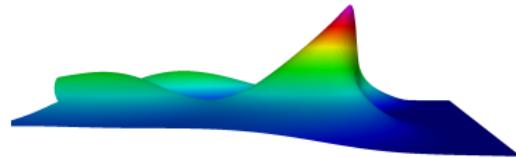


The mathematics behind nonlinear sound waves

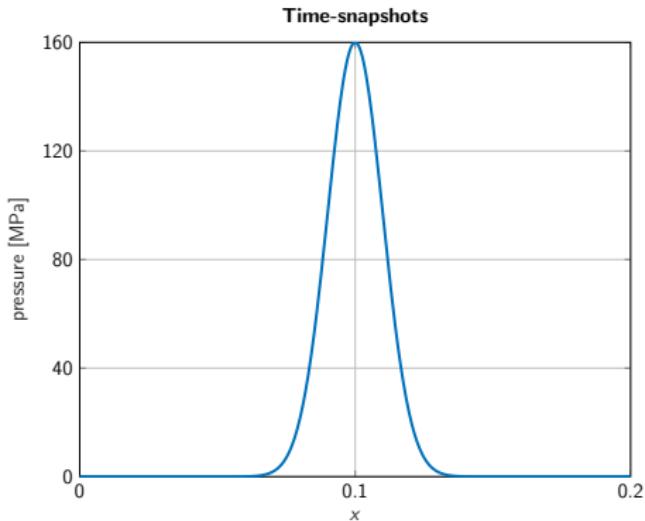
Vanja Nikolić

Radboud University

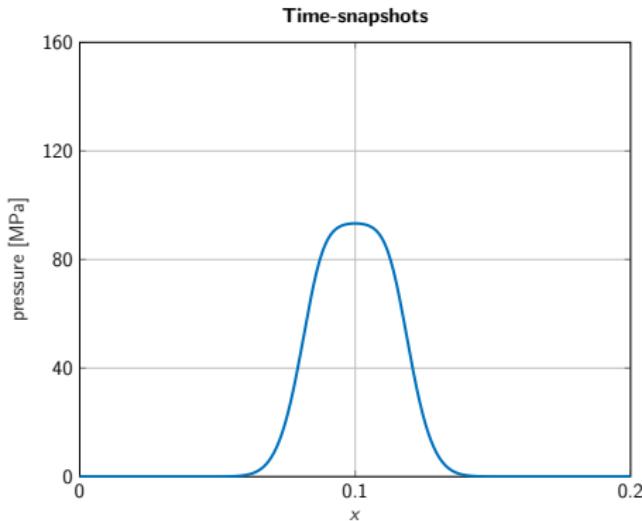


VU Amsterdam General Mathematics Colloquium, March 2021

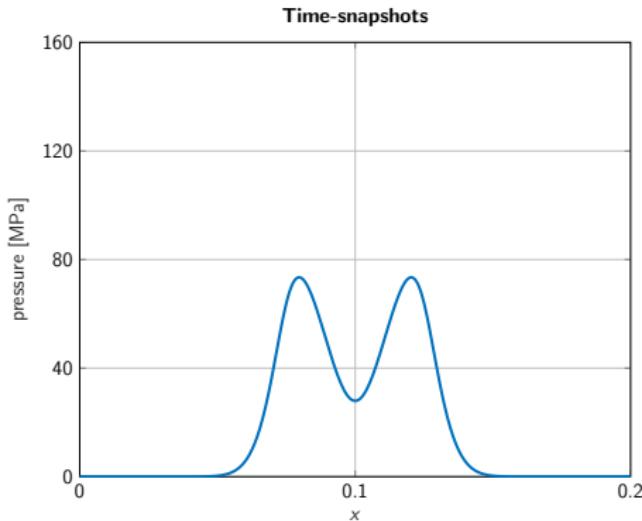
Nonlinear sound propagation



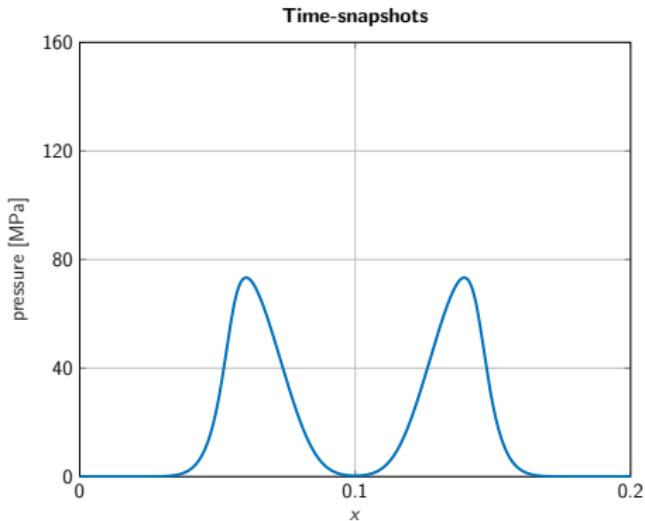
Nonlinear sound propagation



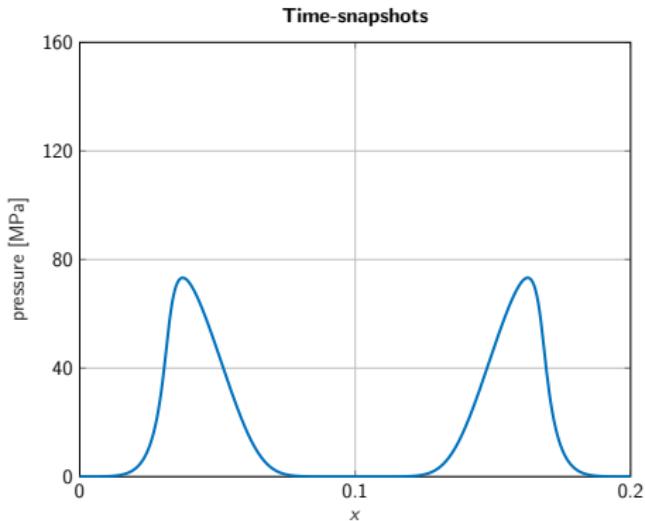
Nonlinear sound propagation



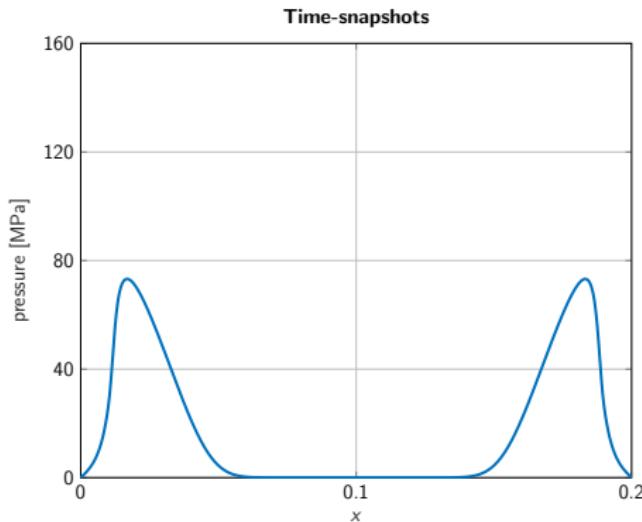
