Counter-Example Guided Training of Neural Network Controllers - Appendix

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Abstract

In this Appendix, we provide the proof of Proposition 2 in the submission entitled "Counter-Example Guided Training of Neural Network Controllers" (Track Name: NeurIPS 2020, Paper ID: 9059).

1 Proof of Proposition 2

We first recall the proposition. We consider a plant Π with dynamics:

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)), \\ y(t) &= w(x(t)). \end{aligned} \tag{1}$$

The control input u is piece-wise constant and computed by a discrete-time controller implemented as a neural net $\mathcal{N}$. The neural net is trained using a set of desired closed-loop behaviours. Note that in our setting, the desired closed-loop behaviours are assumed to be produced by a nominal controller or by combining a number of (local) nominal controllers. Proposition 2 shows that it is possible to train the neural net so as it reproduces a given desired behaviour as closely as possible. More formally, we can restate Proposition 2 as follows.

Proposition 2. *Let $\mathcal{N}$ be a neural net trained using the data set $\mathcal{D}$. Let $\mathcal{D}' = \mathcal{D} \cup d_{\gamma_s}$ where d_{γ_s} be data corresponding to a new desired closed-loop behaviour γ_s under a control setting s. Let $\mathcal{N}'$ be the new net trained using $\mathcal{D}'$ and γ_s' be the behaviour of $(\mathcal{N}'||\Pi)$ under the control setting s. If the training error of $\mathcal{N}'$ is bounded by ϵ, then there exists bounded η depending on ϵ such that $\mu(\gamma_s, \gamma_s') \leq \eta$.*

Proof. Let T_h be the horizon of closed-loop behaviours under consideration. The distance metric between γ_s, γ_s' is

$$\mu(\gamma_s, \gamma_s') = sup_{0 \leq t \leq T_h} |\gamma_s(t) - \gamma_s'(t)|$$

where $|\cdot|$ denotes the infinity norm and sup the supremum.

Let f be L_u-Lipschitz with respect to u and L_x-Lipschitz with respect to x. It then follows that

$$|f(x,u) - f(x,u')| \leq L_u|u - u'|.$$

The above inequality gives an upper bound on the deviation in the dynamics of the plants induced by a deviation in the control input. Note that by the definition of the neural net training error, under the same control setting s, the distance between the control input produced by the newly retrained neural net and the control input of the desired behaviour is bounded by ϵ. Now to determine the distance between x_s and x_s' which are the solutions to (1) under the same control setting s but different control input signals u and u' respectively, we use the fundamental inequality in the theory of differential equations (see [1]) which bounds the distance between the solutions of two differential equations under a bounded difference in the initial conditions and a bounded difference in the derivatives. A direct application of this classic result gives:

$$\forall t \in [0, T_h] \ |x_s(t) - x'_s(t)| \leq \frac{\epsilon L_u}{L_x}(e^{L_x t} - 1).$$

The term ϵL_u corresponds to an upper bound on the difference in the derivatives. The right-hand side of the above inequality gives the bound η (stated in the proposition) on the distance between the behaviour of the plant with the neural net controller and the corresponding desired behaviour . $\square$

References

[1] J. Dieudonné, *Calcul Infinitésimal*. Hermann, Paris, 1968.