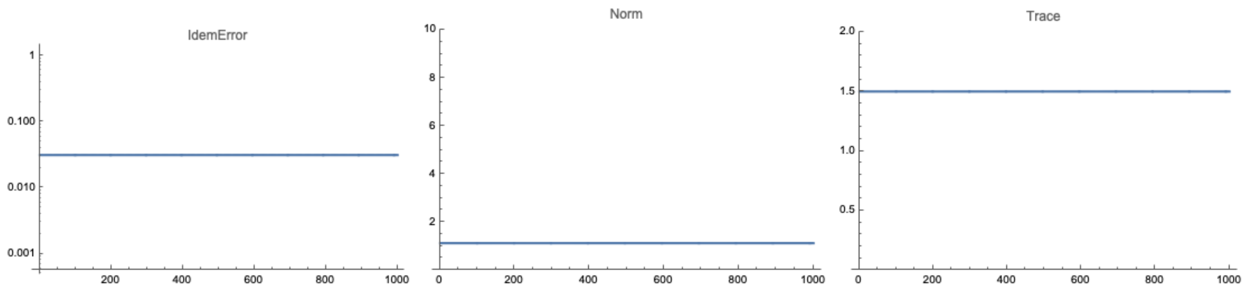
```
m = {{0.5, 0}, {0, 1}};
inp = m + 0*RandomReal[{-1, 1}, Dimensions[m]];
data = TestUpdateRule[inp, R1, 1000];
out = Last[data];

Print[Row[{"Original: ", MatrixForm[m], ", Eigenvalues: ", Eigenvalues[m]}]];
Print[Row[{"Noisy input: ", MatrixForm[inp], ", Eigenvalues: ", Eigenvalues[inp]}]];
Print[Row[{"Output: ", MatrixForm[out], ", Eigenvalues: ", Eigenvalues[out]}]];
PlotError[data, {0, 10}]
```

Original: $\begin{pmatrix} 0.5 & 0 \\ 0 & 1 \end{pmatrix}$, Eigenvalues: {1., 0.5}

Noisy input: $\begin{pmatrix} 0.5 & 0. \\ 0. & 1. \end{pmatrix}$, Eigenvalues: {1., 0.5}

Output: $\begin{pmatrix} 0.5 & 0. \\ 0. & 1. \end{pmatrix}$, Eigenvalues: {1., 0.5}



We can visualize the repellant nature of the matrix in (7). Here I add a tiny bit of random noise, but the matrix $\mathbf{A}$ is not recovered:

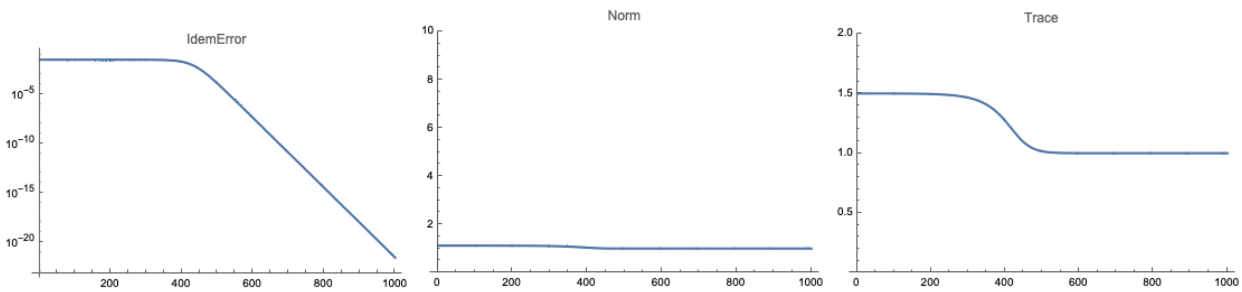
```
m = {{0.5, 0}, {0, 1}};
inp = m + 0.0001*RandomReal[{-1, 1}, Dimensions[m]];
data = TestUpdateRule[inp, R1, 1000];
out = Last[data];

Print[Row[{"Original: ", MatrixForm[m], ", Eigenvalues: ", Eigenvalues[m]}]];
Print[Row[{"Noisy input: ", MatrixForm[inp], ", Eigenvalues: ", Eigenvalues[inp]}]];
Print[Row[{"Output: ", MatrixForm[out], ", Eigenvalues: ", Eigenvalues[out]}]];
PlotError[data, {0, 10}]
```