Nonlinear sound propagation



Nonlinear sound propagation



Nonlinear sound propagation

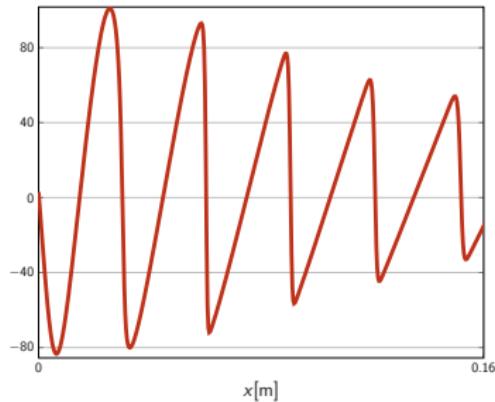
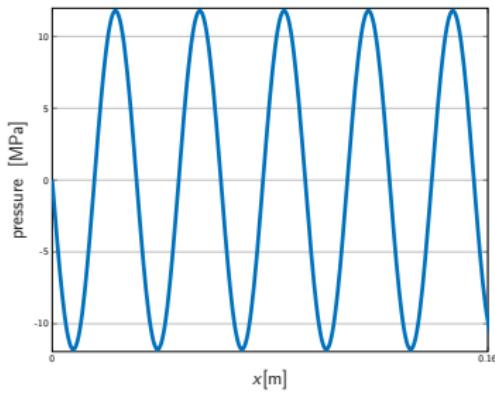


1 Nonlinear acoustics

2 Mathematical analysis in nonlinear acoustics

3 Numerical approximation

Linear vs. nonlinear waves



- Differences are pronounced with **high amplitude-to-wavelength ratio**

Possible applications

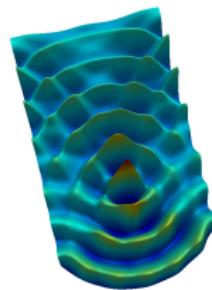
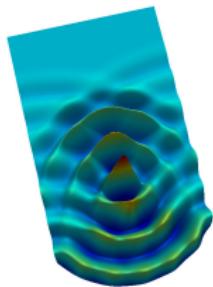
- High-Intensity Focused Ultrasound (**HIFU**)

Possible applications

- High-Intensity Focused Ultrasound (**HIFU**)

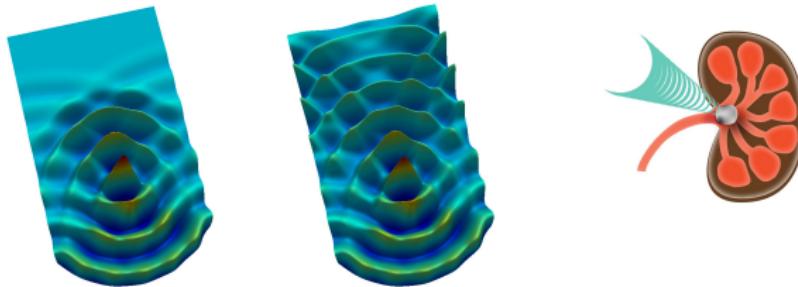
Possible applications

- Shock-wave lithotripsy



Possible applications

- Shock-wave lithotripsy



- Cancer treatments (under research)

A selection of references

- Modeling

[Westervelt 1963], [Blackstock 1963], [Kuznetsov 1970], [Szabo 1993], [Jordan 2008],
[Prieur, Holm 2011], [Christov, Christov, Jordan 2015], ...

A selection of references

- Modeling

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[Hughes, Kato, Marsden 1977], [Kawashima, Shibata 1992], [Mizohata, Ukai 1993],
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Schnaubelt 2015], ...

- Numerical analysis

[Kaltenbacher, N., Thalhammer 2015], [N., Wohlmuth 2019], [Antonietti, Mazzieri, Muhr,
N., Wohlmuth 2020], [Maier 2020], ...

Mathematical models in acoustics

Wave equation

$$u_{tt} - c^2 \Delta u = 0$$

Mathematical models in acoustics

Wave equation

$$u_{tt} - c^2 \Delta u - (\text{damping}) = (\text{nonlinear effects})$$

Mathematical models in acoustics

Westervelt's equation

$$u_{tt} - c^2 \Delta u - b \Delta u_t = (k u^2)_{tt}$$

Mathematical models in acoustics

↑ local nonlinear effects, relaxing/heterogeneous media, ...

Westervelt's equation

$$u_{tt} - c^2 \Delta u - b \Delta u_t = (k u^2)_{tt}$$

↓ restricted propagation direction

Mathematical models in acoustics

Westervelt's equation

$$u_{tt} - c^2 \Delta u - b \Delta u_t = (k u^2)_{tt}$$

$c > 0$ speed of sound, $b \geq 0$ sound diffusivity,

$k = \beta_a / (\rho c^2)$, β_a coefficient of nonlinearity, ρ mass density

Physical background

Thermoviscous Navier–Stokes–Fourier system

- ① Conservation of mass, momentum
- ② Temperature law, Entropy production equation
- ③ Nonlinear pressure-density relation

Physical background

Thermoviscous Navier–Stokes–Fourier system

- ➊ Conservation of mass, momentum
 - ➋ Temperature law, Entropy production equation
 - ➌ Nonlinear pressure-density relation
-
- ➍ Weakly-nonlinear theory
 - ~ The governing system is approximated by one equation

More general models

- The Kuznetsov equation

$$\psi_{tt} - c^2 \Delta \psi - b \Delta \psi_t = (\kappa \psi_t^2 + |\nabla \psi|^2)_t$$

where $u = \varrho \psi_t$

- Incorporates local nonlinear effects

More general models

- The Jordan–Moore–Gibson–Thompson equation

$$\tau \psi_{ttt} + \psi_{tt} - c^2 \Delta \psi - (b + \tau c^2) \Delta \psi_t = (\kappa(\psi_t)^2 + |\nabla \psi|^2)_t$$

- Incorporates thermal relaxation

 1 Nonlinear acoustics

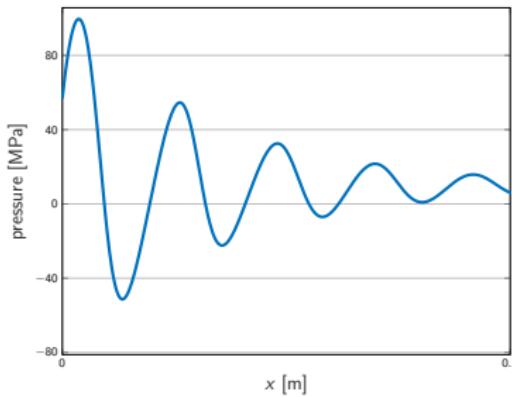
 2 Mathematical analysis in nonlinear acoustics