Original: $\begin{pmatrix} 0.5 & 0 \\ 0 & 1 \end{pmatrix}$, Eigenvalues: {1., 0.5}

Noisy input: $\begin{pmatrix} 0.499906 & -0.0000564943 \\ 0.0000893316 & 0.999966 \end{pmatrix}$, Eigenvalues: {0.999966, 0.499906}

Output: $\begin{pmatrix} -2.01597\times 10^{-8} & -0.000112975 \\ 0.000178642 & 1. \end{pmatrix}$, Eigenvalues: $\{1., 2.23197\times 10^{-11}\}$



For a different repelling example, consider this (4×4) matrix:

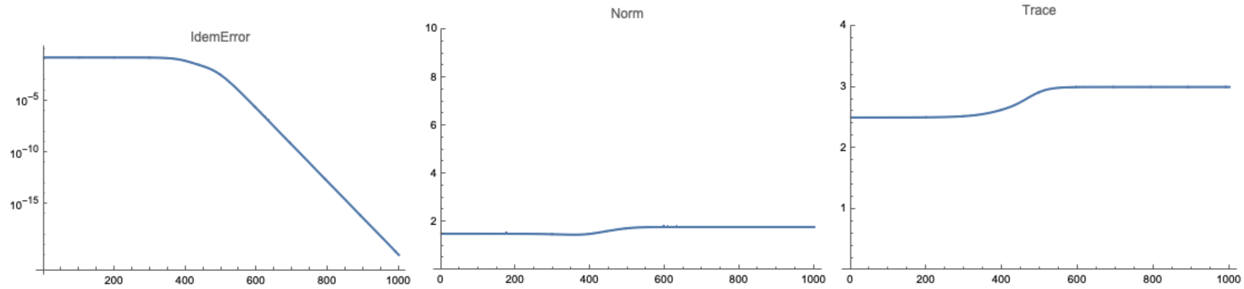
```
m = randomMatrixWithEigenvalues[{1, 0.5, 0.5, 0.5}];
inp = m + 0.0001*RandomReal[{-1, 1}, Dimensions[m]];
data = TestUpdateRule[inp, R1, 1000];
out = Last[data];

Print[Row[{"Original: ", MatrixForm[m], ", Eigenvalues: ", Eigenvalues[m]}]];
Print[Row[{"Noisy input: ", MatrixForm[inp], ", Eigenvalues: ", Eigenvalues[inp]}]];
Print[Row[{"Output: ", MatrixForm[out], ", Eigenvalues: ", Eigenvalues[out]}]];
PlotError[data, {0, 10}]
```

Original: $\begin{pmatrix} 0.625 & 0.125 & 0.125 & 0.125 \\ 0.375 & 0.875 & 0.375 & 0.375 \\ -0.125 & -0.125 & 0.375 & -0.125 \\ 0.125 & 0.125 & 0.125 & 0.625 \end{pmatrix}$, Eigenvalues: $\{1., 0.5, 0.5, 0.5\}$

Noisy input: $\begin{pmatrix} 0.625025 & 0.125064 & 0.125071 & 0.124951 \\ 0.375093 & 0.874907 & 0.374994 & 0.374918 \\ -0.12498 & -0.125048 & 0.375086 & -0.124942 \\ 0.12495 & 0.125026 & 0.12502 & 0.624952 \end{pmatrix}$, Eigenvalues: $\{0.999911, 0.500158, 0.500038, 0.499863\}$

Output: $\begin{pmatrix} 0.622251 & 0.236769 & 0.0472422 & -0.285127 \\ 0.518682 & 0.674896 & -0.0648676 & 0.391504 \\ 0.216781 & -0.135876 & 0.972889 & 0.163628 \\ -0.357755 & 0.224237 & 0.0447417 & 0.729965 \end{pmatrix}$, Eigenvalues: $\{1., 1., 1., 1.03457 \times 10^{-11}\}$



Experiments show that any amount of noise great enough to offset **A** by machine precision is enough to cause the algorithm to find an alternative solution.