 3 Numerical approximation

Analysis of nonlinear acoustic models

$$u_{tt} - c^2 \Delta u - b \Delta u_t = (k u^2)_{tt}$$

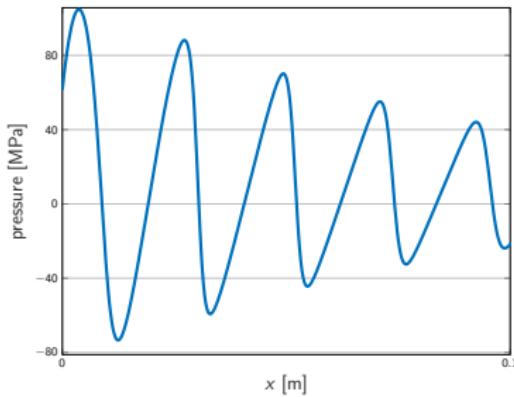
- Influence of the damping $b = 1 \text{ m}^2/\text{s}$



Analysis of nonlinear acoustic models

$$u_{tt} - c^2 \Delta u - b \Delta u_t = (k u^2)_{tt}$$

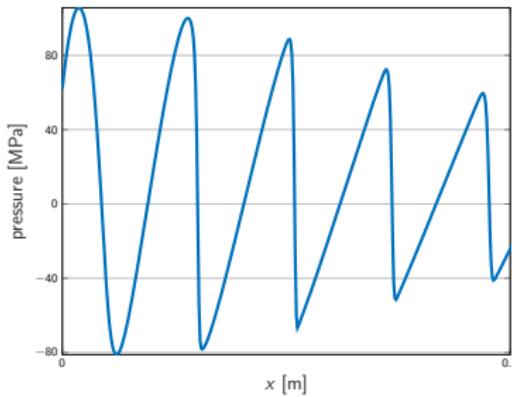
- Influence of the damping $b = 0.1 \text{ m}^2/\text{s}$



Analysis of nonlinear acoustic models

$$u_{tt} - c^2 \Delta u - b \Delta u_t = (k u^2)_{tt}$$

- Influence of the damping $b = 0 \text{ m}^2/\text{s}$



Limiting behavior

- Sound diffusivity b is in practice relatively **small**
- **Q:** Can we characterize the behavior of sound waves as $b \rightarrow 0$?

Challenges in the analysis

Westervelt's equation

$$u_{tt} - c^2 \Delta u - b \Delta u_t = (k u^2)_{tt}$$

Challenges in the analysis

Westervelt's equation

$$u_{tt} - c^2 \Delta u - b \Delta u_t = 2k u_t^2 + 2k u u_{tt}$$

Challenges in the analysis

Westervelt's equation

$$(1 - 2ku)u_{tt} - c^2 \Delta u - b\Delta u_t = 2ku_t^2$$

Challenges in the analysis

Westervelt's equation

$$(1 - 2ku)u_{tt} - c^2 \Delta u - b\Delta u_t = 2ku_t^2$$

- Quasi-linear wave equation
- Degenerates if $1 - 2ku = 0$
 - ~~~ The pressure needs to be below $1/(2k)$

Limiting behavior

- Wave energy at time t

$$\|u\|_E^2 = \frac{1}{2} \int_{\Omega} |u_t|^2 \, dx + \frac{c^2}{2} \int_{\Omega} |\nabla u|^2 \, dx$$

Limiting behavior

- Wave energy at time t

$$\|u\|_E^2 = \frac{1}{2} \int_{\Omega} |u_t|^2 \, dx + \frac{c^2}{2} \int_{\Omega} |\nabla u|^2 \, dx$$

- How fast does u converge in the energy norm as $b \rightarrow 0$?

Limiting behavior

- Convergence in the energy norm

$$\sup_{t \in (0, T)} \|u^{(b)}(t) - u^{(0)}(t)\|_E \lesssim b$$

[Kaltenbacher & N. 2020]

- Assumptions:** Smooth and small data, short final time T
- Smallness comes from ensuring that pressure stays below $1/2k$

Limiting behavior of more general models

- The Jordan–More–Gibson–Thompson equation

$$\tau\psi_{ttt} + \psi_{tt} - c^2\Delta\psi - (b + \tau c^2)\Delta\psi_t = f(\psi_t, \psi_{tt}, \nabla\psi, \nabla\psi_t)$$

- Perturbation of the wave speed for

$$z = \tau\psi_t + \psi$$

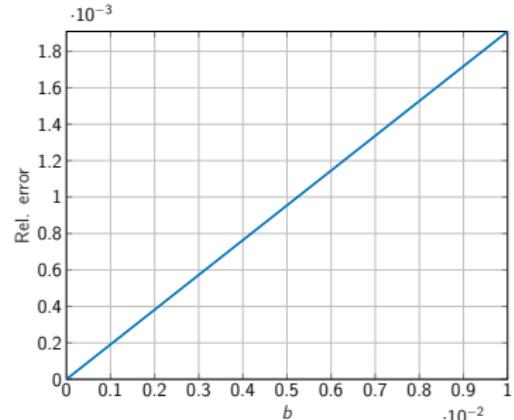
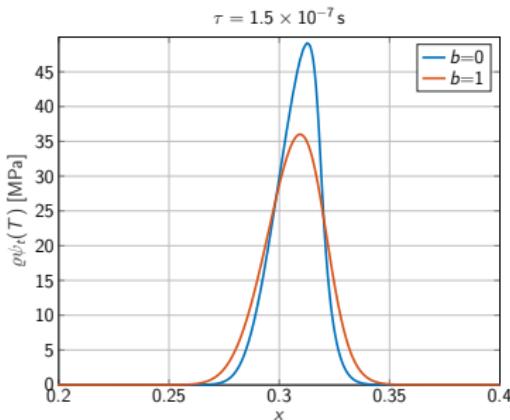
- Compared to Westervelt's equation: Weaker regularity assumptions

Limiting behavior of more general models

- Convergence in the energy norm for the JMGT equation

$$\sup_{t \in (0, T)} \|\psi^{(b)}(t) - \psi^{(0)}(t)\|_E \lesssim b$$

[Kaltenbacher & N. 2021]



1 Nonlinear acoustics

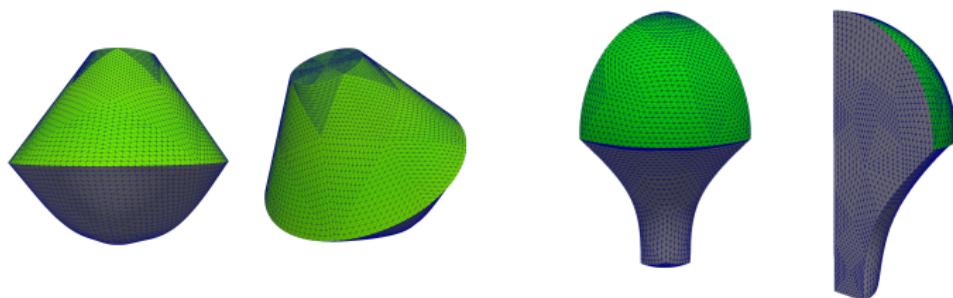
2 Mathematical analysis in nonlinear acoustics

3 Numerical approximation

Numerical approximation

Semi-discrete acoustic models

$$u_{h,tt} - c^2 \Delta_h u_h - (\text{damping})_h = (\text{nonlinear effects})_h$$



Examples of computational domains

Numerical analysis

- ① Does an approximate solution u_h exist?
- ② Is u_h a stable and accurate representation of u ?

Numerical analysis

- ➊ Does an approximate solution u_h exist?
- ➋ Is u_h a stable and accurate representation of u ?
- ➌ How fast does u_h converge as $h \rightarrow 0$?

Challenges

- Approximate pressure is **not very smooth** in general

Challenges

- Approximate pressure is **not very smooth** in general
 - ~~> The arguments from the continuous setting cannot be transferred
- The approximate pressure needs to be **below $1/(2k)$** as well

Strategy: A fixed-point argument

- Linearized problem

$$(1 - 2k \mathbf{w}_h) \mathbf{u}_{h,tt} - c^2 \Delta_h \mathbf{u}_h - b \Delta_h \mathbf{u}_{h,t} = 2k \mathbf{w}_{h,t} \mathbf{u}_{h,t}$$

Define the mapping $\mathcal{F} : \mathbf{w}_h \mapsto \mathbf{u}_h$, where

- \mathbf{w}_h in an h neighborhood of u
- \mathbf{u}_h solves the linearized problem

The **fixed point** $\mathbf{w}_h = \mathbf{u}_h$ solves the nonlinear problem.

Numerical analysis

- A priori error estimate

$$\|u - u_h\|_E \lesssim h^p$$

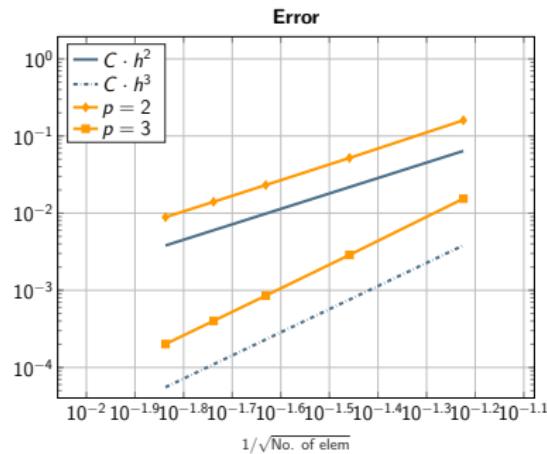
[N. & Wohlmuth 2019], [Antonietti, Mazzieri, Muhr, N., & Wohlmuth 2020]

- Assumptions: Small and smooth data and small enough h

Numerical analysis

- Westervelt's equation with a given source term

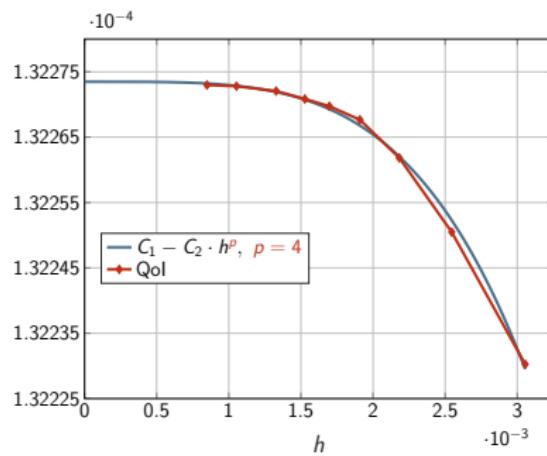
~~~ Exact solution known



## Numerical analysis

- Westervelt's equation with a sinusoidal boundary excitation

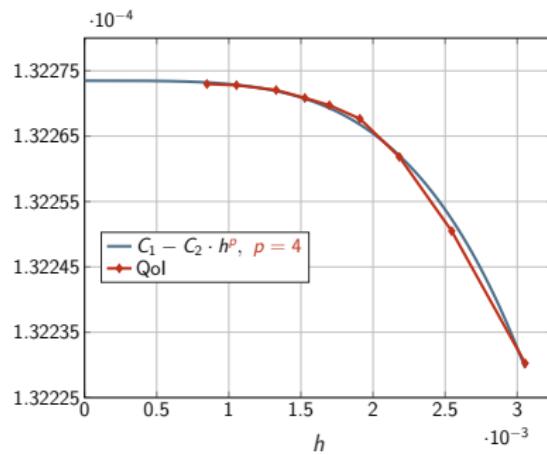
~~~ Unknown solution



Numerical analysis

- Westervelt's equation with a sinusoidal boundary excitation

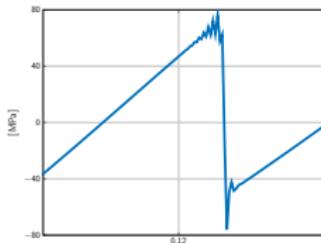
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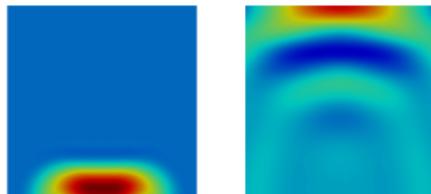
The degeneracy bound  $1/(2k) \approx 214$  MPa

## About numerical simulations

- Computationally expensive

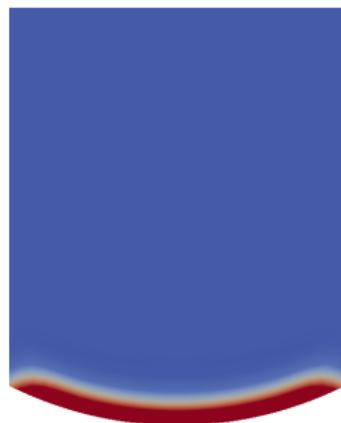
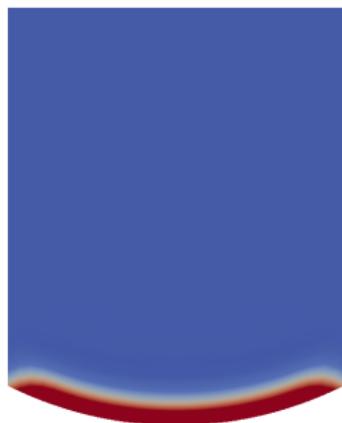


- Non-physical oscillations around peaks

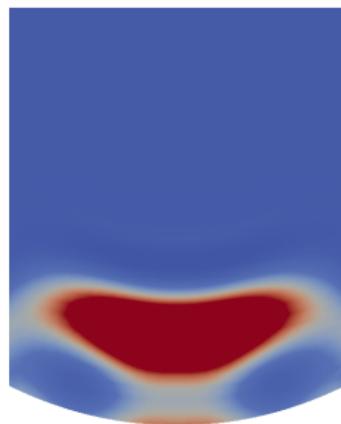
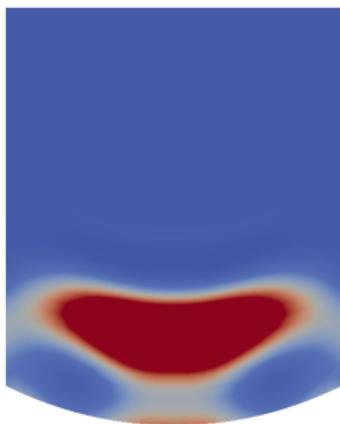


- Unwanted reflections

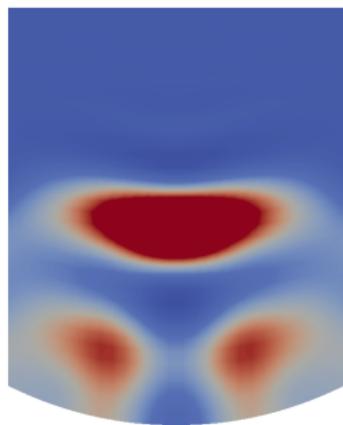
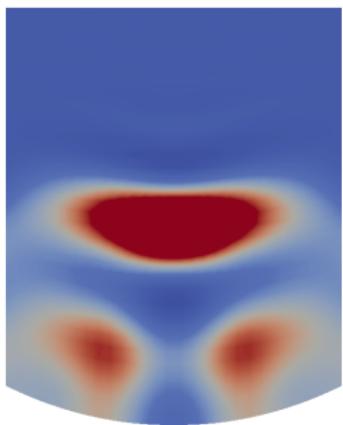
## Reflective vs. absorbing boundaries



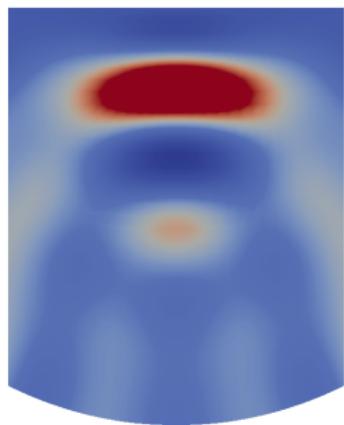
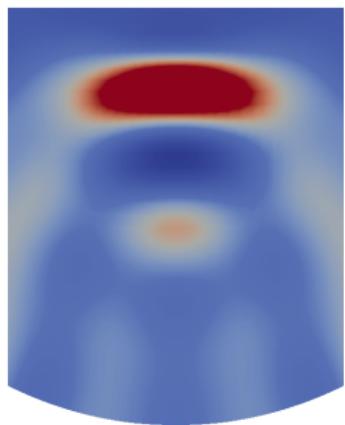
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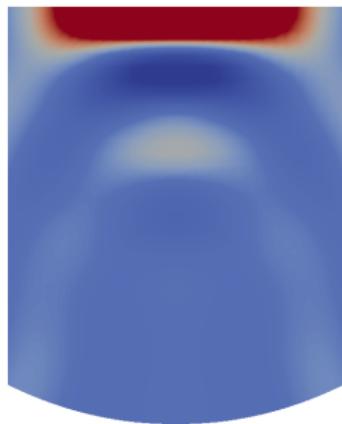
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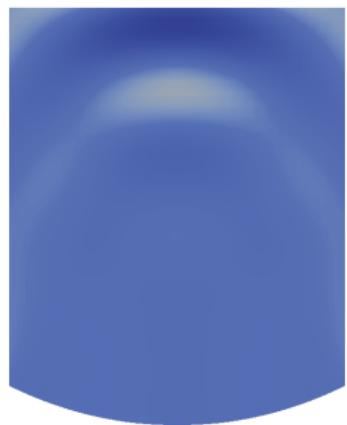
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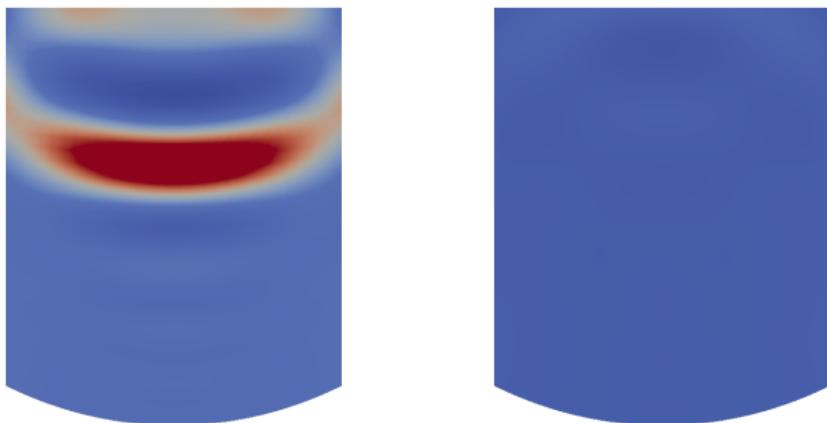
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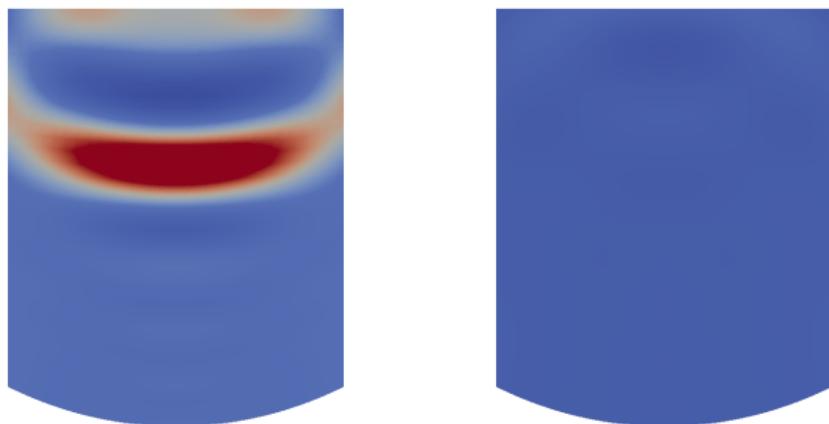


## Reflective vs. absorbing boundaries



Unsuitable conditions  $\rightsquigarrow$  Pollution of the pressure field

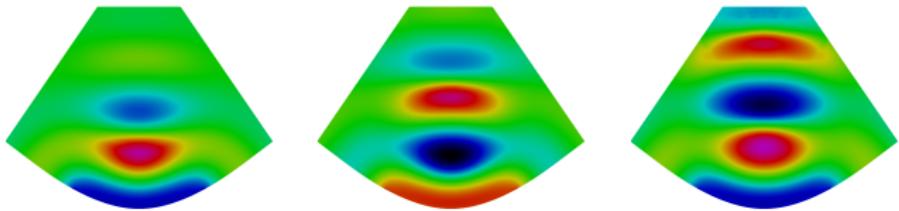
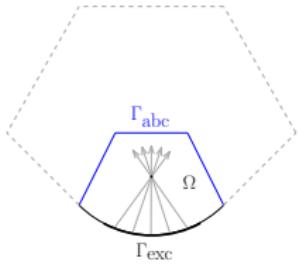
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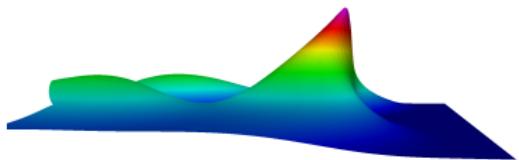
A solution: Problem-tailored absorbing conditions

[Shevchenko, Kaltenbacher 2015], [Muhr, N., Wohlmuth 2019]

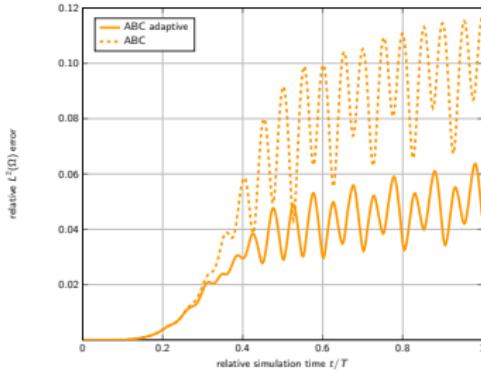
## Combined with adaptivity



- Focusing



- Improvement in the relative error: 4% vs. 8%



## Summary

- PDE-based analysis & simulation offer a way of better understanding (nonlinear) ultrasonic waves

## Further topics

- Coupling strategies (elasto-acoustic)
- Propagation in general lossy media
- Manipulation of sound waves



## Based on

- V. Nikolić and B. Wohlmuth,  
[A priori error estimates for the finite element approximation of Westervelt's quasi-linear acoustic wave equation](#),  
SIAM J. Numer. Anal., 2019.
- P. F. Antonietti, I. Mazzieri, M. Muhr, V. Nikolić and B. Wohlmuth,  
[A high-order discontinuous Galerkin method for nonlinear sound waves](#),  
J. Comput. Phys., 2020.
- B. Kaltenbacher and V. Nikolić,  
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